

Preface

John von Neumann developed much of the rigorous formalism for quantum mechanics during the late 1920s [339], including a theory of unbounded operators in a Hilbert space that led to a mathematical justification of Heisenberg’s uncertainty principle. In his autobiography [356, p. 97 ff.], Norbert Wiener gave a poignant account of his own role in advancing mathematical ideas parallel to von Neumann’s during Wiener’s visit to Göttingen in 1925. In contrast to Heisenberg, Wiener was motivated by limitations on simultaneous localization in time and frequency, or *duration and bandwidth limiting* pertaining to macroscopic physical systems. Meanwhile, across the Atlantic, Nyquist [246] and Hartley [137], working at Bell Labs, were laying the foundations for Shannon’s communication theory in their accounts of transmission of information over physical communications channels. Hartley also quantified limitations on joint localization in time and frequency, perhaps in a more positive light than suggested by Wiener, by defining a bound on the rate at which information might be transmitted over a band-limited channel. Eventually this bound, expressed in terms of the *time–bandwidth product*, was formalized in the *Bell Labs theory*—a series of papers by Landau, Slepian, and Pollak [195, 196, 303, 309] appearing in the *Bell System Technical Journal* in the early 1960s, with additional follow-on work extending through 1980.

The Bell Labs theory identified those band-limited functions—the *prolate spheroidal wave functions* (PSWFs)—that have the largest proportions of their energies localized in a given time interval. It also quantified the eigenvalue behavior of the time- and frequency-localization operator “ $P_{\Omega}Q_T$ ” that gives rise to these eigenfunctions. There are approximately $2\Omega T$ orthogonal functions that are band limited to the frequency band $[0, \Omega]$, while having most of their energies localized in the time interval $[-T, T]$. Perhaps the most satisfying account of this $2\Omega T$ principle, at least from a purely mathematical perspective, is due to Landau and Widom (Theorem 1.3.1 in this book). Their result quantified asymptotically, but precisely, the number of eigenvalues of $P_{\Omega}Q_T$ larger than any given $\alpha \in (0, 1)$. The masterful use of the technical machinery of compact self-adjoint operators that went into this result arguably places the Bell Labs theory, as a foundational affirmation of what

is observed in the physical universe, alongside von Neumann's work. However, the theory did not lead to immediate widespread use of prolates in signal processing.

This was partly due to the fact that the appropriate parallel development in the *discrete domain* had not yet been made. During the period from the late 1970s to the early 1980s, a number of contributions provided this discrete parallel theory of time and band limiting, e.g., [307], the basic tool being the *discrete prolate spheroidal sequences* (DPSSs). Applications then emerged. Thomson [323] used this discrete theory in his *multitaper* method for spectrum estimation, and Jain and Ranganath [162] formalized the use of DPSSs in signal extrapolation. Ideas of Grünbaum [126] and others continued to provide a firm foundation for further potential applications.

While Thomson's methods caught on quickly, particularly in the geosciences, applications of the theory of time and band limiting to the modeling of communications channels, numerical analysis, and other promising areas, such as tomography, had to wait. There were several reasons for this delay: wavelets attracted much of the attention in the 1980s, and mobile communications were still in their lag phase.

Applications of time and band limiting appear now to be entering a log phase. The IEEE literature on this topic has grown quickly, largely due to potential uses in communications, where multiband signals are also playing a wider role, and in compressed sensing. At the same time, potential uses in numerical analysis are also being identified, stemming from the observation that the PSWFs have zeros that are more evenly spaced than their (Chebyshev and Legendre) polynomial counterparts, making them attractive for spectral element methods. Extension and refinement of the mathematical ideas underlying the Bell Labs theory, particularly in the context of multiband signals, but also involving higher order approximations, will continue in the coming decades.

The need to have a common source for the Bell Labs theory and its extensions, together with a resource for related methods that have been developed in the mathematics and engineering literature since 2000, motivated us to put together this monograph. Here is an outline of the contents.

Chapter 1 lays out the fundamental results of the Bell Labs theory, including a discussion of properties of the PSWFs and properties of the eigenvalues of $P_{\Omega}Q_T$. A discussion of the parallel theory of DPSSs is also included. Further discussion in the *chapter notes* focuses primarily on the application of the DPSSs to signal extrapolation.

The PSWFs and their values were important enough in classical physics to devote large portions of lengthy monographs to tables of their values, e.g., [54, 231, 318]. Modern applications require not only accuracy, but also computational efficiency. Since the largest eigenvalues of $P_{\Omega}Q_T$ are very close to one, they are also close to one another. This means that standard numerical tools for estimating finite-dimensional analogues of projections onto individual eigenspaces, such as the singular value decomposition, are numerically unstable. This is one among a raft of issues that led several researchers to reconsider numerical methods for estimating and applying PSWFs. Chapter 2 addresses such numerical analytical aspects of PSWFs, including quadrature formulas associated with prolates and approximations of func-

tions by prolate series. Along with classical references, sources for more recent advances here include [35, 36, 41–44, 123, 178, 280, 296, 331, 364].

Chapter 3 contains a detailed review of Thomson’s multitaper method. Through multitapering, PSWFs can be used to model channel characteristics; see, e.g., [124, 179, 211, 294]. However, if such a channel model is intended to be used by a single mobile device under severe power usage constraints, it might be preferable to employ a fixed prior basis in order to encode channel information in a parsimonious fashion. Zemen and Mecklenbräuker [370, 372] proposed variants of the DPS sequences for various approaches to this problem. These variants will also be outlined in Chap. 3.

Chapter 4 contains a development of parallel extensions of the Bell Labs theory to the case of multiband signals. Multiband signals play an increasing role in radio frequency (RF) communications as RF spectrum usage increases. We outline Landau and Widom’s [197] extension of the $2\Omega T$ theorem to the case in which the time- and frequency-localization sets S and Σ are finite unions of intervals. Other *non-asymptotic* estimates are given for the number of eigenvalues of $P_{\Sigma}Q_S$ that are larger than one-half, quantified in terms of the linear distributions of S and Σ .

The second part of Chap. 4 addresses a related problem in the theory of time and multiband limiting of finite signals (i.e., signals defined on a finite set of integers). Candès, Romberg, and Tao [57–60] proved *probabilistic* estimates on the norm of the corresponding time- and band-limiting operator. The main estimate states that if the time- and frequency-localization sets are *sparse* in a suitable sense, then, with overwhelming probability, the localization operator has norm smaller than one-half. The estimates were developed for applications to compressed sensing, in particular to probabilistic recovery of signals with sparse Fourier spectra from random measurements. The techniques involve intricate combinatorial extensions of the same approach that was employed by Landau and Widom in their $2\Omega T$ theorem.

Ever since Shannon’s seminal work [293], sampling theory has played a central role in the theory of band-limited functions. Aspects of sampling theory related to numerical quadrature are developed in Sect. 2.3. Chapter 5 reviews other aspects of sampling of band-limited signals that are, conceptually, closely tied to time and band limiting. This includes Landau’s necessary conditions for sampling and interpolation [188, 189], phrased in terms of the Beurling densities, whose proofs use the same techniques as those employed in Chap. 4 to estimate eigenvalues for time and band limiting. Results quantifying the structure of sets of sampling and interpolation for Paley–Wiener spaces of band-limited functions are surveyed. The latter part of Chap. 5 addresses methods for sampling multiband signals. Several such methods have been published since the 1990s. A few of these approaches will be addressed in detail, with other approaches outlined in the chapter notes.

Chapter 6 contains several results that represent steps toward a general theory that envelops sampling and time and band limiting, both for band-limited and for multiband signals. The core of the chapter builds on work of Walter and Shen [347] and Khare and George [177], who observed that the integer samples of suitably normalized PSWFs form eigenvectors of the discrete matrix whose entries are correlations of time localizations of shifts of the sinc kernel—the reproducing kernel

for the space of band-limited signals. This fact permits a method for constructing approximate time- and band-limiting projections from samples. Other discrete methods for building time- and multiband-limiting projection operators from constituent components are also presented in Chap. 6.

This text is primarily a mathematical monograph. While applications to signal processing play a vital motivational role, those applications also involve technical details whose descriptions would detract from the emphasis on mathematical methods. However, we also want this book to be useful to practitioners who do not prove theorems for a living—so not all of the theorems that are discussed are proved in full. In choosing which results to present in finer detail, our goal has been to maintain focus on central concepts of time and band limiting. We also emphasize real-variable methods over those of complex function theory, in several cases favoring the more intuitive result or argument over the most complete result or rigorous proof. We hope that this approach serves to make the book more accessible to a broader audience, even if it makes the reading less satisfying to a stalwart mathematician.

The core material in each chapter is supplemented to some extent with *notes and auxiliary results* that outline further major mathematical contributions and primary applications. In some cases, proofs of results auxiliary to the core theorems are also included with this supplementary material. Even so, we have made choices regarding which contributions to mention. We easily could have cited over 1000 contributions without having mentioned every important theorem and application pertinent to time and band limiting.

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