

Preface

This book is designed for a one-semester, post-calculus linear algebra course, primarily intended for mathematics, physics, and computer science majors. While basic calculus is a prerequisite for such a course, very little of it is used in the book. Certainly, multivariable calculus is not required. Vectors are treated fully in Chapter 1, but for classes familiar with them, this chapter may be skipped or just reviewed briefly. Complex numbers, series, and exponentials are presented briefly in an appendix, but they are needed only in Section 7.4, which may not be covered in some courses.

The selection of topics conforms to a large extent to the recommendations of the Linear Algebra Curriculum Study Group.¹ The main differences are that the book begins with a chapter on Euclidean vector geometry, mostly in three dimensions; determinants are treated more fully and are placed just before eigenvalues, which is where they are needed; the LU factorization is relegated to Chapter 8 on numerical methods; and the facts about linear transformations are collected in one chapter and are treated in more detail.

This book is considerably shorter than the 400 to 800 pages of most introductory linear algebra books, which are more suitable for two- or three-semester courses.

While many applications are presented, they are mostly taken from physics, and several new ones have been added in the second edition. However, these examples give only a glimpse of how the subject is used in other fields, and further details are left to texts in those fields. There is, though, a section on computer graphics and a chapter on numerical methods. Also, most sections contain MATLAB[®] exercises. On the other hand, we hope that the student's interest will be aroused not only by the possible applications, but also by the geometrical background and the beautiful structure of linear algebra. Nevertheless, for readers especially interested in applications, a list of the ones discussed follows this preface.

The more difficult exercises and theorems are marked by an asterisk. Some exercises are used to develop new topics, whose inclusion in the main text would have disrupted the flow of ideas. The symbols ■ and ♦ are used to indicate the end of proofs and examples, respectively.

¹ David Carlson, Charles R. Johnson, David C. Lay, A. Duane Porter. The Linear Algebra Curriculum Study Group Recommendations for the First Course in Linear Algebra. *College Mathematics Journal*, 24:1 (1993) 41–46.

In this second edition, in response to the concerns of some users of the first edition, many of the earlier proofs and explanations have been expanded and a few new ones added. Also, exercises involving laborious computations have been replaced by simpler ones, and some new ones have been added.

Foreword to Instructors

- The brevity mentioned above makes the book easier to use. Important points are not drowned in a sea of detail, and instructors and students do not have to search for what to keep and what to omit. In a minimal course, however, the following sections may be omitted entirely: Section 4.3 on computer graphics, Section 5.1 on orthogonal projections and least squares, Section 6.2 on cofactor expansions of determinants, Cramer's rule, etc., Section 6.3 on the cross product, Sections 7.3 and 7.4 on principal axes and complex matrices, and Chapter 8 on numerical methods. Theorem 3.4.8 (The Exchange Theorem) may also be omitted, since an alternative direct proof of the dimension theorem is provided in the new edition.
- The geometric content is heavily emphasized. In fact, as mentioned above, the book begins with a chapter on Euclidean vector geometry, mostly in three dimensions. Most other similar textbooks start with the solution of linear systems. We believe that this early introduction of the geometrical background helps students to visualize the concepts of linear algebra and provides easy concrete examples. Additionally, many students in this course, e.g., computer science majors, are not required to take multivariable calculus, and do not see this important material anywhere else.
- In the first chapter, the equations of planes are given in both parametric and nonparametric form, in contrast to most calculus books, which present only the nonparametric form. Many examples and exercises illustrate the transition from one form to the other. However, we avoid using the cross product at this stage, because it is only available in \mathbb{R}^3 . We use the method of solving simultaneous equations to obtain a normal vector to a plane, and this topic is revisited as an example to Gaussian elimination. On the other hand, Section 6.3 is devoted to the cross product as an illustration of the use of determinants, and it is only at that point that it is used to obtain a normal vector to a plane.
- The “back and forth” process between parametric and nonparametric equations for lines and planes lays the groundwork for the same transition between describing a subspace of \mathbb{R}^n as a set of linear combinations or as the solution set of a homogeneous system of linear equations, that is, as the column space of a matrix or the null space of another matrix. Another generalization of this issue is finding orthogonal complements of subspaces of \mathbb{R}^n given in either form.
- Many books use the notation $\|\mathbf{p}\|$ for the length of a vector \mathbf{p} in \mathbb{R}^n , but we prefer $|\mathbf{p}|$, because in \mathbb{R}^1 length is the absolute value, and there is no

reason to change notation for higher dimensions, just as there was none in using $+$ for addition of both scalars and of vectors. The notation $\|\mathbf{p}\|$ is left for other norms.

- Important concepts are presented as definitions and theorems. Students are advised to memorize them. It is not enough just to understand the material; the main concepts must be remembered well to be able to build on them.
- Except for the Spectral Theorem in the complex case and theorems from other fields of mathematics, all theorems are proved. It is thus left to the instructor to adjust the level of the course from the computational to the fairly theoretical by omitting as many or as few proofs as desired.
- Great care has been taken to motivate every new concept, even those that many books do not, such as dot product, matrix operations, linear independence (not just in two or three dimensions), determinants, eigenvalues, and eigenvectors.
- The letter symbols are selected to reflect the connections between related quantities, a principle often ignored in other linear algebra books. Vectors and their components, matrices and their column and row vectors and entries are denoted by the same letters with different fonts, like \mathbf{v} , v_i and A , \mathbf{a}_i , \mathbf{a}^j , a_{ij} . The main exception is the unit matrix, which is, bowing to tradition, denoted by I , its columns by \mathbf{e}_i , and its entries by δ_{ij} .
- Only standard notation is used, so that students who study further, will have no difficulty in reading applied or more advanced texts. Nonstandard notation, such as the use of a list in parentheses for column vectors and in brackets for row vectors, or $\vec{\mathbf{a}}_i$ or \mathbf{A}_i for a row vector of a matrix, found in some other introductory linear algebra books, is avoided. We use \mathbf{a}_i for the column vectors of a matrix A and \mathbf{a}^i for its row vectors. This is standard notation in more advanced books. (See, e.g., *Introduction to Linear and Nonlinear Programming* by David G. Luenberger, Addison-Wesley, 1973.) We also use $\mathbf{x}_A = (x_{A1}, x_{A2}, \dots, x_{An})^T$ for the coordinate vector of a vector \mathbf{x} relative to an ordered basis or basis matrix A . (Compare this, e.g., with the notation $[\mathbf{x}]_{\mathcal{B}} = (c_1, c_2, \dots, c_n)^T$ of *Linear Algebra and Its Applications* by David Lay, Addison-Wesley, 1993, where the brackets on the left are superfluous, the coordinates of \mathbf{x} are denoted by the unrelated letter c , and the basis \mathcal{B} is not indicated on the right, not to mention that we need an ordered basis or basis matrix here.) Our notation makes the notoriously messy topic of change of basis much simpler.
- Similarity of matrices is introduced in the context of changing bases.
- Most introductory linear algebra books introduce determinants by unmotivated formulas. This book introduces them by three simple properties, expanding on the approach in Strang.²

² Gilbert Strang, *Linear Algebra and its Applications*, 3rd ed. Harcourt Brace, San Diego, 1988.

- MATLAB exercises at the end of most sections reinforce and expand the linear algebra material. They also provide some introduction to MATLAB, but should be used in conjunction with a MATLAB manual.
- The appendix on implication and equivalence introduces the student in an informal way to certain crucial elements of proofs, and is highly recommended reading for most.
- All displayed equations are numbered, and in the new edition, mnemonic headings are appended to all definitions, theorems, figures, and examples. These numbers and headings should make references to these items easier and make their connections more transparent.

Foreword to Students

Linear algebra is probably your first mathematics course in which the theory is just as important as the computations. To study from this book you have to carefully read the text with paper and pencil in hand.

The book starts out gently, with analytic geometry, but soon the algebra takes over and the subject becomes more abstract, which may cause some difficulty for some of you.

Studying this kind of mathematics involves three interwoven steps:

1. You must understand the material.
2. You must learn the concepts thoroughly so that you remember them and can apply them knowledgeably.
3. You must practice it, doing exercises.

Each of these steps is necessary and supports the others.

In many other subjects, understanding is not a problem, and so many students believe that once they pass that hurdle, they have done enough. Not true: If you understand something in class, that does not mean you will know it the next day. You must study after every class and make sure that you are able to explain the material in your own words so that you do not forget it. If you don't, then you have to start over again on your own, with the class attendance wasted. You will need to study several hours after every class. This is especially important, because most concepts are built upon each other. For instance, vectors, introduced in Section 1.1, are used throughout the book; matrices introduced in Section 2.2 are used throughout the rest of the book, and so on.

On the other hand, you cannot do mathematics by rote memorization without understanding, because the subject is generally too complicated for that. Also, doing that would defeat the whole purpose of studying mathematics, which is the comprehension of its logic and the ability to use it in applications—not just in those that were presented, but in other similar (or even somewhat different) applications.

Working out solutions to the exercises reinforces both the learning and the understanding of the material and is often also useful in its own right, because many exercises involve important applications of the theory.

In studying linear algebra, you have to thoroughly understand and remember the definitions first, since everything else is built on them. If you don't remember a definition, you cannot possibly understand the theory that depends on it and the exercises that make use of it.

Next in importance come the theorems, lemmas (minor or auxiliary theorems), and corollaries. These are usually preceded by introductory examples and followed by further examples that illuminate various aspects and applications of the theorems. You must study these examples together with the theorems and their proofs. It is permissible to read everything just superficially at first, to get a basic understanding, but after that, you must study it again in detail. When studying a theorem, isolate the conditions or hypotheses which make it tick. Try to see where these conditions are used in the proof, and what would happen if a condition were changed or omitted. After pinpointing the conditions, do the same for the conclusions, and last, try to follow the steps of the proof. This is where the paper and pencil come in: Write these steps down. Close the book and write down the conditions, the conclusions, or the whole statement that you are studying. Try to fill in steps that are just briefly indicated in the proofs. If the proof has a reference to some earlier material, be sure to look it up and explain to yourself how it is used. The same advice applies to the follow-up examples as well: make sure you see where the conditions of the theorem are used and why they are necessary, and follow the computations on paper.

There is an appendix on implication and equivalence, which introduces in an informal way certain crucial elements of proofs. It is highly recommended reading for all those who have not seen this material before.

Finally, after you have gone through the steps listed above, you will be ready to tackle exercises. The odd-numbered ones have solutions available in a Students' Solution Manual on the book's webpage. Do those exercises first; they are usually similar to examples in the text. Don't look at the solution before making a really serious attempt to solve a problem on your own. If a problem looks too difficult at first, then look at a similar example in the text or go back and review the definition or theorem that the problem is intended to illustrate. A problem that you have solved stays much better in your mind than one that you have merely read, and its structure becomes much clearer. But, of course, once you have solved a problem, there is no harm in looking up the solution. You may even learn a different way of solving it, or find an error in your solution (or perhaps in the solution manual).

If you follow the advice above, you will probably find linear algebra to be a very interesting and enjoyable subject, but if you don't, then it may become an unpleasant chore.

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Géza Schay

List of Applications

1. Center of mass. Exercises 1.1.7, 1.1.8.
2. Work as a dot product. Example 1.2.4.
3. Equations of lines and planes. Section 1.3.
4. An electrical network, Kirchhof's laws. Example 2.3.2.
5. A connection matrix for an airline. Example 2.4.8.
6. Population changes. Example 2.4.9.
7. The structure of the system expressing Kirchhof's laws. Example 3.5.7.
8. Hermite polynomials. Example 3.6.4.
9. Legendre polynomials. Exercise 3.6.11.
10. Computer graphics. Section 4.3.
11. Orthogonal projections and least-squares approximations. Section 5.1.
12. Coriolis force. Example 6.3.5.
13. Lorentz force. Example 6.3.6.
14. Systems of difference and differential equations. Section 7.2.
15. Population growth. Example 7.2.1.
16. An electric circuit with resistor, condenser, and coil. Example 7.2.2.
17. A predator-prey population model. Exercise 7.2.11.
18. Conic sections and quadric surfaces. Section 7.3.



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