Preface to the Second Edition

The original edition of this book was written to be accessible to sophomore-level undergraduates, who had only seen one year of a standard calculus sequence. In particular, for most of the text, no prior exposure to multivariable calculus, vector calculus, or linear algebra was assumed. After receiving lots of reader feedback to the first edition, it became clear that the people reading this book were more advanced. Hence, for the second edition, most of the material on basic topics such as partial derivatives and multiple integrals has been removed, and more advanced applications of differential forms have been added.

The largest of the new additions is a chapter containing an introduction to differential geometry, based on the machinery of differential forms. At all times, we have made an effort to be consistent with the rest of the text here, so that the material is presented in $\mathbb{R}^3$ for concreteness, but all definitions are formulated to easily generalize to arbitrary dimensions. This is perhaps a unique approach to this material.

Other smaller additions worth noting include new sections on *linking number* and the *Hopf Invariant*. These join the host of brief applications of differential forms which originally appeared in the first edition, including *Maxwell's Equations*, *DeRham Cohomology*, *foliations*, *contact structures*, and the *Godbillon–Vey Invariant*. While the treatment given here of these topics is far from exhaustive, we feel it is important to include these teasers to give the reader some hint at the usefulness of differential forms.

Finally, other changes worth noting are a rearrangement of some of the original sections for clarity, and the addition of several new problems and examples.

Claremont, CA   *David Bachman*
January, 2011

Preface to the First Edition[1]

The present work is not meant to contain any new material about differential forms. There are many good books out there which give complete treatments of the subject. Rather, the goal here is to make the topic of differential forms accessible to the sophomore-level undergraduate while still providing material that will be of interest to more advanced students.

There are three tracks through this text. The first is a course in *Multivariable Calculus*, suitable for the third semester in a standard calculus sequence. The second track is a sophomore-level *Vector Calculus* class. The last track is for advanced undergraduates, or even beginning graduate students. At many institutions, a course in linear algebra is not a prerequisite for either multivariable calculus or vector calculus. Consequently, this book has been written so that the earlier chapters do not require many concepts from linear algebra. What little is needed is covered in the first section.

The book begins with basic concepts from multivariable calculus such as partial derivatives, gradients and multiple integrals. All of these topics are introduced in an informal, pictorial way to quickly get students to the point where they can do basic calculations and understand what they mean. The second chapter focuses on parameterizations of curves, surfaces and three-dimensional regions. We spend considerable time here developing tools which will help students find parameterizations on their own, as this is a common stumbling block.

Chapter 1 is purely motivational. It is included to help students understand why differential forms arise naturally when integrating over parameterized domains.

The heart of this text is Chapters 3 through 6. In these chapters, the entire machinery of differential forms is developed from a geometric standpoint. New ideas are always introduced with a picture. Verbal descriptions of geometric actions are set out in boxes.

[1] Chapter and Section references have been updated to conform to the present edition.

Chapter 6 focuses on the development of the generalized Stokes' Theorem. This is really the centerpiece of the text. Everything that precedes it is there for the sole purpose of its development. Everything that follows is an application. The equation is simple:

$$\int_{\partial C} \omega = \int_{C} d\omega.$$

Yet it implies, for example, all integral theorems of classical vector analysis. Its simplicity is precisely why it is easier for students to understand and remember than these classical results.

Chapter 6 concludes with a discussion on how to recover all of vector calculus from the generalized Stokes' Theorem. By the time students get through this they tend to be more proficient at vector integration than after traditional classes in vector calculus. Perhaps this will allay some of the concerns many will have in adopting this textbook for traditional classes.

Chapter ∗∗[2] contains further applications of differential forms. These include Maxwell's Equations and an introduction to the theory of foliations and contact structures. This material should be accessible to anyone who has worked through Chapter 6.

Chapter 7 is intended for advanced undergraduate and beginning graduate students. It focuses on generalizing the theory of differential forms to the setting of abstract manifolds. The final section contains a brief introduction to DeRham Cohomology.

We now describe the three primary tracks through this text.

Track 1. Multivariable Calculus (Calculus III).[3] For such a course, one should focus on the definitions of n-forms on $\mathbb{R}^m$, where n and m are at most 3. The following chapters/sections are suggested:

- Chapter 2, perhaps supplementing Section 2.2 with additional material on max/min problems,
- Chapter ∗∗[4],
- Chapter 3, excluding Sections 3.4 and 3.5 due to time constraints,
- Chapters 4–6,[5]
- Appendix A.[6]

Track 2. Vector Calculus. In this course, one should mention that for n-forms on $\mathbb{R}^m$, the numbers n and m could be anything, although in practice

[2] This material has been added to the end of Chapter 5 in the present edition.

[3] The present edition may not be quite as suitable for such a course as the first edition was.

[4] Material has been removed in the present edition.

[5] The material in Sections 5.6, 5.7 and 6.4 were not originally included in these chapters.

[6] This is now Section 4.8.

it is difficult to work examples when either is bigger than 4. The following chapters/sections are suggested:

- Section $**$ (unless Linear Algebra is a prerequisite),[7]
- Chapter 1 (one lecture),
- Chapter 2,
- Chapters 3–6,
- Sections 6.4 through 5.7, as time permits.

Track 3. Upper Division Course. Students should have had linear algebra, and perhaps even basic courses in group theory and topology.

- Chapter 1 (perhaps as a reading assignment),
- Chapters 3–6 (relatively quickly),
- Sections 6.4 through 5.7 and Chapter 7.

The original motivation for this book came from [GP10], the text I learned differential topology from as a graduate student. In that text, differential forms are defined in a highly algebraic manner, which left me craving something more intuitiuve. In searching for a more geometric interpretation I came across Chapter 7 of Arnold's text on classical mechanics [Arn97], where there is a wonderful introduction to differential forms given from a geometric viewpoint. In some sense, the present work is an expansion of the presentation given there. Hubbard and Hubbard's text [HH99] was also a helpful reference during the preparation of this manuscript.

The writing of this book began with a set of lecture notes from an introductory course on differential forms, given at Portland State University, during the summer of 2000. The notes were then revised for subsequent courses on multivariable calculus and vector calculus at California Polytechnic State University, San Luis Obispo and Pitzer College.

I thank several people. First and foremost, I am grateful to all those students who survived the earlier versions of this book. I would also like to thank several of my colleagues for giving me helpful comments. Most notably, Don Hartig, Matthew White and Jim Hoste had several comments after using earlier versions of this text for vector or multivariable calculus courses. John Etnyre and Danny Calegari gave me feedback regarding Section 5.6 and Saul Schleimer suggested Example 27. Other helpful suggestions were provided by Ryan Derby–Talbot. Alvin Bachman suggested some of the formatting of the text. Finally, the idea to write this text came from conversations with Robert Ghrist while I was a graduate student at the University of Texas at Austin.

Claremont, CA *David Bachman*
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[7] Material has been removed in the present edition.

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