Chapter 2
Discrete-Time Inventory Problems with Lead-Time and Order-Time Constraint

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2.1 Introduction

Demand uncertainty of the various items in the supply chain plays an important role in the control of material flow. Moreover, information on the suppliers mode of deliveries is key in selecting appropriate policies for inventory management. In some situations, orders for products cannot be placed while waiting for a delivery of previous orders. Thus placing an order-time constraint on deliveries. This applies where production capacity at the supplier side is limited. This situation may also arise in certain organizations where transportation capacity is limited: see Bensoussan, Çakanyidirim, and Moussaoui [4].

This chapter considers the basic inventory model of Scarf [16] and modifies it by including lead time with order-time constraints of the type mentioned above. It is well known that an (s,S) policy is optimal for Scarf's model for the infinite horizon stationary model: see Igelhart [10], Veinott [18], Beyer and Sethi [6], and Benkherouf [3]. Also, (s,S) policies remain optimal in the presence of lead time: see [18]. This chapter shows that (s,S) policies are still optimal with the additional constraints on order time. A result, which seems to follow straightforwardly from existing results in the literature, turned out to be delicate and needing some care.

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A related model to this chapter's is discussed in [4] for a discrete-state, continuous-time inventory model with infinite horizon where the demand process is assumed to be generated by some Poisson process.

The main ingredients for the problem under consideration in this chapter are given below:

The demand process is assumed to be composed of a sequence of i.i.d variables,

$$D_1, \dots, D_n, \dots,$$

constructed on a probability space $(\Omega, \mathscr{A}, P)$, where D_n represents the demand at time n. Let $\mathscr{F}^n = \sigma(D_1, \dots, D_n)$ be the $\sigma-$algebra generated by the demand process and $\mathscr{F}^0 = (\Omega, \emptyset)$.

An ordering policy V is composed of a sequence of ordering times

$$\tau_1, \dots, \tau_j, \dots,$$

with τ_j being $\mathscr{F}^{\tau_{j-1}}$ measurable. To each ordering time τ_j, one associates a quantity v_{τ_j} which represents the amount ordered. The new element here is that the usual condition $\tau_j \leq \tau_{j+1}$ is replaced with the constraint

$$\tau_{j+1} \geq \tau_j + L, \tag{2.1}$$

in which L is the lead time, $L \geq 1$, and integer.

The inventory evolves as follows:

$$y_{n+1} = y_n + v_n - D_n,$$

with $y_1 = x$, $v_n = 0$, if $n \neq \tau_j$, for some $j \geq L$, and y_n is the inventory level at time n.

To define the objective function, we introduce the cost in a single period. It is composed of a cost on the inventory

$$l(x) = hx^+ + px^-, \tag{2.2}$$

and an order cost

$$C(v) = K\mathbb{1}_{v>0} + cv, \tag{2.3}$$

where h, p, K, c are strictly positive known constants and $x^+ = \max\{0, x\}$, $x^- = -\min\{0, x\}$, with

$$\mathbb{1}_{v>0} = \begin{cases} 1, & \text{If } v > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Costs are assumed to be additive and discounted geometrically at a known rate α, $0 < \alpha < 1$, and that unmet demand is completely backlogged.

The objective function is given by the formula

$$J_x(V) = \sum_{n=1}^{\infty} \alpha^{n-1}(E[l(y_n) + C(v_n)]), \tag{2.4}$$

and the value function is

$$u(x) = \inf_V J_x(V), \tag{2.5}$$

where E is the usual expectation operator.

Inventory models with deterministic lead time can be found in [18], Archibald [1], Sahin [13, 14], and Fegergruen and Schechner [7]. These papers, but [18], are concerned with the computations of system performance measures for (s,S) policies. The closest to the present work is [1], where at most one outstanding order is permitted. This is equivalent to constraint (2.1). However, no proof of the optimality of (s,S) policies was provided. In fact the notion of policies with at most one outstanding order is not new. It can be found in Hadley and Within [9], where it was discussed in the context of lost sales models: see also Kim and Park [11]. This chapter, unlike those that deals with order-time constraint, but [4], provides a proof of the optimality of (s,S) policies.

In the next section, we shall derive the Bellman equation associated to $u(x)$. Section 2.3 is concerned with ground preparation for showing the derivation of (s,S) policies. Section 2.4 deals with existence and derivation of the pair (s,S). The optimality of (s,S) policies is shown, under some technical conditions, in Sect. 2.5. Section 2.6 treats the special case where the demand in each period is exponentially distributed. Further, this section contains some general remarks and a conclusion.

2.2 Dynamic Programming

It is easy to figure out the Bellman equation, considering the possibilities at the initial time. If there is no order at time 0, then (by (2.5) and (2.4)) the best which can be achieved over an infinite planning horizon is

$$l(x) + \alpha E u(x - D).$$

On the other hand if an order of size v is made at time 0, which is possible since there is no pending order, then the inventory evolves as follows:

$$\begin{aligned} y_1 &= \quad x, \\ y_2 &= \quad x - D_1, \\ &\cdots \\ y_{L+1} &= \quad x - (D_1 + \cdots + D_L) + v, \end{aligned}$$

and the best which can be achieved is

$$K + cv + l(x) + \sum_{j=2}^{L} \alpha^{j-1} E l(x - (D_1 + \cdots + D_{j-1}))$$
$$+ \alpha^L E u(x + v - (D_1 + \cdots + D_L)).$$

From these considerations, we can easily derive Bellman equation

$$u(x) = \min\Bigg\{ l(x) + \alpha E u(x-D),$$

$$K - cx + \sum_{j=0}^{L-1} \alpha^j E l(x - D^{(j)}) + \inf_{\eta > x}[c\eta + \alpha^L E u(\eta - D^{(L)})]\Bigg\}, \tag{2.6}$$

in which we use the notation

$$D^{(j)} = \begin{cases} D_1 + \cdots + D_j, & \text{if } j \geq 1 \\ 0, & \text{if } j = 0 \end{cases}.$$

In the next section, we shall recast the function u given in (2.6) in a form which we shall find useful later on in our analysis.

2.3 *(s,S)* Policy

We first transform equation (2.6), by introducing a constant $s \in R$, and changing $u(x)$ into $H_s(x)$, using the formula

$$u(x) = -cx + \sum_{j=0}^{L-1} \alpha^j E l(x - D^{(j)}) + C_s + H_s(x), \tag{2.7}$$

where C_s will be defined below. In fact, let

$$g(x) = (1-\alpha)cx + \alpha^L E l(x - D^{(L)}), \tag{2.8}$$

then we take

$$C_s = \frac{g(s) + \alpha c \bar{D}}{1-\alpha},$$

where $\bar{D}$ is the expected value of D, with $0 < \bar{D} < \infty$.

We note the formula

$$\sum_{j=0}^{L-1} \alpha^j E g(x - D^{(j)})$$

$$= cx(1-\alpha^L) - \alpha c \bar{D} \frac{1-\alpha^L}{1-\alpha} + \alpha^L c \bar{D} L + \sum_{j=0}^{L-1} \alpha^{j+L} E l(x - D^{(j+L)}).$$

We then check that $H_s(x)$ is the solution of

$$H_s(x) = \min \Bigg\{ g(x) - g(s) + \alpha E H_s(x - D), \\ K + \inf_{\eta > x} \left[\sum_{j=0}^{L-1} \alpha^j E(g(\eta - D^{(j)}) - g(s)) + \alpha^L E H_s(\eta - D^{(L)}) \right] \Bigg\}. \quad (2.9)$$

On the other hand, for any s we define

$$H_s(x) = \begin{cases} g(x) - g(s) + \alpha E H_s(x - D), & x \geq s \\ 0, & x < s \end{cases}. \quad (2.10)$$

A solution H_s satisfying (2.10) exists, is unique, and is continuous in $\mathbb{R}$.

We have used the same notation $H_s(x)$ between (2.9) and (2.10), because we want to find s so that the solution of (2.10) is also a solution of (2.9).

Let us set

$$g_s(x) = (g(x) - g(s))\mathbb{1}_{x \geq s},$$

then the solution of (2.10) satisfies, for all x,

$$H_s(x) = g_s(x) + \alpha E H_s(x - D). \quad (2.11)$$

From this relation we deduce, after iterating that

$$H_s(x) = \sum_{j=0}^{L-1} \alpha^j E g_s(x - D^{(j)}) + \alpha^L E H_s(x - D^{(L)}),$$

therefore also

$$\sum_{j=0}^{L-1} \alpha^j E(g(x - D^{(j)}) - g(s)) + \alpha^L E H_s(x - D^{(L)}) \\ = H_s(x) + \sum_{j=0}^{L-1} \alpha^j E(g(x - D^{(j)}) - g(s))\mathbb{1}_{x - D^{(j)} < s}.$$

Define

$$\Psi_s(x) = H_s(x) + \sum_{j=0}^{L-1} \alpha^j E(g(x - D^{(j)}) - g(s))\mathbb{1}_{x - D^{(j)} < s}, \quad (2.12)$$

then if we want $H_s(x)$ to satisfy (2.9), we must have

$$H_s(x) = \min\left\{g(x) - g(s) + \alpha E H_s(x-D), K + \inf_{\eta > x} \Psi_s(\eta)\right\}. \tag{2.13}$$

This relation motivates the choice of s. We want to find s so that

$$K + \inf_{\eta > s} \Psi_s(\eta) = 0. \tag{2.14}$$

If such an s exists, we define $S = S(s)$ as the point where the infimum is attained.

$$\inf_{\eta > s} \Psi_s(\eta) = \Psi_s(S(s)). \tag{2.15}$$

Note, from (2.13), that $\Psi_s(x)$ coincides with $H_s(x)$, for $x > s$, when $L = 1$.

In the next section we shall show the existence of the pair (s,S) satisfying (2.14) and (2.15).

2.4 Existence and Characterization of the Pair (s,S)

Let $F^{(L)}(x)$ and $f^{(L)}(x)$ be the cumulative distribution and the density function of $D^{(L)}$ respectively and $\bar{F}^{(L)}(x) = 1 - F^{(L)}(x)$, with $F^{(0)}(x) = 1$. Then (2.8) and (2.2) give

$$g(x) = (1-\alpha)cx + \alpha^L h \int_0^x (x-\xi) f^{(L)}(\xi)\mathrm{d}\xi + \alpha^L p \int_x^\infty (\xi - x) f^{(L)}(\xi)\mathrm{d}\xi.$$

It follows that $\mu(x) = g'(x)$ is given by

$$\mu(x) = (1-\alpha)c + \alpha^L h - \alpha^L (h+p)\bar{F}^{(L)}(x). \tag{2.16}$$

We assume that

$$(1-\alpha)c - \alpha^L p < 0. \tag{2.17}$$

There exists a unique $\bar{s} > 0$ such that

$$\begin{aligned} &\mu(x) \le 0, \quad \text{if } x \le \bar{s} \\ &\mu(x) \ge 0, \quad \text{if } x \ge \bar{s} \end{aligned}. \tag{2.18}$$

Note that if $L = 0$, (2.17) reduces to the classical condition of optimality of (s,S) policies with no lead time.

Lemma 2.4.1. *The solution H_s of (2.10) satisfies for $s \le x$,*

$$\begin{aligned} &H_s(x) + \sum_{j=0}^{L-1} \alpha^j \int_s^x \bar{F}^{(j)}(x-\xi)\mu(\xi)\mathrm{d}\xi \\ &\quad = \frac{1-\alpha^L}{1-\alpha}(g(x)-g(s)) + \sum_{j=L}^{\infty} \alpha^j \int_s^x F^{(j)}(x-\xi)\mu(\xi)\mathrm{d}\xi. \end{aligned}$$

Furthermore, the equation is still valid for $s > x$ by dropping the integral term on the right-hand side.

Proof. The proof follows immediately by recalling the definition of μ in (2.16) and checking that H_s given by

$$H_s(x) = \sum_{j=0}^{\infty} \alpha^j \int_s^x F^{(j)}(x-\xi)\mu(\xi)\mathrm{d}\xi, \tag{2.19}$$

solves (2.10). □

Lemma 2.4.2. *The function Ψ_s defined in (2.12) is given by*

$$\begin{aligned} \Psi_s(x) = &\frac{1-\alpha^L}{1-\alpha}(g(x)-g(s)) + \sum_{j=L}^{\infty} \alpha^j \int_s^x F^{(j)}(x-\xi)\mu(\xi)\mathrm{d}\xi \\ &-((1-\alpha)c+\alpha^L h)\bar{D}\sum_{j=1}^{L-1} j\alpha^j \\ &+\alpha^L(h+p)\sum_{j=1}^{L-1}\alpha^j \int_0^{+\infty} \bar{F}^{(j)}(\zeta)\bar{F}^{(L)}(x-\zeta)\mathrm{d}\zeta. \end{aligned} \tag{2.20}$$

Proof. Note that

$$\begin{aligned} &E(g(x-D^{(j)})-g(s))\mathbb{1}_{x-D^{(j)}<s} \\ &\quad = -E\mathbb{1}_{D^{(j)}>x-s}\left(\int_{x-D^{(j)}}^{s} \mu(\xi)\mathrm{d}\xi\right) \\ &\quad = -\int_{x-s}^{+\infty}\left(\int_{x-\zeta}^{s}\mu(\xi)d\xi\right)\mathrm{d}F^{(j)}(\zeta) \\ &\quad = -\int_{x-s}^{+\infty}\bar{F}^{(j)}(\zeta)\mu(x-\zeta)\mathrm{d}\zeta \\ &\quad = -\int_{-\infty}^{s}\bar{F}^{(j)}(x-\xi)\mu(\xi)\mathrm{d}\xi \\ &\quad = -\int_{-\infty}^{x}\bar{F}^{(j)}(x-\xi)\mu(\xi)\mathrm{d}\xi + \int_s^x \bar{F}^{(j)}(x-\xi)\mu(\xi)\mathrm{d}\xi. \end{aligned}$$

The Lemma is now immediate from further direct computations on $\int_{-\infty}^{x} \bar{F}^{(j)}(x-\xi)\mu(\xi)\mathrm{d}\xi$, (2.16), Lemma 2.4.1, and the definition of Ψ_s. This finishes the proof. □

It is easy to deduce from Lemma 2.4.2 that $\Psi_s(x)$ is bounded below and tends to $+\infty$ as $x \to +\infty$. Therefore the infimum over $x \geq s$ is attained and we can define $S(s)$ as the smallest infimum.

Proposition 2.4.1. *We assume that (2.17) holds, then one has*

1.

$$\Psi_s(S(s)) \to -((1-\alpha)c + \alpha^L h)\bar{D}\sum_{j=1}^{L-1} j\alpha^j, \text{ as } s \to +\infty. \tag{2.21}$$

2.

$$\max_s \Psi_s(S(s)) = \Psi_{\bar{s}}(S(\bar{s})) \geq 0, \tag{2.22}$$

3.

$$\Psi_s(S(s)) \to -\infty, \text{ as } s \to -\infty, \tag{2.23}$$

4. There exists one and only one solution of (2.14) such that $s \leq \bar{s}$.

Proof. We compute, by Lemma 2.4.2,

$$\begin{aligned} \Psi_s'(x) = {} & \frac{1-\alpha^L}{1-\alpha}\mu(x) + \sum_{j=L}^{\infty} \alpha^j \int_s^x f^{(j)}(x-\xi)\mu(\xi)\mathrm{d}\xi \\ & - \alpha^L(h+p)\sum_{j=1}^{L-1}\alpha^j \int_0^x \bar{F}^{(j)}(\zeta) f^{(L)}(x-\zeta)\mathrm{d}\zeta, \end{aligned} \tag{2.24}$$

which can also be written as

$$\Psi_s'(x) = \gamma(x) + \sum_{j=L}^{\infty} \alpha^j \int_s^x f^{(j)}(x-\xi)\mu(\xi)\mathrm{d}\xi, \tag{2.25}$$

with

$$\gamma(x) = \frac{1-\alpha^L}{1-\alpha}((1-\alpha)c - \alpha^L p) + \alpha^L(h+p)\sum_{j=0}^{L-1}\alpha^j \int_0^x F^{(j)}(x-\xi) f^{(L)}(\xi)\mathrm{d}\xi. \tag{2.26}$$

The function γ is increasing in x and

$$\begin{aligned} \gamma(x) &= \frac{1-\alpha^L}{1-\alpha}((1-\alpha)c - \alpha^L p) < 0, \text{ for } x \leq 0, \\ \gamma(\infty) &= \frac{1-\alpha^L}{1-\alpha}((1-\alpha)c + \alpha^L h), \end{aligned}$$

therefore there exists a unique s^* such that

$$\gamma(x) < 0, \quad \text{for } x < s^*,$$
$$\gamma(x) > 0, \quad \text{for } x > s^*,$$
$$\gamma(s^*) = 0, \; s^* > 0.$$

Note that

$$\frac{1-\alpha^L}{1-\alpha}\mu(x) - \gamma(x) = \alpha^L(h+p)\sum_{j=0}^{L-1}\alpha^j \int_0^x \bar{F}^{(j)}(x-\xi) f^{(L)}(\xi)\mathrm{d}\xi. \tag{2.27}$$

The quantity on the right-hand side of the above equality vanishes for $x \leq 0$ or $L = 1$. Otherwise

$$\frac{1-\alpha^L}{1-\alpha}\mu(x) - \gamma(x) > 0, \quad \text{for } x > 0, \; L \geq 2.$$

Since $\gamma(s^*) = 0$, we have using (2.18), $\mu(s^*) > 0$, hence $0 < \bar{s} < s^*$.

Next, if $s \geq s^*$, $\Psi_s'(x) \geq 0$; for $x \geq s$ by (2.25); hence $S(s) = s$. Therefore, we get by (2.20) that

$$\Psi_s(S(s)) = -((1-\alpha)c + \alpha^L h)\bar{D}\sum_{j=1}^{L-1} j\alpha^j$$
$$+ \alpha^L(h+p)\sum_{j=1}^{L-1}\alpha^j \int_0^{+\infty} \bar{F}^{(j)}(\zeta)\bar{F}^{(L)}(s-\zeta)\mathrm{d}\zeta, \tag{2.28}$$

this function decreases in s and converges to $-((1-\alpha)c + \alpha^L h)\bar{D}\sum_{j=1}^{L-1} j\alpha^j$ as $s \to +\infty$. This proves part (1).

Consider now $\bar{s} < s < s^*$. We note that

$$\Psi_s'(s) = \gamma(s) < 0,$$
$$\Psi_s'(s^*) = \sum_{j=L}^{\infty}\alpha^j \int_s^{s^*} f^{(j)}(s^*-\xi)\mu(\xi)\mathrm{d}\xi > 0,$$

hence in this case

$$\bar{s} < s < S(s) < s^*.$$

If $s < \bar{s}$, then we can claim that

$$s < \bar{s} < S(s).$$

Indeed, for $s < x < \bar{s}$, we can see from formula (2.25) and (2.18) that $\Psi_s'(x) < 0$. However, we cannot compare $S(s)$ with s^* in this case.

We then study the behavior of $\Psi_s(S(s))$. We have already seen that, for $s > s^*$ the function $\Psi_s(S(s))$ is decreasing to the negative constant $-((1-\alpha)c + \alpha^L h)\bar{D}\sum_{j=1}^{L-1} j\alpha^j$. In this case, it follows from (2.24) that

$$\frac{\mathrm{d}}{\mathrm{d}s}\Psi_s(S(s)) = -\alpha^L(h+p)\sum_{j=1}^{L-1}\alpha^j\int_0^s \bar{F}^{(j)}(\zeta)f^{(L)}(s-\zeta)\mathrm{d}\zeta,\ s > s^*. \tag{2.29}$$

Note that $s > 0$. If $s < s^*$, then $s < S(s)$; therefore

$$\frac{\mathrm{d}}{\mathrm{d}s}\Psi_s(S(s)) = \frac{\partial \Psi_s}{\partial s}(S(s)),$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}s}\Psi_s(S(s)) = -\mu(s)\left[\frac{1-\alpha^L}{1-\alpha} + \sum_{j=L}^{\infty}\alpha^j F^{(j)}(S(s)-s)\right]. \tag{2.30}$$

Note that at $s = s^*$ the two formulas in (2.29) and (2.30) coincide, since

$$-\frac{1-\alpha^L}{1-\alpha}\mu(s^*) = -\alpha^L(h+p)\sum_{j=1}^{L-1}\alpha^j\int_0^{s^*}\bar{F}^{(j)}(\zeta)f^{(L)}(s^*-\zeta)\mathrm{d}\zeta,$$

which can be easily checked by remembering the definition of $s^*(\gamma(s^*) = 0)$, and (2.27). It follows clearly that $\Psi_s(S(s))$ decreases on $(\bar{s}, s^*)$ and increases on $(-\infty, \bar{s})$. Finally $\Psi_s(S(s))$ decreases on $(\bar{s}, +\infty)$ and increases on $(-\infty, \bar{s})$. So it attains its maximum at $\bar{s}$. From the formula for $\Psi_{\bar{s}}(x); H_{\bar{s}}(x)$; for $x > \bar{s}$ and the fact that $\bar{s}$ is the minimum of g, we get immediately that $\Psi_{\bar{s}}(S(\bar{s})) \geq 0$. This shows part (2) of the proposition.

Finally, for $s < 0$ we have

$$\begin{aligned}
\Psi_s(S(s)) &\leq \Psi_s(0) \\
&= \frac{1-\alpha^L}{1-\alpha}(g(0)-g(s)) + \sum_{j=L}^{\infty}\alpha^j\int_s^0 F^{(j)}(-\xi)\mathrm{d}\xi \\
&\quad - ((1-\alpha)c - \alpha^L h)\bar{D}\sum_{j=1}^{L-1} j\alpha^j,
\end{aligned}$$

which tends to be $-\infty$ as $s \to -\infty$. This proves (3). Therefore $\Psi_s(S(s))$ is increasing from $-\infty$ to a positive number as s grows from $-\infty$ to $\bar{s}$. Therefore there is one and only one $s < \bar{s}$ satisfying (2.14). The proof has been completed. □

2.5 Solution as an (s,S) Policy

It remains to see whether the solution H_s of equation (2.10) where s is the solution of (2.14) is indeed a solution of (2.13). It is useful to use, instead of $\Psi_s(x)$ a different function, namely

$$\Phi_s(x) = H_s(x) + \sum_{j=1}^{L-1} \alpha^j E(g(x - D^{(j)}) - g(s))\mathbb{1}_{x-D^{(j)}<s}, \tag{2.31}$$

which differs from $\Psi_s(x)$ by simply deleting the term corresponding to $j = 0$.

Clearly

$$\Phi_s(x) = \Psi_s(x), \quad \text{for } x \geq s.$$

However, when $x < s$,

$$\Psi_s(x) = \Phi_s(x) + g(x) - g(s),$$

Note that in finding S it is indifferent to work with one or the other function. Note also that, when $L = 1$, $\Phi_s(x) = H_s(x)$, for all x. Now we claim that

$$\inf_{\eta > x} \Psi_s(\eta) = \inf_{\eta > x} \Phi_s(\eta).$$

This equality is obvious when $x > s$, since the functions are identical. If $x < s$ we have

$$\inf_{\eta > x} \Psi_s(\eta) = \inf_{\eta > x} \Phi_s(\eta) = \inf_{\eta > s} \Phi_s(\eta) = -K.$$

Indeed,

$$\inf_{\eta > x} \Psi_s(\eta) = \min \left[\inf_{\eta > s} \Psi_s(\eta), \inf_{x < \eta < s} \Psi_s(\eta) \right],$$

and $\inf_{\eta > s} \Psi_s(\eta) = -K$, whereas (2.10) and (2.13) give

$$\inf_{x < \eta < s} \Psi_s(\eta) = \inf_{x < \eta < s} \sum_{j=0}^{L-1} \alpha^j E(g(\eta - D^{(j)}) - g(s))\mathbb{1}_{x-D^{(j)}<s} > 0,$$

hence the claim is true. Therefore (2.13) becomes

$$H_s(x) = \min \left\{ g(x) - g(s) + \alpha E H_s(x - D), K + \inf_{\eta > x} \Phi_s(\eta) \right\}. \tag{2.32}$$

For $x < s$, this relation reduces to

$$0 = \min[g(x) - g(s), 0],$$

which is true since $g(x)$ is decreasing for $x < s$.We then consider $x > s$. Since $H_s(x)$ is equal to the first term of the bracket. Therefore, what we have to prove is

$$H_s(x) \leq K + \inf_{\eta > x} \Phi_s(\eta), \quad \text{for } x > s. \tag{2.33}$$

In fact, we have not been able to prove (2.33) for all values of L. We know it is true for $L = 1$. We will prove it afterwards for exponential demands. For general demand distributions we have:

Proposition 2.5.2. *We assume*

$$\alpha^L((1-\alpha)c - \alpha^L p)(L-1)\bar{D} + (1-\alpha)K \geq 0, \tag{2.34}$$

then property (2.33) is satisfied.

Proof. We note that this result includes the case $L = 1$ in which the condition is automatically satisfied. The proof is similar to that of the case $L = 1$.

We recall from (2.10) that

$$H_s(x) - \alpha E H_s(x-D) = g_s(x), \quad \text{for all } x. \tag{2.35}$$

We then find a similar equation for $\Phi_s(x)$. This is where it is important to consider $\Phi_s(x)$ and not $\Psi_s(x)$, since we write the equation for any x and not just for $x > s$. It is easy to verify, using (2.31), that $\Phi_s(x)$ is the solution of

$$\begin{aligned}
&\Phi_s(x) - \alpha E \Phi_s(x-D) \\
&\quad = g_s(x) + \alpha E(g(x-D) - g(s))\mathbb{1}_{x-D<s} \\
&\qquad - \alpha^L E(g(x - D^{(L)}) - g(s))\mathbb{1}_{x-D^{(L)}<s},
\end{aligned} \tag{2.36}$$

and again we check that this equation coincides with (2.35) when $L = 1$. Going back to (2.33) we recall that

$$K + \inf_{\eta > x} \Phi_s(\eta) \geq K + \inf_{\eta > s} \Phi_s(\eta) = 0, \quad \text{for all } x > s. \tag{2.37}$$

So it is sufficient to prove (2.33) for $x > x_0 > s$, where x_0 is the first value x such that $H_s(x) \geq 0$. We necessarily have $H_s(x_0) = 0$. We have $s < \bar{s} < x_0$. Let us fix $\xi > 0$ and consider the domain $x \geq x_0 - \xi$. We can write, using (2.35), that for all x

$$H_s(x) - \alpha E H_s(x-D)\mathbb{1}_{x-D \geq x_0 - \xi} \leq g_s(x), \tag{2.38}$$

using the fact that

$$E H_s(x-D)\mathbb{1}_{x-D<x_0-\xi} \leq 0.$$

Define next

$$M_s(x) = \Phi_s(x+\xi) + K.$$

We note that $M_s(x) > 0$ for all x. We then state, by (2.37), that

$$\begin{aligned} &M_s(x) - \alpha E M_s(x-D) \\ &\quad = g_s(x+\xi) + \alpha E(g(x+\xi-D) - g(s))\mathbb{1}_{x+\xi-D<s} \\ &\qquad - \alpha^L E(g(x+\xi-D^{(L)}) - g(s))\mathbb{1}_{x+\xi-D^{(L)}<s} + (1-\alpha)K, \end{aligned}$$

and using the positivity of M we can assert that

$$\begin{aligned} &M_s(x) - \alpha E M_s(x-D)\mathbb{1}_{x-D\geq x_0-\xi} \\ &\quad \geq g_s(x+\xi) + \alpha E(g(x+\xi-D) - g(s))\mathbb{1}_{x+\xi-D<s} \\ &\qquad - \alpha^L E(g(x+\xi-D^{(L)}) - g(s))\mathbb{1}_{x+\xi-D^{(L)}<s} + (1-\alpha)K. \end{aligned} \tag{2.39}$$

We now consider the difference $Y_s(x) = H_s(x) - M_s(x)$, in the domain $x \geq x_0 - \xi$. We have by (2.38) and (2.39) that

$$\begin{aligned} &Y_s(x) - \alpha E Y_s(x-D)\mathbb{1}_{x-D\geq x_0-\xi} \\ &\quad \leq g_s(x) - g_s(x+\xi) - \alpha E(g(x+\xi-D) - g(s))\mathbb{1}_{x+\xi-D<s} \\ &\qquad + \alpha^L E(g(x+\xi-D^{(L)}) - g(s))\mathbb{1}_{x+\xi-D^{(L)}<s} - (1-\alpha)K. \end{aligned} \tag{2.40}$$

We first check easily, that

$$g_s(x+\xi) - g_s(x) \geq 0, \quad \text{for all } x \geq x_0 - \xi. \tag{2.41}$$

Consider next the function

$$\chi_s(y) = \alpha E(g(y-D) - g(s))\mathbb{1}_{y-D<s} - \alpha^L E(g(y-D^{(L)}) - g(s))\mathbb{1}_{y-D^{(L)}<s},$$

for $y \geq x_0$. We check that

$$\chi_s(y) = \int_{-\infty}^{s} \left(\int_{y-\zeta}^{+\infty} \left(-\alpha f(\eta) + \alpha^L f^{(L)}(\eta) \right) \mathrm{d}\eta \right) \mu(\zeta)\mathrm{d}\zeta,$$

so in fact

$$\chi_s(y) = \int_{-\infty}^{s} (-\alpha \bar{F}(y-\zeta) + \alpha^L \bar{F}^{(L)}(y-\zeta))\mu(\zeta)\mathrm{d}\zeta. \tag{2.42}$$

Note that in the integral, $\mu(\zeta) < 0$, since $s < \bar{s}$. We deduce that

$$\chi_s(y) \geq \alpha \int_{-\infty}^{s} (\bar{F}^{(L)}(y-\zeta) - \bar{F}(y-\zeta))\mu(\zeta)\mathrm{d}\zeta.$$

Further, note that $\bar{F}^{(L)}(y-\zeta) - \bar{F}(y-\zeta) \geq 0$, and $\mu(\zeta) \geq ((1-\alpha)c - \alpha^L p)$. Therefore

$$\chi_s(y) \geq \alpha^L((1-\alpha)c - \alpha^L p) \int_{-\infty}^{s} (\bar{F}^{(L)}(y-\zeta) - \bar{F}(y-\zeta))\mathrm{d}\zeta.$$

It follows that for $y \geq s$,

$$\begin{aligned}
\chi_s(y) &\geq \alpha^L((1-\alpha)c - \alpha^L p) \int_{-\infty}^{y} (\bar{F}^{(L)}(y-\zeta) - \bar{F}(y-\zeta))\mathrm{d}\zeta \\
&= \alpha^L((1-\alpha)c - \alpha^L p) \int_{0}^{\infty} (\bar{F}^{(L)}(u) - \bar{F}(u))\mathrm{d}u \\
&= \alpha^L((1-\alpha)c - \alpha^L p)(L-1)\bar{D}.
\end{aligned}$$

Thanks to assumption (2.34) we can assert from (2.40) and (2.42) that

$$Y_s(x) - \alpha E Y_s(x-D)\mathbb{1}_{x-D\geq x_0-\xi} \leq 0, \quad \text{for all } x \geq x_0 - \xi.$$

Also

$$Y_s(x_0-\xi) \leq -\Phi_s(x_0) - K \leq -K,$$

since $\Phi_s(x_0) \geq H_s(x_0) = 0$. It follows that

$$Y_s(x) \leq 0, \quad \text{for all} x \geq x_0 - \xi,$$

which is the desired result. □

2.6 The Exponential Case

In this section, we shall examine the special case when the demand D is exponentially distributed. That is,

$$f(x) = \begin{cases} \beta \exp(-\beta x), & \text{if } x \geq 0, \\ 0, & \text{otherwise} \end{cases}, \tag{2.43}$$

for some $\beta > 0$. The next proposition shows that property (2.33) automatically holds in this case.

Proposition 2.6.3. *We assume that the demand is distributed according to an exponential distribution, then (2.33) holds.*

Before we proceed to the proof of Proposition 2.6.3, the following result is needed.

Lemma 2.6.3. *For $s \leq \bar{s}$, the solution H_s of (2.10) satisfies the following properties:*

1. *Is constant on $(-\infty, s]$,*
2. *Strictly decreasing on $(s, \bar{s}]$,*
3. *$H_s(x) \to \infty$, as $x \to \infty$*

Proof. It clear that (1) follows from (2.10). Property (2) can be easily deduced from (2.18) and (2.19).
We claim that

$$H_s(x) \geq \frac{1}{1-\alpha}(g(\bar{s}) - g(s)). \tag{2.44}$$

Assume otherwise and define

$$A_s = \{x \in \mathbb{R},\ s \leq x \leq \bar{s}\}.$$

Let

$$x^* = \min\left\{x \in A_s : H_s(x^*) = \frac{1}{1-\alpha}(g(\bar{s}) - g(s))\right\}.$$

If follows from (2.10), the definitions of g, x^* and (2) that

$$H_s(x^*) = g(x^*) - g(s) + \alpha E H_s(x^* - D) > (g(\bar{s}) - g(s)) + \alpha H_s(x^*).$$

Hence,

$$H_s(x^*) > \frac{1}{1-\alpha}(g(\bar{s}) - g(s)).$$

This is in contradiction with the definition of x^*. Therefore (2.44) is true. This leads by (2.10) to

$$H_s(x) \geq g(x) - g(s)) + \alpha \frac{1}{1-\alpha}(g(\bar{s}) - g(s)).$$

It is then immediate to see that (3) holds. This finishes the proof. □

Proof (Proposition 2.6.3). We are going to show that

$$H_s'(x) \geq 0, \quad \text{if } x \geq x_0, \tag{2.45}$$

then (2.33) will follow immediately, since for $x \geq x_0$

$$H_s(x) \leq H_s(x+\xi) \leq \Phi_s(x+\xi) \leq \Phi_s(x+\xi) + K, \quad \text{for } \xi \geq 0.$$

It can be shown using integration by parts that if f is given (2.43), then H_s is the solution of the differential equation

$$(1-\alpha)\beta H_s(x) + H_s'(x) = G(x), \tag{2.46}$$

where $H_s(s) = 0$ and $G(x) = \beta(g(x) - g(s)) + \mu(x)$. In fact the function H_s can be written explicitly. However, we content ourselves with (2.46) since this will be sufficient to proceed further in the proof.

We claim that H_s has a unique minimum point.

Indeed, Lemma 2.6.3 implies H_s attains a minimum belonging to the interval $(\bar{s}, \infty)$. Further, H_s has a local minimum on (s, x_0). Moreover, it is easy to show that H_s is twice differentiable on (s, ∞). Assume now that H_s has two minima x_1 and x_2, with $x_1 < x_2$. It is clear that we can select x_1 such that $s < x_1 < x_0$. It follows that there exists an $x^* \in (x_1, x_2)$ such that x^* is a local maximum. Therefore, $H'(x^*) = 0$, and $H''(x^*) \leq 0$. Differentiating both sides of (2.46), we get that

$$H_s''(x^*) = G'(x^*) = \beta(\mu(x^*) + \mu'(x^*)).$$

The definition of μ in (2.16) with (2.18) and the fact $x^* > \bar{s}$ imply that $H_s''(x^*) > 0$. This leads to a contradiction. Therefore, H_s has a unique minimum and this minimum belongs to the interval (s, x_0). Consequently, (2.45) is true. The result has been proven. $\square$

In this chapter a discrete-time continuous-state inventory model, with a fixed lead time of several periods, was considered. Orders for products cannot be placed while waiting for a delivery of previous orders. It was shown that the policy which minimizes the total expected inventory costs over an infinite planning horizon is of an (s, S) type: see Propositions 2.5.2 and 2.6.3.

It seems natural to ask if (s, S) policies are still optimal for continuous-state, continuous-time models for inventory models with constraint (2.1). A possible starting point for such investigation is the work of Bensoussan, Liu, and Sethi [5]: see also Benkherouf and Bensoussan [2]. The answer to this question remains open. Another interesting problem is to see if requirement (2.34) can be weakened. Moreover, since the present work seems to be among the first attempts at examining the optimality of (s, S) policies for inventory models with lead-time and order-time constraints and has only dealt with the basic stationary model of Scarf, then it may be worthwhile to look at possible extensions of the present work to the large body of existing inventory models in the literature. These may include models with Markovian demand as discussed in Sethi and Feng [15], or demand dependent on the environment as found in Song and Zipkin [17], or models with two suppliers as treated in Fox, Metters, and Semple [8].

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