Preface

In the present book we study the pullback equation for differential forms

$$\varphi^*(g) = f,$$

namely, given two differential k-forms f and g we want to discuss the equivalence of such forms. This turns out to be a system of nonlinear first-order partial differential equations in the unknown map φ.

The problem that we study here is a particular case of the equivalence of tensors which has received considerable attention. However, the pullback equation for differential forms has quite different features than those for symmetric tensors, such as Riemannian metrics, which has also been studied a great deal. In more physical terms, the problem of equivalence of forms can also be seen as a problem of mass transportation.

This is an important problem in geometry and in analysis. It has been extensively studied, in the cases $k = 2$ and $k = n$, but much less when $3 \leq k \leq n-1$. The problem considered here of finding normal forms (Darboux theorem, Pfaff normal form) is a fundamental question in symplectic and contact geometry. With respect to the literature in geometry, the main emphasis of the book is on regularity and boundary conditions. Indeed, special attention has been given to getting optimal regularity; this is a particularly delicate point and requires estimates for elliptic equations and fine properties of Hölder spaces.

In the case of volume forms (i.e., $k = n$), our problem is clearly related to the widely studied subject of optimal mass transportation. However, our analysis is not in this framework. As stated before, the two main points of our analysis are that we provide optimal regularity in Hölder spaces and, at the same time, we are able to handle boundary conditions.

Our book will hopefully appeal to both geometers and analysts. In order to make the subject more easily attractive for the analysts, we have reduced as much as possible the notations of geometry. For example, we have restricted our attention to domains in $\mathbb{R}^n$, but it goes without saying that all results generalize to manifolds with or without boundary.

In Part I we gather some basic facts about exterior and differential forms that are used throughout Parts II and IV. Most of the results are standard, but they are presented so that the reader may be able to grasp the main results of the subject without getting too involved with the terminology and concepts of differential geometry.

Part II presents the classical Hodge decomposition following the proof of Morrey, but with some variants, notably in our way of deriving the Gaffney inequality. We also give applications to several versions of the Poincaré lemma that are constantly used in the other parts of the book. Part II can be of interest independently of the main subject of the book.

Part III discusses the case $k = n$. We have tried in this part to make it, as much as possible, independent of the machinery of differential forms. Indeed, Part III can essentially be read with no reference to the other parts of the work, except for the properties of Hölder spaces presented in Part V.

Part IV deals with the general case. Emphasis in this part is given to the symplectic case $k = 2$. We also briefly deal with the simpler cases $k = 0, 1, n-1$. The case $3 \leq k \leq n-2$ is much harder and we are able to obtain results only for forms having a special structure. The difficulty is already at the algebraic level.

In Part V we gather several basic properties of Hölder spaces that are used extensively throughout the book. Due to the nonlinearity of the pullback equation, Hölder spaces are much better adapted than Sobolev spaces. The literature on Hölder spaces is considerably smaller than the one on Sobolev spaces. Moreover, the results presented here cannot be found solely in a single reference. We hope that this part will be useful to mathematicians well beyond those who are primarily interested in the pullback equation.

Acknowledgments Several results of Part IV find their origins from joint works and discussions with S. Bandyopadhyay. During the preparation of the manuscript, we have benefited from many helpful comments by P. Bousquet, G. Cupini, W. Gangbo, N. Kamran, T. Ratiu, K.D. Semmler, D. Serre and D. Ye. The discussions with M. Troyanov have been particularly fruitful. We also thank H. Brézis for accepting, with enthusiasm, our book in the Birkhäuser series that he edits.

The research of the third author has been, in part, subsidized by a grant of the Fonds National Suisse de la Recherche Scientifique.

Lausanne

CSATÓ Gyula
DACOROGNA Bernard
KNEUSS Olivier

http://www.springer.com/978-0-8176-8312-2

The Pullback Equation for Differential Forms
Csató, G.; Dacorogna, B.; Kneuss, O.
2012, XI, 436 p., Hardcover
ISBN: 978-0-8176-8312-2
A product of Birkhäuser Basel