

Chapter 2

Numerical Aspects of Time and Band Limiting

This chapter is concerned with the role of the prolates in numerical analysis—particularly their approximation properties and application in the numerical solution of differential equations. The utility of the prolates in these contexts is due principally to the fact that they form a *Markov system* (see Defn. 2.1.6) of functions on $[-1, 1]$, a property that stems from their status as eigenfunctions of the differential operator $\mathcal{P}$ of (1.6), and allows the full force of the Sturm–Liouville theory to be applied. The Markov property immediately gives the orthogonality of the prolates on $[-1, 1]$ (previously observed in Sect. 1.2 as the *double orthogonality* property) and also a remarkable collection of results regarding the zeros of the prolates as well as quadrature properties that are central to applications in numerical analysis.

In Sect. 2.1 we give an account of the classical theory on which so many modern developments hinge. In particular, the theory of *oscillating kernels* is explored, a key result being that the eigenfunctions of such kernels form Markov systems. Much of this material is adapted from the book of Gantmacher and Krein [109]. Properties of the differential operator $\mathcal{P}$ of (1.6) are investigated. It is shown that $\mathcal{P}$ is inverted by an integral operator with oscillating kernel, from which we conclude that the eigenfunctions of $\mathcal{P}$ (i.e., the prolates) form a Markov system on $[-1, 1]$.

Section 2.2 is devoted to the quadrature properties of the prolates, which are inherited from the Markov property. It is shown how generalized Gaussian quadratures for the prolates may be effectively parlayed into approximate quadrature results for band-limited functions. Of particular interest is the remarkable result of Xiao, Rokhlin, and Yarvin [364] in which the Euclidean division algorithm for polynomials is generalized to band-limited functions, where the role of the degree of the polynomial is played by the function’s band limit. As a corollary we give an alternative approximate quadrature result for band-limited functions in which the nodes are the zeros of the N th prolate rather than generalized Gaussian quadrature nodes, whose construction is cumbersome and achieved only through the approximate solution of a system of nonlinear equations.

In Sect. 2.3 we give approximate local sampling expansions of band-limited functions as corollaries of the quadrature results of the previous section. When applying the classical Shannon theorem (see Theorem 5.1.3), local approximations of a band-

limited function may be obtained from uniformly spaced samples. The slow decay of the sinc function, however, means that many such samples may need to be taken (far from the region in which the approximation is required) to achieve reasonable fidelity. In contradistinction to this situation, the local sampling results of this section achieve local representations with high fidelity using only a few samples taken on the interval of interest.

The use of prolates as spectral elements in numerical schemes for the solution of differential equations is the subject of Sect. 2.4. Of special importance here is the observation that the band limit parameter c of the prolates may be adjusted to achieve optimal convergence of expansions and uniformity of quadrature nodes, this latter property giving a clear benefit in using the prolates for such purposes rather than the more familiar bases of Legendre or Chebyshev polynomials.

Section 2.5 deals with the approximation of non-band-limited functions, in particular those analytic on $[-1, 1]$ and those in Sobolev spaces $H^s[-1, 1]$. We outline the work done by Boyd in quantifying the performance of prolate expansions (in terms of decay of coefficients) as compared to that of Chebyshev and Legendre expansions.

A remark on notation is in order. The eigenfunction solutions $\phi_n(t; c)$ of (1.6) were normalized to have unit norm in $L^2(\mathbb{R})$. The functions $\bar{\phi}_n = \bar{\phi}_n^{(c)}$ used primarily in this chapter are normalized such that $\|\bar{\phi}_n\|_{L^2[-1, 1]} = 1$. That is, $\bar{\phi}_n^{(c)}(t) = \phi_n(t; c) / \sqrt{\lambda_n(c)}$. The “bar” thus refers to normalization, not conjugation. While this notation is slightly irritating, in the limit $c \rightarrow 0$ it is consistent with the somewhat standard notation $\bar{P}_n$ for the $L^2[-1, 1]$ -normalized Legendre polynomials.

Since this chapter addresses primarily matters of numerical analysis, we will also define band-limitedness in this chapter in terms of the numerical analyst’s Fourier transform $\sqrt{2\pi}\mathcal{F}_{2\pi}$ with kernel $e^{-ix\xi}$. This usage will not be explicit. Instead, we will say that a function $f \in L^2(\mathbb{R})$ is c -band limited in this chapter only if f is in the range of the operator $F_{2c/\pi}$ as introduced on p. 12, that is, $f(t) = \int_{-1}^1 e^{icst} h(s) ds$ for some $h \in L^2([-1, 1])$.

2.1 Oscillation Properties

We aim to show here that the prolates $\{\bar{\phi}_n^{(c)}\}_{n=0}^\infty$ form a *Markov system* on $[-1, 1]$, from which we deduce many properties of their zeros. The Markov property follows from the fact that the prolates are eigenfunctions of a certain integral operator with symmetric *oscillating* kernel which we show is another consequence of Slepian’s “lucky accident.” The results in this section are classical and many have been adapted from the book of Gantmacher and Krein [109].

2.1.1 Chebyshev and Markov Systems

Definition 2.1.1. A system of continuous functions $\{\varphi_i\}_{i=1}^n$ defined on an interval I is said to be a Chebyshev system on I if any non-trivial linear combination $\varphi(x) = \sum_{i=1}^n c_i \varphi_i(x)$ vanishes on I no more than $n - 1$ times.

Given functions $\{\varphi_i\}_{i=1}^n$ defined on I as in Definition 2.1.1 and $\{x_i\}_{i=1}^n \subset I$, we denote the $n \times n$ determinant of the matrix with (i, j) th entry $\varphi_i(x_j)$ as

$$\Delta \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} = \det(\varphi_i(x_j))_{i,j=1}^n.$$

With this notation, Chebyshev systems are characterized as follows.

Lemma 2.1.2. A system of continuous functions $\{\varphi_i\}_{i=1}^n$ defined on an interval I forms a Chebyshev system if and only if for all $x_1, x_2, \dots, x_n \in I$ with $x_1 < x_2 < \cdots < x_n$ we have

$$\Delta \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} \neq 0.$$

Proof. The determinant $\det_{i,j}(\varphi_i(x_j))$ vanishes if and only if there exists a non-trivial solution $(c_1, c_2, \dots, c_n)$ of the system of equations

$$\sum_{i=1}^n c_i \varphi_i(x_j) = 0 \quad (j = 1, 2, \dots, n).$$

Equivalently, the determinant is zero if and only if there exist constants $c_1, c_2, \dots, c_n$ not all zero such that the function $\varphi(x) = \sum_{i=1}^n c_i \varphi_i(x)$ vanishes at the distinct points $x_1, x_2, \dots, x_n$, that is, $\{\varphi_i\}_{i=1}^n$ is not a Chebyshev system. $\square$

We classify the isolated zeros of a function as either *nodes* or *anti-nodes* as follows.

Definition 2.1.3. Let f be defined on the open interval (a, b) and $c \in (a, b)$ be a zero of f . Then c is a node of f if in every neighborhood of c there exists x_1, x_2 such that $x_1 < c < x_2$ and $f(x_1)f(x_2) < 0$. Also, c is an anti-node of f if there is a neighborhood U of c such that whenever $x_1, x_2 \in U$ and $x_1 < c < x_2$, one has $f(x_1)f(x_2) > 0$.

Corollary 2.1.4. Suppose the functions $\{\varphi_i\}_{i=1}^n$ form a Chebyshev system on (a, b) and $\varphi(x) = \sum_{i=1}^n c_i \varphi_i(x)$ with the c_i not all zero. Suppose also that φ has p nodes and q anti-nodes in (a, b) . Then $p + 2q \leq n - 1$.

Proof. Let $a < x_1 < x_2 < \cdots < x_m < b$. We say $X_m = \{x_k\}_{k=1}^m$ is φ -oscillating if

$$(-1)^{\ell+k} \varphi(x_k) \geq 0 \tag{2.1}$$

for some integer ℓ and all $k = 1, 2, \dots, m$. If X_m is φ -oscillating and α is an anti-node of φ then there exists $\varepsilon > 0$ such that either $X_{m+2}^- = X_m \cup \{\alpha, \alpha - \varepsilon\}$ or $X_{m+2}^+ =$

$X_m \cup \{\alpha, \alpha + \varepsilon\}$ is φ -oscillating. Let the p nodes of φ be $y_1, y_2, \dots, y_p \in (a, b)$. Then there exists $\beta_1, \beta_2, \dots, \beta_{p+1} \in (a, b)$ such that

$$\beta_1 < y_1 < \beta_2 < \dots < \beta_p < y_p < \beta_{p+1}$$

and $\varphi(\beta_k)\varphi(\beta_{k+1}) < 0$ for $k = 1, 2, \dots, p$. The collection $\{\beta_j\}_{j=1}^{p+1}$ is φ -oscillating. Since φ has q anti-nodes, we can find a φ -oscillating collection $\{x_k\}_{k=1}^{p+2q+1} \subset (a, b)$. Suppose $p + 2q \geq n$. Then there is an integer ℓ such that $(-1)^{\ell+k}\varphi(x_k) \geq 0$ for $k = 1, 2, \dots, n+1$. However,

$$\Delta \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n & \varphi \\ x_1 & x_2 & \cdots & x_n & x_{n+1} \end{pmatrix} = \sum_{i=1}^n c_i \Delta \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n & \varphi_i \\ x_1 & x_2 & \cdots & x_n & x_i \end{pmatrix} = 0 \quad (2.2)$$

since each of the determinants in the sum is zero, the last row being identical to the i th row. On the other hand, expanding the determinant on the left-hand side of (2.2) along the last row and applying (2.2) gives

$$\sum_{i=1}^{n+1} (-1)^{n+1+i} \varphi(x_i) \Delta \begin{pmatrix} \varphi_1 & \cdots & \varphi_{i-1} & \varphi_i & \cdots & \varphi_n \\ x_1 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{n+1} \end{pmatrix} = 0. \quad (2.3)$$

However, since $\{\varphi_i\}_{i=1}^n$ is a Chebyshev system on (a, b) , each of the determinants on the left-hand side of (2.3) has the same sign. At the same time, the collection $\{x_1, x_2, \dots, x_n\}$ is φ -oscillating, so that each term in the sum on the left-hand side of (2.3) has the same sign, from which we conclude that each term is zero. Again, since $\{\varphi_i\}_{i=1}^n$ is a Chebyshev system, the determinants in (2.3) are not zero, hence $\varphi(x_i) = 0$ for $i = 1, 2, \dots, n+1$, thus contradicting the Chebyshev property. We conclude that $p + 2q \leq n - 1$. $\square$

Lemma 2.1.5. *Let f, g be continuous on (a, b) and suppose*

- (i) *f, g have respectively p and $p + 1$ nodes in (a, b) and no other zeros inside (a, b) ;*
- (ii) *For all real t , the number of zeros of $f_t(x) = f(x) - tg(x)$ in (a, b) is either p or $p + 1$ and all those zeros are nodes.*

Then the zeros of f and g alternate.

Proof. Let $\alpha_1 < \alpha_2 < \dots < \alpha_{p+1}$ be the nodes of g . They partition (a, b) into $p + 2$ subintervals $(a, \alpha_1), (\alpha_1, \alpha_2), \dots, (\alpha_{p+1}, b)$. Inside each of these subintervals, the function $\psi(x) = f(x)/g(x)$ is continuous since f, g are continuous and g does not vanish. Furthermore, ψ is monotone on each subinterval. To see this, suppose the contrary, that is, there is a subinterval (α_i, α_{i+1}) and $x_1, x_2, x_3 \in (\alpha_i, \alpha_{i+1})$ with $x_1 < x_2 < x_3$ and $[\psi(x_2) - \psi(x_1)][\psi(x_3) - \psi(x_2)] < 0$. Without loss of generality we may assume $\psi(x_1) < \psi(x_2)$ so that $\psi(x_3) < \psi(x_2)$. Since ψ is continuous on the closed interval $[x_1, x_3]$, it attains its maximum value on $[x_1, x_3]$ at some interior point $x_0 \in (x_2, x_3)$. Let $t_0 = \psi(x_0)$. Then $\psi(x) - \psi(x_0) \leq 0$ on $[x_1, x_3]$ and

$$f_{i_0}(x) = f(x) - t_0 g(x) = f(x) - \frac{f(x_0)}{g(x_0)} g(x) = g(x)(\psi(x) - \psi(x_0))$$

has a zero but not a node at x_0 since the sign of $g(x)$ is constant on (α_i, α_{i+1}) , thus contradicting the assumption that each function f_i has no anti-nodes. We conclude that ψ is monotone on each subinterval (α_i, α_{i+1}) .

The monotonicity of ψ implies the existence of (possibly infinite) limits ℓ_i^+ and ℓ_i^- defined by

$$\begin{aligned}\ell_i^+ &= \lim_{x \rightarrow \alpha_i^+} \psi(x) \quad (i = 0, 1, \dots, p+1, \alpha_0 = a) \\ \ell_i^- &= \lim_{x \rightarrow \alpha_i^-} \psi(x) \quad (i = 1, 2, \dots, p+2, \alpha_{p+2} = b).\end{aligned}$$

We claim that none of the internal limits $\ell_1^\pm, \dots, \ell_{p+1}^\pm$ is finite. To see this, suppose ℓ_i^- is finite for some $1 \leq i \leq p+1$. Then $f(\alpha_i) = 0$ so that each α_i is a common zero of f and g . Suppose ψ does not change sign as x passes through α_i , that is, the limits ℓ_i^+ and ℓ_i^- have the same sign. There are four possible cases:

Case 1: $\ell_i^\pm = \pm\infty$.

Case 2: $\ell_i^\pm$ is finite and $\ell_i^- \neq \ell_i^+$.

Case 3: $\ell_i^+ = \ell_i^-$ and ψ remains monotone as x passes through α_i .

Case 4: $\ell_i^\pm = \ell_i^-$ and ψ has an extremum at α_i .

In each of cases 1, 2, and 3, we may choose a number t_1 such that in a neighborhood of $x = \alpha_i$, the graph of ψ lies on different sides of the horizontal line $y = t_1$. Consider the function

$$f_{t_1}(x) = f(x) - t_1 g(x) = g(x)(\psi(x) - t_1).$$

Then α_i is an anti-node of f_{t_1} since $f_{t_1}(\alpha_i) = 0$ and both $g(x)$ and $\psi(x) - t_1$ change sign as x passes through α_i , thus contradicting the assumption that each f_i has no anti-nodes. In case 4, let $t_1 = \ell_i^-$. Then there exists $t_2 < t_1$ such that $f_{t_2}(x) = g(x)(\psi(x) - t_2)$ has two nodes, one on either side of α_i and retains its node at α_i since $g(\alpha_i) = 0$, thus contradicting the assumption that the number of nodes of both f_{t_1} and f_{t_2} is either p or $p+1$. We conclude that the internal limits $\ell_i^\pm$ must be infinite. Since ψ varies monotonically on (α_i, α_{i+1}) , we conclude that either $\ell_i^\pm = \pm\infty$ or $\ell_i^\pm = \mp\infty$. Consequently, since ψ is continuous on (α_i, α_{i+1}) , there exists exactly one point $\beta_i \in (\alpha_i, \alpha_{i+1})$ for which $\psi(\beta_i) = 0$. Hence $f(\beta_i) = 0$. This proves the lemma. $\square$

Definition 2.1.6. A sequence of functions $\{\varphi_i\}_{i=0}^\infty$ is said to form a *Markov system* on the interval (a, b) if, for all integers $n \geq 1$, the finite collection $\{\varphi_i\}_{i=0}^n$ forms a Chebyshev system on (a, b) .

Theorem 2.1.7. If an orthonormal sequence of functions $\{\varphi_i\}_{i=0}^\infty$ forms a Markov system on the interval (a, b) , then

- (a) The function φ_0 has no zeros on (a, b) ;
- (b) For all $j \geq 0$, φ_j has j nodes in (a, b) and no other zeros in (a, b) ;

- (c) For all integers k and m with $0 < k < m$ and arbitrary constants $c_k, c_{k+1}, \dots, c_m$ (not all zero), the linear combination $\varphi(x) = \sum_{i=k}^m c_i \varphi_i(x)$ has at least k nodes and at most m zeros in (a, b) (where each anti-node is counted as two zeros). In particular, if φ has m distinct zeros in (a, b) , then all are nodes;
- (d) The nodes of any consecutive functions φ_j and φ_{j+1} alternate.

Proof. For each positive integer m , the functions $\{\varphi_i\}_{i=0}^m$ form a Chebyshev system on (a, b) . By Corollary 2.1.4, the function $\varphi(x) = \sum_{i=k}^m c_i \varphi_i(x)$ has at most m zeros on (a, b) (where we count each anti-node twice). Suppose φ has p nodes $\xi_1, \dots, \xi_p$ with $a < \xi_1 < \dots < \xi_p < b$ and consider the function

$$\psi(x) = \Delta \begin{pmatrix} \varphi_0 & \cdots & \varphi_{p-1} & \varphi_p \\ \xi_1 & \cdots & \xi_p & x \end{pmatrix} \quad (a \leq x \leq b). \quad (2.4)$$

By Lemma 2.1.2, $\psi(x) \neq 0$ if $x \notin \{\xi_1, \dots, \xi_p\}$ and changes sign as x passes through each of these points, so that ψ and φ have the same nodes $\xi_1, \dots, \xi_p$. Consequently, $\langle \varphi, \psi \rangle \neq 0$. However, by expanding the determinant (2.4) along the last column we find that ψ may be written in the form $\psi(x) = \sum_{i=0}^p d_i \varphi_i(x)$. By the orthogonality of $\{\varphi_i\}_{i=0}^\infty$, we see that $\langle \varphi, \psi \rangle = 0$ if $p < k$, and conclude that $p \geq k$. This completes the proof of statement (c).

To prove statement (b), we apply statement (c) with $k = m = j$. From this we see that φ_j has at least j nodes and at most j zeros. Hence each zero is a node and there are precisely j such points in (a, b) .

To prove statement (d) observe that from statement (b), φ_j and φ_{j+1} have j and $j+1$ nodes respectively and no other zeros in (a, b) . Also, given a real number t , the function $\mu(x) = \varphi_j(x) - t\varphi_{j+1}(x)$ has, by part (c), at least j nodes and its q anti-nodes must satisfy $j+2q \leq j+1$, from which we conclude that $q = 0$ and that μ has exactly j zeros, all of which are nodes. Hence φ_j and φ_{j+1} satisfy the conditions of Lemma 2.1.5, from which we conclude that the nodes of φ_j and φ_{j+1} alternate. $\square$

2.1.2 Oscillating Matrices and Kernels

We now gather relevant results in matrix algebra which will be essential in what follows. The proofs can be found in [109].

Given an $n \times n$ matrix A , an integer $1 \leq p \leq n$ and two collections of integers $i_1, i_2, \dots, i_p$ and $k_1, k_2, \dots, k_p$ satisfying $1 \leq i_1 < i_2 < \dots < i_p \leq n$, $1 \leq k_1 < k_2 < \dots < k_p \leq n$, let $A \begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix}$ be the determinant of the $p \times p$ sub-matrix of A with (ℓ, m) th entry $a_{i_\ell k_m}$, that is, the determinant of the $p \times p$ submatrix of A obtained by retaining only the rows of A numbered $i_1, i_2, \dots, i_p$ and the columns numbered $k_1, k_2, \dots, k_p$.

Definition 2.1.8. A matrix $A = (a_{ij})_{i,j=1}^n$ is said to be *completely non-negative* (respectively *completely positive*) if all its minors of any order are non-negative (respectively positive):

$$A \begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix} \geq 0 \quad (\text{resp. } > 0)$$

for $1 \leq i_1 < i_2 < \cdots < i_p \leq n$, $1 \leq k_1 < k_2 < \cdots < k_p \leq n$, $p = 1, 2, \dots, n$.

A is said to be *oscillating* if it is completely non-negative and there exists a positive integer κ such that A^κ is completely positive. The smallest such κ is known as the exponent of A . Completely positive matrices are therefore oscillating matrices with exponent 1.

Definition 2.1.9. The square matrix L with (i, k) th entry ℓ_{ik} is a *one-pair matrix* if there are numbers $\{\psi_i\}_{i=1}^n$ and $\{\chi_i\}_{i=1}^n$ such that

$$\ell_{ik} = \begin{cases} \psi_i \chi_k & \text{if } i \leq k \\ \psi_k \chi_i & \text{if } i \geq k. \end{cases}$$

Of particular interest to us will be those one-pair matrices which are also oscillating. The class of such matrices is characterized as follows [109].

Theorem 2.1.10. *In order for the one-pair matrix of Definition 2.1.9 to be oscillating, it is necessary and sufficient that the following conditions hold:*

- (i) *The $2n$ numbers $\{\psi_i\}_{i=1}^n$, $\{\chi_i\}_{i=1}^n$ are nonzero and have the same sign;*
- (ii) *The quotients ψ_i/χ_i are increasing: $\psi_1/\chi_1 < \psi_2/\chi_2 < \cdots < \psi_n/\chi_n$.*

Let I, J be intervals and $K: I \times J \rightarrow \mathbb{R}$. Given $x_1, x_2, \dots, x_n \in I$ and $s_1, s_2, \dots, s_n \in J$, we define $K \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ s_1 & s_2 & \cdots & s_n \end{pmatrix}$ by

$$K \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ s_1 & s_2 & \cdots & s_n \end{pmatrix} = \det_{i,j} (K(x_i, s_j))_{i,j=1}^n.$$

Let $K = K(x, s)$ be defined on $[a, b] \times [a, b]$. Associated with K and $[a, b]$ we define an interval I_K by

$$I_K = \begin{cases} (a, b) & \text{if } K(a, a) = 0 = K(b, b) \\ [a, b) & \text{if } K(a, a) \neq 0 = K(b, b) \\ (a, b] & \text{if } K(a, a) = 0 \neq K(b, b). \end{cases}$$

Definition 2.1.11. The kernel $K(x, s)$ defined on $[a, b] \times [a, b]$ is said to be *oscillating* if the following three conditions are satisfied:

- (i) $K(x, s) > 0$ for $x, s \in I_K$, $\{x, s\} \neq \{a, b\}$;
- (ii) $K \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ s_1 & s_2 & \cdots & s_n \end{pmatrix} \geq 0$ for all $a < x_1 < x_2 < \cdots < x_n < b$ and $a < s_1 < s_2 < \cdots < s_n < b$ ($n = 1, 2, \dots$);
- (iii) $K \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} > 0$ for $a < x_1 < x_2 < \cdots < x_n < b$ ($n = 1, 2, \dots$).

The condition $\{x, s\} \neq \{a, b\}$ means that x and s cannot simultaneously equal the two ends a and b of the interval. In the case of a symmetric kernel this condition can be written $a < x < b, s \in I_K$.

The next result provides an alternative definition of oscillating kernels.

Lemma 2.1.12. *The kernel $K(x, s)$ defined on $[a, b] \times [a, b]$ is oscillating if and only if for any $x_1, x_2, \dots, x_n \in I_K$, at least one of which is an interior point, the $n \times n$ matrix A with (i, k) th entry $a_{ik} = K(x_i, x_k)$ is oscillating.*

Definition 2.1.13. A one-pair kernel K defined on $[a, b] \times [a, b]$ is one of the form

$$K(x, s) = \begin{cases} \psi(x)\chi(s) & \text{if } x \leq s \\ \psi(s)\chi(x) & \text{if } s \leq x \end{cases}$$

where ψ and χ are continuous functions.

As a corollary of Theorem 2.1.10, we have the following characterization of one-pair oscillating kernels.

Theorem 2.1.14. *The one-pair kernel of Definition 2.1.13 is oscillating if and only if the following conditions are satisfied:*

- (i) $\psi(x)\chi(x) > 0$ for all $a < x < b$;
- (ii) The function $\psi(x)/\chi(x)$ increases monotonically on the interval (a, b) .

Consider now the eigenvalue equation

$$\int_a^b K(x, s)\varphi(s) ds = \lambda \varphi(x). \quad (2.5)$$

The main result of this section is the following.

Theorem 2.1.15. *Let K be a continuous oscillating symmetric kernel on $[a, b] \times [a, b]$. Then*

- (i) *The eigenvalues λ of (2.5) are all positive and simple: $\lambda_0 > \lambda_1 > \dots > 0$.*
- (ii) *The fundamental eigenfunction φ_0 corresponding to the largest eigenvalue λ_0 has no zeros in the interval I_K .*
- (iii) *For any $j \geq 1$, the eigenfunction φ_j corresponding to the eigenvalue λ_j has exactly j nodes in I_K .*
- (iv) *Given integers m, k with $0 \leq k \leq m$ and real numbers $c_k, c_{k+1}, \dots, c_m$ not all zero, the linear combination $\varphi(x) = \sum_{i=k}^m c_i \varphi_i(x)$ has no more than m zeros and no fewer than k nodes in the interval I_K .*
- (v) *The nodes of two neighboring eigenfunctions φ_j and φ_{j+1} alternate.*

We will show that the prolates are eigenfunctions of a continuous oscillating symmetric kernel and hence that they inherit the properties outlined in Theorem 2.1.15. The proof of Theorem 2.1.15 is postponed until Sect. 2.6.2, as it requires several new concepts and intermediate results.

2.1.3 Sturm–Liouville Theory

Given a constant $c > 0$, we consider here the differential operator $\mathcal{P}$ of (1.6), the eigenfunctions of which are the prolates $\{\bar{\phi}_n^{(c)}\}_{n=0}^\infty$. We aim to show that $\mathcal{P}$ is inverted by an integral operator $\mathcal{G}$ associated with a continuous, symmetric, oscillating kernel. The eigenfunctions of $\mathcal{P}$ therefore coincide with those of $\mathcal{G}$, and an application of Theorem 2.1.15 will reveal that the prolates form a Markov system on $[-1, 1]$.

The bulk of this section comes from private communication between the authors and Mark Tygert. The main result of this section is as follows.

Theorem 2.1.16. *The differential operator $\mathcal{P}$ of (1.6) has infinitely many eigenvalues $\{\chi_n\}_{n=0}^\infty$ with $0 < \chi_0 < \chi_1 < \dots$. The corresponding eigenfunctions $\bar{\phi}_n = \bar{\phi}_n^{(c)}$ are orthonormal and complete in $L^2[-1, 1]$ and form a Markov system on $[-1, 1]$.*

The proof of Theorem 2.1.16 is deferred until after the proof of Lemma 2.1.19, at which time we will have developed the required machinery.

The standard form of the equation $\mathcal{P}u = 0$ is $u'' - P(x)u' + Q(x)u = 0$ with coefficient functions $P(x) = -2x/(1-x^2)$ and $Q(x) = c^2x^2/(1-x^2)$. Since $(1-x)P(x)$ and $(1-x)^2Q(x)$ are analytic at $x = 1$, the point $x = 1$ is a *regular singular point*, implying that there exists a power series solution (Frobenius solution) of the form

$$u(x) = \sum_{n=0}^{\infty} a_n(x-1)^{n+r} \quad (2.6)$$

with $a_0 \neq 0$. Rewriting the equation $\mathcal{P}u = 0$ in the form

$$(1-x)u'' - \frac{2x}{1+x}u' - \frac{c^2x^2}{1+x}u = 0 \quad (2.7)$$

and substituting (2.6) into (2.7) gives

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)(x-1)^{n+r-1} + \frac{2x}{1+x} \sum_{n=0}^{\infty} a_n(n+r)(x-1)^{n+r-1} \\ + \frac{c^2x^2}{1+x} \sum_{n=0}^{\infty} a_n(x-1)^{n+r} = 0. \end{aligned}$$

Extracting the lowest order term (the coefficient of $(x-1)^{r-1}$) gives the *indicial equation* $r^2 = 0$, from which we conclude that there is a Frobenius series solution of the form $u_1(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$ centered at $x = 1$ with $a_0 \neq 0$. To obtain another solution centered at $x = 1$, we use the method of *reduction of order* to produce a solution of the form $u_2 = vu_1$ with $v = C \log(x-1) + \sum_{n=0}^{\infty} b_n(x-1)^{n+1}$. From the form of this solution we see that u_2 has at most a logarithmic singularity at $x = 1$. Since the coefficient functions in (2.7) are analytic at $x = 1$ and with power series having radius of convergence at least two, the Frobenius solution u_1 centered at $x = 1$ will also have radius of convergence two. Hence u_1 is also a power series

solution for $\mathcal{P}u = 0$ centered at $x = -1$. However, by the previous argument, the equation $\mathcal{P}u = 0$ has a Frobenius solution w_1 centered at $x = -1$ with radius of convergence two and another solution w_2 with a logarithmic singularity at $x = -1$. We conclude that u_1 has at worst a logarithmic singularity at $x = -1$ and w_1 has at worst a logarithmic singularity at $x = 1$. Let

$$C(x) = (1 - x^2)(w_1(x)u_1'(x) - u_1(x)w_1'(x)). \quad (2.8)$$

Since $\mathcal{P}u_1 = \mathcal{P}w_1 = 0$, we have

$$\begin{aligned} \frac{dC}{dx} &= -2x(w_1u_1' - u_1w_1') + (1 - x^2)(w_1u_1'' - u_1w_1'') \\ &= w_1((1 - x^2)u_1'' - 2xu_1') - u_1((1 - x^2)w_1'' - 2xw_1') \\ &= w_1(c^2x^2u_1) - u_1(c^2x^2w_1) = 0 \end{aligned}$$

which shows that $C(x) = C$ is a constant. Suppose that $C = 0$. Then, from (2.8), $w_1'/w_1 = u_1'/u_1$ so that $|u_1|$ and $|w_1|$ have the same logarithmic derivative. This implies that $|u_1| = \lambda |w_1|$ for some constant λ (provided $u_1, w_1 \neq 0$). Suppose $w_1(x) = 0$ and $u_1(x) \neq 0$. Then the Wronskian equation $w_1(x)u_1'(x) - u_1(x)w_1'(x) = 0$ shows that $w_1'(x) = 0$. Successively differentiating the Wronskian equation and using the hypothesis $u_1(x) \neq 0$ shows that all the derivatives of w_1 are zero, so that $w_1 \equiv 0$, a contradiction. Hence $w_1(x) = 0$ implies $u_1(x) = 0$. Further, $u_1(x) = \pm \lambda w_1(x)$ on any open interval over which w_1 is non-vanishing. Since $u_1 \in C^2(-1, 1]$, $w_1 \in C^2[-1, 1)$, and $u_1 = \lambda w_1$, we see that $u_1, w_1 \in C^2[-1, 1]$. Since $p(x) = 1 - x^2$ vanishes at $x = \pm 1$ and $\mathcal{P}u_1 = 0$, an integration by parts yields

$$\begin{aligned} 0 &= \int_{-1}^1 (\mathcal{P}u_1)(x)u_1(x) dx = \int_{-1}^1 (((1 - x^2)u_1'(x))' - c^2x^2u_1(x))u_1(x) dx \\ &= (1 - x^2)u_1'(x)u_1(x) \Big|_{-1}^1 - \left(\int_{-1}^1 (1 - x^2)((u_1'(x))^2 + c^2x^2u_1(x)^2) dx \right) \end{aligned}$$

from which we conclude that $\int_{-1}^1 (1 - x^2)u_1'(x)^2 dx = c^2 \int_{-1}^1 x^2u_1(x)^2 dx = 0$. But the integrands are non-negative, so $u_1 \equiv 0$, a contradiction. Thus, $C \neq 0$.

Let the nonzero constant C be as in (2.8) and define G on the square $Q = [-1, 1] \times [-1, 1]$ by

$$G(x, s) = \begin{cases} w_1(x)u_1(s)/C & \text{if } x < s \\ w_1(s)u_1(x)/C & \text{if } s < x \\ w_1(x)u_1(x)/C & \text{if } s = x \in (-1, 1) \\ \infty & \text{if } x = s = 1 \text{ or } x = s = -1. \end{cases} \quad (2.9)$$

Lemma 2.1.17. *With G as in (2.9), let $\mathcal{G}$ be the integral operator*

$$(\mathcal{G}f)(x) = \int_{-1}^1 G(x, s) f(s) \, ds.$$

Then $\mathcal{G} : L^2[-1, 1] \rightarrow L^2[-1, 1]$ is a Hilbert–Schmidt operator.

Proof. Direct calculation gives

$$\mathcal{P}_x G(x, s) = 0 \text{ if } x < s \text{ and } \mathcal{P}_s G(x, s) = 0 \text{ if } s < x$$

and also

$$\begin{aligned} \lim_{x \rightarrow s^+} \frac{\partial G}{\partial x}(x, s) - \lim_{x \rightarrow s^-} \frac{\partial G}{\partial x}(x, s) &= \lim_{x \rightarrow s^+} \frac{w_1'(x)u_1(s)}{C} - \lim_{x \rightarrow s^-} \frac{w_1(s)u_1'(x)}{C} \\ &= \frac{w_1'(s)u_1(s) - w_1(s)u_1'(s)}{C} = \frac{1}{1-s^2}, \end{aligned}$$

the last equality being a consequence of the definition (2.8) of the constant C .

Since u_1, w_1 have singularities at $x = -1$ and $x = 1$ respectively which are at worst logarithmic, there exists a constant B such that

$$|u_1(x)| \leq B(1 + \log|1+x|) \text{ and } |w_1(x)| \leq B(1 + \log|1-x|).$$

Consequently,

$$\begin{aligned} \iint_Q |G(x, s)|^2 \, dx \, ds &\leq \frac{B^4}{C^2} \int_{-1}^1 (1 + \log|1-x|)^2 \int_x^1 (1 + \log|1+s|)^2 \, ds \, dx \\ &\quad + \frac{B^4}{C^2} \int_{-1}^1 (1 + \log|1+x|)^2 \int_{-1}^x (1 + \log|1-s|)^2 \, ds \, dx < \infty \end{aligned}$$

from which the result follows. $\square$

As an immediate consequence of Lemma 2.1.17, the eigenfunctions of $\mathcal{G}$ form a complete orthonormal basis for $L^2[-1, 1]$.

Lemma 2.1.18. *The integral operator $\mathcal{G}$ in Lemma 2.1.17 is the inverse of the differential operator $\mathcal{P}$ in (1.6).*

Proof. First note that

$$\begin{aligned} (\mathcal{P}\mathcal{G}f)(x) &= \mathcal{P}_x \int_{-1}^1 G(x, s) f(s) \, ds \\ &= \frac{1}{C} \left(\mathcal{P}_x \int_{-1}^x w_1(s) u_1(x) f(s) \, ds + \mathcal{P}_x \int_x^1 w_1(x) u_1(s) f(s) \, ds \right). \end{aligned} \quad (2.10)$$

Direct computation, using the fact that $\mathcal{P}u_1 = 0$, gives

$$\begin{aligned}
& \mathcal{P}_x \int_{-1}^x w_1(s) u_1(x) f(s) ds \\
&= (1-x^2) \left(u_1''(x) \int_{-1}^x w_1(s) f(s) ds + u_1'(x) w_1(x) f(x) + (u_1 w_1 f)'(x) \right) \\
&- 2x \left(u_1'(x) \int_{-1}^x w_1(s) f(s) ds + u_1(x) w_1(x) f(x) \right) \\
&\quad + c^2 x^2 u_1(x) \int_{-1}^x w_1(s) f(s) ds \\
&= (1-x^2) (u_1'(x) w_1(x) f(x) + (u_1 w_1 f)'(x)) - 2x u_1(x) w_1(x) f(x). \tag{2.11}
\end{aligned}$$

By a similar calculation,

$$\begin{aligned}
& \mathcal{P}_x \int_x^1 w_1(x) u_1(s) f(s) ds \\
&= (1-x^2) (-w_1'(x) u_1(x) f(x) - (w_1 u_1 f)'(x)) + 2x w_1(x) u_1(x) f(x). \tag{2.12}
\end{aligned}$$

Combining (2.10), (2.11), and (2.12) and applying (2.8) gives

$$(\mathcal{P}\mathcal{G}f)(x) = \frac{(1-x^2)}{C} (u_1'(x) w_1(x) - w_1'(x) u_1(x)) f(x) = f(x),$$

that is, $\mathcal{P}\mathcal{G} = I$. Furthermore,

$$\begin{aligned}
(\mathcal{G}\mathcal{P}f)(x) &= \int_{-1}^1 G(x,s) (\mathcal{P}f)(s) ds \\
&= \frac{u_1(x)}{C} \int_{-1}^x w_1(s) (\mathcal{P}f)(s) ds + \frac{w_1(x)}{C} \int_x^1 u_1(s) (\mathcal{P}f)(s) ds. \tag{2.13}
\end{aligned}$$

Integrating the first integral on the right-hand side of (2.13) by parts and using the fact that $\mathcal{P}u_1 = 0$ gives

$$\int_{-1}^x w_1(s) \mathcal{P}f(s) ds = (1-x^2) (w_1(x) f'(x) - w_1'(x) f(x)). \tag{2.14}$$

Similarly, for the second integral on the right-hand side of (2.13), we have

$$\int_x^1 u_1(s) \mathcal{P}f(s) ds = (1-x^2) (u_1'(x) f(x) - u_1(x) f'(x)). \tag{2.15}$$

Combining (2.13), (2.14), and (2.15) gives

$$(\mathcal{G}\mathcal{P}f)(x) = \frac{(1-x^2)}{C} (u_1'(x) w_1(x) - w_1'(x) u_1(x)) f(x) = f(x),$$

that is, $\mathcal{G}\mathcal{P} = I$. Hence $\mathcal{G} = \mathcal{P}^{-1}$. □

As an immediate consequence of Lemma 2.1.18 we see that $\mathcal{G}$ and $\mathcal{P}$ have the same collection of eigenfunctions, namely the prolates $\{\bar{\phi}_n^{(c)}\}_{n=0}^\infty$.

Lemma 2.1.19. *The one-pair kernel G is oscillating.*

Proof. We first show that u_1, w_1 have no zeros on $[-1, 1]$. The proof of Lemma 2.6.8 shows that $u_1(1) \neq 0$. Suppose now that $u_1(x_0) = 0$ for some $x_0 \in (-1, 1)$. Since $\mathcal{P}u_1 = 0$, an integration by parts gives

$$0 = \int_{x_0}^1 (\mathcal{P}u_1)(x)u_1(x) dx = \int_{x_0}^1 [(1-x^2)u_1'(x)^2 + c^2x^2u_1(x)^2] dx,$$

so that $u_1(x) = 0$ on $[x_0, 1]$ which in turn implies that $u_1 \equiv 0$, a contradiction. We conclude that $u_1(x) \neq 0$ for all $x \in (-1, 1]$. Similarly, $w_1(x) \neq 0$ for all $x \in [-1, 1)$, that is,

$$|u_1(x)w_1(x)| > 0 \text{ for all } x \in (-1, 1).$$

Furthermore,

$$\left| \frac{d}{dx} \left(\frac{u_1(x)}{w_1(x)} \right) \right| = \left| \frac{w_1(x)u_1'(x) - u_1(x)w_1'(x)}{w_1(x)^2} \right| = \frac{C}{(1-x^2)w_1(x)^2} > 0.$$

Now we apply Theorem 2.1.14 to conclude that the kernel G is oscillating. $\square$

Proof (of Theorem 2.1.16). Since the eigenfunctions of $\mathcal{P}$ coincide with those of $\mathcal{G}$, the orthonormality and completeness of the eigenfunctions follows from the fact that $\mathcal{G}$ is self-adjoint and Hilbert–Schmidt (hence compact). The simplicity of the eigenvalues and the fact that they form a Markov system now follow by appealing to Lemma 2.1.19 and Theorem 2.1.15. $\square$

2.2 Quadratures

Quadrature involves the estimation of integrals $\int_a^b \varphi(x)\omega(x) dx$ for a given function (or class of functions) φ and fixed integrable non-negative weight function ω by sums of the form $\sum_{k=1}^n w_k \varphi(x_k)$. The points $\{x_k\}_{k=1}^n$ are the *nodes* of the quadrature while the coefficients $\{w_k\}_{k=1}^n$ are its *weights*. Classically, quadrature rules consist of n nodes and weights and integrate the monomials $\{x^k\}_{k=0}^{2n-1}$ exactly. By the linearity of such schemes, they integrate polynomials of degree up to $2n - 1$ exactly.

Definition 2.2.1. A quadrature formula is said to be *generalized Gaussian* with respect to a collection of $2N$ functions $\varphi_j : [a, b] \rightarrow \mathbb{R}$ ($j = 1, 2, \dots, 2N$) and non-negative weight function ω if it consists of N weights and nodes and integrates the functions $\{\varphi_j\}_{j=1}^{2N}$ exactly, that is,

$$\int_a^b \varphi_j(x)\omega(x) dx = \sum_{k=1}^N w_k \varphi_j(x_k) \quad (j = 1, 2, \dots, 2N). \quad (2.16)$$

The w_k and x_k are called generalized Gaussian weights and nodes respectively.

The following general result about Chebyshev systems is due to Markov [222, 223]. Other proofs can be found in [170, 186].

Theorem 2.2.2. *Suppose $\{\phi_i\}_{i=1}^{2n}$ is a Chebyshev system on $[a, b]$ and ω is non-negative and integrable on $[a, b]$. Then there exists a unique generalized Gaussian quadrature for $\{\phi_i\}_{i=1}^{2n}$ on $[a, b]$ with respect to ω . The weights of the quadrature are positive.*

By Theorems 2.2.2 and 2.1.16, there exists a unique N -point generalized Gaussian quadrature for the first $2N$ prolates $\{\bar{\phi}_n\}_{n=0}^{2N-1}$. Computing the weights and nodes of a generalized Gaussian quadrature involves solving the $2N$ equations (2.16). While these equations are linear in the weights $\{w_k\}_{k=1}^N$, they are nonlinear in the nodes $\{x_k\}_{k=1}^N$. In [72] a Newton iteration for the approximate solution of these equations for arbitrary Chebyshev systems is given. The algorithm is very similar to an algorithm for the computation of nodes and weights of *generalized Gauss–Lobatto quadratures* which we discuss in Sect. 2.4. In the case of the prolates, estimates of the accuracy of such quadratures when applied to arbitrary functions with band limit c typically rely on the eigenfunction property of Corollary 1.2.6, which may be written in the form

$$\mu_n(2c/\pi)\bar{\phi}_n^{(c)}(x) = \int_{-1}^1 e^{-icx\tau} \bar{\phi}_n^{(c)}(t) dt \quad (2.17)$$

where $\mu_n(a) = 2i^n \sqrt{\lambda_n(a)/a}$. From now on we abbreviate

$$\mu_n(2c/\pi) = i^n \sqrt{2\pi\lambda_n(2c/\pi)/c} = \mu_n.$$

Since classical Gaussian quadrature schemes integrate polynomials of degree up to $2N - 1$ exactly, they may be used to estimate integrals of functions that are effectively approximated by polynomials. Similarly, generalized Gaussian quadratures for the prolates $\{\bar{\phi}_n^{(c)}\}_{n=0}^{2N-1}$ may be used to approximate integrals of functions that are effectively approximated by this collection. As the following result [364] shows, when N is sufficiently large, the class of functions that are effectively approximated by $\{\bar{\phi}_n^{(c)}\}_{n=0}^{2N-1}$ includes functions of band limit less than or equal to c .

Theorem 2.2.3. *Let $\{x_k\}_{k=1}^N$ be nodes and $\{w_k\}_{k=1}^N$ be non-negative weights for the generalized Gaussian quadrature for the prolates $\{\bar{\phi}_n^{(c)}\}_{n=0}^{2N-1}$ with $N > c/\pi$. Then for any function f on the line with band limit c ,*

$$\left| \sum_{k=1}^N w_k f(x_k) - \int_{-1}^1 f(x) dx \right| \leq C |\mu_{2N-1}| \|f\|_{L^2(\mathbb{R})} \quad (2.18)$$

with C a constant independent of N .

Proof. Let $\bar{\phi}_n = \bar{\phi}_n^{(c)}$ and $\mu_n = \mu_n(2c/\pi)$. Since $\{\bar{\phi}_n\}_{n=0}^\infty$ is an orthonormal basis for $L^2[-1, 1]$, for each $x, t \in [-1, 1]$ we have the expansion

$$e^{-icxt} = \sum_{n=0}^{\infty} \left(\int_{-1}^1 e^{-icxu} \bar{\phi}_n(u) du \right) \bar{\phi}_n(t) = \sum_{n=0}^{\infty} \mu_n \bar{\phi}_n(x) \bar{\phi}_n(t) \quad (2.19)$$

where we have used the eigenfunction property (2.17). The error incurred when the quadrature is applied to an exponential e^{-icxt} ($|t| < 1$ fixed) is

$$\begin{aligned} E(t) &= \sum_{k=1}^N w_k e^{-icx_k t} - \int_{-1}^1 e^{-icxt} dx \\ &= \sum_{k=1}^N w_k \left(\sum_{n=0}^{\infty} \mu_n \bar{\phi}_n(x_k) \bar{\phi}_n(t) \right) - \int_{-1}^1 \sum_{n=0}^{\infty} \mu_n \bar{\phi}_n(x) \bar{\phi}_n(t) dx \\ &= \sum_{n=2N}^{\infty} \mu_n \bar{\phi}_n(t) \left(\sum_{k=1}^N w_k \bar{\phi}_n(x_k) - \int_{-1}^1 \bar{\phi}_n(x) dx \right). \end{aligned} \quad (2.20)$$

However, as stated in [296],

$$M_N^c = \max_{0 \leq n \leq N} \max_{|x| \leq 1} |\bar{\phi}_n^{(c)}(x)| \leq 2\sqrt{N}. \quad (2.21)$$

Furthermore, the weights w_k of a generalized Gaussian quadrature are non-negative with the sums $\sum_{k=1}^N w_k$ bounded independent of N . Consequently, (2.20) and the asymptotic formula of Corollary 1.2.11 may be combined to give the estimate

$$|E(t)| \leq C \sum_{n=2N}^{\infty} \sqrt{n} |\mu_n| |\bar{\phi}_n(t)| \leq C \sum_{n=2N}^{\infty} n |\mu_n| \leq C |\mu_{2N-1}| \quad (2.22)$$

since $N > c/\pi$ is large enough so that $\{\mu_n\}_{n=2N}^\infty$ decreases super-exponentially.

Suppose now that f is a function on the line with band limit c , that is,

$$f(x) = \int_{-1}^1 \sigma(t) e^{icxt} dt \quad (2.23)$$

for some $\sigma \in L^2[-1, 1]$. Then, applying the quadrature to f yields an error E_f given by

$$\begin{aligned} E_f &= \sum_{k=1}^N w_k f(x_k) - \int_{-1}^1 f(x) dx \\ &= \int_{-1}^1 \sigma(t) \left(\sum_{k=1}^N w_k e^{-icx_k t} - \int_{-1}^1 e^{-icxt} dx \right) dt = \int_{-1}^1 \sigma(t) E_t dt, \end{aligned}$$

from which we conclude that $|E_f| \leq C |\mu_{2N-1}| \|\sigma\|_{L^2[-1,1]} \leq C |\mu_{2N-1}| \|f\|_{L^2(\mathbb{R})}$. $\square$

In order to obtain estimates of the form (2.18) for functions f on the line of band limit c , it is not necessary that the quadrature rule integrate the first N prolates exactly. With this in mind we make the following definition.

Definition 2.2.4. Let N be a positive integer, $c > 0$, and $\varepsilon \geq 0$. The collection of nodes $\{x_k\}_{k=1}^N \subset [-1, 1]$ and positive weights $\{w_k\}_{k=1}^N$ are said to integrate functions on $[-1, 1]$ with band limit c to precision ε if for all functions f with band limit c of the form (2.23) we have

$$\left| \int_{-1}^1 f(x) dx - \sum_{k=1}^N w_k f(x_k) \right| \leq \varepsilon \|\sigma\|_{L^1[-1,1]}.$$

A collection of such weights and nodes will be referred to as an *approximate generalized Gaussian quadrature* of band limit c and precision ε . The class of such quadratures is denoted $\text{GGQ}(c, \varepsilon)$.

This definition weakens the condition on the nodes and weights of the quadrature and allows for more flexible constructions. The question then becomes: What are appropriate weights and nodes? In Theorem 2.2.6 below, we show that the nodes may be chosen to be the zeros of a prolate $\tilde{\phi}_N^{(c)}$ with N sufficiently large. Before that, however, we require the following generalization of the Euclidean division algorithm due to Xiao, Rokhlin, and Yarvin [364].

Theorem 2.2.5. Let $\sigma, \rho \in C^1[-1, 1]$ with $\sigma(\pm 1) = 0 \neq \rho(\pm 1)$. Let $f = F_{-4c/\pi}(\sigma)$ and $p = F_{-2c/\pi}(\rho)$. Then there exist unique η, ξ such that with $q = F_{-2c/\pi}(\eta)$ and $r = F_{-2c/\pi}(\xi)$ we have $f = pq + r$.

The role of the degrees of the polynomials in the classical Euclidean algorithm is replaced in Theorem 2.2.5 by the band limits of the functions f , p , q , and r .

Proof. The proof is constructive. Let p, q be as in the theorem. Then

$$\begin{aligned} p(x)q(x) &= \int_{-1}^1 \int_{-1}^1 \rho(t)\eta(s)e^{-icx(t+s)} dt ds \\ &= \int_0^2 e^{-icxu} \int_{u-1}^1 \rho(u-s)\eta(s) ds du + \int_{-2}^0 e^{-icxu} \int_{-1}^{u+1} \rho(u-s)\eta(s) ds du \\ &= \int_{-2}^2 e^{-icxu} \int_{\max(-1, u-1)}^{\min(1, u+1)} \rho(u-s)\eta(s) ds du. \end{aligned}$$

Consequently,

$$\begin{aligned}
f(x) - p(x)q(x) &= \frac{1}{2} \int_{-2}^2 \sigma\left(\frac{u}{2}\right) e^{-icxu} du - \int_{-2}^2 e^{-icxu} \int_{\max(-1, u-1)}^{\min(1, u+1)} \rho(u-s)\eta(s) ds du \\
&= \frac{1}{2} \int_{|u| \leq 1} \sigma\left(\frac{u}{2}\right) e^{-icxu} du - \int_{|u| \leq 1} e^{-icxu} \int_{\max(-1, u-1)}^{\min(1, u+1)} \rho(u-s)\eta(s) ds du \\
&\quad + \frac{1}{2} \int_{1 \leq |u| \leq 2} \sigma\left(\frac{u}{2}\right) e^{-icxu} du - \int_{1 \leq |u| \leq 2} e^{-icxu} \int_{\max(-1, u-1)}^{\min(1, u+1)} \rho(u-s)\eta(s) ds du.
\end{aligned}$$

We now want to choose η so that

$$\frac{1}{2} \sigma\left(\frac{u}{2}\right) = \int_{\max(-1, u-1)}^{\min(1, u+1)} \rho(u-s)\eta(s) ds, \quad 1 \leq |u| \leq 2.$$

The two cases $1 \leq u \leq 2$ and $-2 \leq u \leq -1$ give the two equations:

$$\frac{1}{2} \sigma\left(\frac{u}{2}\right) = \begin{cases} \int_{u-1}^1 \rho(u-s)\eta(s) ds & (1 \leq u \leq 2) \\ \int_{-1}^{u+1} \rho(u-s)\eta(s) ds & (1 \leq -u \leq 2). \end{cases} \quad (2.24)$$

Differentiating both of these equations with respect to u gives

$$\frac{1}{4} \sigma'\left(\frac{u}{2}\right) = \begin{cases} -\rho(1)\eta(u-1) + \int_{u-1}^1 \rho'(u-s)\eta(s) ds & (1 \leq u \leq 2) \\ \rho(-1)\eta(u+1) + \int_{-1}^{u+1} \rho'(u-s)\eta(s) ds & (1 \leq -u \leq 2). \end{cases} \quad (2.25)$$

Defining integral operators $V_1 : L^2[0, 1] \rightarrow L^2[0, 1]$ and $V_2 : L^2[-1, 0] \rightarrow L^2[-1, 0]$ by

$$V_1 f(v) = \int_v^1 \rho'(v+1-s)f(s) ds; \quad V_2 f(w) = \int_{-1}^w \rho'(w-1-s)f(s) ds,$$

equation (2.25) above becomes

$$(V_1 - \rho(1)I)\eta(v) = \frac{1}{4} \sigma'\left(\frac{v+1}{2}\right); \quad (V_2 + \rho(-1)I)\eta(w) = \frac{1}{4} \sigma'\left(\frac{w-1}{2}\right) \quad (2.26)$$

for $v \in [0, 1]$ and $w \in [-1, 0]$. V_1 and V_2 are Volterra integral operators and $V_1 + \lambda I$, $V_2 + \lambda I$ have bounded inverses if and only if $\lambda \neq 0$. Therefore, to solve the equations (2.26) we require $\rho(1)\rho(-1) \neq 0$. With this proviso, (2.26) provides the definition of η on $[-1, 1]$. Integrating (2.25) with respect to u gives

$$\frac{1}{2} \sigma\left(\frac{u}{2}\right) = \begin{cases} \int_{u-1}^1 \rho(u-s)\eta(s) ds + C_1 & (1 \leq u \leq 2) \\ \int_{-1}^{u+1} \rho(u-s)\eta(s) ds + C_2 & (1 \leq -u \leq 2) \end{cases}$$

with C_1, C_2 arbitrary constants. However, by evaluating the first of these equations at $u = 2$ and the second at $u = -2$ we find that $C_1 = \sigma(1)/2$ and $C_2 = \sigma(-1)/2$. Hence, with η so defined, we have a solution of (2.24) only when $\sigma(1) = \sigma(-1) = 0$. Once this is assumed we define ξ by

$$\xi(u) = \frac{1}{2} \sigma\left(\frac{u}{2}\right) - \int_{\max(-1, u-1)}^{\min(1, u+1)} \rho(u-s) \eta(s) ds \quad (|u| \leq 1).$$

Then $f = pq + r$ as required. $\square$

We now apply the theorem to the construction of approximate generalized Gaussian quadratures for band-limited functions. Recall that by Theorems 2.1.7 and 2.1.16, the N th prolate $\bar{\phi}_N^{(c)}$ has precisely N nodes in $(-1, 1)$. Given a sequence $-1 < x_0 < x_1 < \dots < x_{N-1} < 1$, the invertibility of the $N \times N$ matrix $(\bar{\phi}_j(x_k))_{j,k=0}^{N-1}$ ensures the existence of real weights $\{w_k\}_{k=0}^{N-1}$ for which

$$\sum_{k=0}^{N-1} w_k \bar{\phi}_j(x_k) = \int_{-1}^1 \bar{\phi}_j(x) dx \quad (0 \leq j \leq N-1). \quad (2.27)$$

When the nodes $\{x_k\}_{k=0}^{N-1}$ are the N zeros of $\bar{\phi}_N$ in $(-1, 1)$, then the weights are positive and their sum is bounded independently of N and c . To see this, observe that the constant function 1 has prolate expansion $1 = \sum_{j=0}^{\infty} \bar{\phi}_j(x) \int_{-1}^1 \bar{\phi}_j(t) dt = \sum_{j=0}^{\infty} \mu_j \bar{\phi}_j(0) \bar{\phi}_j(x)$. Evaluating this equation at $x = x_k$, multiplying both sides by w_k , and summing over k gives

$$\begin{aligned} \sum_{k=0}^{N-1} w_k &= \sum_{k=0}^{N-1} \sum_{j=0}^{\infty} \mu_j \bar{\phi}_j(0) w_k \bar{\phi}_j(x_k) \\ &= \sum_{j=0}^{N-1} \mu_j \bar{\phi}_j(0) \sum_{k=0}^{N-1} w_k \bar{\phi}_j(x_k) + \sum_{j=N}^{\infty} \mu_j \bar{\phi}_j(0) \sum_{k=0}^{N-1} w_k \bar{\phi}_j(x_k) \\ &\leq \sum_{j=0}^{\infty} |\mu_j|^2 \bar{\phi}_j(0)^2 + \sum_{k=0}^{N-1} w_k \sum_{j=N}^{\infty} \mu_j \bar{\phi}_j(0) \bar{\phi}_j(x_k) \\ &\leq 2 + \sum_{k=0}^{N-1} w_k \sum_{j=N}^{\infty} 4j |\mu_j| \leq 2 + C |\mu_{N-1}| \sum_{k=0}^{N-1} w_k, \end{aligned}$$

where we have used the Plancherel theorem for prolate expansions ($2 = \int_{-1}^1 dt = \sum_{j=0}^{\infty} |\langle 1, \bar{\phi}_j \rangle|^2 = \sum_{j=0}^{\infty} |\mu_j|^2 \bar{\phi}_j(0)^2$) and the estimate (2.21). Rearranging this inequality gives $\sum_{k=0}^{N-1} w_k \leq 2/(1 - C |\mu_{N-1}|) \approx 2$ when N is significantly larger than $2c/\pi$. These ingredients allow for the following approximate generalized quadrature result [364].

Theorem 2.2.6. *Let $\{x_k\}_{k=0}^{N-1} \subset (-1, 1)$ be the nodes of the N th prolate function $\bar{\phi}_N = \bar{\phi}_N^{(c)}$ with bandwidth c , and let $\{w_k\}_{k=0}^{N-1}$ be real weights satisfying (2.27). Then for any function f with band limit $2c$ satisfying the conditions of Theorem 2.2.5 we have the quadrature error estimate*

$$E_f = \left| \sum_{k=0}^{N-1} w_k f(x_k) - \int_{-1}^1 f(x) dx \right| \leq C |\mu_{N-1}| (\|\eta\|_{L^2[-1,1]} + \|\xi\|_{L^2[-1,1]})$$

with η, ξ as in Theorem 2.2.5.

Proof. Since $\bar{\phi}_N(\pm 1) \neq 0$ by Lemma 2.6.8 and $\bar{\phi}_N(x) = \int_{-1}^1 \rho(t) e^{-icx} dt$ with $\rho(t) = \bar{\phi}_N(t)/\mu_N$, we may apply Theorem 2.2.5 to produce functions q and r of band limit c with $f = \bar{\phi}_N q + r$. Let $q = F_{-2c/\pi}(\eta)$ and $r = F_{-2c/\pi}(\xi)$. Then, since $\bar{\phi}_N(x_k) = 0$ for all nodes x_k , we have

$$\begin{aligned} E_f &= \left| \sum_{k=0}^{N-1} w_k f(x_k) - \int_{-1}^1 f(x) dx \right| \\ &= \left| \sum_{k=0}^{N-1} w_k [\bar{\phi}_N(x_k) q(x_k) + r(x_k)] - \int_{-1}^1 [\bar{\phi}_N(x) q(x) + r(x)] dx \right| \\ &\leq \left| \sum_{k=0}^{N-1} w_k r(x_k) - \int_{-1}^1 r(x) dx \right| + \left| \int_{-1}^1 \bar{\phi}_N(x) q(x) dx \right|. \end{aligned} \quad (2.28)$$

By the definition of q and the eigenfunction property (2.17) of the prolates,

$$\begin{aligned} \left| \int_{-1}^1 \bar{\phi}_N(x) q(x) dx \right| &= \left| \int_{-1}^1 \eta(t) \int_{-1}^1 \bar{\phi}_N(x) e^{icx} dx dt \right| \\ &= \left| \int_{-1}^1 \eta(t) \mu_N \bar{\phi}_N(t) dt \right| \leq |\mu_N| \|\eta\|_{L^2[-1,1]}. \end{aligned} \quad (2.29)$$

Furthermore, we may apply (2.22) with $2N$ replaced by N to obtain

$$\begin{aligned} \left| \sum_{k=0}^{N-1} w_k r(x_k) - \int_{-1}^1 r(x) dx \right| &= \left| \int_{-1}^1 \xi(t) \left(\sum_{k=0}^{N-1} w_k e^{-icx_k t} - \int_{-1}^1 e^{-icx} dx \right) dt \right| \\ &\leq C |\mu_{N-1}| \|\xi\|_{L^2[-1,1]}. \end{aligned} \quad (2.30)$$

Applying (2.29) and (2.30) to (2.28) now gives the required estimate. $\square$

The conditions $\sigma(\pm 1) = 0$ of Theorem 2.2.6 can be removed at the expense of a slower rate of convergence as the following corollary, whose proof can be found in Sect. 2.6.3, shows.

Corollary 2.2.7. *Let $\{x_k\}_{k=0}^{N-1}$, $\{w_k\}_{k=0}^{N-1}$, $\bar{\phi}_N$ be as in Theorem 2.2.6. Then for any function f with band limit $2c$ of the form $f(x) = \int_{-1}^1 \sigma(t) e^{-2icx} dt$ with $\sigma, \sigma' \in L^2[-1, 1]$, we have*

$$E_f = \left| \sum_{k=0}^{N-1} w_k f(x_k) - \int_{-1}^1 f(x) dx \right| \leq C \sqrt{|\mu_{N-1}|} (\|\sigma'\|_{L^2[-1,1]} + \|\sigma\|_{L^2[-1,1]}).$$

Discussion

It is important, when applying Theorem 2.2.6, that the norms $\|\eta\|_{L^2[-1,1]}$ and $\|\xi\|_{L^2[-1,1]}$ be not significantly larger than $\|\sigma\|_{L^2[-1,1]} + \|\sigma'\|_{L^2[-1,1]}$, for otherwise

the accuracy of the quadrature is compromised. In [364] it is claimed that such estimates have been obtained in the case where $n > 2c/\pi + 10\log(c)$. Numerical evidence of the accuracy of these procedures is also given in [364], where it is observed that the number of quadrature nodes required to accurately integrate band-limited functions is roughly $\pi/2$ times fewer than the number of classical Gaussian nodes needed to attain the same accuracy.

2.3 Local Sampling Formulas

2.3.1 Uniform Oversampling

In order to compute a representation of a band-limited function f that is valid on some interval I by means of the classical Shannon formula (Theorem 5.1.3), samples of f are required far from I . This is because of the slow decay of the cardinal sine function. One remedy to this situation is to replace the cardinal sine with an interpolating function having faster decay. This requires *oversampling*. If $f \in \text{PW}_\Omega$ and $\varphi \in L^2(\mathbb{R})$ is chosen so that

$$\hat{\varphi}(\xi) = \begin{cases} 1 & \text{if } |\xi| < \Omega/2 \\ 0 & \text{if } |\xi| > \Omega \end{cases} \quad (2.31)$$

then f can be reconstructed from the samples $\{f(n/(2\Omega))\}_{n=-\infty}^{\infty}$ via

$$f(t) = \frac{1}{2\Omega} \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2\Omega}\right) \varphi\left(t - \frac{n}{2\Omega}\right).$$

The sampling rate in this case is twice the *Nyquist* rate of the Shannon theorem, but may be reduced to any rate greater than or equal to the Nyquist rate by reducing the support of $\hat{\varphi}$. When the rate is strictly greater than Ω samples per unit time, φ may be chosen to have faster decay in time. How fast can a function φ satisfying (2.31) decay in time? There is no L^2 function satisfying (2.31) with exponential decay in time: the Fourier transform of such a function would be analytic in a strip in the complex plane containing the real line, a condition that would preclude the compact support of $\hat{\varphi}$ in (2.31).

It is simple to construct a function φ with inverse polynomial decay ($|\varphi(t)| \leq C(1+|t|)^{-N}$ for some fixed positive N) whose Fourier transform $\hat{\varphi}$ is a piecewise polynomial satisfying (2.31). In [244], functions having *root exponential decay* in time, that is, $|\varphi(t)| \leq Ce^{-\sqrt{2\pi}|t|}$, and satisfying (2.31) were constructed. In [93] Dziubański and Hernández showed that *sub-exponential decay* is possible: given any $\varepsilon > 0$, one can construct a function φ whose Fourier transform satisfies (2.31) and such that $|\varphi(t)| \leq C_\varepsilon e^{-\pi|t|^{1-\varepsilon}}$. The constant C_ε blows up as $\varepsilon \rightarrow 0$. Finally, in an unpublished manuscript, Güntürk and DeVore gave a sophisticated construction

that achieves perhaps the optimal outcome—a function φ with Fourier transform $\hat{\varphi}$ satisfying (2.31) such that $|\varphi(t)| \leq C \exp(-\alpha|x|/\log^\gamma|x|)$ for $\alpha > 0$ and $\gamma > 1$. This construction is outlined in Sect. 6.3.2.

Unfortunately, apart from the spline-based examples, none of the constructions just mentioned generate sampling functions φ that can be expressed in closed form—a serious disadvantage when compared with the simplicity of the Shannon theorem. In any event, the oversampling results require knowledge of samples taken off (though near) the interval on which an approximation is desired. In summary, if we are prepared to oversample—at any rate higher than that prescribed by the Shannon theorem—then uniform samples can be used to construct local approximations with samples taken on and near the interval of interest. But the absence of closed forms for the sampling functions of such schemes renders them impractical.

More is possible if we are prepared to forego the uniformity of the samples. As we shall see, relatively few nonuniform samples taken only on the interval of interest may be combined using prolates to construct accurate quadrature-based approximate sampling schemes.

2.3.2 Local Approximate Sampling with Prolates

In [296], the authors generated localized sampling formulas for band-limited functions on the line through the use of approximate quadrature schemes. In particular, given a function f with band limit $c > 0$ of the form (2.23), approximations of f via truncations of its prolate series are made, that is, $f(x) \approx \sum_{n=0}^N a_n \tilde{\phi}_n^{(c)}(x)$. Here the coefficients $\{a_n\}_{n=0}^N$ are obtained from samples $\{f(x_k)\}_{k=0}^{M-1}$ where $\{x_k\}_{k=0}^{M-1}$ are quadrature nodes associated with quadrature weights $\{w_k\}_{k=0}^{M-1}$ for which

$$\sum_{k=0}^{M-1} w_k g(x_k) \approx \int_{-1}^1 g(x) dx$$

for all functions on the line with band limit $2c$.

Since $\{\tilde{\phi}_n\}_{n=0}^\infty$ is a complete orthonormal system in $L^2[-1, 1]$, any $f \in L^2[-1, 1]$ admits a prolate expansion of the form

$$f(x) = \sum_{n=0}^{\infty} b_n \tilde{\phi}_n(x) \quad (2.32)$$

where $b_n = \langle f, \tilde{\phi}_n \rangle = \int_{-1}^1 f(x) \tilde{\phi}_n(x) dx$. We now develop error bounds for truncations of the series (2.32) for functions f with band limit $c > 0$. The starting point is the following estimate of the decay of the coefficients in these expansions [296].

Lemma 2.3.1. *Let f be a function on the line with band limit c as in (2.23) and $M_n = M_n^c$ be as in (2.21). Then*

$$\left| \int_{-1}^1 f(x) \bar{\phi}_n(x) dx \right| \leq |\mu_n| M_n \|\sigma\|_{L^1[-1,1]}.$$

Proof. With an application of (2.17) we have

$$\begin{aligned} \left| \int_{-1}^1 f(x) \bar{\phi}_n(x) dx \right| &= \left| \int_{-1}^1 \left(\int_{-1}^1 \sigma(t) e^{-icxt} dt \right) \bar{\phi}_n(x) dx \right| \\ &= \left| \int_{-1}^1 \sigma(t) \int_{-1}^1 \bar{\phi}_n(x) e^{-icxt} dx dt \right| \\ &= |\mu_n| \left| \int_{-1}^1 \sigma(t) \bar{\phi}_n(t) dt \right| \leq |\mu_n| M_n \|\sigma\|_{L^1[-1,1]} \quad \square \end{aligned}$$

Convergence of the series (2.32) is now investigated by estimating its tail. The following result is a variation on Lemma 3.2 of [296].

Lemma 2.3.2. *Let $c > 0$ and $N \geq 2c$ be an integer. Then*

$$\sum_{n=N+1}^{\infty} |\mu_n| (M_n)^2 \leq C(N+1)^{3/2} (0.386)^N.$$

Proof. By Lemma 2.6.7, for all n we have

$$|\mu_n| \leq \frac{\sqrt{\pi} c^n (n!)^2}{(2n)! \Gamma(n+3/2)} \leq \frac{4c^n}{2^{2n} (n-1)!}, \quad (2.33)$$

so that by (2.21),

$$\sum_{n=N+1}^{\infty} |\mu_n| (M_n)^2 \leq 16 \sum_{n=N+1}^{\infty} \frac{(c/4)^n n}{(n-1)!} \quad (2.34)$$

Let $h(x) = x(x+1)e^x = \sum_{n=1}^{\infty} nx^n/(n-1)!$. With $S_N(x)$ the N th partial sum of this series and $R_N(x) = h(x) - S_N(x)$ the N th remainder, the Taylor remainder theorem gives

$$\begin{aligned} |R_N(x)| &\leq \sup_{|t| \leq |x|} |h^{(N+1)}(t)| \frac{|x|^{N+1}}{(N+1)!} \\ &\leq (|x|^2 + (2N+3)|x| + (N+1)^2) e^{|x|} \frac{|x|^{N+1}}{(N+1)!}. \end{aligned}$$

Consequently, since $c \leq N/2$, the sum in (2.34) may be estimated by

$$\begin{aligned}
\sum_{n=N+1}^{\infty} |\mu_n|(M_n)^2 &\leq 16R_N(c/4) \\
&\leq 16 \left(\left(\frac{c}{4} \right)^2 + (2N+3) \left(\frac{c}{4} \right) + (N+1)^2 \right) e^{c/4} \frac{(c/4)^{N+1}}{(N+1)!} \\
&\leq \left(\frac{81}{4} \right) (N+1) e^{N/8} \frac{(N/8)^{N+1}}{N!}.
\end{aligned}$$

An application of the Stirling inequality $n^{n+1/2}\sqrt{2\pi} \leq e^n n!$ now gives

$$\sum_{n=N+1}^{\infty} |\mu_n|(M_n)^2 \leq \frac{81}{32\sqrt{2\pi}} (N+1) \sqrt{N} \left(\frac{e^{9/8}}{8} \right)^N$$

from which the result follows. $\square$

Lemma 2.3.2 remains true with the condition $N \geq (0.9)c$ in the sense that $\sum_{n=N+1}^{\infty} |\mu_n|(M_n)^2 \leq C(N+1)^{3/2} \beta^N$ with $0 < \beta < 1$.

These estimates allow us to investigate the nature of the convergence of the prolate series (2.32). The following result appears in [296].

Corollary 2.3.3. *Let $c > 0$ and f be a function on the line with band limit c of the form (2.23) for some $\sigma \in L^2[-1, 1]$. Then the prolate series (2.32) converges uniformly to f on $[-1, 1]$.*

Proof. Given non-negative integers $M \leq N$, with M_n as in (2.21),

$$\left| \sum_{n=M}^N \left(\int_{-1}^1 f(t) \bar{\phi}_n(t) dt \right) \bar{\phi}_n(x) \right| \leq \|\sigma\|_{L^1[-1,1]} \sum_{n=M}^{\infty} |\mu_n|(M_n)^2 \rightarrow 0$$

as $M, N \rightarrow \infty$ due to Lemma 2.3.2. Hence the series (2.32) satisfies the Cauchy criterion for convergence and consequently converges uniformly. Since it also converges in $L^2[-1, 1]$ to f , it must converge uniformly to f on $[-1, 1]$. $\square$

As a corollary of Lemma 2.3.1, if $c > 0$, N is a non-negative integer, and f is a band-limited function of the form (2.23), then for all $x \in [-1, 1]$,

$$\left| f(x) - \sum_{n=0}^N \langle f, \bar{\phi}_n \rangle \bar{\phi}_n(x) \right| \leq \left(\sum_{n=N+1}^{\infty} |\mu_n|(M_n)^2 \right) \|\sigma\|_{L^1[-1,1]}. \quad (2.35)$$

Lemma 2.3.2 may be applied to the right-hand side of (2.35) to show that if $N \geq 2c$ then the truncation error that appears on the left-hand side of (2.35) is bounded by $C(N+1)^{3/2} (0.386)^N \|\sigma\|_{L^1[-1,1]}$.

Given a function f with band limit c as in (2.23) and $n \geq 0$, the product $f \bar{\phi}_n^{(c)}$ has band limit $2c$. In fact, $f(x) \bar{\phi}_n^{(c)}(x) = \int_{-1}^1 \rho_n(r) e^{2icxr} dr$ with

$$\rho_n(r) = \frac{2}{\mu_n} \int_{-1}^1 \bar{\phi}_n^{(c)}(t) \sigma(2r-t) dt = \frac{2}{\mu_n} \bar{\phi}_n^{(c)} * \sigma(2r), \quad (2.36)$$

from which we obtain

$$\|\rho_n\|_{L^1[-1,1]} \leq \frac{1}{|\mu_n|} \|\bar{\phi}_n^{(c)}\|_{L^1[-1,1]} \|\sigma\|_{L^1[-1,1]} \leq \frac{\sqrt{2}}{|\mu_n|} \|\sigma\|_{L^1[-1,1]}. \quad (2.37)$$

Given weights $\{w_k\}_{k=0}^{K-1}$ and nodes $\{x_k\}_{k=0}^{K-1}$ which together form a quadrature in $GGQ(2c, \varepsilon)$ for some $\varepsilon > 0$ (see Definition 2.2.4), and a band-limited function f of the form (2.23), we may accurately estimate the $L^2[-1, 1]$ inner products $b_n = \langle f, \bar{\phi}_n \rangle$ that appear in (2.32) by the appropriate quadrature sum as in [296].

Lemma 2.3.4. *Let $c > 0$, N, K be positive integers, and suppose the nodes $\{x_k\}_{k=0}^{K-1}$ and weights $\{w_k\}_{k=0}^{K-1}$ form an element of $GGQ(2c, |\mu_N|^2)$. If f is a band-limited function on the line with band limit c as in (2.23), then*

$$\left| \langle f, \bar{\phi}_n \rangle - \sum_{k=0}^{K-1} w_k f(x_k) \bar{\phi}_n(x_k) \right| \leq \sqrt{2} |\mu_N| \|\sigma\|_{L^1[-1,1]}. \quad (2.38)$$

Proof. Since $f(x) \bar{\phi}_n(x) = \int_{-1}^1 \rho_n(t) e^{-2icxt} dt$ with ρ_n as in (2.36), by (2.37) we have

$$\begin{aligned} \left| \langle f, \bar{\phi}_n \rangle - \sum_{k=0}^{K-1} w_k f(x_k) \bar{\phi}_n(x_k) \right| &\leq |\mu_N|^2 \|\rho_n\|_{L^1[-1,1]} \\ &\leq \sqrt{2} \frac{|\mu_N|^2}{|\mu_n|} \|\sigma\|_{L^1[-1,1]} \leq \sqrt{2} |\mu_N| \|\sigma\|_{L^1[-1,1]}. \quad \square \end{aligned}$$

As shown in [296], the quadratures outlined above yield accurate localized sampling formulas for functions with band limit c .

Theorem 2.3.5. *Let $c \geq 0$, N, K be positive integers, and suppose the nodes $\{x_k\}_{k=0}^{K-1}$ and weights $\{w_k\}_{k=0}^{K-1}$ form an element of $GGQ(2c, |\mu_N|^2)$. For each integer $0 \leq k \leq K-1$, define the interpolating functions S_k by*

$$S_k(x) = w_k \sum_{n=0}^N \bar{\phi}_n(x_k) \bar{\phi}_n(x). \quad (2.39)$$

Then if $N > 2c$, for each band-limited function on the line of the form (2.23) and each $x \in [-1, 1]$ we have the estimate

$$\left| f(x) - \sum_{k=0}^{K-1} f(x_k) S_k(x) \right| \leq C(N+1)^{3/2} (0.386)^N \|\sigma\|_{L^1[-1,1]}.$$

Proof. We write

$$\begin{aligned} \left| f(x) - \sum_{k=0}^{K-1} f(x_k) S_k(x) \right| &\leq \left| f(x) - \sum_{n=0}^N \langle f, \bar{\phi}_n \rangle \bar{\phi}_n(x) \right| \\ &\quad + \left| \sum_{n=0}^N \langle f, \bar{\phi}_n \rangle \bar{\phi}_n(x) - \sum_{k=0}^{K-1} f(x_k) S_k(x) \right| = A + B. \end{aligned} \quad (2.40)$$

With an application of (2.35) and Lemma 2.3.2, A can be estimated by

$$A \leq \sum_{n=N+1}^{\infty} |\mu_n| (M_n)^2 \|\sigma\|_{L^1[-1,1]} \leq C(N+1)^{3/2} (0.386)^N \|\sigma\|_{L^1[-1,1]}. \quad (2.41)$$

Also, by Lemma 2.3.4 and the estimates (2.33) and (2.21),

$$\begin{aligned} B &\leq \sum_{n=0}^N \left| \langle f, \bar{\phi}_n \rangle - \sum_{k=0}^{K-1} f(x_k) w_k \bar{\phi}_n(x_k) \right| |\bar{\phi}_n(x)| \\ &\leq 2\sqrt{2} |\mu_N| \sum_{n=0}^N \sqrt{n} \|\sigma\|_{L^1[-1,1]} \\ &\leq C \frac{c^N (N+1)^{3/2}}{2^{2N} (N-1)!} \|\sigma\|_{L^1[-1,1]} \leq C(N+1)^2 (e/8)^N \|\sigma\|_{L^1[-1,1]} \end{aligned} \quad (2.42)$$

where we have used the Stirling inequality. Combining (2.40), (2.41), and (2.42) now gives the result. $\square$

2.4 Spectral Elements for Differential Equations

Boyd [44] considered the possibility of improving the efficiency of spectral element methods by replacing the Legendre polynomials in such algorithms by prolates. The unevenly spaced collocation grids associated with the Legendre polynomials translate into suboptimal efficiency of Legendre-based spectral elements. For appropriate values of the band-limit parameter c , the spacing of the prolate grid points is more uniform, allowing for improved accuracy and longer time steps when prolate elements are combined with time marching.

2.4.1 Modal and Nodal Bases

In either the Legendre or prolate formulations, *nodal* or *cardinal* functions C_ℓ satisfy $C_\ell(x_k) = \delta_{\ell k}$. Here $\{x_k\}_{k=1}^N$ is the collection of collocation points. The basis-dependent information in an N th order spectral method includes:

- (i) N collocation points $\{x_k\}_{k=1}^N$;
- (ii) N quadrature weights $\{w_k\}_{k=1}^N$;

- (iii) matrices $\alpha_{k\ell} = dC_\ell/dx|_{x=x_k}$ (for first-order equations) and $\beta_{k\ell} = d^2C_\ell/dx^2|_{x=x_k}$ (for second-order equations).

Changing the basis from Legendre polynomials to prolates or vice versa is equivalent to changing the collocation points, quadrature weights, and the matrices $(\alpha_{k\ell})$ and $(\beta_{k\ell})$.

Quadrature weights and nodes for the Legendre polynomials are well known, but computing them in the prolate case is far more complicated and requires solving a system of nonlinear equations via Newton iteration. For small values of c , it is appropriate to initialize the iteration at the Legendre weights and nodes. However, when N and c are both large, these nodes are not sufficiently accurate for the iteration to converge.

It is desirable, for ease of implementation, that *cardinal* prolate functions be constructed, that is, linear combinations $C_\ell = \sum_{n=0}^{N-1} a_{\ell n} \bar{\phi}_n^{(c)}$ with the property that $C_\ell(x_k) = \delta_{k\ell}$ where $x_1, x_2, \dots, x_N$ are predetermined nodes. The collection $\{C_\ell\}_{\ell=1}^N$ is known as the *nodal basis*. Returning to (1.18), that is, $\bar{\phi}_n = \sum_{m=0}^{\infty} \beta_{nm} \bar{P}_m$, we find that the cardinality condition is equivalent to the matrix equation

$$A\Phi = I$$

where $A, \Phi \in M_{N,N}$ with $A_{jn} = a_{jn}$ the unknown coefficient matrix and $\Phi_{nk} = \bar{\phi}_n^{(c)}(x_k)$, so that $A = \Phi^{-1}$. Hence, simple matrix inversion performs the change of basis from the *modal* basis $\{\bar{\phi}_n\}_{n=0}^{N-1}$ to the nodal basis $\{C_\ell\}_{\ell=1}^N$. On the other hand, by evaluating (1.18) at the nodes $\{x_k\}_{k=1}^N$ we have

$$\Phi = B\mathcal{L}$$

with $B_{nj} = \beta_{nj}$ the coefficients in (1.18) and $\mathcal{L}_{jk} = \bar{P}_j(x_k)$. With $\Phi', \Phi'' \in M_{N \times N}$ and $\mathcal{L}', \mathcal{L}'' \in M_{N \times N}$ derivative matrices with entries $(\Phi')_{nk} = d\bar{\phi}_n^{(c)}/dx|_{x=x_k}$ and $(\Phi'')_{nk} = d^2\bar{\phi}_n^{(c)}/dx^2|_{x=x_k}$, $(\mathcal{L}')_{nk} = d\bar{P}_n/dx|_{x=x_k}$ and $(\mathcal{L}'')_{nk} = d^2\bar{P}_n/dx^2|_{x=x_k}$, we also have the relations

$$\Phi' = B\mathcal{L}'; \quad \Phi'' = B\mathcal{L}'' . \quad (2.43)$$

The matrices $\mathcal{L}'$ and $\mathcal{L}''$ are easy to compute given the Legendre recurrence relation (1.13) and its derivatives or, alternatively, the differential recurrence relation (1.16).

When solving second-order differential equations via the spectral method using cardinal prolates, we require the derivative matrices $\mathcal{C}'$ and $\mathcal{C}'' \in M_{N \times N}$ given by $(\mathcal{C}')_{\ell k} = dC_\ell/dx|_{x=x_k}$ and $(\mathcal{C}'')_{\ell k} = d^2C_\ell/dx^2|_{x=x_k}$. From (2.43),

$$\mathcal{C}' = A\Phi' = AB\mathcal{L}'; \quad \mathcal{C}'' = A\Phi'' = AB\mathcal{L}'' .$$

The matrix Φ whose inverse A transforms the modal basis $\{\bar{\phi}_n\}_{n=0}^{N-1}$ to the nodal basis $\{C_j\}_{j=1}^N$ has condition number less than 10^{-4} for all c and $N \leq 30$ [44], but

becomes ill conditioned for larger values of N , in which case it is recommended that the modal basis be employed.

2.4.2 Computing Quadratures

A Gauss–Lobatto quadrature is a collection of nodes $\{x_k\}_{k=1}^N$ and weights $\{w_k\}_{k=1}^N$ such that $\int_{-1}^1 p(x) dx = \sum_{k=1}^N w_k p(x_k)$ for all polynomials p of degree up to $2N-3$. Here $x_1 = -1$, $x_N = 1$ and for $2 \leq i \leq N-1$, x_k is the $(i-1)$ th root of P'_{N-1} while $w_k = 2/(N(N-1)P_{N-1}(x_k)^2)$ for $1 \leq k \leq N$. We generalize this notion as follows.

Definition 2.4.1. An N -point *generalized Gauss–Lobatto quadrature* (GGLQ) for a set of $2N-2$ basis functions $\{\varphi_n\}_{n=0}^{2N-3}$ defined on $[-1, 1]$ is a collection of nodes $\{x_k\}_{k=1}^N$ and weights $\{w_k\}_{k=1}^N$ with $x_1 = -1$, $x_N = 1$ and $\int_{-1}^1 \varphi_n(x) dx = \sum_{k=1}^N w_k \varphi_n(x_k)$ for $0 \leq n \leq 2N-3$.

An N -point GGLQ for the prolates therefore has the property that with β_{nj} the Legendre coefficients of $\bar{\phi}_n$ as in (1.18), the $2N-2$ residuals

$$\rho_n = \int_{-1}^1 \bar{\phi}_n(x) dx - \sum_{k=1}^N w_k \bar{\phi}_n(x_k) = \sqrt{2} \beta_{n0} - \sum_{k=1}^N w_k \bar{\phi}_n(x_k) = 0$$

for $0 \leq n \leq 2N-3$. Since each ρ_n is a nonlinear function of the nodes $\{x_k\}_{k=1}^N$, the solution of these simultaneous equations must be approximated via Newton iteration.

We now assume that the nodes x_k are distributed symmetrically about the origin and the weights are symmetric, that is, if N is even then $x = 0$ is not a node and $x_k = x_{N-k+1}$ and $w_k = w_{N-k+1}$ for $1 \leq i \leq N/2$, while if N is odd then $x_{(N+1)/2} = 0$, $x_k = x_{N-k+1}$, $w_k = w_{N-k+1}$ for $1 \leq i \leq (N-1)/2$. Under these assumptions, the observation that $\bar{\phi}_{2j+1}^{(c)}$ is an odd function for all j implies that the odd residuals $\rho_{2j+1} = 0$. Consequently, we need only insist on the vanishing of the $N-1$ even residuals $r_j = \rho_{2j}$ ($0 \leq j \leq N-2$).

If N is even, the unknowns are the $N/2-1$ nodes $x_{(N+2)/2}, \dots, x_{N-1}$ and the $(N+1)/2$ weights $w_{(N+2)/2}, \dots, w_N$ while, if N is odd, the unknowns are the $(N-3)/2$ nodes $x_{(N+3)/2}, \dots, x_{N-1}$ and the $(N+1)/2$ weights $w_{(N+1)/2}, \dots, w_N$. In either case there are $N-1$ unknowns to be determined. The number of nodes to be determined is

$$M = \frac{1}{4}(2N-5+(-1)^N) = \begin{cases} (N-2)/2 & \text{if } N \text{ is even} \\ (N-3)/2 & \text{if } N \text{ is odd.} \end{cases}$$

Let $\mathbf{u} = (u_0, u_1, \dots, u_{N-2}) \in \mathbb{R}^{N-1}$ be the vector of unknowns

$$\mathbf{u} = \begin{cases} (x_{N-M}, x_{N-M+1}, \dots, x_{N-1}, w_{N-M}, w_{N-M+1}, \dots, w_N) & \text{if } N \text{ is even} \\ (x_{N-M}, x_{N-M+1}, \dots, x_{N-1}, w_{N-M-1}, w_{N-M}, \dots, w_N) & \text{if } N \text{ is odd,} \end{cases}$$

and $\mathbf{r} = \mathbf{r}(\mathbf{u})$ be the residual vector $\mathbf{r}(\mathbf{u}) = (r_0(\mathbf{u}), r_1(\mathbf{u}), \dots, r_{N-2}(\mathbf{u}))$. Thus we have a mapping $\mathbf{r} : \mathbb{R}^{N-1} \rightarrow \mathbb{R}^{N-1}$. Given an initial estimate $\mathbf{u}^{(0)}$ for the unknown weights and nodes, we expand $\mathbf{r}$ as a multivariate Taylor series about $\mathbf{u}^{(0)}$, truncate the series at the linear term, and solve the resulting matrix equation to obtain a second estimate $\mathbf{u}^{(1)}$. This gives the Newton iteration:

$$J(\mathbf{r}(\mathbf{u}^{(n)}))\mathbf{u}^{(n+1)} = -\mathbf{r}(\mathbf{u}^{(n)}) + J(\mathbf{r}(\mathbf{u}^{(n)}))\mathbf{u}^{(n)} \quad (2.44)$$

with $\mathbf{u}^{(n)}$ the n th iterate and $J(\mathbf{r}(\mathbf{u}))$ the $(N-1) \times (N-1)$ Jacobian matrix with (j, k) th entry $[J(\mathbf{r}(\mathbf{u}))]_{jk} = \frac{\partial r_j}{\partial u_k}(\mathbf{u})$ ($0 \leq j, k \leq N-2$). Because of the symmetry assumptions, each residual r_j may be written as

$$r_j = \sqrt{2}\beta_{2j,0} - 2 \sum_{k=N-M}^N w_k \bar{\phi}_{2j}(x_k) + s w_{N-M-1} \bar{\phi}_{2j}(0)$$

with $s = (1 + (-1)^{N+1})/2$, so that the entries of the Jacobian matrix become

$$[J(\mathbf{u})]_{jk} = \frac{\partial r_j}{\partial u_k} = \begin{cases} -2w_{N-M+k}(\bar{\phi}_{2j})'(x_{N-M+k}), & \text{if } 0 \leq k \leq M-1 \\ -(2-s)\bar{\phi}_{2j}(x_{N-M-s}), & \text{if } k = M \\ -2\bar{\phi}_{2j}(x_{N-2M-s+k}), & \text{if } M+1 \leq k \leq N-2. \end{cases}$$

In [44], Boyd describes a variation of this procedure known as *underrelaxation*, which has been shown to expand the domain of convergence, that is, the set of initializations $\mathbf{u}^{(0)}$ for which the algorithm will converge. The following comments summarize the discussion in [44].

When c is small, the Legendre–Lobatto nodes and weights form a good initialization for the iteration. However, in this situation it is usually sufficient to start with more simply defined nodes and weights $\tilde{x}_k, \tilde{w}_k, k = 1, \dots, N$ given by

$$\tilde{x}_k = -\cos\left(\frac{\pi(k-1)}{N-1}\right); \quad \tilde{w}_k = \begin{cases} \frac{\pi}{N} \sin\left(\frac{\pi(k-1)}{N-1}\right), & (1 < k < N) \\ \frac{2}{N(N-1)}, & k = 1 \text{ or } N. \end{cases}$$

With $c^*(N) = (N+1/2)\pi/2$ the *transition bandwidth*, for $N \leq 12$, the above definitions give satisfactory initializations in the range $0 \leq c \leq c^*(N)$. For $N > 12$, the iteration will converge on a smaller range of values of c (see [44]). For large N and c near $c^*(N)$, a better initialization is required. In [44] it is shown that the variation of nodes with c is well approximated by a parabola symmetric about $c = 0$ for all $0 \leq c \leq c^*(N)$. This suggests that if $\mu/c^*(N)$ is small enough so that the Legendre–Lobatto points provide a successful initialization of the Newton iteration for the prolates $\{\bar{\phi}_n^{(\mu)}\}$, then a good initialization for a Newton iteration for the prolates $\{\bar{\phi}_n^{(c)}\}$ with $c > \mu$ can be obtained by the quadratic extrapolation $x_k(c) \approx x_k(0) + (x_k(\mu) - x_k(0))^2 c^2 / \mu^2$.

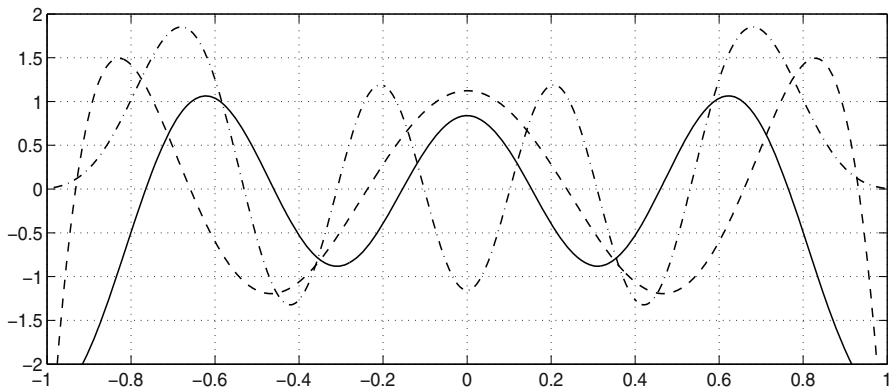


Fig. 2.1 Plots of $\bar{\phi}_6^{(c)}$ for $c = 1$ (dashed), $c = 20$ (dots) and $c = c^*(6) \approx 10.2$ (solid), the transition bandwidth. The zeros are evenly distributed for $c = c^*(6)$

An explanation of the more uniform distribution of the zeros of the prolates relative to that of the Legendre polynomials was given by Boyd in [43], where he observed that if $c < c^*(n)$ then for all $x \in [-1, 1]$,

$$\bar{\phi}_n^{(c)}(x) \sim \frac{\sqrt{\mathcal{E}(x; m)}}{(1-x^2)^{1/4}(1-mx^2)^{1/4}} J_0\left(\frac{c}{\sqrt{m}} \mathcal{E}(x; m)\right)$$

where J_0 is the Bessel function of order zero, $m = c^2/\chi_n$ with χ_n the n th eigenvalue of the prolate operator $\mathcal{P}$ as in (1.6), and $\mathcal{E}(x; m) = \int_x^1 (\sqrt{1-mt^2}/\sqrt{1-t^2}) dt$ is the difference of the complete and incomplete elliptic integrals of the second kind. Outside small neighborhoods of ± 1 of width $O(c^{-1/2})$, J_0 can be approximated to obtain

$$\bar{\phi}_n^{(c)}(x) \sim \sqrt{\frac{2}{\pi \chi_n}} \frac{m^{1/4}}{(1-x^2)^{1/4}(1-mx^2)^{1/4}} \cos(cm^{-1/2} \mathcal{E} - \pi/4).$$

Since $\chi_n(c^*(n)) = c^*(n)^2(1 + O(n^{-2}))$, we have

$$\bar{\phi}_n^{c^*(n)}(x) \sim \sqrt{\frac{2}{\pi}} \frac{1}{c} (1-x^2)^{-1/2} \cos(n\pi(1-x)/2),$$

from which we see that when $c \sim c^*(n)$, $\bar{\phi}_n^{(c)}$ oscillates like $\cos(n\pi(1-x)/2)$ so that the zeros of these prolates are nearly uniformly distributed.

A plot of the prolates $\bar{\phi}_6^{(c)}$ for $c = 1, 20$ and the transition bandwidth $c = c^*(6) = \pi(6 + 1/2)/2 \approx 10.2$ appears in Fig. 2.1. Observe the more uniform arrangement of the zeros of $\bar{\phi}_6^{(c^*)}$ over those of $\bar{\phi}_6^{(1)}$ and $\bar{\phi}_6^{(20)}$.

2.4.3 Convergence and Continuously Adjusted Spectral Parameters

Given a series $S = \sum_{n=0}^{\infty} f_n$ with $\{f_n\}_{n=0}^{\infty}$ functions defined on $[a, b]$, we say that the error in the sequence of partial sums $S_N = \sum_{n=0}^N f_n$ has asymptotic rate of geometric convergence $\mu > 0$ if

$$\overline{\lim}_{N \rightarrow \infty} \frac{\log E(N)}{N} = -\mu$$

where $E(N) = \|S - S_N\|_{L^\infty[a, b]}$. Series with $\|S - S_N\|_{L^\infty[a, b]} \leq A(N)e^{-N\mu}$ with A an algebraic function of N are examples of such series.

When the basis functions contain an internal parameter (such as the band limit parameter c of the prolate basis $\{\tilde{\phi}_n^{(c)}\}_{n=0}^{\infty}$) it is useful to generalize the notion of convergence to allow for variations of this internal parameter.

Definition 2.4.2 (Continuously adjusted asymptotic rate of geometric convergence). Suppose that for each τ in an interval I , the collection $\{\varphi_n^{(\tau)}\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2[a, b]$ and for each integer $n \geq 1$, the mapping $\tau \mapsto \varphi_n^{(\tau)}$ from I to $L^2[a, b]$ is continuous. Let $c : \mathbb{N} \rightarrow I$. Given $f \in L^2[a, b]$ and an integer $N \geq 1$, let

$$S_N = S_N^{(c)} = \sum_{n=1}^N \langle f, \varphi_n^{(c(N))} \rangle \varphi_n^{(c(N))}$$

and $E^{(c)}(N) = \|S - S_N^{(c)}\|_{L^\infty[a, b]}$. We say that the partial sums $\{S_N^{(c)}\}_{N=1}^{\infty}$ have continuously adjusted asymptotic rate of geometric convergence ν if

$$\overline{\lim}_{N \rightarrow \infty} \frac{\log E^{(c)}(N)}{N} = -\nu.$$

In [43], Boyd investigated a two-dimensional array of errors $E^{(c)}(N)$ for prolate expansions of particular functions f as N and c vary and argued that c should be chosen smaller than $c^*(N)$ for accurate representation.

2.5 Prolate Expansions of Non-band-limited Functions

The key to approximating non-band-limited functions by prolate series is first to approximate them by band-limited functions. The following result from [41] deals with approximations of analytic functions by band-limited functions.

Theorem 2.5.1 (Constructive band-limited approximation). Suppose f is analytic in the rectangle $[-1 - \delta, 1 + \delta] \times [-\delta, \delta] \subset \mathbb{C}$ for some $\delta > 0$. Let $c > 0$ and $f^{(c)}$ be the function on the line defined by

$$f^{(c)}(x) = \frac{1}{2\pi} \int_{-c}^c \left(\int_{-\infty}^{\infty} f(t)b(t)e^{-it\xi} dt \right) e^{i\xi x} d\xi \quad (2.45)$$

where $b(x) \geq 0$ is C^∞ and satisfies $b(x) = 1$ if $|x| \leq 1$ and $b(x) = 0$ if $|x| > 1 + \delta$. Then

- (i) $f^{(c)}$ is band limited with band limit c , and
- (ii) for all integers $n > 0$, there exists a constant $A_n > 0$ such that

$$\max_{|x| \leq 1} |f(x) - f^{(c)}(x)| \leq A_n c^{-n}.$$

Proof. Claim (i) is clear from the definition (2.45). To see (ii), note that $f^{(c)}(x) = (\widehat{[-c, c] f b(\cdot/2\pi)})^\vee(x/2\pi)/(2\pi)$ and $|x| < 1$. Then

$$\begin{aligned} |f(x) - f^{(c)}(x)| &= |f(x)b(x) - f^{(c)}(x)| \\ &\leq \int_{-\infty}^{\infty} |\widehat{fb}(\xi) - \widehat{f^{(c)}}(\xi)| d\xi \\ &= \int_{-\infty}^{\infty} |\widehat{bf}(\xi) - \widehat{[-\frac{c}{2\pi}, \frac{c}{2\pi}]}(\xi) \widehat{bf}(\xi)| d\xi = \int_{|\xi| > \frac{c}{2\pi}} |\widehat{bf}(\xi)| d\xi. \end{aligned}$$

But $f \in C^\infty$, so $bf \in C_c^\infty \subset \mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing functions. Since the Fourier transform preserves $\mathcal{S}$, $\widehat{bf} \in \mathcal{S}$ and for all $n \geq 0$ there exists $B_n > 0$ such that $|\widehat{bf}(\xi)| \leq B_n |\xi|^{-n-1}$. Hence $|f(x) - f^{(c)}(x)| \leq \int_{|\xi| > c/2\pi} B_n |\xi|^{-n-1} d\xi = A_n c^{-n}$. $\square$

If f is analytic on the rectangle $[-1 - \delta, 1 + \delta] \times [-\delta, \delta] \subset \mathbb{C}$ for some $\delta > 0$ then the Legendre and prolate series for f on $[-1, 1]$ converge geometrically. In fact, when $f \in H^s[-1, 1]$ (the Sobolev space of functions in $L^2[-1, 1]$ with square-integrable distributional derivatives up to order s) we have the following variation on Theorem 3.1 of [69].

Theorem 2.5.2. *Let $f \in H^s[-1, 1]$, b_n ($n \geq 0$) be as in (2.32), $\{\chi_n\}_{n=0}^\infty$ be the corresponding eigenvalue of the differential operator $\mathcal{P}$ in (1.6), and suppose $\chi_N \geq 400$. Then there exists a positive constant D such that*

$$|b_N| \leq D \left[\left(\frac{\chi_N}{16} \right)^{-s/2} \|f\|_{H^s[-1, 1]} + \left(\frac{(3.75)c}{\sqrt{\chi_N}} \right)^{\sqrt{\chi_N}/4} \|f\|_{L^2[-1, 1]} \right]. \quad (2.46)$$

Since $\chi_n \sim n(n+1)[1 + O(c^2/2n^2)]$ (see [43]), (2.46) forces the rate of decay of the prolate coefficients of $f \in H^s[-1, 1]$ to be geometric. However, (2.46) also forces the rate of decay of the prolate coefficients of $f \in C^\infty[-1, 1]$ (in particular for band-limited f) to be roughly of the order of n^{-n} once $\chi_N \geq (3.75)c$.

The result of Chen, Gottlieb, and Hesthaven [69] gives decay of the form

$$|b_N| \leq C(N^{-2s/3} \|f\|_{H^s[-1, 1]} + (c/\sqrt{\chi_N})^{\delta N} \|f\|_{L^2[-1, 1]})$$

with $\delta > 0$, while Wang and Zhang [349] work in a different Sobolev-type space, showing that

$$|b_N| \leq C(N^{-s} \|(1-x^2)^{s/2} D_x^s f\|_{L^2[-1,1]} + (2^{1/6} c / \sqrt{\chi_N})^{\delta N} \|f\|_{L^2[-1,1]}).$$

The proof of Theorem 2.5.2 can be reduced to the following lemmas, whose proofs may be found in Sect. 2.6.4.

Lemma 2.5.3. Suppose $\bar{\phi}_N^{(c)}$ has the Legendre expansion $\bar{\phi}_N^{(c)} = \sum_{k=0}^{\infty} \beta_k \bar{P}_k$ and $q = q_N = c/\sqrt{\chi_N} < 1$. Let $m = [\sqrt{\chi_N}/4]$ be the largest integer less than or equal to $\sqrt{\chi_N}/4$. Then there exists a positive constant D (independent of k and m) such that for all $k \leq 2m$ and with $\alpha = 2.07$,

$$|\beta_k| \leq \begin{cases} D(\alpha/q)^k |\beta_0| & \text{if } k \text{ is even} \\ D(\alpha/q)^k |\beta_1| & \text{if } k \text{ is odd.} \end{cases} \quad (2.47)$$

Let $A_k = \int_{-1}^1 x^k \bar{\phi}_N^{(c)}(x) dx$. Then $A_0 = \sqrt{2}\beta_0$ and $A_1 = \sqrt{2/3}\beta_1$.

Lemma 2.5.4. Suppose $\chi_N \geq 400$. Let $m = [\sqrt{\chi_N}/4]$ and $q = c/\sqrt{\chi_N} < 1$. Then there exists a constant C independent of m for which

$$|A_0| \leq C q^{2m-2} \gamma^m \frac{(3/7)^{\sqrt{\chi_N}/2}}{\sqrt{2m-3/2}} \quad \text{and} \quad |A_1| \leq C q^{2m-2} \gamma^m \frac{(3/7)^{\sqrt{\chi_N}/2}}{\sqrt{2m-1/2}} \quad (2.48)$$

with $\gamma = 4e^2/3$.

Proof (of Theorem 2.5.2). Let f have Legendre expansion $f = \sum_{k=0}^{\infty} a_k \bar{P}_k$ and prolate expansion (2.32). Then $b_N = \int_{-1}^1 f(x) \bar{\phi}_N^{(c)}(x) dx = \sum_{k=0}^{\infty} a_k \beta_k$. With $m = [\sqrt{\chi_N}/4]$, let $f_m = \sum_{k=0}^m a_k \bar{P}_k$ be the m th partial sum of the Legendre expansion of f . Then

$$b_N = \int_{-1}^1 f_m(x) \bar{\phi}_N^{(c)}(x) dx + \int_{-1}^1 [f(x) - f_m(x)] \bar{\phi}_N^{(c)}(x) dx = I + II. \quad (2.49)$$

The truncation error in the Legendre expansion of f satisfies $\|f - f_m\|_{L^2[-1,1]} \leq D m^{-s} \|f\|_{H^s[-1,1]}$ (see [105], [61]), so with an application of the Cauchy–Schwarz inequality, the second integral in (2.49) is bounded by

$$|II| \leq \|f - f_m\|_{L^2[-1,1]} \leq \frac{D}{m^s} \|f\|_{H^s[-1,1]} \leq D \left(\frac{16}{\chi_N} \right)^{s/2} \|f\|_{H^s[-1,1]}. \quad (2.50)$$

Let the constants α and γ be as in Lemma 2.5.3 and Lemma 2.5.4. Application of Cauchy–Schwarz and Lemma 2.5.3 show that the first integral in (2.49) is bounded by

$$\begin{aligned}
|I| &= \left| \sum_{k=0}^m a_k \beta_k \right| \leq \left(\sum_{k=0}^m |a_k|^2 \right)^{1/2} \left(\sum_{k=0}^m |\beta_k|^2 \right)^{1/2} \\
&\leq D \|f\|_{L^2[-1,1]} \left(\sum_{k=0}^m \left(\frac{\alpha}{q} \right)^{2k} \right)^{1/2} \max\{|\beta_0|, |\beta_1|\} \\
&\leq D \|f\|_{L^2[-1,1]} \left(\frac{\alpha}{q} \right)^{m+1} \max\{|\beta_0|, |\beta_1|\}.
\end{aligned}$$

Since $\beta_0 = A_0/\sqrt{2}$ and $\beta_1 = \sqrt{3/2}A_1$, Lemma 2.5.4 gives

$$|I| \leq D \|f\|_{L^2[-1,1]} \alpha^{m+1} \gamma^m q^{m-3} \frac{(3/7)^{\sqrt{\chi_N}/2}}{\sqrt{2m-1/2}}. \quad (2.51)$$

Since $\sqrt{\chi_N}/4 - 1 < m \leq \sqrt{\chi_N}/4$ and $\alpha\gamma(3/7)^2 \leq 3.75$, combining (2.49), (2.50), and (2.51) now gives the result. $\square$

In [43], Boyd examined numerically the rate of convergence of prolate series of representative proptotype functions. For fixed $\xi \in \mathbb{R}$, the coefficients of the function $e^{-ix\xi}$ in the Chebyshev, Legendre, and prolate series respectively are

$$\begin{aligned}
b_n^{\text{Cheb}}(\xi) &= \int_{-1}^1 T_n(x) (1-x^2)^{-1/2} e^{-ix\xi} dx = \pi i^n J_n(-\xi), \\
b_n^{\text{Leg}}(\xi) &= \int_{-1}^1 \bar{P}_n(x) e^{-ix\xi} dx = \frac{\pi^{3/2}}{2} i^n \left(\frac{2}{\xi} \right)^{1/2} (n+1/2) J_{n+1/2}(\xi), \\
b_n^{\text{pro}}(\xi) &= \int_{-1}^1 \bar{\phi}_n^{(c)}(x) e^{-ix\xi} dx = \mu_n \bar{\phi}_n^{(c)}(\xi/c).
\end{aligned}$$

Coefficients of the Chebyshev and Legendre series can be analyzed through well-known asymptotics of Bessel functions of integer and half-integer order respectively. These oscillate until $n \approx \xi$ and then fall sharply as $n \rightarrow \infty$. The prolate eigenvalues μ_n are approximately $\sqrt{2\pi/c}$ for $n < 2c/\pi$ and decay super-exponentially as $n \rightarrow \infty$. This leads to the following heuristic:

In the expansions of $e^{-ix\xi}$ ($\xi \in \mathbb{R}$ fixed and $|x| < 1$):

- (i) Accurate truncations of Chebyshev and Legendre series require $N \geq \xi$;
- (ii) Accurate truncations of prolate series with bandwidth parameter $c = \xi$ require $N \geq 2\xi/\pi$.

Hence, one can approximate a sinusoidal signal accurately using $2/\pi$ as many prolates as with either Legendre or Chebyshev polynomial bases.

We have seen that expansions of complex exponentials lead to expansions of band-limited functions and that by approximating analytic functions by band-limited functions we may write expansions which accurately approximate analytic functions. Boyd [43] quantified the superior performance of prolate expansions over Chebyshev and Legendre expansions, at least in the case of band-limited functions in the following terms.

Let $f \in L^2(\mathbb{R})$ have Fourier transform $\hat{f}$ and $\{g_n\}_{n=1}^\infty$ be either the Chebyshev, Legendre, or prolate orthonormal basis for $L^2[-1, 1]$. By the Plancherel theorem, the n th coefficient a_n of the series $f(x) = \sum_{n=0}^\infty a_n g_n(x)$ ($|x| \leq 1$) may be written as $a_n = \int_{-\infty}^\infty \hat{f}(\xi) b_n(\xi/2\pi) d\xi$ with $b_n(\xi) = b_n^{\text{Cheb}}(\xi)$ or $b_n^{\text{Leg}}(\xi)$ or $b_n^{\text{pro}}(\xi)$ the Chebyshev, Legendre, or prolate coefficients respectively of the function $e^{-ix\xi}$ (ξ fixed) as above. If f is band limited with band limit c , then

- (i) The Chebyshev and Legendre coefficients of f fall super-geometrically for all $|\xi| < c$ when $n > c$, and
- (ii) The prolate coefficients with respect to $\{\bar{\phi}_n^{(c)}\}_{n=0}^\infty$ fall super-exponentially for $n > 2c/\pi$.

2.6 Notes and Auxiliary Results

2.6.1 Miscellaneous Results on Prolates

Derivative Matrices for Spectral Elements

In Sect. 2.4, formulas are given for first and second derivative matrices for prolate series and cardinal prolate series. This is achieved through prior knowledge of the equivalent matrices for Legendre polynomials and the Legendre expansion of the prolates. However, the derivative matrices for prolate series can also be computed directly without direct reference to the Legendre basis, as the following series of results from [364] shows.

Theorem 2.6.1. *Let $c > 0$ and let m, n be non-negative integers. Then with $\bar{\phi}_n = \bar{\phi}_n^{(c)}$,*

$$\int_{-1}^1 \bar{\phi}_n'(x) \bar{\phi}_m(x) dx = \begin{cases} 0 & \text{if } m \equiv n \pmod{2} \\ \frac{2\mu_m^2}{\mu_m^2 + \mu_n^2} \bar{\phi}_m(1) \bar{\phi}_n(1) & \text{if } m \not\equiv n \pmod{2} \end{cases} \quad (2.52)$$

and

$$\int_{-1}^1 x \bar{\phi}_m(x) \bar{\phi}_n(x) dx = \begin{cases} 0 & \text{if } m \equiv n \pmod{2} \\ \frac{2}{ic} \frac{\mu_m \mu_n}{\mu_m^2 + \mu_n^2} \bar{\phi}_m(1) \bar{\phi}_n(1) & \text{if } m \not\equiv n \pmod{2}. \end{cases} \quad (2.53)$$

Proof. Since $\bar{\phi}_n$ is even if n is even and odd if n is odd, the integrals $\int_{-1}^1 \bar{\phi}_m'(x) \bar{\phi}_n(x) dx$ and $\int_{-1}^1 x \bar{\phi}_m(x) \bar{\phi}_n(x) dx$ are zero if $m \equiv n \pmod{2}$ since the integrands are odd. Suppose then that $m \not\equiv n \pmod{2}$. Differentiating (2.17) with respect to x gives $\mu_n \bar{\phi}_n'(x) = ic \int_{-1}^1 t e^{icxt} \bar{\phi}_n(t) dt$. Multiplying both sides of this equation by $\bar{\phi}_m(x)$ and integrating with respect to x over $[-1, 1]$ gives

$$\begin{aligned} \mu_n \int_{-1}^1 \bar{\phi}_n'(x) \bar{\phi}_m(x) dx &= ic \int_{-1}^1 \bar{\phi}_m(x) \int_{-1}^1 t e^{icxt} \bar{\phi}_n(t) dt dx \\ &= ic \int_{-1}^1 t \bar{\phi}_n(t) \int_{-1}^1 e^{icxt} \bar{\phi}_m(x) dx dt = ic \mu_m \int_{-1}^1 t \bar{\phi}_n(t) \bar{\phi}_m(t) dt. \end{aligned} \quad (2.54)$$

However, by interchanging the roles of m and n in (2.54) we obtain

$$\mu_m \int_{-1}^1 \bar{\phi}'_m(x) \bar{\phi}_n(x) dx = ic\mu_n \int_{-1}^1 t \bar{\phi}_n(t) \bar{\phi}_m(t) dt \quad (2.55)$$

and, since the integrals on the right-hand sides of (2.54) and (2.55) are the same, an integration by parts yields

$$\begin{aligned} \int_{-1}^1 \bar{\phi}'_m(x) \bar{\phi}_n(x) dx &= \frac{\mu_n^2}{\mu_m^2} \int_{-1}^1 \bar{\phi}_m(x) \bar{\phi}'_n(x) dx \\ &= \frac{\mu_n^2}{\mu_m^2} \left(\bar{\phi}_m(1) \bar{\phi}_n(1) - \bar{\phi}_m(-1) \bar{\phi}_n(-1) - \int_{-1}^1 \bar{\phi}'_m(x) \bar{\phi}_n(x) dx \right) \\ &= \frac{\mu_n^2}{\mu_m^2} \left(2 \bar{\phi}_m(1) \bar{\phi}_n(1) - \int_{-1}^1 \bar{\phi}'_m(x) \bar{\phi}_n(x) dx \right) \end{aligned} \quad (2.56)$$

since $m \not\equiv n \pmod{2}$, and thus $\bar{\phi}_m(-1) \bar{\phi}_n(-1) = -\bar{\phi}_m(1) \bar{\phi}_n(1)$. Rearranging (2.56) gives (2.52). Combining (2.52) and (2.54) gives (2.53). $\square$

It is clear from symmetry conditions that if $m \not\equiv n \pmod{2}$ then

$$\begin{aligned} \int_{-1}^1 x \bar{\phi}'_n(x) \bar{\phi}_m(x) dx &= \int_{-1}^1 x^2 \bar{\phi}''_n(x) \bar{\phi}_m(x) dx = \int_{-1}^1 x^2 \bar{\phi}'_n(x) \bar{\phi}'_m(x) dx \\ &= \int_{-1}^1 \bar{\phi}''_n(x) \bar{\phi}_m(x) dx = \int_{-1}^1 x^2 \bar{\phi}_n(x) \bar{\phi}_m(x) dx = 0. \end{aligned}$$

When $m \equiv n \pmod{2}$, the (nonzero) values of the above integrals are computed in [364] in terms of the values of $\bar{\phi}_n$, $\bar{\phi}_m$ and various derivatives of these functions at $x = 1$, as well as the eigenvalues μ_n , μ_m , χ_n , χ_m . Here is a typical example.

Theorem 2.6.2. *Let $c > 0$ and let m, n be non-negative integers. Then with $\bar{\phi}_n = \bar{\phi}_n^{(c)}$, if $m \equiv n \pmod{2}$, then*

$$\begin{aligned} \int_{-1}^1 x^2 \bar{\phi}''_n(x) \bar{\phi}_m(x) dx &= \frac{2\mu_n(\bar{\phi}'_n(1) \bar{\phi}_m(1) - \bar{\phi}'_m(1) \bar{\phi}_n(1))}{\mu_m - \mu_n} - \frac{4\mu_n \bar{\phi}_n(1) \bar{\phi}_m(1)}{\mu_m + \mu_n} \\ &= \frac{\mu_n(\chi_n - \chi_m) \bar{\phi}_n(1) \bar{\phi}_m(1)}{\mu_m - \mu_n} - \frac{4\mu_n \bar{\phi}_n(1) \bar{\phi}_m(1)}{\mu_m + \mu_n}. \end{aligned}$$

Numerical Evaluation of μ_n

The methods of Bouwkamp and Boyd for the evaluation of prolates as described in Sect. 1.2.4 also produce the associated eigenvalues $\{\chi_n\}_{n=0}^\infty$ of the differential operator $\mathcal{P}$ of (1.6). From this information we may apply (2.17) to estimate the eigenvalues $\{\mu_n\}_{n=0}^\infty$ of the integral operator F_c defined in Corollary 1.2.6 and hence the eigenvalues $\{\lambda_n\}_{n=0}^\infty$ of the iterated projections $P_{2c/\pi} Q$ of Sect. 1.2. To see this, observe that putting $n = 0$ and $x = 0$ in (2.17) gives

$$\mu_0 \bar{\phi}_0(0) = \int_{-1}^1 \bar{\phi}_0(x) dx.$$

Given that the values of $\bar{\phi}_0$ have already been computed accurately by the Bouwkamp or Boyd methods and that $\bar{\phi}_0(0)$ is not small, estimating the integral on the right-hand side numerically gives μ_0 . Putting $n = 0$, $m = 1$ in (2.53) and numerically computing the integral on the left-hand side of that equation now gives μ_1 . For $m \geq 2$ odd, μ_m may be obtained from (2.53) with $n = 0$, while if $m \geq 2$ is even, we obtain μ_m by applying (2.53) with $n = 1$.

Lemma 2.6.3. *For all non-negative integers n ,*

$$\int_{-1}^1 t^n P_n(t) dt = \frac{2^{n+1} (n!)^2}{(2n+1)!} = \frac{\sqrt{\pi} n!}{2^n \Gamma(n+3/2)}. \quad (2.57)$$

The proof is by induction, the inductive step being provided by the recurrence relation (1.13) and the fact that $\int_{-1}^1 t^m P_n(t) dt = 0$ for all integers $0 \leq m < n$.

Decay of Eigenvalues μ_n

The following lemma appears in Fuchs's paper [103].

Lemma 2.6.4. *The eigenvalues $\mu_n = \mu_n(2c/\pi)$ associated with the eigenfunctions $\tilde{\phi}_n^{(c)}$ as in (2.17) satisfy the differential equation*

$$\frac{d}{dc} \mu_n(2c/\pi) = \frac{\mu_n(2c/\pi)}{2c} [2\tilde{\phi}_n^{(c)}(1)^2 - 1]. \quad (2.58)$$

Proof. Let $\varphi_n^{(c)}(t) = c^{-1/4} \tilde{\phi}_n^{(c)}(t/\sqrt{c})$ ($n \geq 0$). Then

$$\int_{-\sqrt{c}}^{\sqrt{c}} \varphi_n^{(c)}(t) \varphi_m^{(c)}(t) dt = \int_{-1}^1 \tilde{\phi}_n^{(c)}(t) \tilde{\phi}_m^{(c)}(t) dt = \delta_{nm},$$

and, because of (2.17),

$$\int_{-\sqrt{c}}^{\sqrt{c}} e^{ist} \varphi_n^{(c)}(s) ds = \mu_n(2c/\pi) \sqrt{c} \varphi_n^{(c)}(t). \quad (2.59)$$

Multiplying both sides of (2.59) by $\varphi_n^{(b)}(t)$ (with $b > c$) and integrating both sides from $-\sqrt{b}$ to $\sqrt{b}$ gives

$$\begin{aligned} \sqrt{c} \mu_n(2c/\pi) \int_{-\sqrt{b}}^{\sqrt{b}} \varphi_n^{(c)}(t) \varphi_n^{(b)}(t) dt &= \int_{-\sqrt{c}}^{\sqrt{c}} \varphi_n^{(c)}(s) \int_{-\sqrt{b}}^{\sqrt{b}} \varphi_n^{(b)}(t) e^{ist} dt ds \\ &= \sqrt{b} \mu_n(2b/\pi) \int_{-\sqrt{c}}^{\sqrt{c}} \varphi_n^{(c)}(s) \varphi_n^{(b)}(s) ds. \end{aligned} \quad (2.60)$$

Since $b > c$ and the integrands $\varphi_n^{(c)}(s) \varphi_n^{(b)}(s)$ are even functions for all n , (2.60) may be rewritten as

$$(\sqrt{b} \mu_n(2b/\pi) - \sqrt{c} \mu_n(2c/\pi)) \int_{-\sqrt{c}}^{\sqrt{c}} \varphi_n^{(c)}(t) \varphi_n^{(b)}(t) dt = 2\sqrt{c} \mu_n(2c/\pi) \int_{\sqrt{c}}^{\sqrt{b}} \varphi_n^{(c)}(t) \varphi_n^{(b)}(t) dt.$$

Dividing both sides of this last equation by $b - c$, allowing $b \rightarrow c^+$, and applying the orthonormality of $\{\varphi_n^{(c)}\}_{n=0}^\infty$ gives

$$\frac{d^+}{dc} (\sqrt{c} \mu_n(2c/\pi)) = \mu_n(2c/\pi) \varphi_n^{(c)}(\sqrt{c})^2 = \frac{\mu_n(2c/\pi)}{\sqrt{c}} \tilde{\phi}_n^{(c)}(1)^2. \quad (2.61)$$

Reversing the roles of b and c gives

$$\frac{d^-}{dc} (\sqrt{c} \mu_n(2c/\pi)) = \mu_n(2c/\pi) \varphi_n^{(c)}(\sqrt{c})^2 = \frac{\mu_n(2c/\pi)}{\sqrt{c}} \tilde{\phi}_n^{(c)}(1)^2$$

so that $\frac{d}{dc}(\sqrt{c}\mu_n(2c/\pi))$ exists and is equal to the right-hand side of (2.61). Rearranging the resulting equation gives (2.58). $\square$

The following lemma appears in [280].

Lemma 2.6.5. *Let $\{\mu_n(2c/\pi)\}$ be the collection of eigenvalues appearing in (2.17). Then*

$$\lim_{c \rightarrow 0^+} \frac{\mu_n(2c/\pi)}{c^n} = \frac{i^n \sqrt{\pi}(n!)^2}{\Gamma(n+3/2)(2n)!}. \quad (2.62)$$

Proof. Differentiating both sides of (2.17) n times with respect to x and evaluating at $x = 0$ gives

$$\mu_n(2c/\pi) D^n \bar{\phi}_n^{(c)}(0) = (ic)^n \int_{-1}^1 t^n \bar{\phi}_n^{(c)}(t) dt. \quad (2.63)$$

Furthermore, $\bar{\phi}_n^{(c)}(x) = \bar{P}_n(x) + O(c^2)$. The uniform convergence of $\bar{\phi}_n^{(c)}$ to $\bar{P}_n$ and the analyticity of $\bar{\phi}_n^{(c)}(x)$ as a function of both x and c now give the uniform convergence of $D_x^n \bar{\phi}_n^{(c)}(x)$ to $D_x^n \bar{P}_n(x) = \frac{(2n)! \sqrt{n+1/2}}{2^n n!}$. Hence, allowing $c \rightarrow 0^+$ in (2.63) gives $\lim_{c \rightarrow 0^+} \frac{\mu_n(2c/\pi)}{c^n} = \frac{i^n 2^n n!}{(2n)!} \int_{-1}^1 t^n P_n(t) dt$. Applying Lemma 2.6.3 now gives the result. $\square$

The following result appears in [280].

Lemma 2.6.6. *Let $c > 0$ and let n be a non-negative integer. Then*

$$\mu_n(2c/\pi) = \frac{i^n \sqrt{\pi} c^n (n!)^2}{(2n)! \Gamma(n+3/2)} \exp\left(\int_0^c \left(\frac{2\bar{\phi}_n^{(\tau)}(1)^2 - 1}{2\tau} - \frac{n}{\tau}\right) d\tau\right).$$

Proof. Let $v_n = i^{-n} \mu_n$ so that each $v_n = \sqrt{2\pi\lambda_n/c}$ is positive. We first rearrange (2.58) so that

$$\frac{\frac{d}{d\tau}(v_n(2\tau/\pi))}{v_n(2\tau/\pi)} = \frac{2\bar{\phi}_n^{(\tau)}(1)^2 - 1}{2\tau}.$$

With $0 < c_0 < c$, we integrate both sides of this equation with respect to τ from c_0 to c to obtain

$$\ln(v_n(2c/\pi)) - \ln(v_n(2c_0/\pi)) = \int_{c_0}^c \frac{2\bar{\phi}_n^{(\tau)}(1)^2 - 1}{2\tau} d\tau.$$

Since $\bar{\phi}_n^{(c)}(x) = \bar{P}_n(x) + O(c^2)$ and $\bar{P}_n(1) = \sqrt{n+1/2}$ we have $\frac{2\bar{\phi}_n^{(\tau)}(1)^2 - 1}{2\tau} = \frac{n}{\tau} + q_n(\tau)$ with q_n smooth. Hence

$$\ln(v_n(2c/\pi)) = \ln\left(\frac{v_n(2c_0/\pi)}{c_0^n}\right) + n \ln c + \int_{c_0}^c q_n(\tau) d\tau.$$

Exponentiating both sides of this equation, allowing $c_0 \rightarrow 0^+$, and applying (2.62) now gives the result. $\square$

Rokhlin and Xiao [280] also showed that the prolates satisfy the bound

$$|\bar{\phi}_n^{(c)}(1)| < \sqrt{n+1/2}. \quad (2.64)$$

As a consequence they gave the following estimate of the rate of decay of the eigenvalues $\mu_n(2c/\pi)$.

Lemma 2.6.7. *Let $c > 0$ and let n be a non-negative integer. Then*

$$|\mu_n(2c/\pi)| \leq \frac{\sqrt{\pi}(n!)^2 c^n}{\Gamma(n+3/2)(2n)!}.$$

Proof. Define the function F by

$$F(c) = \int_0^c \left(\frac{2\bar{\phi}_n^{(\tau)}(1)^2 - 1}{2\tau} - \frac{n}{\tau} \right) d\tau.$$

Then, by Lemma 2.6.6, $\frac{\mu_n(2c/\pi)}{i^n c^n} = \frac{\sqrt{\pi}(n!)^2}{(2n)!\Gamma(n+3/2)} \exp(F(c))$. However, by (2.64), $F'(c) = \frac{2\bar{\phi}_n^{(c)}(1)^2 - 1}{2c} - \frac{n}{c} < 0$, so $\frac{\mu_n(2c/\pi)}{i^n c^n}$ is decreasing, since

$$\frac{d}{dc} \left(\frac{\mu_n(2c/\pi)}{i^n c^n} \right) = \frac{\sqrt{\pi}(n!)^2}{(2n)!\Gamma(n+3/2)} F'(c) \exp(F(c)) < 0.$$

By (2.62), $\frac{\mu_n(2c/\pi)}{i^n c^n} \leq \lim_{c \rightarrow 0^+} \frac{\mu_n(2c/\pi)}{i^n c^n} = \frac{\sqrt{\pi}(n!)^2}{\Gamma(n+3/2)(2n)!}$ and the result follows. $\square$

Lemma 2.6.7 should be compared to Corollary 1.2.11. Recalling that $\mu_n(a) = 2i^n \sqrt{\lambda_n(a)/a}$, it is clear that both results provide roughly the same upper bound. The difference is that the more sophisticated (and more difficult) Corollary 1.2.11 also proves an asymptotic lower bound.

Values of $\bar{\phi}_n(1)$, $\bar{\phi}_n(0)$

Lemma 2.6.8. *For any $c > 0$ and integer $n \geq 0$ we have $\bar{\phi}_n(1) \neq 0$.*

Proof. Suppose that for some $n \geq 0$, $\bar{\phi}_n(1) = 0$. Each $\bar{\phi}_n$ satisfies the differential equation

$$(1-x^2)\bar{\phi}_n''(x) - 2x\bar{\phi}_n'(x) + (\chi_n - c^2 x^2)\bar{\phi}_n(x) = 0, \quad (2.65)$$

and evaluating both sides of (2.65) at $x = 1$ gives

$$-2\bar{\phi}_n'(1) + (\chi_n - c^2)\bar{\phi}_n(1) = 0 \quad (2.66)$$

from which we see that $\bar{\phi}_n'(1) = 0$. Differentiating (2.65) and evaluating both sides at $x = 1$ gives

$$-4\bar{\phi}_n''(1) + (\chi_n - 2 - c^2)\bar{\phi}_n'(1) - 2c^2\bar{\phi}_n(1) = 0$$

from which we conclude that $\bar{\phi}_n''(1) = 0$. For each integer $k \geq 2$, the k th derivative of (2.65) becomes

$$\begin{aligned} (1-x^2)\bar{\phi}_n^{(k+2)}(x) - 2(k+1)x\bar{\phi}_n^{(k+1)}(x) + (\chi_n - k(k+1) - c^2 x^2)\bar{\phi}_n^{(k)}(x) \\ - 2c^2 kx\bar{\phi}_n^{(k-1)}(x) - c^2 k(k-1)\bar{\phi}_n^{(k-2)}(x) = 0, \end{aligned}$$

and evaluating this equation at $x = 1$ gives

$$-2(k+1)\bar{\phi}_n^{(k+1)}(1) + (\chi_n - k(k+1) - c^2)\bar{\phi}_n^{(k)}(1) - 2c^2 k\bar{\phi}_n^{(k-1)}(1) - c^2 k(k-1)\bar{\phi}_n^{(k-2)}(1) = 0.$$

An inductive argument now gives $\bar{\phi}_n^{(k)}(1) = 0$ for all integers $k \geq 0$. Since each $\bar{\phi}_n$ is analytic on the complex plane, we conclude that $\bar{\phi} \equiv 0$, a contradiction. $\square$

A similar argument can be used to show that if n is even then $\bar{\phi}_n(0) \neq 0$. The following is a consequence of (2.66).

Corollary 2.6.9. *With $c > 0$ and integers $m, n \geq 0$,*

$$\bar{\phi}'_m(1)\bar{\phi}_n(1) - \bar{\phi}'_n(1)\bar{\phi}_m(1) = \frac{1}{2}(\chi_m - \chi_n)\bar{\phi}_n(1)\bar{\phi}_m(1).$$

2.6.2 Proof of Theorem 2.1.15

Definition 2.6.10. Given $a < b$, the n -dimensional simplex $M_n = M_n(a, b)$ is defined by

$$M_n = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n; a \leq x_1 \leq x_2 \leq \dots \leq x_n \leq b\}.$$

Note that $M^1(a, b) = [a, b]$.

Lemma 2.6.11. *Let $K(X, S)$ ($X, S \in M_n$) be a continuous symmetric kernel with*

$$K(X, S) \geq 0, \quad K(X, X) > 0 \text{ for all } X, S \in M_n.$$

Then the eigenvalue λ_0 of the operator $T_K : L^2(M_n) \rightarrow L^2(M_n)$ defined by

$$(T_K F)(X) = \int_{M_n} K(X, S)F(S) dS$$

of largest absolute value is positive, simple, and not equal to the remaining eigenvalues. The eigenfunction Φ_0 corresponding to λ_0 has no zeros inside M_n .

Proof. The supremum of $\{\langle T_K \Phi, \Phi \rangle : \|\Phi\| = 1\}$ is attained at an eigenvalue λ_0 of T_K . If $\Phi(X_0) > 0$ at some $X_0 \in M_n$, then the continuity of K and the conditions $K(X_0, S) \geq 0$, $K(X_0, X_0) > 0$ imply that $\lambda_0 > 0$ and λ_0 is the largest eigenvalue of T_K . Let Φ_0 be a corresponding normalized eigenfunction, that is,

$$\lambda_0 = \sup_{\|\Phi\|=1} \langle T_K \Phi, \Phi \rangle = \langle T_K \Phi_0, \Phi_0 \rangle; \|\Phi_0\| = 1.$$

We claim that Φ_0 has no zeros in M_n . Since $\lambda_0 > 0$ and $K(X, S) \geq 0$,

$$\begin{aligned} \lambda_0 &= |\lambda_0| = |\langle T_K \Phi_0, \Phi_0 \rangle| = \left| \int_{M_n} \int_{M_n} K(X, S) \Phi_0(S) dS \Phi_0(X) dX \right| \\ &\leq \int_{M_n} \int_{M_n} K(X, S) |\Phi_0(S)| dS |\Phi_0(X)| dX = \langle T_K |\Phi_0|, |\Phi_0| \rangle. \end{aligned} \quad (2.67)$$

Since λ_0 is maximal, we must have equality in (2.67). Thus, $|\Phi_0|$ is also an eigenfunction of T_K corresponding to λ_0 , that is,

$$\lambda_0 |\Phi_0(X)| = \int_{M_n} K(X, S) |\Phi_0(S)| dS.$$

Now suppose Φ_0 has a zero on M_n and $X_1 \in M_n$ is a zero of Φ_0 such that in any neighborhood U of X_1 there exists $X \in U$ with $\Phi_0(X) \neq 0$. If none of the zeros of Φ_0 has this property then $\Phi_0 \equiv 0$. Since K is continuous and $K(X_1, X_1) > 0$, there is a neighborhood V of X_1 for which $K(X_1, S) > 0$ for all $S \in V$. Hence

$$0 = \lambda_0 |\Phi_0(X_1)| = \int_{M_n} K(X_1, S) |\Phi_0(S)| dS \geq \int_{U \cap V} K(X, S) |\Phi_0(S)| dS > 0,$$

a contradiction. Hence Φ_0 has no zeros in M_n . Also, any eigenfunction with eigenvalue λ_0 has constant sign on M_n and consequently no two such functions can be orthogonal. We conclude that λ_0 is a simple eigenvalue.

By the maximum principle, λ_0 is larger than all positive eigenvalues of T_K . We wish to show that $\lambda_0 > |\lambda'|$ for all negative eigenvalues λ' . Suppose λ' is such a negative eigenvalue, Ψ is an eigenfunction of T_K with eigenvalue λ' , and $\|\Psi\| = 1$, that is,

$$\lambda' \Psi(X) = \int_{M_n} K(X, S) \Psi(S) dS. \quad (2.68)$$

Then

$$|\lambda'| |\Psi(X)| \leq \int_{M_n} K(X, S) |\Psi(S)| dS, \quad (2.69)$$

and we claim that the inequality (2.69) is strict at every $X \in M_n$ for which $\Psi(X) \neq 0$. To see this, suppose that $\Psi(X_1) \neq 0$ and equality holds in (2.69). Then, as a function of S , $K(X_1, S) \Psi(S)$ must have the same sign on all of M_n . However, from (2.68),

$$\begin{aligned} \operatorname{sgn}(\Psi(X_1)) &= -\operatorname{sgn}(\lambda' \Psi(X_1)) = -\operatorname{sgn}\left(\int_{M_n} K(X_1, S) \Psi(S) dS\right) \\ &= -\operatorname{sgn}(K(X_1, X_1) \Psi(X_1)) = -\operatorname{sgn}(\Psi(X_1)), \end{aligned}$$

a contradiction. Hence the inequality in (2.69) is strict. Multiplying both sides of (2.69) by $|\Psi(X)|$ and integrating over M_n gives

$$\begin{aligned} |\lambda'| &= |\lambda'| \|\Psi\|^2 < \int_{M_n} \int_{M_n} K(X, S) |\Psi(S)| dS |\Psi(X)| dX \\ &= \langle T_K |\Psi|, |\Psi| \rangle \leq \langle T_K \Phi_0, \Phi_0 \rangle = \lambda \|\Phi_0\|^2 = \lambda_0, \end{aligned}$$

as required. The proof is complete. $\square$

Definition 2.6.12. Given a kernel $K(x, s)$ ($x, s \in (a, b)$), and integers $n, j \geq 1$, the n th associated kernel $K_n(X, S)$ ($X = (x_1, x_2, \dots, x_n)$, $S = (s_1, s_2, \dots, s_n) \in M_n(a, b)$) is

$$K_n(X, S) = K \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ s_1 & s_2 & \cdots & s_n \end{pmatrix} = \det_{i,j} (K(x_i, s_j))_{i,j=1}^n.$$

The j th iterated kernel $K^j(x, s)$ associated with K is given by $K^1 = K$ and, when $j \geq 2$,

$$K^j(x, s) = \int_a^b \cdots \int_a^b K(x, t_1) K(t_1, t_2) \cdots K(t_{j-1}, s) dt_1 dt_2 \cdots dt_{j-1},$$

that is, K^j is the kernel associated with the operator $T_{K^j} = (T_K)^j = T_K \circ T_K \circ \cdots \circ T_K$ (j factors).

Lemma 2.6.13. Let the kernels K , L , and N defined on $(a, b) \times (a, b)$ be related by

$$N(x, s) = \int_a^b K(x, t) L(t, s) dt,$$

or equivalently, $T_N = T_K \circ T_L$. Then

(i) The n th associated kernels K_n , L_n , and N_n defined on $M_j(a, b) \times M_j(a, b)$ are related by

$$N_n(X, S) = \int_{M_n} K_n(X, T) L_n(T, S) dT.$$

(ii) If K^j is the j th iteration of the kernel K , then $[K_n]^j = [K^j]_n$, that is, the n th associated kernel of the j th iteration of K is the j th iteration of the n th associated kernel of K .

Proof. First, if f is defined on the n -dimensional cube $Q^n(a, b) = (a, b)^n$, then

$$\int_{Q^n(a, b)} f(t_1, t_2, \dots, t_n) dt_1 dt_2 \cdots dt_n = \sum_{\sigma \in \Sigma_n} \int_{M_n(a, b)} f(t_{\sigma(1)}, t_{\sigma(2)}, \dots, t_{\sigma(n)}) dt_1 dt_2 \cdots dt_n$$

where Σ_n is the symmetric group of permutations of $\{1, 2, \dots, n\}$. Given two collections of functions $\{\phi_1, \phi_2, \dots, \phi_n\}$ and $\{\psi_1, \psi_2, \dots, \psi_n\}$ defined on $[a, b]$, the behavior of determinants under row or column swaps gives

$$\begin{aligned} & \int_{Q^n} \Delta \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} \Delta \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} dt_1 dt_2 \cdots dt_n \\ &= \sum_{\sigma \in \Sigma_n} \int_{M_n} \Delta \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ t_{\sigma(1)} & t_{\sigma(2)} & \cdots & t_{\sigma(n)} \end{pmatrix} \Delta \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \\ t_{\sigma(1)} & t_{\sigma(2)} & \cdots & t_{\sigma(n)} \end{pmatrix} dt_1 dt_2 \cdots dt_n \\ &= n! \int_{M_n} \Delta \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} \Delta \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} dt_1 dt_2 \cdots dt_n. \end{aligned}$$

Given an $n \times n$ matrix A with (i, j) th entry a_{ij} , its determinant may be computed by the Leibniz formula $\det(A) = \sum_{\omega \in \Sigma_n} \text{sgn}(\omega) \prod_{i=1}^n a_{i, \omega(i)}$. Here $\text{sgn}(\omega) = \pm 1$ is the *sign* of the permutation ω . Consequently, by the product formula for determinants,

$$\begin{aligned} & \int_{M_n} \Delta \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} \Delta \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} dt_1 dt_2 \cdots dt_n \\ &= \frac{1}{n!} \int_{Q^n} \det(\phi_i(t_k))_{i,k} \det(\psi_j(t_k))_{j,k} dt_1 dt_2 \cdots dt_n \\ &= \frac{1}{n!} \int_{Q^n} \det \left(\sum_{m=1}^n \phi_i(t_m) \psi_j(t_m) \right)_{i,j} dt_1 dt_2 \cdots dt_n \\ &= \frac{1}{n!} \sum_{m_1=1}^n \sum_{m_2=1}^n \cdots \sum_{m_n=1}^n \int_{Q^n} \det(\phi_i(t_{m_i}) \psi_j(t_{m_i}))_{i,j} dt_1 dt_2 \cdots dt_n \\ &= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \int_{Q^n} \det(\phi_i(t_{\sigma(i)}) \psi_j(t_{\sigma(i)}))_{i,j} dt_1 dt_2 \cdots dt_n \\ &= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sum_{\omega \in \Sigma_n} \text{sgn}(\omega) \int_{Q^n} \prod_{\ell=1}^n \phi_{\ell}(t_{\sigma(\ell)}) \psi_{\omega(\ell)}(t_{\sigma(\ell)}) dt_1 dt_2 \cdots dt_n \\ &= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sum_{\omega \in \Sigma_n} \text{sgn}(\omega) \prod_{\ell=1}^n \langle \phi_{\ell}, \psi_{\omega(\ell)} \rangle = \det(\langle \phi_i, \psi_j \rangle). \end{aligned} \tag{2.70}$$

Putting $\psi_i(t) = K(x_i, t)$ and $\phi_i(t) = L(t, s_i)$ now gives the result, since if $X = (x_1, x_2, \dots, x_n)$, $T = (t_1, t_2, \dots, t_n)$, $S = (s_1, s_2, \dots, s_n) \in M_n$, then $\Delta \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} = K_n(X, T)$, $\Delta \begin{pmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} = L_n(T, S)$, and by (2.70),

$$\begin{aligned}
& \int_{M^n} K_n(X, T) L_n(T, S) dT \\
&= \int_{M^n} \Delta \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} \Delta \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} dt_1 dt_2 \cdots dt_n \\
&= \det_{i,j} \left(\int_a^b K(x_i, t) L(t, s_j) dt \right) = \det_{i,j} (N(x_i, s_j)) = N_n(X, S).
\end{aligned}$$

This completes the proof of statement (i), from which statement (ii) immediately follows. $\square$

Let $\Lambda_n = \{I = (i_1, i_2, \dots, i_n) \in \mathbb{Z}^n : 1 \leq i_1 < i_2 < \cdots < i_n\}$.

Lemma 2.6.14. *Let $\{\varphi_i\}_{i=1}^\infty$ be a complete orthonormal system of eigenfunctions of an integral operator T_K with symmetric kernel $K(x, s)$ ($x, s \in [a, b]$) with $\{\lambda_i\}_{i=1}^\infty$ the associated eigenvalues. Then the functions*

$$\Phi_I(X) = \Delta \begin{pmatrix} \varphi_{i_1} & \varphi_{i_2} & \cdots & \varphi_{i_n} \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}$$

(where $X = (x_1, x_2, \dots, x_n) \in M_n$ and $I = (i_1, i_2, \dots, i_n) \in \Lambda_n$) form a complete orthonormal system of eigenfunctions of the operator T_{K_n} acting on $L^2(M_n)$ by $(T_{K_n} F)(X) = \int_{M^n} K_n(X, S) F(S) dS$ with associated eigenvalues $\lambda_I = \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_n}$, that is, $T_{K_n} \Phi_I = \lambda_I \Phi_I$.

Proof. As in the proof of the previous lemma, putting $\psi_k(t) = K(x_k, t)$ gives $\Delta \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \\ t_1 & t_2 & \cdots & t_n \end{pmatrix} = K_n(X, T)$, and by (2.70),

$$\begin{aligned}
\int_{M_n} K_n(X, S) \Phi_I(S) dS &= \int_{M_n} \Delta \begin{pmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \\ s_1 & s_2 & \cdots & s_n \end{pmatrix} \Delta \begin{pmatrix} \varphi_{i_1} & \varphi_{i_2} & \cdots & \varphi_{i_n} \\ s_1 & s_2 & \cdots & s_n \end{pmatrix} ds_1 ds_2 \cdots ds_n \\
&= \det_{k,j} (\langle \psi_k, \varphi_{i_j} \rangle) = \det_{k,j} \left(\int_a^b K(x_k, t) \varphi_{i_j}(t) dt \right) \\
&= \det_{k,j} (\lambda_{i_j} \varphi_{i_j}(x_k)) = \lambda_I \det_{i,j} (\varphi_{i_j}(x_i)) = \lambda_I \Phi_I(X),
\end{aligned}$$

that is, the functions Φ_I are eigenfunctions of T_{K_n} with associated eigenvalues λ_I . The orthogonality of this collection is an immediate consequence of (2.70):

$$\langle \Phi_I, \Phi_J \rangle = \int_{M_n} \Phi_I(X) \Phi_J(X) dX = \det_{\ell,k} (\langle \varphi_{i_\ell}, \varphi_{j_k} \rangle) = \det_{\ell,k} (\delta_{i_\ell, j_k}) = \delta_{I,J}.$$

Finally, we show that the collection $\{\Phi_I\}_{I \in \Lambda_n}$ forms a complete set in $L^2(M_n)$. The kernel admits the eigenfunction (Mercer) expansion $K(x, s) = \sum_{\ell=1}^\infty \lambda_\ell \varphi_\ell(x) \varphi_\ell(s)$, so that the n th associated kernel takes the form

$$\begin{aligned}
K_n(X, S) &= \det(K(x_i, s_j)) = \det \left(\sum_{\ell=1}^{\infty} \lambda_{\ell} \varphi_{\ell}(x_i) \varphi_{\ell}(s_j) \right) \\
&= \sum_{\ell_0=1}^{\infty} \sum_{\ell_1=1}^{\infty} \cdots \sum_{\ell_{n-1}=1}^{\infty} \prod_{j=0}^{n-1} \lambda_{\ell_j} \varphi_{\ell_j}(x_j) \det(\varphi_{\ell_i}(s_j)) \\
&= \sum_{\{\ell_0, \ell_1, \dots, \ell_{n-1}\} \text{ distinct}} \prod_{j=0}^{n-1} \lambda_{\ell_j} \varphi_{\ell_j}(x_j) \det(\varphi_{\ell_i}(s_j)) \\
&= \sum_{\sigma \in \Sigma_n} \sum_{(\ell_0, \ell_1, \dots, \ell_{n-1}) \in \Lambda_n} \prod_{j=0}^{n-1} \lambda_{\ell_{\sigma(j)}} \varphi_{\ell_{\sigma(j)}}(x_j) \det(\varphi_{\ell_{\sigma(i)}}(s_j)) \\
&= \sum_{\sigma \in \Sigma_n} \sum_{(\ell_0, \ell_1, \dots, \ell_{n-1}) \in \Lambda_n} \prod_{j=0}^{n-1} \lambda_{\ell_{\sigma(j)}} \varphi_{\ell_{\sigma(j)}}(x_j) \operatorname{sgn}(\sigma) \det(\varphi_{\ell_i}(s_j)) \\
&= \sum_{(\ell_0, \ell_1, \dots, \ell_{n-1}) \in \Lambda_n} \left(\prod_{k=0}^{n-1} \lambda_{\ell_k} \right) \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) \prod_{j=0}^{n-1} \varphi_{\ell_{\sigma(j)}}(x_j) \det(\varphi_{\ell_i}(s_j)) \\
&= \sum_{L \in \Lambda_n} \lambda_L \Phi_L(X) \Phi_L(S).
\end{aligned}$$

Hence, if $F \in L^2(M_n)$ and $\langle F, \Phi_L \rangle = 0$ for all $L \in \Lambda_n$, then $T_{K^n} F = 0$. Since 0 is not an eigenvalue of T_{K^n} , we have arrived at a contradiction and we conclude that $\{\Phi_L\}_{L \in \Lambda_n}$ is a complete orthonormal system for $L^2(M_n)$. $\square$

We are now in a position to prove Theorem 2.1.15.

Proof (of Theorem 2.1.15). We relabel the eigenvalues of T_K so that $|\lambda_0| \geq |\lambda_1| \geq \cdots$ and denote the corresponding eigenfunctions by $\{\varphi_j\}_{j=0}^{\infty}$. Let K_n be the n th associated kernel of K . By Lemma 2.6.14, the eigenvalues are n -fold products of $\{\lambda_j\}_{j=0}^{\infty}$ (with no repetitions). The product with the largest absolute value will be $\lambda_0 \lambda_1 \cdots \lambda_{n-1}$, which corresponds to the eigenfunction

$$\Phi_{(0,1,\dots,n-1)}(X) = \Delta \begin{pmatrix} \varphi_0 & \varphi_1 & \cdots & \varphi_{n-1} \\ x_1 & x_2 & \cdots & x_n \end{pmatrix} = \det(\varphi_{i-1}(x_j)) \quad (X \in M_n).$$

Since K_n satisfies the conditions of Lemma 2.6.11,

$$\lambda_0 \lambda_1 \cdots \lambda_{n-1} > 0 \quad \text{and} \quad \lambda_0 \lambda_1 \cdots \lambda_{n-1} > |\lambda_0 \lambda_1 \cdots \lambda_{n-2} \lambda_n|.$$

These inequalities hold for all n , so we conclude that $\lambda_0 > \lambda_1 > \cdots > \lambda_n > \cdots > 0$. Also, from Lemma 2.6.11 we conclude that $\Phi_{(0,1,\dots,n-1)}(X) \neq 0$ for all $X \in M^n$. By Lemma 2.1.2, for every n the finite collection $\{\varphi_j\}_{j=0}^{n-1}$ is a Chebyshev system. Equivalently, the infinite collection $\{\varphi_j\}_{j=0}^{\infty}$ is a Markov system on (a, b) . The conclusions of Theorem 2.1.15 now follow from Theorem 2.1.7. The self-adjointness of the operator and the simplicity of the eigenvalues now also imply the orthogonality of the eigenfunctions. $\square$

2.6.3 Proof of Corollary 2.2.7

Proof (of Corollary 2.2.7). Given $0 < \delta < 1/2$, choose a real-valued C^∞ bump function b on the line with the properties (see, e.g., [8])

- (i) $0 \leq b(x) \leq 1$ for all x ,
- (ii) $b(x) = 0$ if $|x| \geq 1$ and $b(x) = 1$ if $|x| < 1 - \delta$,

(iii) $|b'(x)| \leq 2/\delta$ for all x .

An application of Wirtinger's inequality (see Appendix) gives

$$\begin{aligned} \frac{\delta}{2} \|\sigma b'\|_{L^2[0,1]} &\leq \left(\int_{1-\delta}^1 |\sigma(t)|^2 dt \right)^{1/2} \\ &\leq \left(\int_{1-\delta}^1 |\sigma(t) - \sigma(1)|^2 dt \right)^{1/2} + |\sigma(1)|\sqrt{\delta} \\ &\leq \frac{2\delta}{\pi} \left(\int_{1-\delta}^1 |\sigma'(t)|^2 dt \right)^{1/2} + |\sigma(1)|\sqrt{\delta}. \end{aligned}$$

A similar estimate may be made for $\|\sigma b'\|_{L^2[-1,0]}$ so that we have

$$\begin{aligned} \|\sigma b'\|_{L^2[-1,1]} &\leq \frac{2}{\sqrt{\delta}} (|\sigma(1)| + |\sigma(-1)|) + \frac{4\sqrt{2}}{\pi} \left(\int_{1-\delta < |t| < 1} |\sigma'(t)|^2 dt \right)^{1/2} \\ &\leq \frac{2}{\sqrt{\delta}} (|\sigma(1)| + |\sigma(-1)| + \|\sigma'\|_{L^2[-1,1]}). \end{aligned} \quad (2.71)$$

However, $|\sigma(1)| + |\sigma(-1)| \leq 2\sqrt{2}(\|\sigma\|_{L^2[-1,1]} + \|\sigma'\|_{L^2[-1,1]})$, and substituting this inequality into (2.71) gives

$$\|\sigma b'\|_{L^2[-1,1]} \leq \frac{C}{\sqrt{\delta}} (\|\sigma\|_{L^2[-1,1]} + \|\sigma'\|_{L^2[-1,1]}). \quad (2.72)$$

Now $\tilde{\sigma} = \sigma b$ satisfies the conditions of Theorems 2.2.5 and 2.2.6, so that $\tilde{f}$ defined by $\tilde{f}(t) = \int_{-1}^1 \tilde{\sigma}(x) e^{-2icxt} dx$ satisfies the conditions of Theorem 2.2.6. Furthermore,

$$|f(t) - \tilde{f}(t)| \leq \int_{-1}^1 |\sigma(x)| |b(x) - 1| dx \leq \int_{1-\delta \leq |t| \leq 1} |\sigma(t)| dt \leq \sqrt{2\delta} \|\sigma\|_{L^2[-1,1]}.$$

By Theorem 2.2.5 there exist functions $\tilde{q}, \tilde{r}$ of band limit c such that $\tilde{f} = \phi_N \tilde{q} + \tilde{r}$. Let $\tilde{q}(t) = \int_{-1}^1 \tilde{\eta}(x) e^{-icxt} dx$ and $\tilde{r}(t) = \int_{-1}^1 \tilde{\xi}(x) e^{-icxt} dx$. Then

$$\begin{aligned} \|\tilde{\eta}\|_{L^2[-1,1]} + \|\tilde{\xi}\|_{L^2[-1,1]} &\leq C(\|\tilde{\sigma}\|_{L^2[-1,1]} + \|\tilde{\sigma}'\|_{L^2[-1,1]}) \\ &\leq \frac{C}{\sqrt{\delta}} (\|\sigma\|_{L^2[-1,1]} + \|\sigma'\|_{L^2[-1,1]}) \end{aligned}$$

and the error in the quadrature when applied to f may be estimated by

$$\begin{aligned} E_f &= \left| \sum_{k=0}^{N-1} w_k f(x_k) - \int_{-1}^1 f(x) dx \right| \leq \sum_{k=0}^{N-1} w_k |f(x_k) - \tilde{f}(x_k)| \\ &\quad + \left| \sum_{k=0}^{N-1} w_k \tilde{f}(x_k) - \int_{-1}^1 \tilde{f}(x) dx \right| + \int_{-1}^1 |f(x) - \tilde{f}(x)| dx \\ &\leq C\sqrt{2\delta} \|\sigma\|_{L^2[-1,1]} + C|\mu_{N-1}| (\|\tilde{\eta}\|_{L^2[-1,1]} + \|\tilde{\xi}\|_{L^2[-1,1]}) \\ &\leq C\sqrt{2\delta} \|\sigma\|_{L^2[-1,1]} + \frac{C}{\sqrt{\delta}} |\mu_{N-1}| (\|\sigma\|_{L^2[-1,1]} + \|\sigma'\|_{L^2[-1,1]}). \end{aligned}$$

Putting $\delta = |\mu_{N-1}|$ now gives the required estimate. $\square$

2.6.4 Proofs of Other Auxiliary Lemmas for Theorem 2.5.2

Proof (of Lemma 2.5.3). Suppose k is even. Since $\|\bar{P}_k\|_{L^2[-1,1]} = \|\bar{\Phi}_N\|_{L^2[-1,1]} = 1$, $|\beta_k| \leq 1$ for all k . Hence (2.47) is valid for $k = 2, 4$ with $D = \max\{|\beta_4|, |\beta_2|\}/(\alpha^2|\beta_0|)$. Assume now that (2.47) is valid for all even integers less than or equal to k . Setting

$$\gamma(x) = \frac{x(x-1)}{(2x-1)\sqrt{(2x-3)(2x+1)}}; \quad \delta(x) = \frac{2x(x+1)-1}{(2x+3)(2x-1)}$$

the recurrence relation (1.13) may be written

$$\beta_{k+2} = \frac{1}{\gamma(k+2)} \left(\frac{1}{q^2} \left(1 - \frac{k(k+1)}{\chi_N} \right) - \delta(k) \right) \beta_k - \frac{\gamma(k)}{\gamma(k+2)} \beta_{k-2}. \quad (2.73)$$

For $x \geq 2$, $1/4 \leq \gamma(x) \leq 2\sqrt{5}/15$ and $1/2 \leq \delta(x) \leq 11/21$. Thus if $k \leq 2m$,

$$\begin{aligned} \frac{1}{q^2} \left(1 - \frac{k(k+1)}{\chi_N} \right) - \delta(k) &\geq \left(1 - \frac{2m(2m+1)}{\chi_N} \right) - \delta(k) \\ &\geq 1 - \frac{(\sqrt{\chi_N}/2)^2}{\chi_N} - \frac{11}{21} = 1 - \frac{1}{4} - \frac{11}{21} > 0, \end{aligned}$$

that is, the coefficient of β_k in (2.73) is positive. Consequently, from (2.73), the bounds on $\gamma(x)$, $\delta(x)$, and the inductive hypothesis, we have

$$\begin{aligned} |\beta_{k+2}| &\leq \frac{1}{\gamma(k+2)} \left(\frac{1}{q^2} \left(1 - \frac{k(k+1)}{\chi_N} \right) - \delta(k) \right) |\beta_k| + \frac{\gamma(k)}{\gamma(k+2)} |\beta_{k-2}| \\ &\leq \frac{4}{q^2} |\beta_k| + \frac{8\sqrt{5}}{15} |\beta_{k-2}| \leq D \left(\frac{\alpha}{q} \right)^{k+2} |\beta_0| \left(\frac{4}{\alpha^2} + \frac{1}{\alpha^4} \right) < D \left(\frac{\alpha}{q} \right)^{k+2} |\beta_0|. \end{aligned}$$

The result follows by induction. $\square$

Proof (of Lemma 2.5.4). The N th prolate $\bar{\Phi}_N^{(c)}$ satisfies the prolate differential equation

$$((1-x^2)\bar{\Phi}')' + \chi_N(1-q^2x^2)\bar{\Phi} = 0. \quad (2.74)$$

Let $0 \leq \ell \leq m$ be an integer. Multiplying both sides of (2.74) by $x^{2\ell}$ and integrating over $[-1, 1]$ with respect to x gives

$$2\ell(2\ell-1)A_{2\ell-2} + (\chi_N - 2\ell(2\ell+1))A_{2\ell} - \chi_N q^2 A_{2\ell+2} = 0. \quad (2.75)$$

In particular, $A_0 = q^2 A_2$. Since $2m(2m+1) \leq \chi_N$, each of $A_0, A_2, \dots, A_{2m+2}$ has the same sign, so by (2.75)

$$A_{2\ell} = \frac{\chi_N q^2 A_{2\ell+2} - 2\ell(2\ell-1)A_{2\ell-2}}{\chi_N - 2\ell(2\ell+1)} \leq \frac{\chi_N q^2 A_{2\ell+2}}{\chi_N - 2\ell(2\ell+1)} = \frac{q^2 A_{2\ell+2}}{1 - \frac{2\ell(2\ell+1)}{\chi_N}}.$$

Iterating this formula gives

$$A_0 \leq q^{2m-2} A_{2m-2} X_m; \quad X_m = \prod_{\ell=1}^{m-1} \left(1 - \frac{2\ell(2\ell+1)}{\chi_N} \right)^{-1}. \quad (2.76)$$

Now

$$\begin{aligned}
-\log X_m &= \sum_{\ell=1}^{m-1} \log \left(1 - \frac{2\ell(2\ell+1)}{\chi_N} \right) \geq \int_0^m \log \left(1 - \frac{4x^2}{\chi_N} \right) dx \\
&= \left[x \log \left(1 - \frac{4x^2}{\chi_N} \right) - 2x + \sqrt{\chi_N} \tanh^{-1} \left(\frac{2x}{\sqrt{\chi_N}} \right) \right]_{x=1}^{x=m}. \tag{2.77}
\end{aligned}$$

Since $\sqrt{\chi_N}/4 - 1 \leq m \leq \sqrt{\chi_N}/4$, we have $1 - 4m^2/\chi_N \geq 3/4$ and $2m/\sqrt{\chi_N} \geq 1/2 - 2/\sqrt{\chi_N} \geq 2/5$ provided $\chi_N \geq 400$. Consequently,

$$\sqrt{\chi_N} \tanh^{-1}(2m/\sqrt{\chi_N}) \geq \sqrt{\chi_N} \tanh^{-1}(2/5) = \frac{\sqrt{\chi_N}}{2} \log(7/3).$$

Substituting this into (2.77) yields

$$-\log X_m \geq m \log(3/4) - 2m + \frac{\sqrt{\chi_N}}{2} \log(7/3)$$

so that after exponentiating both sides, we have

$$X_m \leq \left(\frac{4e^2}{3} \right)^m \left(\frac{3}{7} \right)^{\sqrt{\chi_N}/2}. \tag{2.78}$$

Combining (2.76) with (2.78) and the inequality $|A_m| \leq (m+1/2)^{-1/2}$ (which follows directly from the Cauchy–Schwarz inequality) gives the first of the inequalities in (2.48). The second inequality may be established in a similar manner. $\square$

Duration and Bandwidth Limiting

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Hogan, J.A.; Lakey, J.D.

2012, XVII, 258 p. 14 illus., Hardcover

ISBN: 978-0-8176-8306-1

A product of Birkhäuser Basel