2. *Systems of Linear Equations, Matrices*

2.1 Gaussian Elimination

Equations of the form $\sum a_i x_i = b$, for unknowns x_i with arbitrary given numbers a_i and b, are called *linear*, and every set of simultaneous linear equations is called a *linear system*. They are generalizations of the equations of lines and planes which we have studied in Section 1.3. In this section, we begin to discuss how to solve them, that is, how to find numerical values for the x_i that satisfy all the equations of a given system. We also examine whether a given system has any solutions and, if so, then how we can describe the set of all solutions.

Linear systems arise in many applications. Examples in which they occur, in addition to lines and planes, are least-squares fitting of lines, planes, or curves to observed data, methods for obtaining approximate solutions of various differential equations, Kirchhoff's equations relating currents and potentials in electrical circuits, and various economic models. In many applications, the number of equations and unknowns can be quite large, sometimes in the hundreds or thousands. Thus it is very important to understand the structure of such systems and to apply systematic and efficient methods for their solution. Even more important is that, as we shall see, studying such systems leads to several new concepts and theories that are at the heart of linear algebra.

We begin with a simple example.

Example 2.1.1. (A System of Three Equations in Three Unknowns with a Unique Solution). Let us solve the following system:

$$\begin{aligned} 2x + 3y - z &= 8 \\ 4x - 2y + z &= 5 \\ x + 5y - 2z &= 9. \end{aligned} \tag{2.1}$$

(Geometrically this problem amounts to finding the point of intersection of three planes.)

We want to proceed as follows: multiply both sides of the first equation by 2 and subtract the result from the second equation to eliminate the $4x$,

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and subtract 1/2 times the first equation from the third equation to eliminate the x. The system is then changed into the new, equivalent[1] system:

$$\begin{aligned} 2x + 3y \;-\; z &= 8 \\ -8y + 3z &= -11 \\ \frac{7}{2}y - \frac{3}{2}z &= 5. \end{aligned} \tag{2.2}$$

As our next step we want to get rid of the $7y/2$ term in the last equation. We can achieve this elimination by multiplying the middle equation by $-7/16$ and subtracting the result from the last equation. Then we get

$$\begin{aligned} 2x + 3y \;-\; z &= 8 \\ -8y + 3z &= -11 \\ \frac{-3}{16}z &= \frac{3}{16}. \end{aligned} \tag{2.3}$$

At this point we can easily find the solution by starting with the last equation and working our way back up. First, we find $z = -1$, and second, substituting this value into the middle equation we get $-8y-3 = -11$, which yields $y = 1$. Last, we enter the values of y and z into the top equation and obtain $2x + 3 + 1 = 8$, hence $x = 2$.

Substituting these values for x, y, z into Equations 2.1 indeed confirms that they are solutions. ♦

The method of solving a linear system used in the example above is called *Gaussian elimination*,[2] and it is the foremost method of solving such systems. However, before discussing it more generally, let us mention that the way the computations were presented was the way a computer would be programmed to do them. For people, slight variations are preferable. We would rather avoid fractions, and if we want to eliminate, say, x from an equation beginning with bx by using an equation beginning with ax, with a and b nonzero integers, then we could multiply the first equation by a and the other by b to get abx in both. Also, we would sometimes add and sometimes subtract, depending on the signs of the terms involved, where computers always subtract. Last, we might rearrange the equations in a different order, if we see that doing so would result in simpler arithmetic.[3] For example, right at the start of the example above, we could have put the last equation on top because it begins

[1] Equivalence of systems will be discussed in detail on page 46.

[2] Named after Carl Friedrich Gauss (1777–1855). It is ironic that in spite of his many great achievements he is best remembered for this simple but widely used method and for the so-called Gaussian distribution in probability and statistics, which was mistakenly attributed to him but had been discovered by Abraham de Moivre in the 1730s.

[3] Computer programs have to reorder the equations sometimes but for different reasons, namely to avoid division by zero and to minimize roundoff errors.

with x rather than $2x$, and used that equation the way we have used the one beginning with $2x$.

The essence of the method is to subtract multiples of the first equation from the others so that the leftmost term in the first equation eliminates all the corresponding terms below it. Then we continue by similarly using the leftmost term in the new second equation to eliminate the corresponding term (or terms if there are more equations) below that, and so on, down to the last equation. Next, we work our way up by solving the last equation first, then substituting its solution into the previous equation, solving that, and so on. The two phases of the method are called *forward elimination* and *back substitution.* As will be seen shortly, a few complications can and do frequently arise, which make the theory that follows even more interesting and necessary. First, however, we introduce a crucial notational simplification.

Notice that in the forward elimination computations of Example 2.1.1 the variables x, y, z were not really used; they were needed only in the back substitution steps used to determine the solution. All the forward elimination computations were done on the coefficients only. In computer programs there is not even a way (and no need either) to enter the variables. In writing, the coefficients are usually arranged in a rectangular array enclosed in parentheses or brackets, called a *matrix* (plural: *matrices*) and designated by a capital letter, as in

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 1 \\ 1 & 5 & -2 \end{bmatrix}. \tag{2.4}$$

This matrix contains the coefficients on the left side of system 2.1 in the same arrangement, and is therefore referred to as the *coefficient matrix* or just the *matrix* of that system. We may include the numbers on the right sides of the equations as well:

$$B = \begin{bmatrix} 2 & 3 & -1 & 8 \\ 4 & -2 & 1 & 5 \\ 1 & 5 & -2 & 9 \end{bmatrix}. \tag{2.5}$$

This is called the *augmented matrix* of the system. It is often written with a vertical line before its last column as

$$B = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 8 \\ 4 & -2 & 1 & 5 \\ 1 & 5 & -2 & 9 \end{array}\right]. \tag{2.6}$$

Example 2.1.2. (Solving the 3×3 *System of Example 2.1.1, Using Augmented Matrix Notation).* We write the computations of Example 2.1.1 as

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & 8 \\ 4 & -2 & 1 & 5 \\ 1 & 5 & -2 & 9 \end{array}\right] \to \left[\begin{array}{rrr|r} 2 & 3 & -1 & 8 \\ 0 & -8 & 3 & -11 \\ 0 & 7/2 & -3/2 & 5 \end{array}\right]$$
$$\to \left[\begin{array}{rrr|r} 2 & 3 & -1 & 8 \\ 0 & -8 & 3 & -11 \\ 0 & 0 & -3/16 & 3/16 \end{array}\right]. \tag{2.7}$$

The arrows between the matrices above do not designate equality, they just indicate the flow of the computation. For two matrices to be equal, all the corresponding entries must be equal, and here they are clearly not equal.

Next, we change from the last augmented matrix to the corresponding system

$$\begin{aligned} 2x + 3y \; - \; z &= 8 \\ -8y + 3z &= -11 \\ \frac{-3}{16} z &= \frac{3}{16}, \end{aligned} \tag{2.8}$$

which we solve as in Example 2.1.1. ♦

The matrix A in Equation 2.4 is a 3×3 (read: "three by three") matrix, and in Equation 2.5, B is a 3×4 matrix. Similarly, if a matrix has m rows and n columns, we call it an $m \times n$ matrix. In describing matrices, we always say *rows first, then columns.*

The general form of an $m \times n$ matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \tag{2.9}$$

where the $a_{11}, a_{12}, \ldots, a_{mn}$ (read "a sub one-one, a sub one-two," etc.) are arbitrary real numbers. They are called the entries of the matrix A, with a_{ij} denoting the entry at the intersection of the ith row and jth column. Thus in the double subscript ij the order is important. Also, the matrix A is often denoted by $[a_{ij}]$ or (a_{ij}).

Two matrices are said to be equal if they have the same shape, that is, the same numbers of rows and columns, and their corresponding entries are equal.

A matrix consisting of a single row is called a *row vector*, and that of a single column, a *column vector*, and, if we want to emphasize the size n, a row n-vector or a column n-vector.

By definition, a system of m linear equations for n unknowns $x_1, x_2, \ldots, x_n$ has the general form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{2.10}$$

with the coefficient matrix A given in Equation 2.9 having arbitrary entries and the b_i denoting arbitrary numbers as well.[4] We shall frequently find it useful to collect the x_i and the b_i values into two *column* vectors and write such systems as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{2.11}$$

or abbreviated as

$$A\mathbf{x} = \mathbf{b}. \tag{2.12}$$

The expression $A\mathbf{x}$ will be discussed in detail in Section 2.3 and generalized in Section 2.4. For now, we shall just use $A\mathbf{x} = \mathbf{b}$ as a compact reference to the system.

The augmented matrix of this general system is written as

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]. \tag{2.13}$$

The reason for using *column* vectors $\mathbf{x}$ and $\mathbf{b}$ will be explained at the end of Section 2.3, although for $\mathbf{b}$ at least, the choice is rather natural since then the right sides of Equations 2.10 and 2.11 match.

Henceforth all vectors will be column vectors unless explicitly designated otherwise, and also $\mathbb{R}^n$, *for every* n, *will be regarded as a space of column vectors.*

In general, if we want to solve a system given as $A\mathbf{x} = \mathbf{b}$, we reduce the corresponding augmented matrix $[A|\mathbf{b}]$ to a simpler form $[U|\mathbf{c}]$ (details will follow), which we change back to a system of equations, $U\mathbf{x} = \mathbf{c}$. We then solve the latter by back substitution, that is, from the bottom up.

[4] Writing any quantity with a general subscript, like the i here in b_i, is general mathematical shorthand for the list of all such quantities, for all possible values of the subscript i, as in this case for the list $b_1, b_2, \ldots, b_m$. Also, it is customary to say "the b_i" instead of "the b_i's" to avoid any possible confusion.

Let us review the steps of Example 2.1.2. We copied the first row, then we took 4/2 times the entries of the first row in order to change the 2 into a 4, and subtracted those multiples from the corresponding entries of the second row. (We express this operation more briefly by saying that we subtracted 4/2 times the first row from the second row.) Then we took 1/2 times the entries of the first row to change the 2 into a 1 and subtracted them from the third row. In all this computation the entry 2 of the first row played a pivotal role and is therefore called the *pivot* for these operations. In general, a pivot is an entry whose multiples are used to obtain zeros below it, and the first nonzero entry remaining in the last nonzero row after the reduction is also called a pivot. (The precise definition will be given on page 55.) Thus, in this calculation the pivots are the numbers $2, -8, -3/16$.

The operations we used are called elementary row operations.

Definition 2.1.1. ***(Elementary Row Operations).*** *We call the following three types of operations on the augmented matrix of a system elementary row operations:*

1. Multiplying a row by a nonzero number.

2. Exchanging two rows.

3. Changing a row by subtracting a nonzero multiple of another row from it.

Definition 2.1.2. ***(Equivalence of Systems and of Matrices).*** *Two systems of equations are called equivalent if their solution sets are the same. Furthermore, the augmented matrices of equivalent systems are called equivalent to each other as well.*

All elimination steps in this section, like the ones above, have been designed to produce equivalent, but simpler, systems.

Theorem 2.1.1. ***(Row Equivalence).*** *Each elementary row operation changes the augmented matrix of every system of linear equations into the augmented matrix of an equivalent system.*

Proof. Consider any two rows of the augmented matrix of the $m \times n$ system 2.10, say the ith and the jth row:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \tag{2.14}$$

and

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j. \tag{2.15}$$

1. If we form a new augmented matrix by multiplying the ith row of the augmented matrix 2.13 by any $c \neq 0$, then the ith row of the corresponding new system is

$$ca_{i1}x_1 + ca_{i2}x_2 + \cdots + ca_{in}x_n = cb_i, \tag{2.16}$$

which is clearly equivalent to Equation 2.14. Furthermore, since all the other equations of the system 2.10 remain unchanged, every solution $\mathbf{x}$ of the old system is also a solution of the new system and vice versa.
2. If we form a new augmented matrix by exchanging the ith row of the augmented matrix 2.13 by its jth row, then the corresponding system of equations remains the same, except that equations 2.14 and 2.15 are switched. Clearly, changing the order of equations does not change the solutions.
3. If we change the jth row of the augmented matrix 2.13 by subtracting c times the ith row from it, for any $c \neq 0$, then the jth row of the corresponding new system becomes

$$(a_{j1} - ca_{i1})\,x_1 + (a_{j2} - ca_{i2})\,x_2 + \cdots + (a_{jn} - ca_{in})\,x_n = b_j - cb_i. \quad (2.17)$$

The other equations of the system, including the ith one, remain unchanged. Clearly, every vector $\mathbf{x}$ that solves the old system, also solves Equation 2.17, and so it solves the whole new system as well. Conversely, if a vector $\mathbf{x}$ solves the new system, then it solves Equation 2.14, and hence also Equation 2.16, as well as Equation 2.17. Adding the latter two together, we find that it solves Equation 2.15, that is, it solves the old system. ■

Hence any two matrices obtainable from each other by a finite number of successive elementary row operations are equivalent, and to indicate that they are related by such row operations, they are said to be *row equivalent.* Column operations would also be possible, but they are rarely used, and we shall not discuss them at all.

We have used only the third type of elementary row operation so far. The first kind is not necessary for Gaussian elimination but will be used later in further reductions. The second kind must be used if we encounter a zero where we need a pivot, as in the following example.

Example 2.1.3. (A 4×3 *System with a Unique Solution and Requiring a Row Exchange).* Let us solve the following system of $m = 4$ equations in $n = 3$ unknowns:

$$\begin{aligned} x_1 + 2x_2 \phantom{{}-2x_3} &= 2 \\ 3x_1 + 6x_2 - x_3 &= 8 \\ x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + 5x_2 - 2x_3 &= 9. \end{aligned} \quad (2.18)$$

We do the computations in matrix form. We indicate the row operations in the rows between the matrices by arrows, which may be read as "becomes" or "is replaced by." For example, $\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_1$ means that row 2 is replaced by the old row 2 minus 3 times row 1. (The rows may be considered to be vectors, and so we designate them by boldface letters.)

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 2 \\ 3 & 6 & -1 & 8 \\ 1 & 2 & 1 & 0 \\ 2 & 5 & -2 & 9 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_1 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1 \\ \mathbf{r}_4 \leftarrow \mathbf{r}_4 - 2\mathbf{r}_1 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & -2 & 5 \end{array}\right]$$
$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_4 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 \\ \mathbf{r}_4 \leftarrow \mathbf{r}_2 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 \\ \mathbf{r}_4 \leftarrow \mathbf{r}_4 + \mathbf{r}_3 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{2.19}$$

The back substitution phase should start with the third row of the last matrix, since the fourth row just expresses the trivial equation $0 = 0$. The third row gives $x_3 = -2$, the second row corresponds to $x_2 - 2x_3 = 5$ and so $x_2 = 1$, and the first row yields $x_1 + 2x_2 = 2$, from which $x_1 = 0$. ♦

As the example above shows, the number m of equations and the number n of unknowns need not be the same. In this case the four equations described four planes in three-dimensional space, having a single point of intersection given by the *unique solution* we have found. Of course, in general, four planes need not have a point of intersection in common or may have an entire line or plane as their intersection (in the latter case the four equations would each describe the same plane). Systems with solutions are called *consistent*. On the other hand, if there is no intersection, then the system has no solution, and it is said to be *inconsistent*. Inconsistency of the system can happen with just two or three planes as well, for instance if two of them are parallel, and also in two dimensions with parallel lines. So before attacking the general theory, we discuss examples of inconsistent systems and systems with infinitely many solutions. Systems with more equations than unknowns are called *overdetermined,* and are usually (though not always, see Example 2.1.3) inconsistent. Systems with fewer equations than unknowns are called *underdetermined*, and they usually (though not always) have infinitely many solutions. For example, two planes in $\mathbb{R}^3$ would usually intersect in a line, but exceptionally they could be parallel and have no intersection. On the other hand, a system with the same number of equations as unknowns is called *determined* and usually (though not always) has a unique solution. For instance, three planes in $\mathbb{R}^3$ would usually intersect in a point, but by exception they could be parallel and have no intersection or intersect in a line or a plane.

Example 2.1.4. (A 3×3 *Inconsistent System).* Consider the system given by the matrix

$$[A|\mathbf{b}] = \left[\begin{array}{rrr|r} 1 & 2 & 0 & 2 \\ 3 & 6 & -1 & 8 \\ 1 & 2 & 1 & 4 \end{array}\right]. \tag{2.20}$$

Subtracting $3\mathbf{r}_1$ from $\mathbf{r}_2$, and $\mathbf{r}_1$ from $\mathbf{r}_3$, we get

$$[A'|\mathbf{b}'] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]. \tag{2.21}$$

The last two rows of $[A'|\mathbf{b}']$ represent the contradictory equations $-x_3 = 2$ and $x_3 = 2$. These two equations describe parallel planes. Thus $[A|\mathbf{b}]$ had to represent an inconsistent system.

The row operations above have produced two equations of new planes, which have turned out to be parallel to each other. The planes corresponding to the rows of the original $[A|\mathbf{b}]$ are, however, not parallel. Instead, only the three lines of intersection of pairs of them are (see Exercise 2.1.16), like the three parallel edges of a prism; that is why there is no point of intersection common to all three planes.

We may carry the matrix reduction one step further and obtain, by adding the second row to the third one,

$$[A''|\mathbf{b}''] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right]. \tag{2.22}$$

This matrix provides the single self-contradictory equation $0 = 4$ from its last row. There is no geometrical interpretation for such an equation, but algebraically it is the best way of establishing the inconsistency. Thus, this is the typical pattern we shall obtain in the general case whenever there is no solution. ♦

Next we modify the matrix of the last example so that all three planes intersect in a single line.

Example 2.1.5. (A 3×3 *System with a Line for Its Solution Set).* Let

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 3 & 6 & -1 & 8 \\ 1 & 2 & 1 & 0 \end{array}\right]. \tag{2.23}$$

We can reduce this matrix to

$$[A'|\mathbf{b}'] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{2.24}$$

The corresponding system is

$$\begin{aligned} x_1 + 2x_2 \qquad &= 2 \\ -x_3 &= 2 \\ 0 &= 0, \end{aligned} \tag{2.25}$$

which represents just two planes, since the last equation has become the trivial identity $0 = 0$. Algebraically, the second row gives $x_3 = -2$, and the first row relates x_1 to x_2. We can choose an arbitrary value for either x_1 or x_2 and solve the first equation of the system 2.25 for the other. In some other examples, however, we have no choice, as between x_1 and x_2 here. However, *since the pivot cannot be zero, we can always solve the pivot's row for the variable corresponding to the pivot, and that is what we always do.* Thus, we set x_2 equal to a parameter t and solve the first equation for x_1, to obtain $x_1 = 2 - 2t$. We can write the solutions in vector form as (remember: the convention is to use column vectors)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}. \tag{2.26}$$

This is a parametric vector equation of the line of intersection L of the three planes defined by the rows of $[A|\mathbf{b}]$. The coordinates of each of L's points make up one of the infinitely many solutions of the system. ♦

Example 2.1.6. (A 3×4 *System with a Line for its Solution Set).* Let us solve the system

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 + 4x_4 &= 4 \\ -6x_1 - 8x_2 + 6x_3 - 2x_4 &= 1 \\ 4x_1 + 4x_2 - 4x_3 - x_4 &= -7. \end{aligned} \tag{2.27}$$

These equations represent three hyperplanes in four dimensions.[5] We can proceed as before:

$$\left[\begin{array}{rrrr|r} 2 & 3 & -2 & 4 & 4 \\ -6 & -8 & 6 & -2 & 1 \\ 4 & 4 & -4 & -1 & -7 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 + 3\mathbf{r}_1 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_1 \end{array} \left[\begin{array}{rrrr|r} 2 & 3 & -2 & 4 & 4 \\ 0 & 1 & 0 & 10 & 13 \\ 0 & -2 & 0 & -9 & -15 \end{array}\right]$$
$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 + 2\mathbf{r}_2 \end{array} \left[\begin{array}{rrrr|r} 2 & 3 & -2 & 4 & 4 \\ 0 & 1 & 0 & 10 & 13. \\ 0 & 0 & 0 & 11 & 11 \end{array}\right]. \tag{2.28}$$

The variables that have pivots as coefficients, x_1, x_2, x_4 in this case, are called *basic variables.* They can be obtained in terms of the other, so-called *free variables* that correspond to the pivot-free columns. The free variables are usually replaced by parameters, but this is just a formality to show that they can be chosen freely.

Thus, we set $x_3 = t$, and find the solutions again as the points of a line, now given by

[5] A hyperplane in $\mathbb{R}^4$ is a copy of $\mathbb{R}^3$, just as a plane in $\mathbb{R}^3$ is a copy of $\mathbb{R}^2$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9/2 \\ 3 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{2.29}$$

♦

Example 2.1.7. (A 3×4 System with a Plane for Its Solution Set). Consider the system

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 + 4x_4 &= 2 \\ -6x_1 - 9x_2 + 7x_3 - 8x_4 &= -3 \\ 4x_1 + 6x_2 - x_3 + 20x_4 &= 13. \end{aligned} \tag{2.30}$$

We solve this system as follows:

$$\left[\begin{array}{rrrr|r} 2 & 3 & -2 & 4 & 2 \\ -6 & -9 & 7 & -8 & -3 \\ 4 & 6 & -1 & 20 & 13 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 + 3\mathbf{r}_1 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_1 \end{array} \left[\begin{array}{rrrr|r} 2 & 3 & -2 & 4 & 2 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 3 & 12 & 9 \end{array}\right]$$

$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\mathbf{r}_2 \end{array} \left[\begin{array}{rrrr|r} 2 & 3 & -2 & 4 & 2 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{2.31}$$

Since the pivots are in columns 1 and 3, the basic variables are x_1 and x_3 and the free variables x_2 and x_4. Thus we use two parameters and set $x_2 = s$ and $x_4 = t$. Then the second row of the last matrix leads to $x_3 = 3-4t$ and the first row to $2x_1 + 3s - 2(3-4t) + 4t = 2$, that is, to $2x_1 = 8 - 3s - 12t$. Putting all these results together, we obtain the two-parameter set of solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \tag{2.32}$$

which is also a parametric vector equation of a plane in $\mathbb{R}^4$. ♦

Exercises

In the first four exercises, find all solutions of the systems by Gaussian elimination.

Exercise 2.1.1.
$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 &= 0 \\ x_1 + 5x_2 + 2x_3 &= 1 \\ -4x_1 + 6x_3 &= 2 \end{aligned}$$

Exercise 2.1.2.
$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 &= 0 \\ x_1 + 5x_2 + 2x_3 &= 0 \\ -4x_1 + 6x_3 &= 0 \end{aligned}$$

Exercise 2.1.3. $$\begin{aligned} 2x_1 + 2x_2 - 3x_3 &= 0 \\ x_1 + 5x_2 + 2x_3 &= 1 \end{aligned}$$

Exercise 2.1.4. $2x_1 + 2x_2 - 3x_3 = 0$

In the next nine exercises use Gaussian elimination to find all solutions of the systems given by their augmented matrices.

Exercise 2.1.5. $$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ -2 & 3 & -1 & 0 \\ -6 & 6 & 0 & 0 \end{array}\right]$$

Exercise 2.1.6. $$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ -2 & 3 & -1 & 0 \\ -6 & 6 & 0 & -2 \end{array}\right]$$

Exercise 2.1.7. $$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ -2 & 3 & -1 & 0 \\ -6 & 6 & 0 & 0 \end{array}\right]$$

Exercise 2.1.8. $$\left[\begin{array}{rrrr|l} 3 & -6 & -1 & 1 & 12 \\ -1 & 2 & 2 & 3 & 1 \\ 6 & -8 & -3 & -2 & 9 \end{array}\right]$$

Exercise 2.1.9. $$\left[\begin{array}{lrrr|r} 1 & 4 & 9 & 2 & 0 \\ 2 & 2 & 6 & -3 & 0 \\ 2 & 7 & 16 & 3 & 0 \end{array}\right]$$

Exercise 2.1.10. $$\left[\begin{array}{lrr|r} 2 & 4 & 1 & 7 \\ 0 & 1 & 3 & 7 \\ 3 & 3 & -1 & 9 \\ 1 & 2 & 3 & 11 \end{array}\right]$$

Exercise 2.1.11. $$\left[\begin{array}{rrrrr|r} 3 & -6 & -1 & 1 & 5 & 0 \\ -1 & 2 & 2 & 3 & 3 & 0 \\ 4 & -8 & -3 & -2 & 1 & 0 \end{array}\right]$$

Exercise 2.1.12. $$\left[\begin{array}{rrrr|r} 3 & -6 & -1 & 1 & 7 \\ -1 & 2 & 2 & 3 & 1 \\ 4 & -8 & -3 & -2 & 6 \end{array}\right]$$

Exercise 2.1.13. $$\left[\begin{array}{rrrr|r} 3 & -6 & -1 & 1 & 5 \\ -1 & 2 & 2 & 3 & 3 \\ 6 & -8 & -3 & -2 & 1 \end{array}\right]$$

Exercise 2.1.14. What is wrong with the following way of "solving" Exercise 2.1.13?

$$\left[\begin{array}{rrrr|r} 3 & -6 & -1 & 1 & 5 \\ -1 & 2 & 2 & 3 & 3 \\ 6 & -8 & -3 & -2 & 1 \end{array}\right] \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 \left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 3 \\ 3 & -6 & -1 & 1 & 5 \\ 6 & -8 & -3 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow 2\mathbf{r}_2 - \mathbf{r}_3 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_2 \end{array} \left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 3 \\ 0 & -4 & 1 & 4 & 9 \\ 0 & 4 & -1 & -4 & -9 \end{array}\right]$$

$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_2 \end{array} \left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 3 \\ 0 & -4 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$x_3 = s$, $x_4 = t$, $-4x_2 + s + 4t = 9$, $x_2 = -\frac{9}{4} + \frac{1}{4}s + t$, $-x_1 + 2x_2 + 2s + 3t = 3$, $x_1 = -3 + 2\left(-\frac{9}{4} + \frac{1}{4}s + t\right) + 2s + 3t = \frac{5}{2}s + 5t - \frac{15}{2}$, and so

$$\mathbf{x} = \begin{bmatrix} -15/2 \\ -9/4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 5/2 \\ 1/4 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 5 \\ 1 \\ 0 \\ 1 \end{bmatrix} t.$$

Exercise 2.1.15. Show that each pair of the three planes defined by the rows of the matrix in Example 2.1.5 on page 49 has the same line of intersection.

Exercise 2.1.16. Show that the three planes defined by the rows of the matrix in Equation 2.20 on page 48 have parallel lines of intersection.

2.2 The Theory of Gaussian Elimination

We are now at a point where we can summarize the lessons from our examples. Given m equations for n unknowns, we consider their augmented matrix,

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right], \tag{2.33}$$

and reduce it using elementary row operations according to the following algorithm:

1. Search the first column from the top down for the first nonzero entry. If all the entries in the first column are zero, then search the second column from the top down, then the third column for the first nonzero entry. Repeat with succeeding columns if necessary, until a nonzero entry is found. The entry thus found is called the current *pivot*. Stop, if no pivot can be found.
2. Put the row containing the current pivot on top (unless it is already there).

3. Subtract appropriate multiples of the first row from each of the lower rows to obtain all zeros below the current pivot in its column (unless there are all zeros there or no lower rows are left).
4. Repeat the previous steps on the submatrix[6] consisting of all those elements of the last matrix that lie lower than and to the right of the last pivot. Stop if no such submatrix is left.

These steps constitute the *forward elimination* phase of Gaussian elimination (the second phase will be discussed following Definition 2.2.2), and they lead to a matrix of the form described below.

Definition 2.2.1. ***(Echelon Matrix).*** *A matrix is said to be in echelon form[7] or an echelon matrix if it has a staircase-like pattern characterized by the following properties:*
a*. The all-zero rows (if any) are at the bottom.*
b*. The leftmost nonzero entry of each nonzero row, called a leading entry, is in a column to the right of the leading entry of every row above it.*

These properties imply that in an echelon matrix U all the entries of a column below a leading entry are zero. If U arises from the reduction of a matrix A by the forward elimination algorithm above, then the pivots of A become the leading entries of U. Also, if we were to apply the algorithm to an echelon matrix, then it would not be changed and we would find that its leading entries are its pivots.

Note that while a given matrix is row equivalent to many different echelon matrices (just multiply any nonzero row of an echelon matrix by 2, for example), the algorithm above leads to a single well-defined echelon matrix in each case. Furthermore, it will be proved in Section 3.3 that the number and locations, although not the values, of the pivots are unique for all echelon matrices obtainable from the same A. Consequently, the results of Theorem 2.2.1 below, even though they depend on the pivots, are valid unambiguously.

Here is a possible $m \times (n+1)$ echelon matrix obtainable from the matrix $[A|\mathbf{b}]$ above:

$$[U|\mathbf{c}] = \left[\begin{array}{ccccccc|c} p_1 & * & * & * & \cdots & * & * & c_1 \\ 0 & p_2 & * & * & \cdots & * & * & c_2 \\ 0 & 0 & 0 & p_3 & & * & * & c_3 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p_r & * & c_r \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_{r+1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right]. \tag{2.34}$$

[6] A submatrix of a given matrix A is a matrix obtained by deleting any number of rows and/or columns of A.
[7] "Echelon" in French means "rung of a ladder," and in English it is used for some ladder-like military formations and rankings.

The first n columns constitute the echelon matrix U obtained from A, and the last column is the corresponding reduction of $\mathbf{b}$. The p_i denote the pivots of A, while the entries denoted by * and by c_i denote numbers that may or may not be zero. The number r is very important, since it determines the character of the solutions, and has a special name.

Definition 2.2.2. ***(Rank).*** *The number r of nonzero rows of an echelon matrix U obtained by the forward elimination phase of the Gaussian elimination algorithm from a matrix A is called the rank of A and will be denoted by* $\operatorname{rank}(A)$.[8,9]

We can now describe the back substitution phase of Gaussian elimination, in which we change the augmented matrix $[U|\mathbf{c}]$ back to a system of equations $U\mathbf{x} = \mathbf{c}$:

5. If $r < m$ and $c_{r+1} \neq 0$ hold, then the row containing c_{r+1} corresponds to the self-contradictory equation $0 = c_{r+1}$, and so the system has no solutions or, in other words, it is *inconsistent.* (This case occurs in Example 2.1.4, where $m = 3,\ r = 2$ and $c_{r+1} = c_3 = 4$.)
6. If $r = m$ or $c_{r+1} = 0$, then the system is *consistent* and, for every i such that the ith column contains no pivot, the variable x_i is a free variable and we set it equal to a parameter s_i. (In Example 2.1.6, for instance, $r = m = 3$ and x_3 is free. In Example 2.1.7 we have $m = 3,\ r = 2$ and $c_{r+1} = c_3 = 0$ and the free variables are x_2 and x_4.) We need to distinguish two subcases here:
   a. If $r = n$, then there are no free variables and the system has a *unique solution.* (In Example 2.1.2, for instance, $r = m = n = 3$.)
   b. If $r < n$, then the system has *infinitely many solutions.* (In Examples 2.1.6 and 2.1.7, for instance, $r = 3$ and $n = 4$.)
7. In any of the cases of Part 6, we solve for the basic variables x_i corresponding to the pivots p_i, starting in the rth row and working our way up row by row.

The Gaussian elimination algorithm proves the following theorem:

Theorem 2.2.1. ***(Summary of Gaussian Elimination).*** *Consider the $m \times n$ system with A an $m \times n$ matrix and $\mathbf{b}$ an n-vector:*

$$A\mathbf{x} = \mathbf{b}. \tag{2.35}$$

Suppose the matrix $[A|\mathbf{b}]$ is reduced by the algorithm above to the echelon matrix $[U|\mathbf{c}]$ with $\operatorname{rank}(U) = r$.

---

[8] Of course, r is also the rank of U, since the algorithm applied to U would leave U unchanged.

[9] Some books call this quantity the *row rank* of A until they define the column rank and show that the two are equal.

If $r = m$, that is, if U has no zero rows, then the system 2.35 is consistent. If $r < m$, then the system is consistent if and only if $c_{r+1} = 0$.

For a consistent system,

a. there is a unique solution if and only if there are no free variables, that is, if $r = n$;

b. if $r < n$, then there is an $(n - r)$-parameter infinite set of solutions of the form

$$\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{n-r} s_i \mathbf{v}_i. \tag{2.36}$$

We may state the uniqueness condition $r = n$ in another way by saying that the pivots are the diagonal entries $u_{11}, u_{22}, \ldots, u_{nn}$ of U, that is, that U has the form

$$U = \begin{bmatrix} p_1 & * & * & \cdots & * \\ 0 & p_2 & * & \cdots & * \\ 0 & 0 & p_3 & & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & p_r \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{2.37}$$

A matrix of this form is called an *upper triangular matrix* and the p_i its diagonal entries. (The pivots are never 0, but in general, an upper triangular matrix is allowed to have 0 diagonal entries as well.)

Note that for every $m \times n$ matrix we have $0 \leq r \leq \min(m, n)$, because r equals the number of pivots and there can be only one pivot in each row and in each column. We have $r = 0$ only for zero matrices. At the other extreme, if, for a matrix A, $r = \min(m, n)$ holds, then A is said to have *full rank*. If $r < \min(m, n)$ holds, then A is said to be *rank deficient*.

Exercises

Exercise 2.2.1. List all possible forms of 2×2 echelon matrices in a manner similar to Equation 2.37, with p_i for the pivots and $*$ for the entries that may or may not be zero.

Exercise 2.2.2. List all possible forms of 3×3 echelon matrices in a manner similar to Equation 2.37, with p_i for the pivots and $*$ for the entries that may or may not be zero. (*Hint*: There are eight distinct forms.)

In the next four exercises find conditions on a general vector $\mathbf{b}$ that would make the equation $A\mathbf{x} = \mathbf{b}$ consistent for the given matrix A. (*Hint*: Reduce the augmented matrix using undetermined components b_i of $\mathbf{b}$, until the A in it is changed to echelon form, and set $c_{r+1} = c_{r+2} = \cdots = 0$.)

Exercise 2.2.3. $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix}$.

Exercise 2.2.4. $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -6 & 12 \end{bmatrix}$.

Exercise 2.2.5. $A = \begin{bmatrix} 1 & 2 & -6 \\ -2 & -4 & 12 \end{bmatrix}$.

Exercise 2.2.6. $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ 3 & -3 & 0 \\ 2 & 0 & -2 \end{bmatrix}$.

Exercise 2.2.7. Prove that the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if A and $[A|\mathbf{b}]$ have the same rank.

2.3 Homogeneous and Inhomogeneous Systems, Gauss–Jordan Elimination

In the sequel, we need to consider the expression $A\mathbf{x}$ as a new kind of product.[10]

Definition 2.3.1. ***(Matrix-Vector Product).*** *For every $m \times n$ matrix A and every column n-vector $\mathbf{x}$, we define $A\mathbf{x}$ as the column m-vector given by*

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}. \tag{2.38}$$

[10] The product $A\mathbf{x}$ is always written just by juxtaposing the two letters; we never use any multiplication sign in it.

Notice that, on the right, the components of the column vector $\mathbf{x}$ show up across every row of $A\mathbf{x}$; they are "flipped." Actually, the rows on the right are the dot products of the row vectors of A with the vector $\mathbf{x}$. It is customary to write $\mathbf{a}^i$ (with a superscript i) for the ith row of A, and $\mathbf{a}^i\mathbf{x}$ (without a dot) for the ith dot product on the right. Thus Equation 2.38 can also be written as

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{a}^1\mathbf{x} \\ \mathbf{a}^2\mathbf{x} \\ \vdots \\ \mathbf{a}^m\mathbf{x} \end{bmatrix}. \tag{2.39}$$

We also need the following simple properties of $A\mathbf{x}$.

Theorem 2.3.1. ***(Properties of the Matrix-Vector Product).*** *If A is an $m \times n$ matrix, $\mathbf{x}$ and $\mathbf{y}$ column n-vectors, and c a scalar, then*

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} \text{ and } A(c\mathbf{x}) = c(A\mathbf{x}). \tag{2.40}$$

Proof. Using Equation 2.39 and the properties of vectors and dot products, we have

$$A(\mathbf{x}+\mathbf{y}) = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} (\mathbf{x}+\mathbf{y}) = \begin{bmatrix} \mathbf{a}^1(\mathbf{x}+\mathbf{y}) \\ \mathbf{a}^2(\mathbf{x}+\mathbf{y}) \\ \vdots \\ \mathbf{a}^m(\mathbf{x}+\mathbf{y}) \end{bmatrix} = \begin{bmatrix} \mathbf{a}^1\mathbf{x}+\mathbf{a}^1\mathbf{y} \\ \mathbf{a}^2\mathbf{x}+\mathbf{a}^2\mathbf{y} \\ \vdots \\ \mathbf{a}^m\mathbf{x}+\mathbf{a}^m\mathbf{y} \end{bmatrix} \tag{2.41}$$

$$= \begin{bmatrix} \mathbf{a}^1\mathbf{x} \\ \mathbf{a}^2\mathbf{x} \\ \vdots \\ \mathbf{a}^m\mathbf{x} \end{bmatrix} + \begin{bmatrix} \mathbf{a}^1\mathbf{y} \\ \mathbf{a}^2\mathbf{y} \\ \vdots \\ \mathbf{a}^m\mathbf{y} \end{bmatrix} = A\mathbf{x} + A\mathbf{y}. \tag{2.42}$$

Similarly,

$$A(c\mathbf{x}) = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} (c\mathbf{x}) = \begin{bmatrix} \mathbf{a}^1(c\mathbf{x}) \\ \mathbf{a}^2(c\mathbf{x}) \\ \vdots \\ \mathbf{a}^m(c\mathbf{x}) \end{bmatrix} = \begin{bmatrix} c(\mathbf{a}^1\mathbf{x}) \\ c(\mathbf{a}^2\mathbf{x}) \\ \vdots \\ c(\mathbf{a}^m\mathbf{x}) \end{bmatrix} = c \begin{bmatrix} \mathbf{a}^1\mathbf{x} \\ \mathbf{a}^2\mathbf{x} \\ \vdots \\ \mathbf{a}^m\mathbf{x} \end{bmatrix} = c(A\mathbf{x}). \tag{2.43}$$

■

If the solutions of $A\mathbf{x} = \mathbf{b}$ are given by Equation 2.36, the latter is called the *general solution* of the system, as opposed to a *particular solution,* which is obtained by substituting particular values for the parameters into Equation 2.36.

It is customary and very useful to distinguish two types of linear systems depending on the choice of $\mathbf{b}$.

Definition 2.3.2. ***(Homogeneous Versus Inhomogeneous Systems).*** *A system of linear equations* $A\mathbf{x} = \mathbf{b}$ *is called homogeneous if* $\mathbf{b} = \mathbf{0}$*, and inhomogeneous if* $\mathbf{b} \neq \mathbf{0}$*.*

We may restate part of Theorem 2.2.1 for homogeneous systems as follows.

Theorem 2.3.2. ***(Solutions of Homogeneous Systems).*** *For any* $m \times n$ *matrix* A*, the homogeneous system*

$$A\mathbf{x} = \mathbf{0} \tag{2.44}$$

is always consistent: it always has the trivial solution $\mathbf{x} = \mathbf{0}$*. If* $r = n$*, then it has only this solution; and if* $m < n$ *or, more generally, if* $r < n$ *holds, then it has nontrivial solutions as well.*

There is an important relationship between the solutions of corresponding homogeneous and inhomogeneous systems, the analog of which is indispensable for solving many differential equations.

Theorem 2.3.3. ***(General and Particular Solutions).*** *For any* $m \times n$ *matrix* A *and any column* m*-vector* $\mathbf{b}$*, if* $\mathbf{x} = \mathbf{x}_b$ *is any particular solution of the inhomogeneous equation*

$$A\mathbf{x} = \mathbf{b}, \tag{2.45}$$

with $\mathbf{b} \neq \mathbf{0}$*, then*

$$\mathbf{x} = \mathbf{x}_b + \mathbf{v} \tag{2.46}$$

is its general solution if and only if

$$\mathbf{v} = \sum_{i=1}^{n-r} s_i \mathbf{v}_i \tag{2.47}$$

is the general solution of the corresponding homogeneous equation

$$A\mathbf{v} = \mathbf{0}. \tag{2.48}$$

Proof. Assume first that 2.47 is the general solution of Equation 2.48. (Certainly, the Gaussian elimination algorithm would give it in this form.) Then applying A to both sides of Equation 2.46, we get

$$A\mathbf{x} = A(\mathbf{x}_b + \mathbf{v}) = A\mathbf{x}_b + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}. \tag{2.49}$$

Thus, every solution of the homogeneous Equation 2.48 leads to a solution of the form 2.46 of the inhomogeneous equation 2.45.

Conversely, assume that 2.46 is a solution of the inhomogeneous equation 2.45. Then

$$A\mathbf{v} = A(\mathbf{x} - \mathbf{x}_b) = A\mathbf{x} - A\mathbf{x}_b = \mathbf{b} - \mathbf{b} = \mathbf{0}. \tag{2.50}$$

This equation shows that the $\mathbf{v}$ given by Equation 2.47 is indeed a solution of Equation 2.48, or, in other words, that a solution of the form 2.46 of the inhomogeneous equation 2.45 leads to a solution of the form 2.47 of the homogeneous equation 2.48. ■

This theorem establishes a one-to-one pairing of the solutions of the two equations 2.45 and 2.48. Geometrically this means that the solutions of $A\mathbf{v} = \mathbf{0}$ are the position vectors of the points of the hyperplane through the origin given by Equation 2.47, and the solutions of $A\mathbf{x} = \mathbf{b}$ are those of a parallel hyperplane obtained from the first one by shifting it by the vector $\mathbf{x}_b$. (See Figure 2.1.) Note that we could have shifted by the coordinate vector of any other point of the second hyperplane, that is, by any other particular solution $\mathbf{x}'_b$ of Equation 2.45 (see Figure 2.2), and we would have obtained the same new hyperplane.

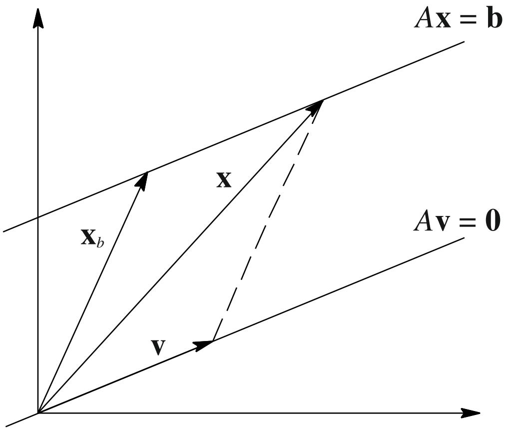

Fig. 2.1. The solution vector $\mathbf{x}$ of the inhomogeneous equation equals the sum of a particular solution and a solution of the corresponding homogeneous equation: $\mathbf{x} = \mathbf{x}_b + \mathbf{v}$

Sometimes the forward elimination procedure is carried further so as to obtain leading entries in the echelon matrix that equal 1, and to obtain 0 entries in the basic columns not just below but also above the pivots. This method is called *Gauss–Jordan elimination* and the final matrix a *reduced echelon matrix* or a *row–reduced echelon matrix*. We give one example of this method.

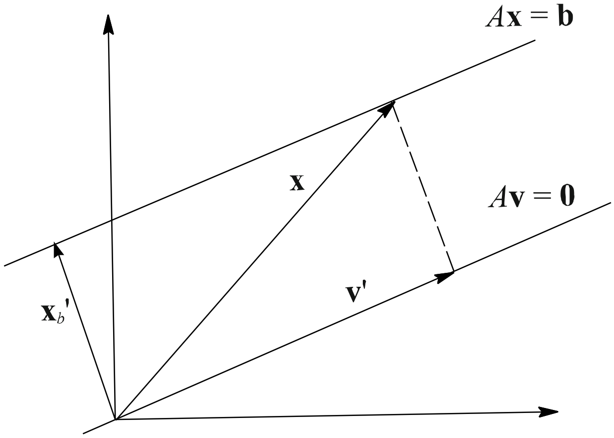

Fig. 2.2. The same solution vector $\mathbf{x}$ of the inhomogeneous equation also equals the sum of another particular solution and another solution of the corresponding homogeneous equation: $\mathbf{x} = \mathbf{x}_b' + \mathbf{v}'$

Example 2.3.1. (Solving Example 2.1.7 by Gauss–Jordan Elimination). Let us continue the reduction of Example 2.1.7, starting with the echelon matrix obtained in the forward elimination phase:

$$\left[\begin{array}{cccc|c} 2 & 3 & -2 & 4 & 2 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1/2 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 \end{array} \left[\begin{array}{cccc|c} 1 & 3/2 & -1 & 2 & 1 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 + \mathbf{r}_2 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 \end{array} \left[\begin{array}{cccc|c} 1 & 3/2 & 0 & 6 & 4 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{2.51}$$

From here on we proceed exactly as in the Gaussian elimination algorithm: we assign parameters s and t to the free variables x_2 and x_4, and solve for the basic variables x_1 and x_3. The latter step is now trivial, since all the work has already been done. The equations corresponding to the final matrix are

$$\begin{aligned} x_1 + \frac{3}{2}s + 6t &= 4 \\ x_3 + 4t &= 3. \end{aligned} \tag{2.52}$$

Thus we find the same general solution as before:

$$\begin{aligned} x_1 &= 4 - \frac{3}{2}s - 6t \\ x_2 &= \phantom{4 - \frac{3}{2}}s \\ x_3 &= 3 \phantom{- \frac{3}{2}s} - 4t \\ x_4 &= \phantom{4 - \frac{3}{2}s - 6} t \end{aligned} \tag{2.53}$$

or in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 0 \\ -4 \\ 1 \end{bmatrix}. \tag{2.54}$$

Notice how the numbers in the first and third rows of this solution correspond to the entries of the last matrix in 2.51, which is in reduced echelon form. ♦

As can be seen from this example, the reduced echelon form combines the results of both the forward elimination and back substitution phases of Gaussian elimination, and the general solution can simply be read from it. In general, if $[R|\mathbf{c}]$ is the reduced echelon matrix corresponding to the system $A\mathbf{x} = \mathbf{b}$, then we assign parameters s_j to the free variables x_j; and if r_{ik} is a pivot of R, that is, a leading entry 1 in the ith row and kth column, then x_k is a basic variable, and the ith row of the reduced system $R\mathbf{x} = \mathbf{c}$ is

$$x_k + \sum_{j>k} r_{ij} s_j = c_i. \tag{2.55}$$

Thus the general solution is given by

$$\begin{array}{ll} x_j = s_j & \text{if } x_j \text{ is free, and} \\ x_k = c_i - \sum_{j>k} r_{ij} s_j & \text{if } x_k \text{ is basic and is in the } i\text{th row.} \end{array} \tag{2.56}$$

Gauss–Jordan elimination is rarely used for the solution of systems, because a variant of Gaussian elimination, which we shall study in Section 8.1, is usually more efficient. However, Gauss–Jordan elimination is the preferred method for inverting matrices, as we shall see in Section 2.3. Also, it is sometimes helpful that the reduced echelon form of a matrix is unique (see Theorem 3.4.2), and that the solution of every system is immediately visible in it.

We conclude this section with an application.

Example 2.3.2. (An Electrical Network). Consider the electrical network shown in Figure 2.3. Here the R_k are positive numbers denoting resistances (unit: ohm (Ω)), the i_k are currents (unit: ampere (A)), and V_1 and V_2 are the voltages (unit: volt (V)) of two batteries represented by the circles. These quantities are related by three laws of physics:

1. **Kirchhof's first law.** The sum of the currents entering a node equals the sum of the currents leaving it.
2. **Kirchhof's second law.** The sum of the voltage drops or potential differences around every loop equals zero.
3. **Ohm's law.** The voltage drop across a resistor R equals Ri, where i is the current flowing through the resistor.

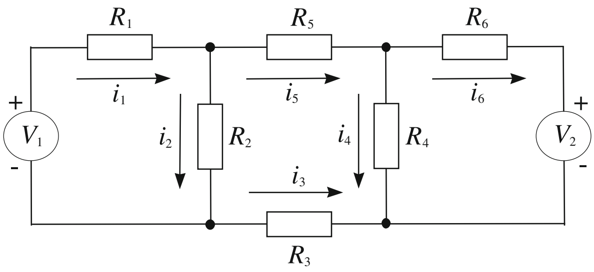

Fig. 2.3. An electrical network with resistors and two voltage sources

By convention, outside a battery the current that it generates flows from the positive terminal to the negative one.[11] However, in a multiloop circuit the directions of the currents are not so obvious. In the circuit above, for instance, the current i_6 is generated by both batteries, and although V_2 would make it flow from right to left, it is possible that the contribution of V_1 would make it flow as the arrow shows. In fact, the arrows for the direction of the currents can be chosen arbitrarily, and if the equations result in a negative value for an i_k, then the current flows in the direction opposite the arrow.

For the circuit of Figure 2.3, Kirchhof's laws give the following six equations for the six unknown currents:[12]

$$
\begin{array}{rcrcrcrcrcrcl}
i_1 & - & i_2 & & & & & - & i_5 & & & = & 0 \\
 & & & & & - & i_4 & + & i_5 & - & i_6 & = & 0 \\
 & & & & i_3 & + & i_4 & & & + & i_6 & = & 0 \\
R_1 i_1 & + & R_2 i_2 & & & & & & & & & = & V_1 \\
 & & R_2 i_2 & + & R_3 i_3 & - & R_4 i_4 & - & R_5 i_5 & & & = & 0 \\
 & & & & & & R_4 i_4 & & & - & R_6 i_6 & = & V_2
\end{array}
\tag{2.57}
$$

Assume that $R_1 = 4\,\Omega$, $R_2 = 24\,\Omega$, $R_3 = 1\,\Omega$, $R_4 = 3\,\Omega, R_5 = 2\,\Omega$, $R_6 = 8\,\Omega$, $V_1 = 80\,\mathrm{V}$, and $V_2 = 62\,\mathrm{V}$. Then the augmented matrix of the system becomes

$$
\left[\begin{array}{rrrrrr|r}
1 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & -1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
4 & 24 & 0 & 0 & 0 & 0 & 80 \\
0 & 24 & 1 & -3 & -2 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & -8 & 62
\end{array}\right],
\tag{2.58}
$$

---

[11] This convention was established before the discovery of electrons, which actually make up the flow by carrying a negative charge around the loop in the opposite direction.

[12] Actually, Kirchhof's laws give more equations than these six. In Example 3.5.7 we shall examine how to select a sufficient set.

and Gaussian elimination gives the solution $i_1 = 8\,\mathrm{A}$, $i_2 = 2\,\mathrm{A}$, $i_3 = -6\,\mathrm{A}$, $i_4 = 10\,\mathrm{A}$, $i_5 = 6\,\mathrm{A}$, $i_6 = -4\,\mathrm{A}$.

In order to give a transparent illustration of Kirchhof's laws, we show this solution in Figure 2.4, with the arrows pointing in the correct directions for the currents.

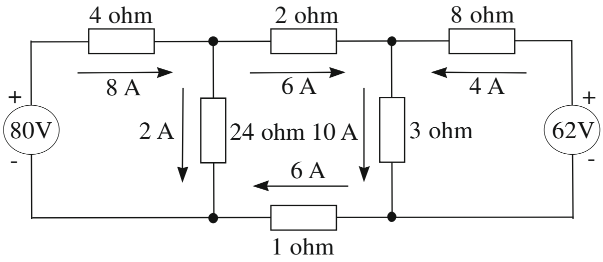

Fig. 2.4. The same circuit as in Figure 2.3, solved

The system above was obtained by what is called the branch method. Another possibility is to use the loop method, which we are now going to illustrate for the same circuit.

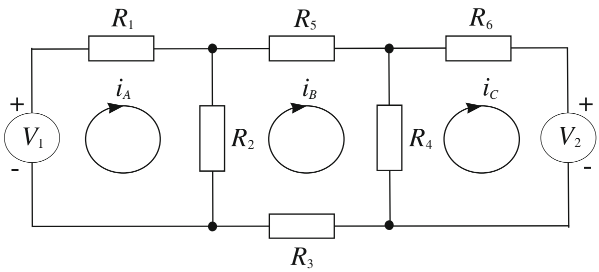

Fig. 2.5. The same circuit with loops shown

We consider only three unknown currents: i_A, i_B, i_C, one for each of the three small loops (see Figure 2.5), with arbitrarily assigned directions. Then, for the resistors shared by two loops, we must use the appropriate superposition of the currents of those loops. Thus the loop equations are

$$\begin{array}{rcl} R_1 i_A + R_2(i_A - i_B) & = & V_1 \\ R_2(i_B - i_A) + R_3 i_B + R_4(i_B - i_C) + R_5 i_B & = & 0 \\ R_4(i_C - i_B) + R_6 i_C & = & -V_2 \end{array} \qquad (2.59)$$

or, equivalently,

$$\begin{array}{rcl} (R_1 + R_2)\, i_A - R_2 i_B & = & V_1 \\ -R_2 i_A + (R_2 + R_3 + R_4 + R_5)\, i_B - R_4 i_C & = & 0 \\ -R_4 i_B + (R_4 + R_6)\, i_C & = & -V_2 \end{array} \qquad (2.60)$$

For the given numerical values, the augmented matrix of this system becomes

$$\left[\begin{array}{rrr|r} 28 & -24 & 0 & 80 \\ -24 & 30 & -3 & 0 \\ 0 & -3 & 11 & -62 \end{array}\right], \tag{2.61}$$

whose solution is $i_A = 8\,\mathrm{A}$, $i_B = 6\,\mathrm{A}$, and $i_C = -4\,\mathrm{A}$. From these loop currents we can easily recover the earlier branch currents as $i_1 = i_A = 8\,\mathrm{A}$, $i_2 = i_A - i_B = 2\,\mathrm{A}$, $i_3 = -i_B = -6\,\mathrm{A}$, $i_4 = i_B - i_C = 10\,\mathrm{A}$, $i_5 = i_B = 6\,\mathrm{A}$, $i_6 = i_C = -4\,\mathrm{A}$. ♦

Exercises

Exercise 2.3.1. List all possible forms of 2×2 reduced echelon matrices.

Exercise 2.3.2. List all possible forms of 3×3 reduced echelon matrices.

Exercise 2.3.3. Solve Exercise 2.1.5 by Gauss–Jordan elimination.

Exercise 2.3.4. Solve Exercise 2.1.8 by Gauss–Jordan elimination.

Exercise 2.3.5. Solve Exercise 2.1.11 by Gauss–Jordan elimination.

Exercise 2.3.6. Solve Exercise 2.1.12 by Gauss–Jordan elimination.

In each of the next two exercises find two particular solutions $\mathbf{x}_b$ and $\mathbf{x}'_b$ of the given system and the general solution $\mathbf{v}$ of the corresponding homogeneous system. Write the general solution of the given system as $\mathbf{x}_b + \mathbf{v}$ and also as $\mathbf{x}'_b + \mathbf{v}$, and show that the two forms are equivalent; that is, that the set of vectors of the form $\mathbf{x}_b + \mathbf{v}$ is identical with the set of vectors of the form $\mathbf{x}'_b + \mathbf{v}$.

Exercise 2.3.7.
$$\begin{aligned} 2x_1 + 3x_2 - 1x_3 &= 4 \\ 3x_1 + 5x_2 + 2x_3 &= 1 \end{aligned}$$

Exercise 2.3.8.
$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 - 2x_4 &= 4 \\ 6x_1 + 6x_2 + 3x_3 + 6x_4 &= 0 \end{aligned}$$

MATLAB Exercises

In MATLAB, linear systems are entered in matrix form. We can enter a matrix by writing its entries between brackets, row by row from left to right, top to bottom, and separating row entries by spaces or commas, and rows by semicolons. For example the command $A = [2, 3; 1, -2]$ would produce the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}.$$

(The size of a matrix is automatic; no size declaration is needed or possible, unlike in most other computer languages.) The entry a_{ij} of the matrix A is denoted by $A(i,j)$ in MATLAB, the ith row by $A(i,:)$ and the jth column by $A(:,j)$.

The vector $\mathbf{b}$ must be entered as a column vector. This can be achieved either by separating its entries by semicolons or by writing a prime after the closing bracket, as in $\mathbf{b} = [1, 2]'$. This would result in the column vector

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The augmented matrix can be formed by the command $[A\ \mathbf{b}]$. Sometimes we may wish to name it as, say, $A\_b = [A\ \mathbf{b}]$ or simply as $C = [A\ \mathbf{b}]$. The command $\mathbf{rref}(C)$ returns the reduced echelon form of C.

The command $\mathbf{x} = A\backslash\mathbf{b}$ always returns a solution of the system $A\mathbf{x} = \mathbf{b}$. This is the unique solution if there is only one; it is a certain particular solution with as many zeros as possible for components of $\mathbf{x}$ with the lowest subscripts, and is the least-squares "solution" (to be discussed in Section 5.1) if the system is inconsistent. This command is the most efficient method of finding a solution and is the one you should use whenever possible. On the other hand, to find the *general solution* of an underdetermined system this method does not work, and you should use $\mathbf{rref}([A\ \mathbf{b}])$ to obtain the reduced echelon matrix, and proceed as in Example 2.3.1 or Equations 2.56.

Exercise 2.3.9.
a. Write MATLAB commands to implement elementary row operations on a 3×6 matrix A.
b. Use these commands to reduce the matrix

$$A = \begin{bmatrix} 3 & -6 & -1 & 1 & 5 & 2 \\ -1 & 2 & 2 & 3 & 3 & 6 \\ 4 & -8 & -3 & -2 & 1 & 0 \end{bmatrix}$$

to reduced echelon form and compare your result to $\mathbf{rref}(A)$.
c. Write MATLAB commands to compute a matrix B with the same rows as the matrix A, but the first two rows switched.
d. Compare $\mathbf{rref}(B)$ with $\mathbf{rref}(A)$. Explain your result.

Exercise 2.3.10. Use MATLAB to find the general solution of $A\mathbf{x} = \mathbf{0}$ for

$$A = \begin{bmatrix} -1 & -2 & -1 & -1 & 1 \\ -1 & -2 & 0 & 3 & -1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 8 & 2 \end{bmatrix}.$$

Exercise 2.3.11. Let A be the same matrix as in the previous exercise and let

$$\mathbf{b} = \begin{bmatrix} 9 \\ 1 \\ -5 \\ -4 \end{bmatrix}.$$

a. Find the general solution of $A\mathbf{x} = \mathbf{b}$ using $\mathbf{rref}([A\ \mathbf{b}])$.
b. Verify that $\mathbf{x} = A\backslash\mathbf{b}$ is indeed a particular solution by computing $A\mathbf{x}$ from it.
c. Find the parameter values in the general solution obtained in Part (a), that give the particular solution of Part (b).
d. To verify the result of Theorem 2.3.3 for this case, show that the general solution of Part (a) equals $\mathbf{x} = A\backslash\mathbf{b}$ plus the general solution of the homogeneous equation found in the previous exercise.

Exercise 2.3.12. Let A and $\mathbf{b}$ be the same as in the last exercise. The command $\mathbf{x} = \mathbf{pinv}(A) * \mathbf{b}$ gives another particular solution of $A\mathbf{x} = \mathbf{b}$. (This solution will be explained in Section 5.1.) Verify Theorem 2.3.3 for this particular solution, as in Part (d) of the previous exercise.

Exercise 2.3.13. Let A be the same matrix as in Exercise 2.3.10 and let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Compute $\mathbf{x} = A\backslash\mathbf{b}$ and substitute this into $A\mathbf{x}$. Explain how your result is possible. (*Hint*: Look at $\mathbf{rref}([A\ \mathbf{b}])$.)

2.4 The Algebra of Matrices

Just as for vectors, we can define algebraic operations for matrices, and these operations will vastly extend their utility.

In order to motivate the forthcoming definitions, it is helpful to interpret matrices as functions or mappings. Thus if A is an $m \times n$ matrix, the matrix-vector product $A\mathbf{u}$ may be regarded as describing a mapping $T_A : \mathbb{R}^n \to \mathbb{R}^m$ of every $\mathbf{u} \in \mathbb{R}^n$ to $A\mathbf{u} \in \mathbb{R}^m$, that is, as $T_A(\mathbf{u}) = A\mathbf{u}$. This is also reflected in the terminology: We frequently read $A\mathbf{u}$ as A being *applied* to $\mathbf{u}$ instead of A *times* $\mathbf{u}$. If $m = n$, we may consider T_A as a transformation of the vectors of $\mathbb{R}^n$ to corresponding vectors in the same space.

Example 2.4.1. (Rotation Matrix). The matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2.62}$$

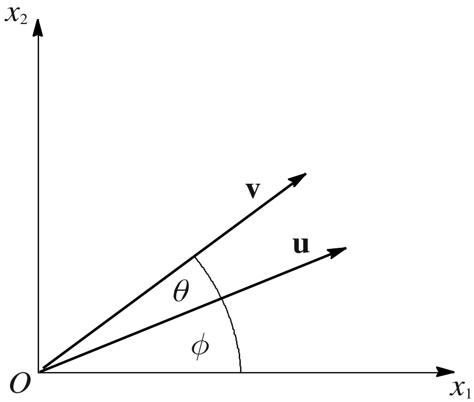

Fig. 2.6. Rotation of $\mathbf{u} \in \mathbb{R}^2$ by the angle θ

represents the rotation T_θ of $\mathbb{R}^2$ around O by the angle θ, as can be seen in the following way. (See Figure 2.6.) Let

$$\mathbf{u} = \begin{bmatrix} |\mathbf{u}| \cos \phi \\ |\mathbf{u}| \sin \phi \end{bmatrix} \tag{2.63}$$

be any nonzero vector in $\mathbb{R}^2$ (see Exercise 1.2.15 on page 26). Then, by Definition 2.3.1,

$$T_\theta(\mathbf{u}) = R_\theta \mathbf{u} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} |\mathbf{u}| \cos \phi \\ |\mathbf{u}| \sin \phi \end{bmatrix} = |\mathbf{u}| \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{bmatrix}, \tag{2.64}$$

and so

$$T_\theta(\mathbf{u}) = R_\theta \mathbf{u} = |\mathbf{u}| \begin{bmatrix} \cos(\phi + \theta) \\ \sin(\phi + \theta) \end{bmatrix}. \tag{2.65}$$

This is indeed a vector of the same length as $\mathbf{u}$ and it encloses the angle $\phi + \theta$ with the x_1-axis. ♦

Such transformations will be discussed in detail in Chapter 4. Here we just present the concept briefly, in order to lay the groundwork for the definitions of matrix operations. These definitions are analogous to the familiar definitions for functions of real variables, where, given functions f and g, their sum $f + g$ is defined as the function such that $(f + g)(x) = f(x) + g(x)$ for every x, and, for any real number c, the product cf is defined as the function such that $(cf)(x) = cf(x)$ for every x.

Definition 2.4.1. ***(Sum and Scalar Multiple of Mappings).*** *Let T_A and T_B be two mappings of $\mathbb{R}^n$ to $\mathbb{R}^m$, for any positive integers m and n.*

We define their sum $T_A + T_B$ *as the mapping that maps every* $\mathbf{x} \in \mathbb{R}^n$ *to* $T_A(\mathbf{x}) + T_B(\mathbf{x}) \in \mathbb{R}^m$ *or, in other words, as the mapping given by*

$$(T_A + T_B)(\mathbf{x}) = T_A(\mathbf{x}) + T_B(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.66}$$

Furthermore, for any scalar c*, the mapping* cT_A *is defined as the mapping that maps every* $\mathbf{x}$ *to* $c(T_A(\mathbf{x})) \in \mathbb{R}^m$*, that is, the mapping for which*

$$(cT_A)(\mathbf{x}) = c(T_A(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.67}$$

Now, let T_A and T_B be two mappings that correspond to two matrices A and B respectively, that is, such that $T_A(\mathbf{x}) = A\mathbf{x}$ and $T_B(\mathbf{x}) = B\mathbf{x}$ for all appropriate $\mathbf{x}$. Then we can use Definition 4.2.4 to define $A + B$ and cA.

Definition 2.4.2. ***(Sum and Scalar Multiple of Matrices).*** *Let* A *and* B *be two* $m \times n$ *matrices, for any positive integers* m *and* n*. We define their sum* $A + B$ *as the matrix that corresponds to* $T_A + T_B$*, or, in other words, as the matrix for which we have*

$$(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.68}$$

Similarly, for any scalar c*, the matrix* cA *is defined as the matrix that corresponds to* cT_A*, that is, as the matrix for which*

$$(cA)\mathbf{x} = c(A\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.69}$$

The mappings $T_A + T_B$ and cT_A both clearly exist, but the existence of corresponding matrices $A + B$ and cA requires proof. Their existence will be proved by Theorem 2.4.1 below, where they will be computed explicitly.

Definition 2.4.2 can be paraphrased as requiring that the order of the operations be reversible: On the right-hand side of Equation 2.68 we first *apply* A and B separately to $\mathbf{x}$ and then *add*, and on the left we first *add* A to B and then *apply* the sum to $\mathbf{x}$. Similarly, on the right-hand side of Equation 2.69 we first apply A to $\mathbf{x}$ and then multiply by c, while on the left this is reversed: A is first multiplied by c and then cA is applied to $\mathbf{x}$. We may also regard Equation 2.68 as a new distributive rule and Equation 2.69 as a new associative rule. Note that Equation 2.69 enables us to drop the parentheses, that is, to write $cA\mathbf{x}$ for $c(A\mathbf{x})$.

Example 2.4.2. (A Matrix Sum). Let

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}. \tag{2.70}$$

Then, applying Definition 2.3.1, for any $\mathbf{x}$ we have

$$A\mathbf{x} = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + 5x_2 \\ 4x_1 + 2x_2 \end{bmatrix}, \tag{2.71}$$

and

$$B\mathbf{x} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 4x_1 + 7x_2 \end{bmatrix}. \tag{2.72}$$

Hence

$$A\mathbf{x}+B\mathbf{x} = \begin{bmatrix} 3x_1 + 5x_2 \\ 4x_1 + 2x_2 \end{bmatrix} + \begin{bmatrix} 2x_1 + 3x_2 \\ 4x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 + 8x_2 \\ 8x_1 + 9x_2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.73}$$

Thus, by Equation 2.68,

$$(A+B)\mathbf{x} = \begin{bmatrix} 5 & 8 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.74}$$

and so,

$$A+B = \begin{bmatrix} 5 & 8 \\ 8 & 9 \end{bmatrix}. \tag{2.75}$$

Here we see that $A+B$ is obtained from A and B by adding corresponding entries. That this addition rule is true in general, and not just for these particular matrices, will be part of Theorem 2.4.1. ♦

Example 2.4.3. (A Scalar Multiple of a Matrix). Let $c = 2$ and

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 2 \end{bmatrix}. \tag{2.76}$$

Then, applying Definition 2.3.1, for every $\mathbf{x}$ we have

$$A\mathbf{x} = \begin{bmatrix} 3 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix}, \tag{2.77}$$

and so,

$$c\,(A\mathbf{x}) = 2\begin{bmatrix} 3x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 + 8x_2 \\ 8x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.78}$$

Thus, by Equation 2.69,

$$(2A)\mathbf{x} = \begin{bmatrix} 6 & 8 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.79}$$

and so,

$$2A = \begin{bmatrix} 6 & 8 \\ 8 & 4 \end{bmatrix}. \tag{2.80}$$

Here we see that $2A$ is obtained by multiplying every entry of A by 2. The next theorem generalizes this multiplication rule to arbitrary c and A. ♦

Theorem 2.4.1. ***(The Sum and Scalar Multiple of Matrices in Terms of Entries).*** *For any two matrices $A = [a_{ik}]$ and $B = [b_{ik}]$ of the same shape, we have*

$$(A + B)_{ik} = a_{ik} + b_{ik} \text{ for all } i, k, \tag{2.81}$$

and for any scalar c, we have

$$(cA)_{ik} = ca_{ik} \text{ for all } i, k. \tag{2.82}$$

Proof. To write Equations 2.68 and 2.69 in terms of components, let us first recall from Definition 2.3.1 that, for all appropriate $\mathbf{x}$, the product $A\mathbf{x}$ is a column m-vector whose ith component is given, for each i, by

$$(A\mathbf{x})_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n, \tag{2.83}$$

which can be abbreviated as

$$(A\mathbf{x})_i = \sum_{j=1}^{n} a_{ij}x_j. \tag{2.84}$$

Applying the same principle to B and $A + B$, we also have, for all i,

$$(B\mathbf{x})_i = \sum_j b_{ij}x_j \tag{2.85}$$

and

$$[(A + B)\mathbf{x}]_i = \sum_j (A + B)_{ij}x_j. \tag{2.86}$$

From Equation 2.68, the definition of vector addition, and the last three equations,

$$\begin{aligned}[(A + B)\mathbf{x}]_i &= (A\mathbf{x} + B\mathbf{x})_i = (A\mathbf{x})_i + (B\mathbf{x})_i \\ &= \sum_j a_{ij}x_j + \sum_j b_{ij}x_j = \sum_j (a_{ij} + b_{ij})x_j.\end{aligned} \tag{2.87}$$

Comparing the two evaluations of $[(A+B)\mathbf{x}]_i$ in Equations 2.86 and 2.87, we obtain

$$\sum_j (A + B)_{ij}x_j = \sum_j (a_{ij} + b_{ij})x_j. \tag{2.88}$$

Equation 2.88 must hold for every choice of $\mathbf{x}$. Choosing $x_k = 1$ for any fixed k, and $x_j = 0$ for all $j \neq k$, yields the first statement of the theorem:

$$(A + B)_{ik} = a_{ik} + b_{ik} \text{ for all } i, k. \tag{2.89}$$

Equation 2.82 can be obtained similarly, and its proof is left as Exercise 2.4.2. ■

This theorem can be paraphrased as: Every entry of a sum of matrices equals the sum of the corresponding entries of the summands; and we multiply a matrix A by a scalar c, by multiplying every entry by c. Notice again, as for vectors, the reversal of operations: "every entry of a sum = sum of corresponding entries" and "every entry of $cA = c \times$ corresponding entry of A."

Let us emphasize that only matrices of the same shape can be added to each other, and that the sum has the same shape, in which case we call them *conformable for addition.* However, for matrices of differing shapes there is no reasonable way of defining a sum.

We can also define multiplication of matrices in certain cases and this will prove to be an enormously fruitful operation. For real-valued functions f and g, their composite $f \circ g$ was defined by $(f \circ g)(x) = f(g(x))$ for all x, and we first define the composite of two mappings similarly, to represent the performance of two mappings in succession.

Definition 2.4.3. ***(Composition of Mappings).*** *Let T_B be a mapping of $\mathbb{R}^n$ to $\mathbb{R}^p$ and T_A be a mapping of $\mathbb{R}^p$ to $\mathbb{R}^m$, for any positive integers $m, p,$ and n. We define the composite $T_A \circ T_B$ as the mapping that maps every $\mathbf{x} \in \mathbb{R}^n$ to $T_A(T_B(\mathbf{x})) \in \mathbb{R}^m$ or, in other words, as the mapping given by*

$$(T_A \circ T_B)(\mathbf{x}) = T_A(T_B(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.90}$$

Next, we define the product of two matrices as the matrix that corresponds to the composite mapping.

Definition 2.4.4. ***(Matrix Multiplication).*** *Let A be an $m \times p$ matrix and B a $p \times n$ matrix, for any positive integers $m, p,$ and n. Let T_A and T_B be the corresponding mappings. That is, let T_B map every $\mathbf{x} \in \mathbb{R}^n$ to a vector $T_B(\mathbf{x}) = B\mathbf{x}$ of $\mathbb{R}^p$ and T_A map every $\mathbf{y} \in \mathbb{R}^p$ to $T_A(\mathbf{y}) = A\mathbf{y}$ of $\mathbb{R}^m$. We define the product AB as the $m \times n$ matrix that corresponds to the composite mapping $T_A \circ T_B$, that is, by the formula*

$$(AB)\mathbf{x} = (T_A \circ T_B)(\mathbf{x}) = A(B\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.91}$$

These mappings and Definition 2.4.4 are illustrated symbolically in Figure 2.7.

That such a matrix always exists will be proved by Theorem 2.4.2, where it will be computed explicitly.

Let us emphasize that only for matrices A and B such that the number of columns of A (the p in the definition) equals the number of rows of B can the product AB be formed, in which case we call them *conformable for multiplication.* Also, we never use any sign for this multiplication, we just write the factors next to each other.

Furthermore, Equation 2.91 can also be viewed as a new associative law or as a reversal of the order of the two multiplications (but not of the factors). Hence, we can drop the parentheses, that is, we can write $AB\mathbf{x}$ for $A(B\mathbf{x})$.

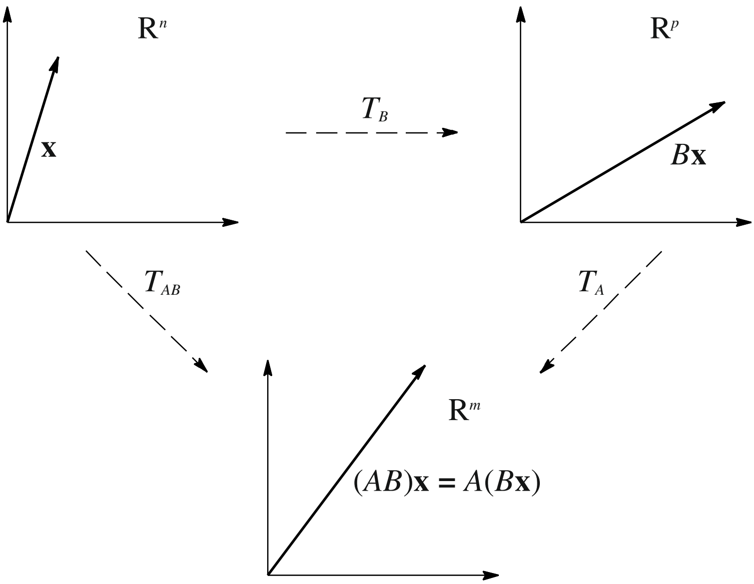

Fig. 2.7. The product of two matrices corresponding to two mappings in succession

Example 2.4.4. (A Matrix Multiplication). Let

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}. \tag{2.92}$$

Then, applying Equation 2.85, for every $\mathbf{x}$ we have

$$B\mathbf{x} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ 3x_2 \end{bmatrix}, \tag{2.93}$$

and similarly

$$A(B\mathbf{x}) = \begin{bmatrix} 3 & 5 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2x_1 + x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 + 18x_2 \\ 2x_1 + \ \ 7x_2 \\ 4x_1 + 14x_2 \end{bmatrix} = \begin{bmatrix} 6 & 18 \\ 2 & 7 \\ 4 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.94}$$

Thus

$$AB = \begin{bmatrix} 6 & 18 \\ 2 & 7 \\ 4 & 14 \end{bmatrix}. \tag{2.95}$$

♦

From the definition we can easily deduce the following rule that gives the entries of AB and shows that the vector $\mathbf{x}$ can be dispensed with in their computation.

Theorem 2.4.2. ***(Matrix Multiplication in Terms of Entries).*** *Let A be an $m \times p$ matrix and B a $p \times n$ matrix. Then the product AB is an $m \times n$ matrix whose entries are given by the formula*

$$(AB)_{ik} = \sum_{j=1}^{p} a_{ij} b_{jk} \textit{ for } i = 1, \ldots, m \textit{ and } k = 1, \ldots, n. \tag{2.96}$$

Proof. The components of $B\mathbf{x}$ can be written as

$$(B\mathbf{x})_j = \sum_{k=1}^{n} b_{jk} x_k \text{ for } j = 1, \ldots, p. \tag{2.97}$$

Also,

$$(A\mathbf{y})_i = \sum_{j=1}^{p} a_{ij} y_j \text{ for } i = 1, \ldots, m. \tag{2.98}$$

Substituting from Equation 2.97 into 2.98, we get

$$(A(B\mathbf{x}))_i = \sum_{j=1}^{p} a_{ij} \left(\sum_{k=1}^{n} b_{jk} x_k \right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{p} a_{ij} b_{jk} \right) x_k. \tag{2.99}$$

On the other hand, we have

$$((AB)\mathbf{x})_i = \sum_{k=1}^{n} (AB)_{ik} x_k. \tag{2.100}$$

In view of Definition 2.4.4 the left-hand sides of Equations 2.99 and 2.100 must be equal, and since the vector $\mathbf{x}$ can be chosen arbitrarily, the coefficients of x_k on the right-hand sides of Equations 2.99 and 2.100 must be equal. This proves the theorem. ■

The special case of Theorem 2.4.2, in which $m = n = 1$, which is also a special case of the definition of $A\mathbf{x}$ (Definition 2.3.1), is worth stating separately:

Corollary 2.4.1. ***(Matrix Products with Row and Column Vectors).*** *If A is a $1 \times p$ matrix, that is, a row p-vector*

$$\mathbf{a} = (a_1, a_2, \ldots, a_p) \tag{2.101}$$

and B a $p \times 1$ matrix, that is, a column p-vector

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix}, \tag{2.102}$$

then their matrix product $\mathbf{ab}$ *is a scalar and is equal to their dot product as vectors, namely*

$$\mathbf{ab} = \sum_{j=1}^{p} a_j b_j. \tag{2.103}$$

Also, if $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ *is a* $p \times n$ *matrix, where the* $\mathbf{b}_i$ *stand for the column* p*-vectors of* B, *then*

$$\mathbf{a}B = (\mathbf{ab}_1, \mathbf{ab}_2, \ldots, \mathbf{ab}_n), \tag{2.104}$$

which is a row n*-vector.*

It is very important to observe that matrix multiplication is *not commutative.* This will be seen by direct computations, but it also follows from the definition as two mappings in succession, since mappings are generally not commutative. The latter is true even in the case of transformations in the same space. Consider, for instance, the effect of a north-south stretch followed by a 90-degree rotation on a car facing north, and of the same operations performed in the reverse order. In the first case we end up with a longer car facing west, and in the second case with a wider car facing west.

In case of the two vectors in Corollary 2.4.1, the product $\mathbf{ba}$ is very different from $\mathbf{ab}$. The latter is a scalar, as given by Equation 2.103. However, if the column vector comes first, then $\mathbf{a}$ and $\mathbf{b}$ do not even have to have the same number of entries. Changing $\mathbf{b}$ in Corollary 2.4.1 to a column m-vector and $\mathbf{a}$ to a row n-vector we get, by Theorem 2.4.2 with $p = 1$,

$$\mathbf{ba} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} (a_1, a_2, \ldots, a_n) = \begin{bmatrix} b_1a_1 & b_1a_2 & \cdots & b_1a_n \\ b_2a_1 & b_2a_2 & \cdots & b_2a_n \\ \vdots & \vdots & \vdots & \vdots \\ b_ma_1 & b_ma_n & \cdots & b_ma_n \end{bmatrix}. \tag{2.105}$$

If $m \neq n$, then $\mathbf{ab}$ does not exist. On the other hand, the $\mathbf{ba}$ above is called the *outer product* of the two vectors, in contrast to the much more important inner product given by Equation 2.103, presumably because the outer product is in the space of $m \times n$ matrices, which contains the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ of the factors, and those spaces, in turn, contain the space $\mathbb{R}^1$ of the inner product.

Even if the product AB is defined, often the product BA is not. For example, if A is 2×3, say, and B is 3×1, then AB is, by Definition 2.4.4, a 2×1 matrix, but BA is not defined since the inside numbers 1 and 2 in 3×1 and 2×3 do not match, as required by Definition 2.4.4.

The interpretation of the product in Corollary 2.4.1 as a dot product suggests that the formula of Theorem 2.4.2 can also be interpreted similarly.

Corollary 2.4.2. ***(Product of Two Matrices in Terms of Their Row and Column Vectors).*** *Let* A *be an* $m \times p$ *matrix and* B *a* $p \times n$ *matrix* *and let us denote the* ith *row of* A *by* $\mathbf{a}^i$ *and the* kth *column of* B *by* $\mathbf{b}_k$*, that is, let*[13]

$$\mathbf{a}^i = (a_{i1}, a_{i2}, \ldots, a_{ip}) \tag{2.106}$$

and

$$\mathbf{b}_k = \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{pk} \end{bmatrix}. \tag{2.107}$$

Then we have

$$(AB)_{ik} = \mathbf{a}^i \mathbf{b}_k \text{ for } i = 1, \ldots, m \text{ and } k = 1, \ldots, n. \tag{2.108}$$

This result may be paraphrased as saying that the entry in the ith *row and* kth *column of* AB *equals the dot product of the* ith *row of* A *with the* kth *column of* B*. Consequently, we may write out the entire product matrix as*

$$AB = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n) = \begin{bmatrix} \mathbf{a}^1\mathbf{b}_1 & \mathbf{a}^1\mathbf{b}_2 & \cdots & \mathbf{a}^1\mathbf{b}_n \\ \mathbf{a}^2\mathbf{b}_1 & \mathbf{a}^2\mathbf{b}_2 & \cdots & \mathbf{a}^2\mathbf{b}_n \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}^m\mathbf{b}_1 & \mathbf{a}^m\mathbf{b}_2 & \cdots & \mathbf{a}^m\mathbf{b}_n \end{bmatrix}. \tag{2.109}$$

The last formula is analogous to the outer product in Equation 2.105, but the entries on the right are inner products of vectors rather than ordinary products of numbers. This corollary is very helpful in the evaluation of matrix products, as will be seen below.

Let us also comment on the use of superscripts and subscripts. The notation we follow for row and column vectors is standard in multilinear algebra (treated in more advanced courses) and will serve us well later, but we have stayed with the more elementary standard usage of just subscripts for matrix elements. Thus our notation is a mixture of two conventions. To be consistent, we should have used a^i_j instead of a_{ij} to denote an entry of A, since then a^i_j could have been properly interpreted as the jth component of the ith row $\mathbf{a}^i$, and also as the ith component of the jth column $\mathbf{a}_j$. However, since here we need no such sophistication, we have adopted the simpler convention.

Example 2.4.5. (A Matrix Product in Terms of Row and Column Vectors). Let

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -1 \\ 5 & 6 \end{bmatrix}. \tag{2.110}$$

[13] The i here is a superscript to distinguish a row of a matrix from a column, which is denoted by a *sub*script, and must not be mistaken for an exponent.

Then

$$AB = \begin{bmatrix} (2\ \ 4)\begin{bmatrix} 3 \\ 5 \end{bmatrix} & (2\ \ 4)\begin{bmatrix} -1 \\ 6 \end{bmatrix} \\ \\ (3\ \ 7)\begin{bmatrix} 3 \\ 5 \end{bmatrix} & (3\ \ 7)\begin{bmatrix} -1 \\ 6 \end{bmatrix} \end{bmatrix} \tag{2.111}$$

and so

$$AB = \begin{bmatrix} 2 \cdot 3 + 4 \cdot 5 & 2 \cdot (-1) + 4 \cdot 6 \\ 3 \cdot 3 + 7 \cdot 5 & 3 \cdot (-1) + 7 \cdot 6 \end{bmatrix} = \begin{bmatrix} 26 & 22 \\ 44 & 39 \end{bmatrix}. \tag{2.112}$$

For further reference, note that we can factor out the column vectors $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 6 \end{bmatrix}$ in the columns of AB as given in Equation 2.111, and write AB as

$$AB = \left[\begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \end{bmatrix} \right] = \left[A \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad A \begin{bmatrix} -1 \\ 6 \end{bmatrix} \right]. \tag{2.113}$$

Thus, in the product AB the matrix A can be distributed over the columns of B. Similarly, we can factor out the row vectors $(2\ \ 4)$ and $(3\ \ 7)$ from the rows of AB as given in Equation 2.111 and write AB also as

$$AB = \begin{bmatrix} (2\ \ 4)B \\ (3\ \ 7)B \end{bmatrix}, \tag{2.114}$$

that is, with the matrix B distributed over the rows of A. ♦

Example 2.4.6. (A Matrix Product in Terms of Entries). Let

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & 3 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 4 & -2 \\ 6 & 3 \end{bmatrix}. \tag{2.115}$$

Then

$$AB = \begin{bmatrix} 2 \cdot 2 - 2 \cdot 4 + 4 \cdot 6 & 2 \cdot (-1) - 2 \cdot (-2) + 4 \cdot 3 \\ 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 & 1 \cdot (-1) + 3 \cdot (-2) + 5 \cdot 3 \end{bmatrix} = \begin{bmatrix} 20 & 14 \\ 44 & 8 \end{bmatrix}. \tag{2.116}$$

♦

Example 2.4.7. (The Product of Two Rotation Matrices). The matrices

$$R_{30} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \tag{2.117}$$

and

$$R_{60} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \tag{2.118}$$

represent rotations by 30° and 60° respectively, according to Example 2.4.1. Their product

$$R_{30}R_{60} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = R_{90} \tag{2.119}$$

represents the rotation by 90°, as it should. ♦

An interesting use of matrices and matrix operations is provided by the following example, typical of a large number of similar applications involving *incidence* or *connection matrices.*

Example 2.4.8. (A Connection Matrix for an Airline). Suppose that an airline has nonstop flights between cities A, B, C, D, E as described by the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}. \tag{2.120}$$

Here the entry m_{ij} is 1 if there is a nonstop connection from city i to city j, and 0 if there is not, with the cities labeled $1, 2, \ldots, 5$ instead of $A, B, \ldots, E$. Then the entries of the matrix

$$M^2 = MM = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{bmatrix} \tag{2.121}$$

show the one-stop connections. Why? Because, if we consider the entry

$$(M^2)_{ik} = \sum_{j=1}^{5} m_{ij}m_{jk} \tag{2.122}$$

of M^2, then the jth term equals 1 in this sum if and only if $m_{ij} = 1$ and $m_{jk} = 1$, that is, if we have a nonstop flight from i to j and another from j to k. If there are two such j values, then the sum will be equal to 2, showing that there are two choices for one-stop flights from i to k. Thus, for instance, $(M^2)_{11} = 2$ shows that there are two one-stop routes from A to A: Indeed, from A one can fly to B or D and back. The entries of the matrix[14]

[14] In matrix expressions with several operations, the precedence rules are analogous to those for numbers: first powers, then products, and last addition and subtraction, unless otherwise indicated by parentheses.

$$M + M^2 = \begin{bmatrix} 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{bmatrix} \tag{2.123}$$

show the number of ways of reaching one city from another with one-leg or two-leg flights. In particular, the zero entries show, for instance, that B and E are not so connected. Evaluating $(M^3)_{25} = (M^3)_{52}$, we would similarly find that even those two cities can be reached from each other with three-leg flights.

What are the vectors on which these matrices act, that is, what meaning can we give to an equation like $\mathbf{y} = M\mathbf{x}$? The answer is that if the components of $\mathbf{x}$ are restricted to just 0 and 1, then $\mathbf{x}$ may be regarded as representing a set of cities and $\mathbf{y}$ the set that can be reached nonstop from $\mathbf{x}$. Thus, for instance,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.124}$$

represents the set $\{A, B\}$, and then

$$\mathbf{y} = M\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{2.125}$$

represents the set $\{A, B, D\}$ that can be reached nonstop from $\{A, B\}$. (Again, if a number greater than 1 were to show up in $\mathbf{y}$, that would indicate that the corresponding city can be reached in more than one way.) ♦

We present one more example, which is a simplified version of a large class of similar applications of matrices.

Example 2.4.9. (A Matrix Description of Population Changes). We want to describe how in a certain town two population groups, those younger than 50 and those 50 or older, change over time. We assume that over every decade, on the one hand, there is a net increase of 10% in the under fifty population, and on the other hand, 20% of the under fifty population becomes fifty or older, while 40% of the initial over fifty population dies. If x_1 and x_2 denote the numbers of people in the two groups at a given time, then their numbers a decade later will be given by the product

$$A\mathbf{x} = \begin{bmatrix} 1.1 & 0 \\ 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.126}$$

Similarly, two decades later the two population groups will be given by

$$A(A\mathbf{x}) = A^2\mathbf{x} = \begin{bmatrix} 1.21 & 0 \\ 0.34 & 0.36 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2.127}$$

and so on. (In Example 7.2.1 we will examine how the two populations change in the long run.) ♦

As we have seen, a matrix can be regarded as a row vector of its columns and also as a column vector of its rows. Making full use of this choice, we can rewrite the product of matrices two more ways, corresponding to the particular cases shown in Equations 2.113 and 2.114. We obtain these new formulas by factoring out the $\mathbf{b}_j$ coefficients in the columns of the matrix on the right of Equation 2.109 and the $\mathbf{a}^i$ coefficients in the rows:

$$\begin{aligned} AB &= \begin{bmatrix} \mathbf{a}^1\mathbf{b}_1 & \mathbf{a}^1\mathbf{b}_2 & \cdots & \mathbf{a}^1\mathbf{b}_n \\ \mathbf{a}^2\mathbf{b}_1 & \mathbf{a}^2\mathbf{b}_2 & \cdots & \mathbf{a}^2\mathbf{b}_n \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}^m\mathbf{b}_1 & \mathbf{a}^m\mathbf{b}_2 & \cdots & \mathbf{a}^m\mathbf{b}_n \end{bmatrix} \\ &= \left[\begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} \mathbf{b}_1, \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} \mathbf{b}_2, \cdots, \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} \mathbf{b}_n \right] \\ &= (A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n) \end{aligned} \tag{2.128}$$

and

$$AB = \begin{bmatrix} \mathbf{a}^1\mathbf{b}_1 & \mathbf{a}^1\mathbf{b}_2 & \cdots & \mathbf{a}^1\mathbf{b}_n \\ \mathbf{a}^2\mathbf{b}_1 & \mathbf{a}^2\mathbf{b}_2 & \cdots & \mathbf{a}^2\mathbf{b}_n \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}^m\mathbf{b}_1 & \mathbf{a}^m\mathbf{b}_2 & \cdots & \mathbf{a}^m\mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}^1(\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n) \\ \mathbf{a}^2(\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n) \\ \vdots \\ \mathbf{a}^m(\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n) \end{bmatrix} = \begin{bmatrix} \mathbf{a}^1 B \\ \mathbf{a}^2 B \\ \vdots \\ \mathbf{a}^m B \end{bmatrix}. \tag{2.129}$$

We summarize these results as follows.

Corollary 2.4.3. ***(Product of Two Matrices in Terms of the Row or Column Vectors of One of Them).*** *Let A and B be as in Corollary 2.4.2. With the same notation for the rows and columns used there, we have*

$$AB = A(\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n) = (A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n) \tag{2.130}$$

and

$$AB = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}^1 B \\ \mathbf{a}^2 B \\ \vdots \\ \mathbf{a}^m B \end{bmatrix}. \tag{2.131}$$

Although the matrix product is not commutative, it still has the other important properties expected of a product, namely associativity and distributivity.

Theorem 2.4.3. ***(Associativity and Distributivity of Matrix Multiplication).*** *Let A, B, and C be arbitrary matrices for which the expressions below all make sense. Then we have the associative law*

$$A(BC) = (AB)C \tag{2.132}$$

and the distributive law

$$A(B + C) = AB + AC. \tag{2.133}$$

Proof. Let A, B, and C be $m \times p$, $p \times q$, and $q \times n$ matrices respectively. Then we may evaluate the left side of Equation 2.132 using Equations 2.130 and Definition 2.4.4 as follows:

$$\begin{aligned} A(BC) &= A(B(\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n)) = A(B\mathbf{c}_1\ B\mathbf{c}_2\ \cdots\ B\mathbf{c}_n) \\ &= (A(B\mathbf{c}_1)\ \cdots\ A(B\mathbf{c}_n)) = ((AB)\mathbf{c}_1\ \ \cdots\ \ (AB)\mathbf{c}_n) \\ &= (AB)(\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n) = (AB)C. \end{aligned} \tag{2.134}$$

We leave the proof of the distributive law to the reader as Exercise 2.4.18. ∎

Note that Equation 2.132 enables us to write ABC, without parentheses, for $A(BC)$ or $(AB)C$.

Once we have defined addition and multiplication of matrices, it is natural to ask what matrices take the place of the special numbers 0 and 1 in the algebra of numbers. Zero is easy: we take every matrix with all entries equal to zero to be a zero matrix. Denoting it by O, regardless of its shape, we have, for every A of the same shape,

$$A + O = A, \tag{2.135}$$

and whenever the product is defined,

$$AO = O \text{ and } OA = O. \tag{2.136}$$

Note that the zero matrices on either side of each of Equations 2.136 may be of different size, although they are usually denoted by the same letter O.

While a little less straightforward, it is still easy to see how to find analogs of 1. For every n, the $n \times n$ matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{2.137}$$

with 1's along its "main diagonal" and zeros everywhere else, has the properties

$$AI = A \text{ and } IA = A, \tag{2.138}$$

whenever the products are defined. This fact can be verified by direct computation in every one of the product's forms, with A in the general form (a_{ij}). We may do it as follows: We write $I = (\delta_{ij})$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } \; i = j \\ 0 & \text{if } \; i \neq j \end{cases} \tag{2.139}$$

is called Kronecker's delta function and is the standard notation for the entries of the matrix I. With this notation, for A and I sized $m \times n$ and $n \times n$ respectively, Theorem 2.4.2 gives

$$(AI)_{ik} = \sum_{j=1}^{n} a_{ij}\delta_{jk} = a_{ik} \text{ for } i = 1, \dots, m \text{ and } k = 1, \dots, n, \tag{2.140}$$

since, by the definition of δ_{jk}, in the sum all terms are zero except the one with $j = k$ and that one gives a_{ik}. This result is, of course, equivalent to $AI = A$. We leave the proof of the other equation of 2.138 to the reader.

For every n the matrix I is called the *unit matrix* or the *identity matrix* of order n. We usually dispense with any indication of its order unless it is important and would be unclear from the context. In such cases we write it as I_n. Notice that the columns of I are the standard vectors $\mathbf{e}_i$ (regarded as column vectors, of course), that is,

$$I = (\mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \dots \;\; \mathbf{e}_n). \tag{2.141}$$

In Section 2.5 we shall see how the inverse of a matrix can be defined in some cases.

In closing this section, we just want to present briefly the promised explanation of the reason for using *column* vectors for $\mathbf{x}$ in the equation $A\mathbf{x} = \mathbf{b}$. In a nutshell, we used column vectors because otherwise the whole formalism of this section would have broken down. The product $A\mathbf{x}$ was used in Definition 2.4.4 of the general matrix product AB, which led to the formula of Theorem 2.4.2 for the components $(AB)_{ik}$. If we want to multiply AB by a third matrix C, we have no problem repeating the previous procedure, that is, form the products $(AB)_{ik}c_{kl}$ and sum over k. However, had we used a *row* vector $\mathbf{x}$ in the beginning, that would have led to the formula $(AB)_{ik} = \sum_{j=1}^{n} a_{ij}b_{kj}$ and then multiplying this result by c_{kl} or c_{lk}, and summing over k, we would have had to use the first subscript of b for summation in this second product, unlike in the first one. Thus the associative law could not be maintained and the nice formulas of Corollary 2.4.3 would also cease to hold. Basically, once we decided to use *rows* of A to multiply $\mathbf{x}$ in the product $A\mathbf{x}$, then we had to make $\mathbf{x}$ a column vector in order to end up with a reasonable formalism.

Exercises

Exercise 2.4.1. Let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}.$$

Find the matrices a. $C = 2A + 3B$, and b. $D = 4A - 3B$.

Exercise 2.4.2. Prove Equation 2.82 of Theorem 2.4.1.

In the next six exercises find the products of the given matrices in both orders, that is, both AB and BA, if possible.

Exercise 2.4.3.

$$A = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Exercise 2.4.4.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \\ 1 & -3 \end{bmatrix}.$$

Exercise 2.4.5.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 3 \\ 3 & -4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \\ 1 & -3 \end{bmatrix}.$$

Exercise 2.4.6.

$$A = \begin{bmatrix} 1 & -2 & 3 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Exercise 2.4.7.

$$A = \begin{bmatrix} 1 & -2 & 3 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \\ 1 & -3 \\ -2 & 5 \end{bmatrix}.$$

Exercise 2.4.8.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \\ 1 & -3 \\ -2 & 5 \end{bmatrix}.$$

Exercise 2.4.9. Verify the associative law for the product of the matrices

$$A = [1 \ -2], \ B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix}.$$

Exercise 2.4.10. With the notation of Example 2.4.1, prove that for every two rotation matrices R_α and R_β we have $R_\alpha R_\beta = R_{\alpha+\beta}$.

Exercise 2.4.11. Find two nonzero 2×2 matrices A and B such that $AB = O$.

Exercise 2.4.12. Show that the cancellation law does not hold for matrix products: Find nonzero 2×2 matrices A, B, C such that $AB = AC$ but $B \neq C$.

Exercise 2.4.13. * Let A be an $m \times p$ matrix and B a $p \times n$ matrix. Show that the product AB can also be written in the following alternative forms:
a. $AB = \mathbf{a}_1\mathbf{b}^1 + \mathbf{a}_2\mathbf{b}^2 + \cdots + \mathbf{a}_p\mathbf{b}^p$,
b. $AB = \left(\sum_{i=1}^p \mathbf{a}_i b_{i1}, \sum_{i=1}^p \mathbf{a}_i b_{i2}, \ldots, \sum_{i=1}^p \mathbf{a}_i b_{in}\right)$ or $(AB)_j = \sum_{i=1}^p \mathbf{a}_i b_{ij}$,

c. $AB = \begin{bmatrix} \sum_{j=1}^p a_{1j}\mathbf{b}^j \\ \sum_{j=1}^p a_{2j}\mathbf{b}^j \\ \vdots \\ \sum_{j=1}^p a_{mj}\mathbf{b}^j \end{bmatrix}$ or $(AB)^i = \sum_{j=1}^p a_{ij}\mathbf{b}^j$.

Exercise 2.4.14. Let A be any $n \times n$ matrix. Its powers, for all nonnegative integer exponents k, are defined by induction as $A^0 = I$ and $A^k = AA^{k-1}$. Show that the rules $A^kA^l = A^{k+l}$ and $(A^k)^l = A^{kl}$ hold, just as for real numbers.

Exercise 2.4.15. Find a nonzero 2×2 matrix A such that $A^2 = O$.

Exercise 2.4.16. Find a 3×3 matrix A such that $A^2 \neq O$ but $A^3 = O$.

Exercise 2.4.17. Find the number of three-leg flights connecting B and D in Example 2.4.8 by evaluating $(M^3)_{24} = (M^3)_{42}$.

Exercise 2.4.18. Prove Equation 2.133 of Theorem 2.4.3.

The next five exercises deal with *block multiplication.*

Exercise 2.4.19. Show that

$$\left[\begin{array}{rr|rr} 1 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]\left[\begin{array}{rr} 0 & 0 \\ 0 & 0 \\ \hline 3 & 2 \\ 1 & -1 \end{array}\right]$$
$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}.$$

Exercise 2.4.20. Show that if two conformable matrices of any size are partitioned, as in the previous exercise, so that the products make sense, then

$$[A \quad B]\begin{bmatrix} C \\ D \end{bmatrix} = [AC + BD].$$

Exercise 2.4.21. Show that if two conformable matrices of any size are partitioned into four submatrices each, so that the products and sums make sense, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}.$$

Exercise 2.4.22. Compute the product by block multiplication, using the result of the previous exercise:

$$\left[\begin{array}{rr|rr} 1 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]\left[\begin{array}{rr|rr} 1 & -2 & 1 & 0 \\ 2 & 0 & -3 & 1 \\ \hline 0 & 0 & 2 & 3 \\ 0 & 0 & 7 & 4 \end{array}\right].$$

Exercise 2.4.23. Partition the first matrix of the previous exercise as

$$\left[\begin{array}{rr|rr} 1 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 \end{array}\right].$$

Find the appropriate corresponding partition of the second matrix, and evaluate the product by using these blocks.

MATLAB Exercises

In MATLAB, the product of matrices is denoted by $*$, and a power like A^k by $A\hat{\ }k$; both the same as for numbers. The unit matrix of order n is denoted by

eye(n), and the $m \times n$ zero matrix by **zeros**(m, n). The command **rand**(m, n) returns an $m \times n$ matrix with random entries uniformly distributed between 0 and 1. The command **round**(A) rounds each entry of A to the nearest integer.

Exercise 2.4.24. As in Example 2.4.1, let $\mathbf{v}$ denote the vector obtained from the vector $\mathbf{u}$ by a rotation through an angle θ.
a. Compute $\mathbf{v}$ for $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and each of $\theta = 15\,^\circ, 30\,^\circ, 45\,^\circ, 60\,^\circ, 75\,^\circ$, and $90\,^\circ$. (MATLAB will compute the trig functions if you use radians.)
b. Use MATLAB to verify that $R_{75^\circ} = R_{25^\circ} * R_{50^\circ}$.

Exercise 2.4.25. Let

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

be the connection matrix of an airline network as in Example 2.4.8.
a. Which cities can be reached from A with exactly two stops?
b. Which cities can be reached from A with two stops or less?
c. What is the number of stops needed to reach all cities from all others?

Exercise 2.4.26. Let $\mathbf{a} = 10 * \mathbf{rand}(1, 4) - 5$ and $\mathbf{b} = 10 * \mathbf{rand}(1, 4) - 5$.
a. Compute $C = \mathbf{a} * \mathbf{b}'$ and **rank**(C) for ten instances of such $\mathbf{a}$ and $\mathbf{b}$. (Use the up-arrow key.)
b. Make a conjecture about **rank**(C) in general.
c. Prove your conjecture.

Exercise 2.4.27. Let $A = 10 * \mathbf{rand}(2, 4) - 5$ and $B = 10 * \mathbf{rand}(4, 2) - 5$.
a. Compute $C = A * B$, $D = B * A$, **rank**(C), and **rank**(D) for ten instances of such A and B.
b. Make a conjecture about **rank**(C) and **rank**(D) in general.

Exercise 2.4.28. In MATLAB you can enter blocks in a matrix in the same way as you enter scalars. Use this method to solve a. Exercise 2.4.19, and b. Exercise 2.4.22.

2.5 The Inverse and the Transpose of a Matrix

While for vectors it is impossible to define division, for matrices it is possible in some very important cases.

We may try to follow the same procedure as for numbers. The fraction b/a has been defined as the solution of the equation $ax = b$, or as b times $1/a$, where $1/a$ is the solution of $ax = 1$. For matrices we mimic the latter formula: To find the inverse of a matrix A, we look for the solution of the matrix equation

$$AX = I, \tag{2.142}$$

where I is the $n \times n$ unit matrix and X an unknown matrix. In terms of mappings, because I represents the identity mapping or no change, this equation means that if a mapping is given by the matrix A, we are looking for the matrix X of the (right) inverse mapping, that is, of the mapping that is undone if followed by the mapping A. (As it turns out, and as should be evident from the geometrical meaning, the order of the factors does not matter if A represents a mapping from $\mathbb{R}^n$ to itself.)

By the definition of the product, if I is $n \times n$, then A must be $n \times p$ and X of size $p \times n$ for some p. Then Equation 2.142 corresponds to n^2 scalar equations for np unknowns. Thus, if $p < n$ holds, then we have fewer unknowns than equations and generally no solutions apart from exceptional cases. On the other hand, if $p > n$ holds, then we have more unknowns than equations, and so generally infinitely many solutions. Since we are interested in finding unique solutions, we restrict our attention to those cases in which $p = n$ holds, or in other words to $n \times n$ or *square* matrices A. (Cases of $p \neq n$ are left to Exercises 2.5.8–2.5.11.) For a square matrix, n is called the order of A. For such A, Equation 2.142 may be written as

$$A(\mathbf{x}_1 \;\; \mathbf{x}_2 \dots \mathbf{x}_n) = (\mathbf{e}_1 \;\; \mathbf{e}_2 \dots \mathbf{e}_n) \tag{2.143}$$

and by Equation 2.130 we can decompose this equation into n separate systems

$$A\mathbf{x}_1 = \mathbf{e}_1, \quad A\mathbf{x}_2 = \mathbf{e}_2, \quad \dots, \quad A\mathbf{x}_n = \mathbf{e}_n \tag{2.144}$$

for the n unknown n-vectors $\mathbf{x}_1, \; \mathbf{x}_2, \dots, \; \mathbf{x}_n$.

Before proceeding further with the general theory, let us consider an example.

Example 2.5.1. (Finding the Inverse of a 2×2 Matrix by Solving Two Systems). Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{2.145}$$

and so let us solve

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.146}$$

or equivalently the separate systems

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.147}$$

Subtracting 3 times the first row from the second in both systems, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.148}$$

Adding the second row to the first and dividing the second row by –2, again in both systems, we obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} -2 \\ 3/2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}. \tag{2.149}$$

Hence

$$x_{11} = -2,\ x_{21} = 3/2,\ x_{12} = 1,\ x_{22} = -1/2 \tag{2.150}$$

or in matrix form

$$X = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}. \tag{2.151}$$

It is easy to check that this X is a solution of $AX = I$, and in fact of $XA = I$, too. Furthermore, since the two systems given by Equations 2.147 have the same matrix A on their left sides, the row reduction steps were exactly the same for both, and can therefore be combined into the reduction of a single augmented matrix with the two columns of I on the right, that is, of $[A|I]$ as follows:

$$\begin{aligned} &\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \to \\ &\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]. \end{aligned} \tag{2.152}$$

♦

We can generalize the results of this example in part as a definition and in part as a theorem.

Definition 2.5.1. ***(The Inverse of a Matrix).*** *A matrix A is called invertible if it is a square matrix and there exists a unique square matrix X of the same size such that $AX = I$ and $XA = I$ hold. Such an X, if one exists, is called the inverse of A and is denoted by A^{-1}.*

Theorem 2.5.1. ***(Inverting a Matrix by Row Reduction).*** *A square matrix is invertible if and only if the augmented matrix $[A|I]$ can be reduced* *by elementary row operations to the form $[I|C]$, and in that case C is the inverse A^{-1} of A.*

Proof. The augmented matrix corresponding to the equation $AX = I$ is $[A|I]$. If the reduction of $[A|I]$ produces the form $[I|C]$, then the matrix equation corresponding to the latter augmented matrix is $IX = C$ or, equivalently, $X = C$. By Theorem 2.1.1, $IX = C$ has the same solution set as $AX = I$, and so C is the unique solution of $AX = I$.

By reversing the elementary row operations, we can undo the above reduction; that is, we can change $[I|C]$ back to $[A|I]$. But then the same steps would change $[C|I]$ into $[I|A]$, which corresponds to solving the matrix equation $CY = I$ for an unknown matrix Y uniquely as $IY = A$, or $Y = A$. Hence, $CA = I$. Thus, if C solves $AX = I$, then it also solves $XA = I$, and it is the only solution of both equations. Thus A is invertible, with C as its inverse A^{-1}.

On the other hand, if $[A|I]$ cannot be reduced to the form $[I|X]$, then the system $AX = I$ has no solution for the following reason: In this case the reduction of A must produce a zero row at the bottom of every corresponding echelon matrix U, because if U had no zero row, then it could be further reduced to I. The last row of every reduction of $[A|I]$ that reduces A to an echelon matrix U with a zero bottom row must be a sum of nonzero multiples of some rows (or maybe just a single row). Suppose this sum contains c times the ith row (with $c \neq 0$). Then the submatrix $[A|\mathbf{e}_i]$ (see footnote 6 on page 54) will be reduced to $[U|c\mathbf{e}_n]$: Since the zero entries of the $\mathbf{e}_i$ column cannot affect the single 1 of it, c times this 1 ends up at the bottom. The matrix $[U|c\mathbf{e}_n]$, however, represents an inconsistent system, because the last row of U is zero, but the last component of $c\mathbf{e}_n$ is not. ∎

Example 2.5.2. (Finding the Inverse of a 2×2 Matrix by Row Reduction). Let us find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \tag{2.153}$$

if it exists.

We form the augmented matrix $[A|I]$ and reduce it as follows:

$$\begin{array}{llll}
\left[\begin{array}{rr|rr} 2 & 3 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right] & \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_1 \end{array} & \left[\begin{array}{rr|rr} 1 & -2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right] & \\
\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\mathbf{r}_1 \end{array} & \left[\begin{array}{rr|rr} 1 & -2 & 0 & 1 \\ 0 & 7 & 1 & -2 \end{array}\right] & \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2/7 \end{array} & \left[\begin{array}{rr|rr} 1 & -2 & 0 & 1 \\ 0 & 1 & 1/7 & -2/7 \end{array}\right] \\
\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 + 2\mathbf{r}_2 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \end{array} & \left[\begin{array}{rr|rr} 1 & 0 & 2/7 & 3/7 \\ 0 & 1 & 1/7 & -2/7 \end{array}\right]. & &
\end{array} \tag{2.154}$$

Thus we can read off the inverse of A as

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}. \tag{2.155}$$

It is easy to check that we do indeed have $AA^{-1} = A^{-1}A = I$. ♦

Example 2.5.3. (Showing Noninvertibility of a 2×2 Matrix by Row Reduction). Here is an example of a noninvertible square matrix. Let us try to compute the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}. \tag{2.156}$$

We form the augmented matrix $[A|I]$ and reduce it as follows:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\mathbf{r}_1 \end{array} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]. \tag{2.157}$$

The corresponding system is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \tag{2.158}$$

and so the second row of $[A|I]$ corresponds to the self-contradictory equations

$$0x_{11} + 0x_{21} = -2 \tag{2.159}$$
$$0x_{12} + 0x_{22} = 1. \tag{2.160}$$

Thus A has no inverse. ♦

Just as for numbers $b/a = a^{-1}b$ is the solution of $ax = b$, for matrix equations we have a similar consequence of Definition 2.5.1.

Theorem 2.5.2. *(Using the Inverse to Solve Matrix Equations).* *If A is an invertible $n \times n$ matrix and B an arbitrary $n \times p$ matrix, then the equation*

$$AX = B \tag{2.161}$$

has the unique solution

$$X = A^{-1}B. \tag{2.162}$$

Proof. That $X = A^{-1}B$ is a solution can be seen easily by substituting it into Equation 2.161:

$$A(A^{-1}B) = (AA^{-1})B = IB = B, \tag{2.163}$$

and that it is the only solution can be seen in this way. Assume that Y is another solution, so that

$$AY = B \tag{2.164}$$

holds. Multiplying both sides of this equation by A^{-1} we get

$$A^{-1}(AY) = A^{-1}B \tag{2.165}$$

and this equation reduces to

$$(A^{-1}A)Y = Y = A^{-1}B, \tag{2.166}$$

which shows that $Y = X$. ■

If $p = 1$ holds, Equation 2.161 becomes our old friend

$$A\mathbf{x} = \mathbf{b}, \tag{2.167}$$

where $\mathbf{x}$ and $\mathbf{b}$ are n-vectors. Thus Theorem 2.5.2 provides a new way of solving this equation. Unfortunately, this technique has little practical significance, since computing the inverse of A is generally more difficult than solving Equation 2.167 by Gaussian elimination. In some theoretical considerations, however, it is useful to know that the solution of Equation 2.167 can be written as

$$\mathbf{x} = A^{-1}\mathbf{b}, \tag{2.168}$$

and if we have several equations like 2.167 with the same left sides, then they can be combined into an equation of the form 2.161 with $p > 1$ and profitably solved by computing the inverse of A and using Theorem 2.5.2.

Example 2.5.4. (Solving an Equation for an Unknown 2×3 *Matrix).* Let us solve

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 2 & 3 & -5 \\ 4 & -1 & 3 \end{bmatrix}. \tag{2.169}$$

From Example 2.5.1 we know that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}. \tag{2.170}$$

Hence, by Theorem 2.5.2, we obtain

$$X = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 & -5 \\ 4 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -7 & 13 \\ 1 & 5 & -9 \end{bmatrix}. \tag{2.171}$$

♦

As we have just seen, if A is invertible, then Equation 2.168 provides the solution of Equation 2.167 for every n-vector $\mathbf{b}$. It is then natural to ask whether the converse is true, that is, whether the existence of a solution of Equation 2.167 for *every* $\mathbf{b}$ implies the invertibility of A. (We know that a *single* **b** is not enough: Equation 2.167 may be solvable for some right-hand sides and not for others; see, e.g., Examples 2.1.4 and 2.1.5.) The answer is yes.

Theorem 2.5.3. ***(Existence of Solutions Criterion for the Invertibility of a Matrix).*** *An $n \times n$ matrix A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has a solution for every n-vector* $\mathbf{b}$.

Proof. The "only if" part of this statement has already been proved; we just included it for the sake of completeness. To prove the "if" part, let us assume that $A\mathbf{x} = \mathbf{b}$ has a solution for every n-vector $\mathbf{b}$. Then it has a solution for each standard vector $\mathbf{e}_i$ in the role of $\mathbf{b}$; that is, each of the equations

$$A\mathbf{x}_1 = \mathbf{e}_1,\ A\mathbf{x}_2 = \mathbf{e}_2, \ldots,\ A\mathbf{x}_n = \mathbf{e}_n \tag{2.172}$$

has a solution by assumption. These equations can, however, be combined into the single equation

$$A\left(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\right) = \left(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\right), \tag{2.173}$$

which can be written as

$$AX = I \tag{2.174}$$

whose augmented matrix is $[A|I]$. From the proof of Theorem 2.5.1 we know that the solution of this equation, if one exists, must be $X = A^{-1}$, and since we have stipulated the existence of a solution, the invertibility of A follows. ■

The condition of solvability of $A\mathbf{x} = \mathbf{b}$ for *every* possible right side can be replaced by the requirement of uniqueness of the solution for a *single* $\mathbf{b}$.

Theorem 2.5.4. ***(Unique-Solution Criterion for the Invertibility of a Matrix).*** *An $n \times n$ matrix A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has a unique solution for some n-vector* $\mathbf{b}$.

Proof. If A is invertible, then, by Theorem 2.5.2, $\mathbf{x} = A^{-1}\mathbf{b}$ gives the unique solution of $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b}$. Conversely, if, for some $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has a unique solution, then Theorem 2.2.1 (page 55) shows that the rank of A equals n and consequently that $AX = I$ also has a unique solution. Of course, this solution must be A^{-1}. ■

The vector $\mathbf{b}$ in Theorem 2.5.4 may be taken to be the zero vector. This case is sufficiently important for special mention.

Corollary 2.5.1. ***(Trivial-Solution Criterion for the Invertibility of a Matrix).*** *A square matrix A is invertible if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

Definition 2.5.2. ***(Singular and Nonsingular Matrices).*** *An $n \times n$ matrix A for which the associated system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every n-vector $\mathbf{b}$ is called nonsingular; otherwise, it is called singular.*

Let us collect some equivalent characterizations of nonsingular square matrices that follow from our considerations up to now.

Theorem 2.5.5. ***(Various Criteria for a Matrix to be Nonsingular).*** *An $n \times n$ matrix A is nonsingular if and only if it has any (and thus all) of the following properties:*
1. A is invertible.
2. The rank of A is n.
3. A is row equivalent to I.
4. $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$.
6. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

For numbers, the product and the inverse are connected by the formula $(ab)^{-1} = a^{-1}b^{-1} = b^{-1}a^{-1}$. For matrices, we have an analogous result, but with the significant difference that the product is noncommutative and the order of the factors on the right must be reversed.

Theorem 2.5.6. ***(Inverse of the Product of Two Matrices).*** *If A and B are invertible matrices of the same size, then so too is AB, and*

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{2.175}$$

Proof. The proof is very simple: Repeated application of the associative law and the definition of I give

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= ((AB)B^{-1})A^{-1} = (A(BB^{-1}))A^{-1} = (AI)\,A^{-1} \\ &= AA^{-1} = I\end{aligned} \tag{2.176}$$

and similarly in the reverse order

$$(B^{-1}A^{-1})(AB) = I. \tag{2.177}$$

■

Another theorem for numbers, namely that $\left(a^{-1}\right)^{-1} = a$, also has an analog for matrices.

Theorem 2.5.7. ***(Inverse of the Inverse of a Matrix).*** *If A is an invertible matrix, then so too is A^{-1} and*

$$(A^{-1})^{-1} = A. \tag{2.178}$$

The proof is left as Exercise 2.5.19.

There exists another simple operation for matrices, one that has no analog for numbers. Although we will not need it until later, we present it here since it rounds out our discussion of the algebra of matrices.

Definition 2.5.3. ***(Transpose of a Matrix).*** *For every $m \times n$ matrix A, we define its transpose A^T as the $n \times m$ matrix obtained from A by making the jth column of A into the jth row of A^T for each j; that is, by defining the jth row of A^T as $(a_{1j}, a_{2j}, \ldots, a_{mj})$. Equivalently,*

$$(A^T)_{ji} = a_{ij} \tag{2.179}$$

for all $1 \le i \le m$ and $1 \le j \le n$.

From this definition it easily follows that the ith row of A becomes the ith column of A^T as well. Also, the transpose of a column n-vector is a row n-vector and vice versa. This fact is often used for avoiding the inconvenient appearance of tall column vectors by writing them as transposed rows:

Example 2.5.5. (Transpose of a Row Vector)

$$(x_1, x_2, \ldots, x_n)^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{2.180}$$

♦

Example 2.5.6. (Transpose of a 2×3 Matrix). Let

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 4 & -1 & 3 \end{bmatrix}. \tag{2.181}$$

Then

$$A^T = \begin{bmatrix} 2 & 4 \\ 3 & -1 \\ -5 & 3 \end{bmatrix}. \tag{2.182}$$

♦

The transpose has some useful properties.

Theorem 2.5.8. ***(Transpose of the Product of Two Matrices and of the Inverse of a Matrix).*** *If A and B are matrices such that their product is defined, then*

$$(AB)^T = B^T A^T, \tag{2.183}$$

and if A is invertible, then so too is A^T and

$$(A^T)^{-1} = (A^{-1})^T. \tag{2.184}$$

Proof. AB is defined when A is $m \times p$ and B is $p \times n$, for arbitrary m, p, and n. Then B^T is $n \times p$ and A^T is $p \times m$, and so $B^T A^T$ is also defined and is $n \times m$, the same size as $(AB)^T$. To prove Equation 2.183, we need only show that corresponding elements of those two products are equal. Indeed, for every $i = 1, \dots, n$ and $j = 1, \dots, m$,

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_{k=1}^{p} a_{jk} b_{ki} = \sum_{k=1}^{p} (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij}. \tag{2.185}$$

Hence $(AB)^T = B^T A^T$.

Next, we prove the second statement of the theorem. If A is invertible, then there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. Applying Equation 2.183, with $B = A^{-1}$, we obtain

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I, \tag{2.186}$$

and also

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I. \tag{2.187}$$

Hence A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$. ∎

Exercises

In the first six exercises find the inverse matrix if possible.

Exercise 2.5.1. $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}.$

Exercise 2.5.2. $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}.$

Exercise 2.5.3. $A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & -1 & 1 \\ 3 & 2 & -2 \end{bmatrix}.$

Exercise 2.5.4. $A = \begin{bmatrix} 0 & -6 & 2 \\ 3 & -1 & 0 \\ 4 & 3 & -2 \end{bmatrix}.$

Exercise 2.5.5. $A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & -1 & 3 \\ 3 & 2 & 5 \end{bmatrix}.$

Exercise 2.5.6. $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix}$.

Exercise 2.5.7. Find two invertible 2×2 matrices A and B such that $A \neq -B$ and $A + B$ is not invertible.

Exercise 2.5.8. a. Given the 2×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 4 & -1 & 1 \end{bmatrix},$$

find all 3×2 matrices X by Gauss–Jordan elimination such that $AX = I$ holds. (Such a matrix is called a *right inverse of* A.)
b. Can you find a 3×2 matrix Y such that $YA = I$ holds?

Exercise 2.5.9. a. Given the 3×2 matrix

$$A = \begin{bmatrix} 2 & -1 \\ 4 & -1 \\ 2 & 2 \end{bmatrix},$$

find all 2×3 matrices X by Gauss–Jordan elimination such that $XA = I$ holds. (Such a matrix is called a *left inverse of* A.)
b. Can you find a 2×3 matrix Y such that $AY = I$ holds?

Exercise 2.5.10. * a. Try to formulate a general rule, based on the results of the last two exercises, for the existence of a right inverse and for the existence of a left inverse of a 2×3 and of a 3×2 matrix.
b. Same as above for an $m \times n$ matrix.
c. When would the right inverse and the left inverse be unique?

Exercise 2.5.11. * Show that if a square matrix has a right inverse X and a left inverse Y, then $Y = X$ must hold. (*Hint*: Modify the second part of the proof of Theorem 2.5.2.)

Exercise 2.5.12. The matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is obtained from the unit matrix I by the elementary row operation of adding c times its first row to its second row. Show that for every 3×3 matrix A the same elementary row operation performed on A results in the product matrix EA. Also, find E^{-1} and describe the elementary row operation it corresponds to. (A matrix that produces the same effect by multiplication as an elementary row operation, like this E and the matrices P in the next two exercises, is called an *elementary matrix*.)

Exercise 2.5.13. Find a matrix P such that, for every 3×3 matrix A, PA equals the matrix obtained from A by multiplying its first row by a nonzero scalar c. (*Hint*: Try $A = I$ first.) Find P^{-1}.

Exercise 2.5.14. Find a matrix P such that, for every 3×3 matrix A, PA equals the matrix obtained from A by exchanging its first and third rows. (*Hint*: Try $A = I$ first.) Find P^{-1}.

Exercise 2.5.15. If A is any invertible matrix and c any nonzero scalar, what is the inverse of cA? Prove your answer.

Exercise 2.5.16. For every invertible matrix A and every positive integer n we define $A^{-n} = (A^{-1})^n$. Show that in this case we also have $A^{-n} = (A^n)^{-1}$ and $A^{-m}A^{-n} = A^{-m-n}$ if m is a positive integer as well.

Exercise 2.5.17. A square matrix with a single 1 in each row and in each column and zeros everywhere else is called a *permutation matrix*.
a. List all six 3×3 permutation matrices P and their inverses.
b. Show that, for every such P and for every $3 \times n$ matrix A, PA equals the matrix obtained from A by the permutation of its rows that is the same as the permutation of the rows of I that results in P.
c. What is BP if B is $n \times 3$?

Exercise 2.5.18. State six conditions corresponding to those of Theorem 2.5.5 for a matrix to be singular.

Exercise 2.5.19. Prove Theorem 2.5.7. (*Hint*: Imitate the proof of Theorem 2.5.2 for the equation $A^{-1}X = I$.)

Exercise 2.5.20. Prove that if A, B, C are invertible matrices of the same size, then so is ABC, and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

MATLAB Exercises

In MATLAB, the transpose of A is denoted by A'. The reduction of $[A|I]$ can be achieved by the command $\mathbf{rref}([A\ \mathbf{eye}(n)])$ or $\mathbf{rref}([A\ \mathbf{eye}(\mathbf{size}(A))])$. A^{-1} can also be obtained by writing $\mathbf{inv}(A)$. These commands or the command $\mathbf{rank}(A)$ can be used to determine whether A is singular or not.

Exercise 2.5.21. Let $A = \mathbf{round}(10 * \mathbf{rand}(4))$, $B = \mathbf{triu}(A)$, and $C = \mathbf{tril}(A)$.
a. Find the inverses of B and C in **format rat** by using **rref** if they exist, and verify that they are inverses indeed.
b. Repeat Part (a) five times. (Use the up-arrow key.)
c. Do you see any pattern? Make a conjecture and prove it.

Exercise 2.5.22. Let $A = \mathbf{round}(10 * \mathbf{rand}(3,5))$.
a. Find a solution for $AX = I$ by using **rref**, or show that no solution exists.
b. If you have found a solution, verify that it satisfies $AX = I$.
c. If there is a solution, compute $A\backslash\mathbf{eye}(3)$ and check whether it is a solution.
d. If there is a solution of $AX = I$, try to find a solution for $YA = I$ by using **rref**.
(*Hint*: Rewrite this equation as $A^T Y^T = I$ first.) Draw a conclusion.
e. Repeat all of the above three times.

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