Chapter 2
Time Response of First- and Second-order Dynamical Systems

Science seems to have uncovered a set of laws that,
within the limits set by the uncertainty principle,
tell us how the universe will develop with time,
if we know its state *at any one time.*

Hawking, S.W., 1988, *A Brief History of Time*,
Bantam Books, Toronto-New York-London-Sydney-Auckland.

2.1 Preamble

How physical systems, e.g., aircraft, *respond* to a given input, such as a gust wind, under given initial conditions, like cruising altitude and cruising speed, is known as the *time response* of the system. Here, we have two major items that come into the picture when determining the time response, namely, the mathematical model of the system and the *history* of the input. The mathematical model, as studied in Chap. 1, is given as an ODE in the generalized coordinate of the system. Moreover, this equation is usually nonlinear, and hence, rather cumbersome to handle with the purpose of predicting how the system will respond under given initial conditions and a given input. However, if we first find the equilibrium states of the system, e.g., the altitude, the aircraft angle of attack, and cruising speed in our example above, and then linearize the model about this equilibrium state, then we can readily obtain the information sought, as described here.

We will thus start by assuming that the system model at hand has been linearized about an equilibrium state. That is, we will be concerned *mainly* with the time response of linear, time-invariant (LTI) dynamical systems, also termed *linear time-invariant systems* (LTIS), when subjected to arbitrary initial conditions and inputs. This class of systems is also known as *stationary systems* and *systems with constant coefficients*, and so, we will use these terms interchangeably. We will discuss in this chapter only first- and second-order systems, i.e., systems that are described by first- and second-order ODEs, respectively, more general systems being discussed in later

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chapters. However, notice that we have emphasized above that we will be concerned mainly with LTIS, thereby indicating other kinds of system models that do not fall into the same category. Such models arise, as we saw in Chap. 1, in the presence of Coulomb friction forces. In this case, the models at hand are not linear but, by adopting a simple model of the friction forces, as we did in Chap. 1, the models thus derived turn out to be *piecewise linear*, and hence, lend themselves to an analysis with the tools of LTI dynamical systems.

Applications of this analysis are numerous, e.g., vibration isolation; prediction of vibration transmitted by a moving base; design of instruments; vehicle dynamics; etc. We will outline applications in these fields.

We consider first systems to which no input is applied, these systems being subject only to initial conditions. The response of such systems, in the realm of system theory, is known as the *zero-input response*. In the language of mechanical vibration, the same goes by the name of *free response*. After this, we will study the response of the same systems to an arbitrary input, with zero initial conditions, which is called, in the realm of system theory again, the *zero-state response*. In some instances, this response is called the *forced response*. Here, note that the terminology of system theory is more accurate, for it presupposes zero initial conditions; the forced response presupposes only a nonzero-input, but says nothing about the initial conditions. Hence, we will adopt the terminology of system theory for the sake of language accuracy. The response of these systems to arbitrary initial conditions and arbitrary input is then obtained by *superposition*, namely, as the sum of their zero-input and zero-state responses.

Since we will study LTIS intensively, we need first a definition of these. In Chap. 1 we introduced a definition of linear systems from the point of view of the structure of their mathematical models, namely, *systems that give rise to governing equations where the generalized coordinate, the generalized speed and the time rate of change of the latter appear linearly, i.e., to the first power, multiplied by constants, that are termed the stiffness, the damping coefficient and the mass of the system, respectively.*

From the viewpoint of their time response, the systems of interest to our study in this chapter have the properties listed below: let the time response of a system to an input $f(t)$, with zero initial conditions, be denoted by $x_{f(t)}(t)$, which will represent the time history of the generalized coordinate $x(t)$, for $t \geq 0$. Thus, the time response of linear time-invariant dynamical systems

1. is *linearly homogeneous*, i.e., for any real number α,

$$x_{\alpha f(t)}(t) = \alpha x_{f(t)}(t)$$

   a property that is called *linear homogeneity*, or *homogeneity*, for brevity;
2. is *additive*, i.e., for a second input $g(t)$,

$$x_{f(t)+g(t)}(t) = x_{f(t)}(t) + x_{g(t)}(t)$$

   a property that is called *additivity*;

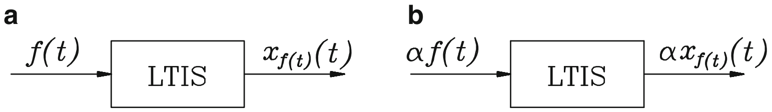


Fig. 2.1 Homogeneity of a LTIS

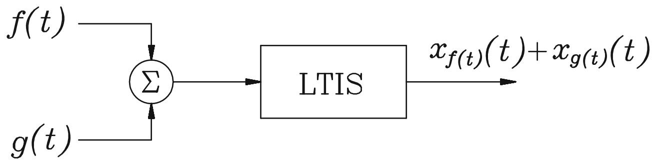


Fig. 2.2 Additivity of a LTIS

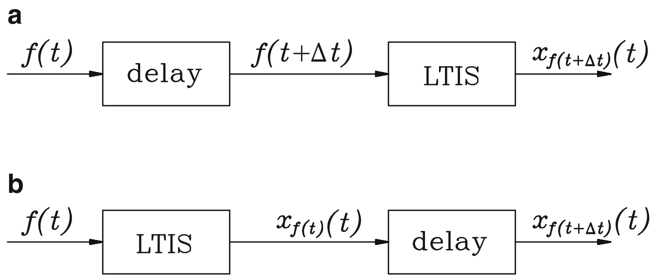


Fig. 2.3 Time invariance of a LTIS

3. is *time invariant*, i.e., for any time-interval Δt,

$$x_{f(t+\Delta t)}(t) = x_{f(t)}(t + \Delta t) \tag{2.1}$$

a property that is called *time invariance*.

The foregoing properties are best illustrated with the aid of a *black-box* representation of a LTIS, as in Fig. 2.1a. In this figure, the system at hand is represented as a block with an input $f(t)$ and a response, also known as *output*, $x_{f(t)}(t)$. Homogeneity is illustrated in Fig. 2.1b, while additivity and time invariance in Figs. 2.2 and 2.3, respectively. In fact, time invariance is best illustrated with the aid of an *artifact* called a *time delay*.

Moreover, let

$$f^{(n)}(t) \equiv \frac{d^n}{dt^n} f(t), \quad F(t) \equiv \underbrace{\int_0^t \cdots \int_0^t}_{n \quad \text{times}} f(\tau) d\tau$$

Then, as a consequence of the additivity property, we have two further properties, namely,

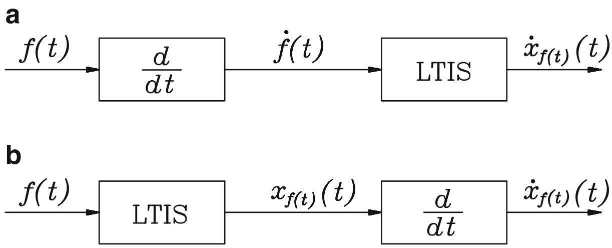


Fig. 2.4 Interchangeability of the differentiation and the input of a LTIS

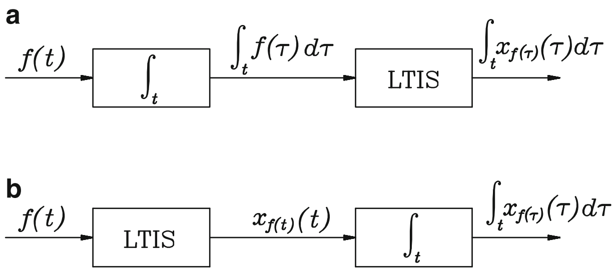


Fig. 2.5 Interchangeability of the integration of the input and the output of a LTIS

$$x_{f^{(n)}(t)}(t) = \frac{d^n}{dt^n} x_{f(t)}(t) \tag{2.2}$$

$$x_{F(t)}(t) = \underbrace{\int_0^t \cdots \int_0^t x_{f(\tau)}(\tau) d\tau}_{n \quad \text{times}} \tag{2.3}$$

We will find that the foregoing properties allow us to obtain the time response of this class of systems in a systematic way, sometimes with substantial time savings and, quite important, in a safer manner, less error prone. The two foregoing properties are illustrated in Figs. 2.4 and 2.5, for one single step of differentiation and integration, respectively.

2.2 The Zero-input Response of First-order LTIS[1]

Consider the first-order, linear, stationary dynamical system described below:

$$\dot{x} = -ax, \quad t > 0, \quad x(0) = x_0 \tag{2.4}$$

[1]In a course focusing on vibrations, this section can be skipped. However, its reading is strongly recommended because it helps gain insight into the response of second-order damped systems, studied in Sect. 2.3.2

where a and x_0 are given constants. We want to determine $x(t)$ *explicitly*; some would say *analytically*, but the latter is vague and very often misused. What we mean here by *explicitly*, as opposed to *numerically*, is also termed *symbolically*, for we are after an algebraic expression of the function $x(t)$. To this end, we assume that this function is *analytic*, i.e., that it has a series expansion of the form:

$$x(t) = x(0) + \dot{x}(0)t + \frac{1}{2}\ddot{x}(0)t^2 + \ldots + \frac{x^{(k)}(0)}{k!}t^k + \ldots \tag{2.5}$$

In order to evaluate the coefficients of the above series, we differentiate both sides of the differential equation of (2.4) infinitely many times with respect to time, which yields

$$\begin{aligned} \dot{x} &= -ax \\ \ddot{x} &= -a\dot{x} = a^2 x \\ x^{(3)} &= a^2\dot{x} = -a^3 x \\ &\vdots \\ x^{(k)} &= (-1)^k a^k x, \quad \text{etc.} \end{aligned} \tag{2.6}$$

Upon evaluation of the foregoing expressions at $t = 0$, we obtain

$$\begin{aligned} \dot{x}(0) &= -ax_0 \\ \ddot{x}(0) &= a^2 x_0 \\ x^{(3)}(0) &= -a^3 x_0 \\ &\vdots \\ x^{(k)}(0) &= (-1)^k a^k x_0, \quad \text{etc.} \end{aligned} \tag{2.7}$$

If we substitute the expressions appearing in Eq. 2.7 into Eq. 2.5, the desired expression is readily derived, namely,

$$x(t) = \left[1 - at + \frac{1}{2}a^2t^2 - \ldots + (-1)^k\frac{a^k t^k}{k!} + \ldots\right]x_0$$

The series appearing inside the brackets in the above equation is readily identified as the exponential of $-at$, i.e.,

$$x(t) = e^{-at}x_0, \quad t \geq 0 \tag{2.8}$$

which is the time response sought.

Needless to say, the time response of Eq. 2.8 is *proportional* to the initial value x_0. Moreover, the time-varying part of the above expression is apparently dependent

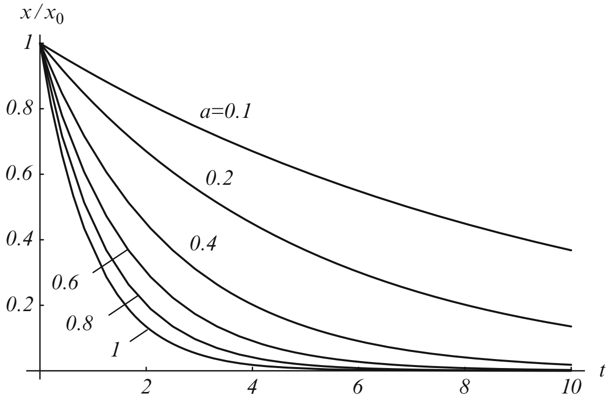
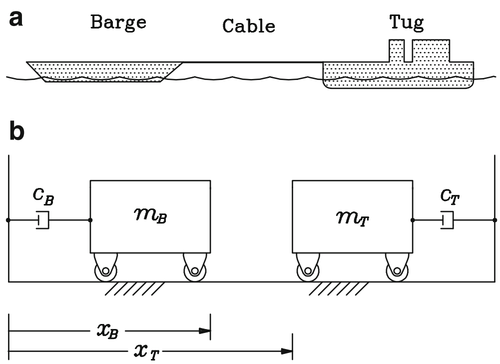


Fig. 2.6 Zero-input response of a first-order system for various values of its time constant

Fig. 2.7 Tugboat towing a barge: (**a**) physical system; and (**b**) iconic model



upon one single parameter, namely, coefficient a, which determines the nature of the response. Note that this coefficient has units of frequency, its reciprocal, labelled τ, having units of time. In fact, when $a > 0$, τ being positive as well, is termed the *time constant* of the system.

Furthermore, from Eq. 2.8, it is apparent that (1) if $a > 0$, the time response decays with time and does so the faster the greater a is, i.e., the time response of a system of this kind fades away faster for smaller time constants τ; (2) if $a < 0$, the time response grows unbounded, i.e., we are in the presence of an unstable system. We show in Fig. 2.6 various plots of the response of LTI first-order dynamical systems for various values of positive a, with $x(t)$ divided by x_0.

Example 2.2.1 (Collision Detection). Figure 2.7a shows a tugboat towing a barge at a uniform speed v_0. Due to an engine failure, the tugboat undergoes a shock that breaks the towing cable. Upon the assumption that the boat and the barge will continue on the same course after failure, an insurance company wants to know under which conditions on the relevant physical parameters involved an eventual collision will not occur.

Solution: Let m_T and m_B denote the mass of the tugboat and the barge, respectively. Moreover, assume that wind and water drag forces are known to be proportional to traveling speeds, the proportionality factors depending on water and wind conditions as well as on hull shape and hull materials. For the prevailing wind and water conditions, and hull properties, the drag constants are denoted by c_T and c_B for the tugboat and barge, respectively. After the failure, each vessel behaves as a first-order system and can be modeled as a mass-dashpot system, as shown in Fig. 2.7b, namely,

$$m_B\dot{v}_B + c_B v_B = 0, \quad m_T\dot{v}_T + c_T v_T = 0, \quad v_B(0) = v_T(0) = v_0$$

or

$$\dot{v}_B + \frac{1}{\tau_B}v_B = 0, \quad \dot{v}_T + \frac{1}{\tau_T}v_T = 0, \quad v_B(0) = v_T(0) = v_0 \tag{2.9}$$

The time constants of the two foregoing systems, τ_B and τ_T, are thus,

$$\tau_B = \frac{m_B}{c_B}, \quad \tau_T = \frac{m_T}{c_T}$$

Obviously, to avoid collisions, the barge speed should decay faster than the tugboat speed. Hence, the company should grant insurance only if the time constant of the barge is smaller than that of the tugboat. The condition sought is, then,

$$\tau_B < \tau_T \quad \text{or} \quad \frac{m_B}{c_B} < \frac{m_T}{c_T}$$

Note that tugboats are usually less massive than barges but, fortunately for insurance companies, barges are usually not as streamlined as tugboats and, hence, present a larger drag coefficient than tugboats, thereby making unlikely a collision under the circumstances described above.

2.3 The Zero-input Response of Second-order LTIS

In this section we are concerned with LTI dynamical systems described by a second-order ODE. We shall thus distinguish among *undamped*, *underdamped*, *critically damped* and *overdamped* systems. For brevity, we focus on the first two types, the last two being left as exercises.

2.3.1 Undamped Systems

The model associated with these systems is displayed below:

$$\ddot{x} = -\omega_n^2 x, \quad t > 0, \quad x(0) = x_0, \quad \dot{x}(0) = v_0 \tag{2.10}$$

which is the equation governing the motion of the *harmonic oscillator*. Again, in order to derive the time response of this system explicitly, we differentiate both sides of the differential equation of (2.10) infinitely many times, which yields

$$
\begin{aligned}
x^{(3)} &= -\omega_n^2 \dot{x} \\
x^{(4)} &= -\omega_n^2 \ddot{x} = \omega_n^4 x \\
x^{(5)} &= \omega_n^4 \dot{x} \\
x^{(6)} &= \omega_n^4 \ddot{x} = -\omega_n^6 x \\
&\vdots \\
x^{(2k)} &= (-1)^k \omega_n^{2k} x \\
x^{(2k+1)} &= (-1)^k \omega_n^{2k} \dot{x}, \quad \text{etc.}
\end{aligned}
\tag{2.11}
$$

Upon evaluation of the foregoing derivatives at $t = 0$, and substitution of these values into the series expansion of $x(t)$, we derive the expression given below:

$$
\begin{aligned}
x(t) = &\left[1 - \frac{\omega_n^2 t^2}{2!} + \frac{\omega_n^4 t^4}{4!} - \ldots + (-1)^k \frac{\omega_n^{2k} t^{2k}}{2k!} + \ldots\right] x_0 \\
&+ \frac{1}{\omega_n}\left[\omega_n t - \frac{\omega_n^3 t^3}{3!} + \frac{\omega_n^5 t^5}{5!} - \ldots + (-1)^k \frac{\omega_n^{2k+1} t^{2k+1}}{(2k+1)!} + \ldots\right] v_0
\end{aligned}
\tag{2.12}
$$

The terms in brackets multiplying x_0 and v_0 in Eq. 2.12 are readily recognized to be the series expansions of $\cos \omega_n t$ and $\sin \omega_n t$, respectively. Hence, the expression sought for $x(t)$ is readily derived as

$$
x(t) = (\cos \omega_n t) x_0 + \frac{1}{\omega_n} (\sin \omega_n t) v_0, \quad t \geq 0 \tag{2.13}
$$

Apparently, the response of undamped second-order systems is a linear combination of the two initial conditions x_0 and v_0, the associated coefficients being time-varying. Moreover, the first coefficient is simply the cosine function with a frequency equal to the natural frequency of the system; the second coefficient is the sine function with the same frequency, but multiplied by the reciprocal of the natural frequency. Note that, regardless of the nature of the generalized coordinate $x(t)$, whether a translational or an angular displacement, the two terms of the expression of Eq. 2.13 are dimensionally homogeneous. Indeed, the first coefficient is dimensionless and multiplies a constant with units of displacement; the second coefficient has units of time but multiplies a constant with units of velocity. As a consequence, the participation of the velocity initial condition on the response becomes more relevant as the ratio v_0/ω_n grows with respect to the initial condition x_0. Shown in Fig. 2.8 are plots of time responses for various values of $v_0/(\omega_n x_0)$.

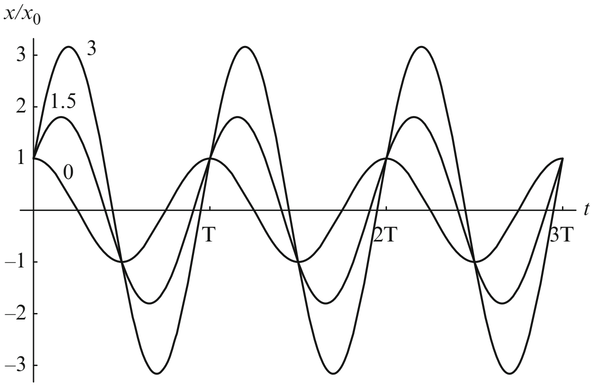


Fig. 2.8 Zero-input response of undamped second-order system for various values of $v_0/\omega_n x_0$

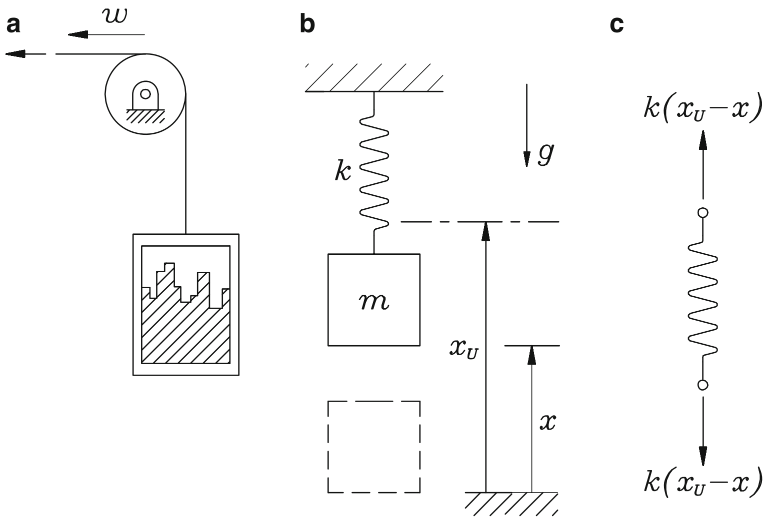


Fig. 2.9 The lifting mechanism of an elevator: (**a**) the mechanical model; (**b**) its iconic model; and (**c**) the FBD of the cable

Example 2.3.1 (Elevator Design). Shown in Fig. 2.9a is a crude iconic model of the lifting mechanism of an elevator. Under the assumption that the damping in the mechanism is negligible, one can model the mechanism-load system as the mass-spring layout shown in Fig. 2.9b. The cable is made of steel fibers that give it a stiffness of 1,000 kN/m, the weight of the empty elevator is 4,905 N and the maximum allowable load is 12 passengers, under the assumption that the average passenger weighs 735.75 N.

(a) Determine the speed w_1 that produces a tension in the cable that is five times the static load in the presence of a sudden arrest.

(b) Under the same sudden-arrest conditions, find the critical speed w_2 beyond which the cable becomes loose and the mechanism can no longer control the elevator.

What conclusions can you draw from this design?

Solution: A crucial step in formulating this problem is the choice of the mass position from which we measure its displacement. Since we want to derive a zero-input mathematical model, we should measure the displacement x from the static equilibrium configuration, and not from the unloaded-spring configuration, as shown in Fig. 2.9b, in light of Example 1.10.4. The governing equation is thus

$$\ddot{x} + \omega_n^2 x = 0, \quad t \geq 0, \quad x(0) = 0, \quad \dot{x}(0) = w_1$$

whose time response is

$$x(t) = \frac{w_1}{\omega_n} \sin \omega_n t$$

From Fig. 2.9c, the tension $F(t)$ in the cable, or in the spring for that matter, is

$$F(t) = k(x_U - x)$$

where x_U, the value of x when the spring is unloaded, is calculated from the condition

$$F(0) = mg, \quad x(0) = 0$$

Hence,

$$x_U = \frac{m}{k} g = \frac{g}{\omega_n^2}$$

In order to calculate the natural frequency of the system, we need k and m, the former being given in the problem statement, the latter derived below from the data:

$$m = \frac{5000 + 750 \times 12}{9.81} = \frac{14000}{9.81} = 1427 \text{ Kg}$$

the natural frequency of the system thus being

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1 \times 10^6}{1427}} = 26.47 \text{ s}^{-1} = 4.213 \text{ Hz}$$

(a) Upon substitution of x and x_U in the expression for $F(t)$, we obtain

$$F(t) = k \left(\frac{g}{\omega_n^2} - \frac{w_1}{\omega_n} \sin \omega_n t \right)$$

which apparently attains a maximum when $\sin \omega_n t$ attains a minimum, i.e., at an instant t_1 at which $\sin \omega_n t_1 = -1$. Therefore,

$$F_{\max} = k\left(\frac{g}{\omega_n^2} + \frac{w_1}{\omega_n}\right)$$

By setting $F_{\max} = 5mg$, we obtain

$$k\left(\frac{g}{\omega_n^2} + \frac{w_1}{\omega_n}\right) = 5mg$$

Therefore,

$$w_1 = \frac{4g}{\omega_n} = 1.4824 \text{ m/s}$$

a speed with which passengers are lifted at the rate of about one-half storey per second.[2]

(b) The above expression for $F(t)$ obviously attains a minimum when $\sin \omega_n t$ attains a maximum, i.e., at an instant t_2 at which $\sin \omega_n t_2 = 1$. Therefore,

$$F_{\min} = k\left(\frac{g}{\omega_n^2} - \frac{w}{\omega_n}\right)$$

Thus, the cable becomes loose when $F_{\min} = 0$, i.e., when

$$\frac{g}{\omega_n^2} = \frac{w}{\omega_n}$$

Hence,

$$w = \frac{g}{\omega_n} = 0.3706 \text{ m/s}$$

which is a speed at which passengers are lifted at the rate of about one storey every 4 s, i.e., unacceptably low. From the above result, it is obvious that the design is poor. Now, if we observe the two expressions for w, it is apparent that this value is inversely proportional to the quantity $\sqrt{k/m}$. Now, the mass is fixed, because the load is specified by the client; what we can do is choose a more compliant cable. For example, if we want to increase the value of w obtained in item (b) by a factor of 25, which would give $w = 9.265$ m/s as a speed that would make the cable loose upon a sudden arrest, then the cable should be specified with a stiffness of 200 kN/m.

Alternatively, we can solve this problem using an energy approach. Indeed, the total energy E_0 of the system can be calculated from the conditions at $t = 0$, while using the position of the mass at this instant as the reference level for the potential

[2] A typical average lifting speed is about 1 storey (≈3 m) per second.

energy due to gravity. Thus, the potential energy due to gravity at $t = 0$ vanishes, but the spring is stretched by a length x_U, and hence,

$$E_0 = \frac{1}{2}mw^2 + \frac{1}{2}kx_U^2 = \frac{1}{2}m\left(w^2 + \frac{g^2}{\omega_n^2}\right)$$

From the expression for $F(t)$ derived above, it is apparent that the maximum value of the tension in the cable is attained at the minimum value of x at the bottom position of the mass, x_b, at which $\dot{x} = 0$. The tension F_b at $x = x_b$ is, thus,

$$F_b = k(x_U - x_b) = 5mg$$

Hence,

$$x_b = x_U - 5\frac{mg}{k} = -4\frac{mg}{k} \equiv -4\frac{g}{\omega_n^2}$$

Therefore, the energy E_b at the bottom position is given by

$$E_b = \frac{1}{2}k(x_U - x_b)^2 + mgx_b = \frac{25}{2}\frac{m^2g^2}{k} - 4\frac{m^2g^2}{k}$$

i.e.,

$$E_b = \frac{17}{2}\frac{m^2}{k}g^2$$

Upon equating E_b with E_0, we obtain

$$\frac{17}{2}\frac{m^2}{k}g^2 = \frac{1}{2}m\left(w^2 + \frac{g^2}{\omega_n^2}\right)$$

Hence,

$$w_1 = 4\frac{g}{\omega_n}$$

which is exactly the same result obtained in item (a). Now, to solve item (b), we note that the cable becomes loose if $x = x_U$ at the top position, in which $\dot{x} = 0$. The energy E_{top} at this position is

$$E_{\text{top}} = mgx_U = mg\left(\frac{m}{k}\right)g$$

Thus, if we equate E_{top} with E_0, we obtain

$$\left(\frac{m^2}{k}\right)g^2 = \frac{1}{2}m\left(w^2 + \frac{g^2}{\omega_n^2}\right)$$

whence

$$w_2 = \frac{g}{\omega_n}$$

a result identical to that obtained in (b).

2.3.2 Damped Systems

Now we consider a damped system, namely,

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \quad t > 0, \quad x(0) = x_0, \quad \dot{x}(0) = v_0 \tag{2.14}$$

with both ζ and ω_n non-negative. In trying to find an expression for $x(t)$ by following the procedure introduced in the first two cases, we would readily find that this does not apply as directly. The reason is that now we would have to factor two infinite series, a task that is extremely cumbersome. However, we can circumvent this problem by writing the given system in *state-variable* form, i.e., by transforming it into a system of two first-order, linear, constant-coefficient ordinary differential equations. This is readily done by letting

$$z_1 \equiv x, \quad z_2 \equiv \dot{x} \tag{2.15}$$

whence the two-dimensional *state-variable* vector $\mathbf{z}(t)$ is defined as

$$\mathbf{z}(t) \equiv \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad \mathbf{z}(0) \equiv \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \tag{2.16}$$

Thus, the system appearing in Eq. 2.14 can be readily expressed as

$$\dot{z}_1 = z_2 \tag{2.17a}$$

$$\dot{z}_2 = -\omega_n^2 z_1 - 2\zeta\omega_n z_2 \tag{2.17b}$$

$$z_1(0) = x_0, \quad z_2(0) = v_0 \tag{2.17c}$$

or, in vector form,

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}, \quad t > 0, \quad \mathbf{z}(0) = \mathbf{z}_0 \tag{2.18a}$$

with matrix $\mathbf{A}$ defined as

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \tag{2.18b}$$

Now, the solution $\mathbf{z}(t)$ can be expressed by means of its series expansion, namely,

$$\mathbf{z}(t) = \mathbf{z}(0) + \dot{\mathbf{z}}(0)t + \frac{1}{2!}\ddot{\mathbf{z}}(0)t^2 + \ldots + \frac{1}{k!}\mathbf{z}^{(k)}(0)t^k + \ldots \tag{2.19}$$

On the other hand, the time derivatives of $\mathbf{z}$ appearing in Eq. 2.19 can be readily derived by successively differentiating both sides of Eq. 2.18a, namely,

$$\begin{aligned}
\dot{\mathbf{z}} &= \mathbf{A}\mathbf{z} \\
\ddot{\mathbf{z}} &= \mathbf{A}\dot{\mathbf{z}} \equiv \mathbf{A}^2\mathbf{z} \\
&\vdots \\
\mathbf{z}^{(k)} &= \mathbf{A}^k\mathbf{z}, \quad \text{etc.}
\end{aligned}$$

Upon evaluation of the foregoing derivatives at $t = 0$ and substitution of these values into Eq. 2.19, a series expansion for $\mathbf{z}(t)$ is obtained, namely,

$$\mathbf{z}(t) = \left(\mathbf{1} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \ldots + \frac{1}{k!}\mathbf{A}^kt^k + \ldots\right)\mathbf{z}_0 \tag{2.20}$$

where $\mathbf{1}$ denotes the 2×2 identity matrix.

The series inside the parentheses multiplying $\mathbf{z}_0$ is readily identified as the exponential[3] of $\mathbf{A}t$, and hence,

$$\mathbf{z}(t) = e^{\mathbf{A}t}\mathbf{z}_0, \quad t \geq 0 \tag{2.21}$$

which is the expression sought.

The problem of computing the time response of the given system has thus been reduced to computing the exponential of $\mathbf{A}t$. This computation can be done in many ways, 19 of which were discussed by Moler and Van Loan [1], but there are many more. Conceptually, the simplest way is via the Cayley-Hamilton Theorem, as discussed in Appendix A. The exponential of $\mathbf{A}t$, as computed therein, is given below. We distinguish here three cases, namely,

1. Underdamped case: $\zeta < 1$. Here we have

$$e^{\mathbf{A}t} = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\begin{bmatrix} \sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t & (\sin\omega_d t)/\omega_n \\ -\omega_n\sin\omega_d t & \sqrt{1-\zeta^2}\cos\omega_d t - \zeta\sin\omega_d t \end{bmatrix} \tag{2.22a}$$

[3]The exponential of a matrix is formally identical to that of a real argument: both have the same expansion.

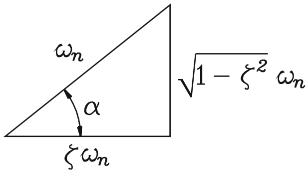


Fig. 2.10 Relation among ω_n, ω_d and ζ

where ω_d, termed the *damped natural frequency*, is defined as

$$\omega_d \equiv \omega_n\sqrt{1-\zeta^2} \tag{2.22b}$$

and hence, the time response of underdamped systems is given by the two components of $\mathbf{z}(t)$, $x(t)$ and $\dot{x}(t)$:

$$x(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right)x_0 + \frac{e^{-\zeta\omega_n t}}{\omega_d}(\sin\omega_d t)v_0 \tag{2.23a}$$

$$\dot{x}(t) = -\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sin\omega_d t)x_0 + \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t - \zeta\sin\omega_d t\right)v_0 \tag{2.23b}$$

The relation among ω_n, ω_d and ζ is best illustrated in Fig. 2.10, whence it is apparent that $\omega_d < \omega_n$.

2. Critically damped case: $\zeta = 1$. Now we have

$$e^{\mathbf{A}t} = e^{-\omega_n t}\begin{bmatrix} 1+\omega_n t & t \\ -\omega_n^2 t & 1-\omega_n t \end{bmatrix} \tag{2.24a}$$

and hence, the time response of critically damped systems is

$$x(t) = e^{-\omega_n t}[(1+\omega_n t)x_0 + v_0 t] \tag{2.24b}$$

$$\dot{x}(t) = e^{-\omega_n t}[-\omega_n^2 t x_0 + (1-\omega_n t)v_0] \tag{2.24c}$$

3. Overdamped case: $\zeta > 1$. In this case,

$$e^{\mathbf{A}t} = \frac{e^{-\zeta\omega_n t}}{r}\begin{bmatrix} r\cosh(r\omega_n t) + \zeta\sinh(r\omega_n t) & \frac{1}{\omega_n}\sinh(r\omega_n t) \\ -\omega_n\sinh(r\omega_n t) & r\cosh(r\omega_n t) - \zeta\sinh(r\omega_n t) \end{bmatrix} \tag{2.25a}$$

where $r \equiv \sqrt{\zeta^2 - 1}$. Hence, the time response of overdamped systems is

$$x(t) = \frac{e^{-\zeta\omega_n t}}{r}\left\{[r\cosh(r\omega_n t) + \zeta\sinh(r\omega_n t)]x_0 + \frac{1}{\omega_n}\sinh(r\omega_n t)v_0\right\} \tag{2.25b}$$

$$\dot{x}(t) = \frac{e^{-\zeta\omega_n t}}{r}\left\{-\omega_n\sinh(r\omega_n t)x_0 + \frac{e^{-\zeta\omega_n t}}{r}[r\cosh(r\omega_n t) - \zeta\sinh(r\omega_n t)]v_0\right\} \tag{2.25c}$$

In all three above cases it is apparent that the response is a linear combination of the two initial conditions x_0 and v_0, such as in the undamped case. Contrary to the undamped case, in the case of damped systems we have one more parameter that comes into the picture, namely, the damping ratio ζ, which brings about another parameter, the damped frequency ω_d of underdamped systems. Shown in Figs. 2.11a, c and e are plots of time responses of underdamped, critically damped and overdamped systems, respectively, for various values of damping ratio, with $x_0 = 1$ and $v_0 = 0$. Figures 2.11b, d and f show the same responses for initial conditions $x_0 = 0$ and $v_0 = 1$. In all cases, x_0 has units of displacement, whether translational or angular, while v_0 has units of the corresponding velocity.

2.3.2.1 Identification of Damping from the Time Response

Knowing the time response of an underdamped system allows us to determine its damping ratio, as we will show presently. Let us take the time response of Fig. 2.11b for $x_0 = 0$ and $v_0 = 1$, and measure the displacements x_k and x_{k+1} at two different instants t_k and t_{k+1}, respectively, separated by a full period T, i.e., $t_{k+1} = t_k + T$, where $T \equiv 2\pi/\omega_d$. We then have, from expression (2.23a),

$$\frac{x_k}{x_{k+1}} = \frac{e^{-\zeta\omega_n t_k}\sin\omega_d t_k}{e^{-\zeta\omega_n t_{k+1}}\sin\omega_d t_{k+1}}$$

However, by virtue of the relationship between t_k and t_{k+1},

$$\sin\omega_d t_{k+1} = \sin(\omega_d t_k + 2\pi) \equiv \sin\omega_d t_k$$

and hence, the foregoing ratio becomes

$$\frac{x_k}{x_{k+1}} = \frac{e^{-\zeta\omega_n t_k}}{e^{-\zeta\omega_n t_{k+1}}} \equiv e^{\zeta\omega_n T}$$

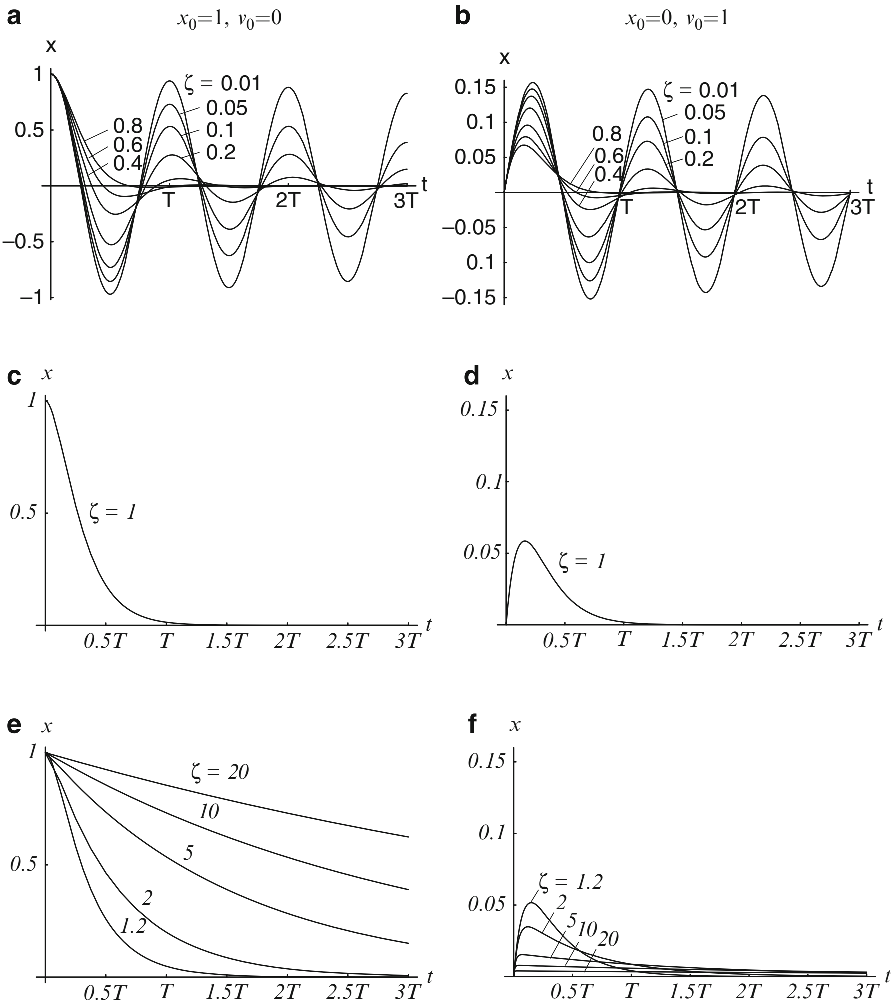


Fig. 2.11 Zero-input response of various second-order systems for various initial conditions

Now, if we take the logarithm of both sides of the foregoing equation, we obtain an interesting relation, namely,

$$\ln\left(\frac{x_k}{x_{k+1}}\right) = \zeta\omega_n T \equiv \frac{2\pi\zeta\omega_n}{\omega_d} \equiv 2\pi\frac{\zeta}{\sqrt{1-\zeta^2}}$$

The left-hand side of the latter expression is known as the *logarithmic decrement* of the motion and is denoted by δ. Note that it is a constant, regardless of the value

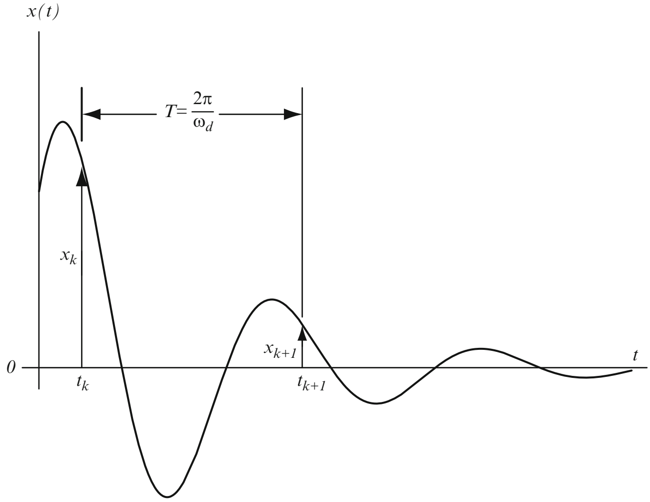


Fig. 2.12 An illustration of the logarithmic decrement

of t_k, and depends solely on the damping ratio ζ, for the natural frequency ω_n is eliminated because $\omega_d T = 2\pi$, and hence,

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{2.26}$$

The ratio x_k/x_{k+1}, from which the logarithmic decrement is computed, is illustrated in Fig. 2.12.

Therefore, knowing the logarithmic decrement from a simple measurement on the time-response plot, we can determine the damping ratio as

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}} \tag{2.27}$$

Note that, for small values of the damping ratio, the denominator of Eq. 2.26 reduces to unity, and hence, the logarithmic decrement and the damping ratio obey a linear relation, namely,

$$\text{For} \quad \zeta \ll 1, \quad \zeta \approx \frac{\delta}{2\pi} \tag{2.28}$$

Finally, note that the above results do not depend on the initial conditions; we should thus be able to derive them with another set of initial conditions. Moreover, if the displacement is measured N cycles apart, it is not difficult to show that the logarithmic decrement can then be expressed as

$$\delta = \frac{1}{N}\ln\left(\frac{x_k}{x_{k+N}}\right) \tag{2.29}$$

The above derivation is left as an **exercise**.

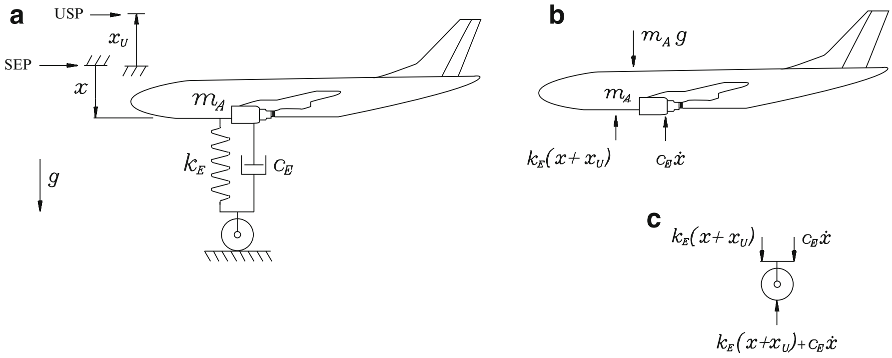


Fig. 2.13 A landing gear: (**a**) its iconic model (USP: unloaded spring position; SEP: static equilibrium position); (**b**) the FBD of the aircraft; and (**c**) the FBD of the landing gear

Example 2.3.2 (Design of a Landing Gear). If we assume that, upon landing, all wheels of an aircraft touch the landing strip simultaneously, and that the aircraft body is rigid, the system consisting of aircraft body and landing gear can be modeled as a mass-spring-dashpot system with a mass m_A accounting for the mass of the aircraft body plus the payload, an equivalent stiffness k_E and an equivalent damping coefficient c_E, as shown in Fig. 2.13a.[4] For an aircraft weighing 1,000 kN, find the equivalent stiffness and the equivalent damping coefficient that, upon landing, will produce oscillations of the body

(a) of 1 Hz and with an amplitude of its second cycle of 5% that of the first cycle.
(b) Moreover, determine the deflection of the landing gear when the oscillations have settled and the system is in equilibrium.
(c) As well, determine the peak force exerted on the ground upon landing, if the vertical approach velocity is of 10 m/s (this value is about 2.5 times that of a rough landing, but we want to design for extreme conditions).

Solution: To begin with, we set up the mathematical model of the system at hand, with the generalized coordinate x measured from the static equilibrium position, the value of x when the spring is unstretched being denoted by x_U, as shown in Fig. 2.13a. The mathematical model thus becomes

$$m_A\ddot{x} + c_E\dot{x} + k_E x = 0, \quad x(0) = x_0, \quad \dot{x}(0) = v_0$$

Now, from the second design requirement, we can readily determine the logarithmic decrement δ by application of relation (2.26), namely,

$$\delta = \ln\left(\frac{100}{5}\right) = 2.9957$$

[4]For this simple model, the wheels are assumed to be massless, rigid disks.

With this value we can now compute the damping ratio ζ from relation (2.27), which yields $\zeta = 0.4304$. Furthermore, knowing the damped frequency and the damping ratio, the natural frequency is readily computed from Eq. 2.22b, namely,

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{2\pi}{0.9026} = 6.9609 \text{ s}^{-1}$$

and hence, from the data and Eqs. 1.56a or 1.58b,

$$k_E = m_A \omega_n^2 = \frac{1000}{9.81} \times 6.9609^2 \text{ kN/m} = 4\,939 \text{ kN/m}$$

and

$$c_E = 2\zeta m_A \omega_n = 2 \times 0.4304 \times \frac{1000}{9.81} \times 6.9609 = 610.8 \text{ kNs/m}$$

thereby determining the design requirements for the overall landing-gear system.[5]

Further, the deflection of the landing-gear system at its equilibrium state is equal to that of the mass-spring-dashpot with which it is modeled. This deflection, denoted x_U, can be readily computed from the relation

$$k_E x_U = m_A g$$

and hence,

$$x_U = \frac{m_A g}{k_E} = \frac{1000}{4\,939} \text{ m} = 0.2024 \text{ m}$$

Now we proceed to determine the peak force transmitted by the landing aircraft to the ground. To this end, we note that the force transmitted by the spring-dashpot array to the mass is equal to that transmitted to the ground, as illustrated in Figs. 2.13b, c.

Thus, if we let $f_T(t)$ denote the transmitted force, we have

$$f_T(t) = c_E \dot{x}(t) + k_E(x + x_U) = c_E \dot{x} + k_E x + m_A g$$

The maximum force, occurring at a time t_M, as yet to be determined, is found upon zeroing the derivative of $f_T(t)$ with respect to time. This derivative is calculated below:

$$\dot{f}_T(t) = c_E \ddot{x} + k_E \dot{x}(t)$$

[5]Aircraft landing gears, like the suspension of terrestrial vehicles are designed with a natural frequency of around 1 Hz.

Upon solving for $\ddot{x}$ from the mathematical model and inserting the result into the foregoing equation, we have, after simplification,

$$\dot{f}_T(t) = m_A \left[\left(\frac{k_E}{m_A} - \frac{c_E^2}{m_A^2} \right) \dot{x} - \frac{c_E}{m_A} \omega_n^2 x \right]$$

Now, if we recall definitions (1.58b), the zeroing of the foregoing expression then leads to

$$(1 - 4\zeta^2)\dot{x} - 2\zeta\omega_n x = 0$$

which is satisfied at time $t = t_M$.

Next, we resort to the time response obtained above, i.e., Eqs. 2.23a, b, and substitute the expressions for $x(t)$ and $\dot{x}(t)$ of those relations into the foregoing equations, which yields, after simplifications,

$$D \sin \omega_d t_M = -\sqrt{1 - \zeta^2} N \cos \omega_d t_M$$

with D and N defined as

$$D \equiv (1 - 2\zeta^2)\omega_n x_0 + \zeta(3 - 4\zeta^2) v_0$$
$$N \equiv 2\zeta\omega_n x_0 - (1 - 4\zeta^2) v_0$$

and hence, t_M is determined as

$$t_M = \frac{1}{\omega_d} \tan^{-1} \left(-\sqrt{1 - \zeta^2} \frac{N}{D} \right)$$

and the maximum transmitted force, f_M, is found as

$$f_M \equiv f_T(t_M) = c_E \dot{x}(t_M) + k_E x(t_M) + m_A g$$

Upon substitution of the relation between x and $\dot{x}$ at $t = t_M$ found above, into the foregoing expression, we obtain

$$f_M = m_A \left(\frac{4\zeta^2}{1 - 4\zeta^2} + 1 \right) \omega_n^2 x(t_M) + m_A g$$

which readily simplifies to

$$f_M = \frac{m_A \omega_n^2}{1 - 4\zeta^2} x(t_M) + m_A g$$

This expression is valid for $1 - 4\zeta^2 \neq 0$. The case $1 - 4\zeta^2 = 0$ is left as an **exercise**.

All we need now is $x(t_M)$, which we evaluate below. We first note that

$$x(t_M) = \frac{e^{-\zeta\omega_n t_M}}{\sqrt{1-\zeta^2}}\left[\left(\sqrt{1-\zeta^2}\cos\omega_d t_M + \zeta\sin\omega_d t_M\right)x_0 + \frac{\sin\omega_d t_M}{\omega_n}v_0\right]$$

Next, we substitute the relation between $\cos\omega_d t_M$ and $\sin\omega_d t_M$ obtained above, which yields

$$x(t_M) = \frac{e^{-\zeta\omega_n t_M}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}(\cos\omega_d t_M)x_0 - \left(\zeta x_0 + \frac{v_0}{\omega_n}\right)\cos\omega_d t_M\right]$$

Upon simplification,

$$x(t_M) = e^{-\zeta\omega_n t_M}\frac{x_0 D - (\zeta x_0 + v_0/\omega_n)N}{D}\cos\omega_d t_M$$

where $\cos\omega_d t_M$ is derived from the value of $\tan\omega_d t_M$ obtained above, i.e.,

$$\cos\omega_d t_M = \frac{D}{\sqrt{(1-\zeta^2)N^2 + D^2}}, \quad 0 \le \omega_d t_M \le \pi$$

Therefore.

$$x(t_M) = e^{-\zeta\omega_n t_M}\frac{x_0 D - (\zeta x_0 + v_0/\omega_n)N}{\sqrt{(1-\zeta^2)N^2 + D^2}}$$

Thus, all we need now to compute $x(t_M)$ is the initial conditions x_0 and v_0. The value of x_0 is the length of the spring upon touching down; at this instant, the spring is stretched a small amount due to the weight of the landing gear, which is not known. However, if we realize that the latter is negligible when compared to the weight of the aircraft body, we can then safely assume that the spring is unloaded upon touching down, and hence,

$$x_0 = -x_U = -0.2024 \text{ m}$$

Further, v_0 is the vertical component of the aircraft velocity upon touching down, i.e.,

$$v_0 = 10 \text{ m/s}$$

which thus yields $x(t_M) = 0.2956$ m, and hence, the peak force is given by

$$f_M = \frac{m_A\omega_n^2}{1-4\zeta^2}x(t_M) + m_A g = \frac{1000\times 6.9609^2}{g(1-4\times 0.4304^2)}\times(0.2956) + 1000 = 6{,}636 \text{ kN}$$

Note that the maximum force exerted by the landing gear on the ground is about seven times the weight of the aircraft, which indicates that dynamical effects are

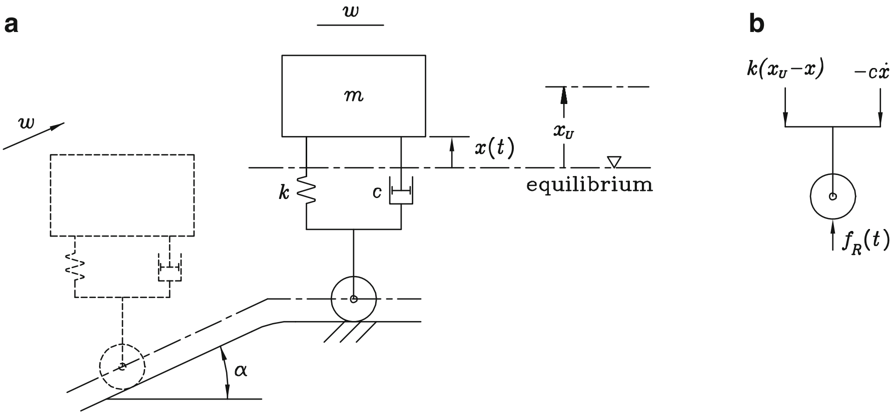


Fig. 2.14 The iconic model of a terrestrial vehicle: (**a**) upon climbing up a slope; and (**b**) the free-body diagram of its suspension

of the utmost importance both for the mechanical engineer designing landing gears and the fuselage structure, and for the civil engineer designing the landing strip.

Example 2.3.3 (Suspension Analysis). A rather crude iconic model of a terrestrial vehicle is shown in Fig. 2.14a, upon overcoming a slope of angle α, assumed to be "small," at a uniform speed w. (a) Determine the force transmitted to the ground *just after* the vehicle has overcome the slope and finds itself on level ground, in terms of the given parameters; (b) for a slope of 2%, $\zeta = \sqrt{2}/2$, and $\omega_n = 1$ Hz, find the speed w that will cause the vehicle "to fly" upon reaching the top of the slope; and (c) for the given numerical values, find the maximum and the minimum values of the force transmitted to the ground and the instants at which these extrema occur, during the first cycle of oscillations.

Solution:

(a) A free-body diagram of the lower part of the suspension is shown in Fig. 2.14b, from which the force $f_T(t)$ transmitted to the ground can be expressed as the negative of the reaction force $f_R(t)$ on the vehicle. Moreover, such as in Example 2.3.2, x_U denotes the value of x when the spring is unloaded, and hence,

$$x_U = \frac{m}{k} g = \frac{g}{\omega_n^2}$$

Therefore,

$$f_R(t) = -f_T(t) = -(kx + c\dot{x}) + mg \equiv -m\left(\omega_n^2 x + 2\zeta\omega_n\dot{x} - g\right)$$

For convenience, we work with $f_R(t)$, which is positive upwards, and hence, this function is non-negative.

Now, since we measure $x(t)$ from the equilibrium configuration, the mathematical model of the system takes the form

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=0, \quad t\geq 0, \quad x(0)=0, \quad \dot{x}(0)=w\alpha$$

where we have approximated $\sin\alpha$ by its argument, by virtue of the small[6] value of α. Therefore, the time response of the system takes the form

$$\begin{aligned} x(t) &= \frac{e^{-\zeta\omega_n t}}{\omega_d}(\sin\omega_d t)w\alpha \\ \dot{x}(t) &= \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t-\zeta\sin\omega_d t\right)w\alpha \end{aligned}$$

Now, the force transmitted to the ground just after the vehicle has overcome the slope is

$$f_R(0)=-m\left[\omega_n^2x(0)+2\zeta\omega_n\dot{x}(0)-g\right]=-m(2\zeta\omega_n w\alpha-g)$$

(b) For the vehicle body "to fly," $f_R(0)$ must vanish, and hence, we must have

$$w=\frac{g}{2\zeta\omega_n\alpha}=\frac{9.81}{2(\sqrt{2}/2)\times 2\pi\times 0.02}=55\text{ m/s}$$

or 198.7 km/h, which is a rather unlikely speed in North-American roads, and hence, the design, defined by the values of ζ and ω_n, is suitable.

(c) In order to determine the maximum—or minimum for that matter—of the force transmitted to the ground, we must set $\dot{f}_R(t)=0$, i.e.,

$$\dot{f}_R(t)=-m(\omega_n^2\dot{x}+2\zeta\omega_n\ddot{x})=0$$

By taking into account the mathematical model of the system at hand into the above equation, we obtain

$$\dot{f}_R(t)=-m\left[\omega_n^2\dot{x}+2\zeta\omega_n\left(-2\zeta\omega_n\dot{x}-\omega_n^2x\right)\right]=0$$

or

$$\left(1-4\zeta^2\right)\dot{x}=2\zeta\omega_n x$$

[6] Roads are typically designed with slopes below 6%, although exceptionally roads with slopes of 10% and even 20% exist. The validity of the approximation $\sin x\approx x$ is limited by $\pi/30$ or $6°$, while a slope of 6% entails an angle $\alpha=3.4°$.

and, since $1-4\zeta^2 = 1-4(1/2) = -1 \neq 0$,

$$\dot{x} = \frac{2\zeta\omega_n x}{1-4\zeta^2}$$

If we now substitute the expressions for $x(t)$ and $\dot{x}(t)$ obtained above, into the foregoing expressions, we have

$$\sqrt{1-\zeta^2}\cos\omega_d t_0 - \zeta\sin\omega_d t_0 = \frac{2\zeta\omega_n}{1-4\zeta^2}\frac{\sin\omega_d t_0}{\omega_n}$$

where t_0 is the instant at which a stationary value of $f_R(t)$ occurs. After simplifications,

$$\tan\omega_d t_0 = \frac{\sqrt{1-\zeta^2}(1-4\zeta^2)}{(3-4\zeta^2)\zeta}$$

For the given numerical value of ζ, $\sqrt{2}/2$, we obtain

$$\tan\omega_d t_0 = -1$$

and hence, stationary values, i.e., maxima and minima, of $f_R(t)$ occur at the values given below:

$$\omega_d t_0 = \frac{3\pi}{4}, \quad \frac{7\pi}{4}, \quad \frac{11\pi}{4}, \ldots, \text{etc.}$$

Now, in order to determine the instants corresponding to the foregoing angular values, we need ω_d, which is readily calculated as

$$\omega_d = \sqrt{1-\zeta^2}\omega_n = \frac{\sqrt{2}}{2}2\pi = \sqrt{2}\pi$$

Thus, $f_R(t)$ attains extremum values—local maxima and minima—at a sequence of values t_i given by

$$t_i = \frac{\sqrt{2}}{2}\left(\frac{3}{4}+i\right), \quad i = 1, 2, \ldots$$

or

$$t_1 = 0.5303 \text{ s}, \quad t_2 = 1.2374 \text{ s}, \quad t_3 = 1.9445 \text{ s}, \ldots, \quad \text{etc.}$$

Now, in order to determine whether a stationary value of $f_R(t)$ is a maximum or a minimum, or even a saddle point, we calculate $\ddot{f}_R(t)$ and investigate its sign. The general expression for the derivative of interest is

$$\ddot{f}_R(t) = -m\left[\left(\omega_n^2 - 4\zeta^2\omega_n^2\right)\ddot{x} - 2\zeta\omega_n^3\dot{x}\right]$$

Upon introducing the mathematical model into the above derivative, we obtain

$$\dddot{f}_R(t) = m\omega_n^3\left[4\zeta(1-2\zeta^2)\dot{x} + (1-4\zeta^2)\omega_n x\right]$$

Now, at a stationary value, i.e., at $t = t_i$, for $i = 1, 2, \ldots,$

$$(1-4\zeta^2)\dot{x}(t_i) = 2\zeta\omega_n x(t_i)$$

and hence,

$$\ddot{f}_R(t_i) = m\frac{\omega_n^4}{1-4\zeta^2}x(t_i) = -19.739x(t_i)$$

whence,

$$\operatorname{sgn}[\ddot{f}_R(t_i)] = -\operatorname{sgn}[x(t_i)]$$

Now we evaluate $x(t_1)$:

$$x(t_1) = x\left(\frac{3\pi}{4\omega_d}\right) = \frac{e^{-\zeta\omega_n t_1}}{\omega_d}\sin\left(\frac{3\pi}{4}\right)w\alpha = \frac{\sqrt{2}}{2}\frac{e^{-\zeta\omega_n t_1}}{\omega_d}w\alpha > 0$$

the sign of $\ddot{f}_R(t_1)$ being therefore negative, which means that $f_R(t_1)$ is a maximum, namely,

$$f_R(t_1) = -m\left[\omega_n^2 x(t_1) + 2\zeta\omega_n\dot{x}(t_1) - g\right]$$

and, if we take into account the relation between x and $\dot{x}$ at stationary values of $f_R(t)$, we obtain

$$f_R(t_1) = -m\left[\frac{\omega_n^2}{1-4\zeta^2}x(t_1) - g\right]$$

where $x(t_1)$ is given by

$$x(t_1) = \frac{e^{-\zeta\omega_n t_1}}{\omega_d}(\sin\omega_d t_1)w\alpha = \frac{e^{-\zeta\omega_d t_1/\sqrt{1-\zeta^2}}}{\omega_d}(\sin\omega_d t_1)w\alpha$$

whence,

$$x(t_1) = 3.0169\times 10^{-4}w$$

Therefore,

$$f_R(t_1) = -m\left(\frac{4\pi^2}{1-2}\times 3.0169\times 10^{-4}w - g\right) = m(0.1191w + g)$$

Now we investigate the sign of $x(t_2)$:

$$x(t_2) = \frac{e^{-\zeta\omega_n t_2}}{\omega_d} \sin\left(\frac{7\pi}{4}\right) w\alpha = -\frac{\sqrt{2}}{2}\frac{e^{-\zeta\omega_n t_2}}{\omega_d} w\alpha < 0$$

Therefore, $f_R(t_2)$ attains a minimum value given by

$$f_R(t_2) = -m\left[\omega_n^2 x(t_2) + 2\zeta\omega_n \dot{x}(t_2) - g\right]$$

Again, upon considering the relation between x and $\dot{x}$ at a stationary value of $f_R(t)$, we obtain

$$f_R(t_2) = m\left[\frac{\omega_n^2 x(t_2)}{4\zeta^2 - 1} + g\right]$$

with $x(t_2)$ readily calculated as

$$x(t_2) = -1.3037 \times 10^{-5} w$$

and hence,

$$f_R(t_2) = \max\{(-5.1468 \times 10^{-4} w + g), 0\}$$

where a provision has been taken in order to avoid negative values of the reaction force, which are physically inadmissible.

2.4 The Zero-State Response of LTIS

The response of a system to the sole action of external excitations, i.e., under the assumption that the system is at rest prior to the application of the excitations, is termed the *zero-state response*. This term indicates that the state of the system—position and velocity in mechanical systems—is equal to zero. In this section, we derive the zero-state response of the systems under study for an arbitrary excitation by resorting to the property of superposition of linear systems. To this end, we decompose the arbitrary input as a continuous train of amplitude-modulated impulses. Hence, we start by studying the response of the systems at hand to a *unit impulse*. It will become apparent that this form of excitation is the simplest one. The unit impulse, like its derivative and its integral, the doublet and the unit step, respectively, belong to a class of functions termed *discontinuous*. These functions pose interesting challenges to the analyst [2], that need not be discussed in an introductory book. These functions will be handled with due care in the balance of the book.

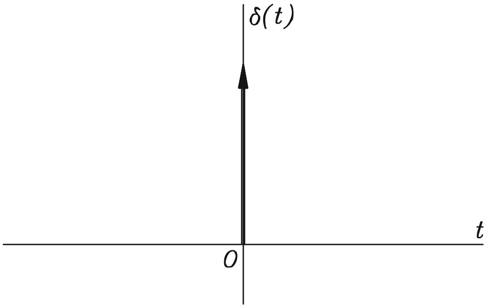


Fig. 2.15 Unit impulse applied at $t = 0$

2.4.1 The Unit Impulse

The unit impulse is defined as a function of time that is zero everywhere, except for an infinitesimally small neighborhood around the origin, in which the function attains unbounded values. However, its time integral from $-\infty$ to $+\infty$ is exactly $+1$. Thus, if we denote by 0^- and 0^+ the instants *just before* and *just after* 0, respectively, the unit-impulse function, represented as $\delta(t)$, is then defined as:

$$\delta(t) \begin{cases} = 0 & \text{for } t \leq 0^-; \\ \to \infty & \text{for } 0^- < t < 0^+; \\ = 0 & \text{for } t \geq 0^+ \end{cases} \tag{2.30}$$

and

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1 \tag{2.31}$$

the unity on the right-hand side of the last equation being dimensionless. As a consequence, the unit-impulse function has units of s^{-1}, i.e., of frequency. This function is also called the *delta function* or the *Dirac function*, and is represented as a vertical arrow at the origin, as in Fig. 2.15, of unit length in the scale adopted.

Note that, if the unit-impulse function is multiplied by a constant A to obtain $A\delta(t)$, it follows from Eq. 2.31 that its time integral from $-\infty$ to $+\infty$ is equal to A. We then say that $A\delta(t)$ is an *impulse function* of magnitude A, the "magnitude" being, in fact, the area under the impulse. Such a non-unit impulse is thus represented as an arrow of height A in the scale adopted. The height of the arrow, then, denotes the time integral of the associated impulse function on the whole real axis.

2.4.2 The Unit Doublet

The time derivative of a unit impulse, $\dot{\delta}(t)$, is called the *doublet* function. It is formally defined as

$$\dot{\delta}(t) = \frac{d}{dt}\delta(t) \tag{2.32}$$

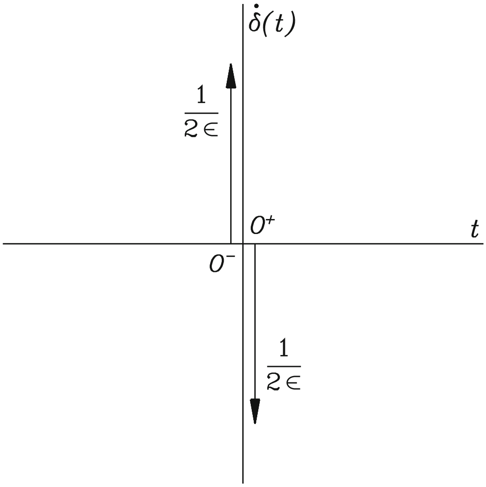


Fig. 2.16 Doublet applied at $t = 0$

Using the definition of the time-derivative, Eq. 2.32 can be written as

$$\dot{\delta}(t) = \lim_{\varepsilon \to 0} \frac{\delta(t+\varepsilon) - \delta(t-\varepsilon)}{2\varepsilon} = \lim_{\varepsilon \to 0} \left[\frac{1}{2\varepsilon}\delta(t+\varepsilon) - \frac{1}{2\varepsilon}\delta(t-\varepsilon) \right]$$

with ε being defined as

$$2\varepsilon \equiv 0^{+} - 0^{-}$$

Therefore, the physical interpretation of a doublet is the limiting case of two impulses of ∞ amplitude, one applied at $t = 0^{-}$ and the other at $t = 0^{+}$, the latter being the negative of the former. The doublet is sketched in Fig. 2.16.

The units of the doublet are, of course, s^{-2}, i.e., frequency-squared. As will be shown later, the doublet can be used to describe inputs that cause the position of the mass of a mass-spring-dashpot system to change instantaneously without undergoing any change in its velocity.

The *triplet*, the *quadruplet*, etc. are functions defined likewise. In general, the $(n-1)$st derivative of the unit impulse, denoted by $\delta^{(n-1)}(t)$, is termed the *n-tuplet* function. Of interest to us are mainly the impulse and the doublet functions.

2.4.3 The Unit Step

Further, the *unit-step* function, or *Heaviside function*, is defined below:

$$u(t) \equiv \begin{cases} 0, & \text{for} \quad t \le 0^{-}, \\ +1, & \text{for} \quad t \ge 0^{+} \end{cases} \tag{2.33}$$

This function is sketched in Fig. 2.17.

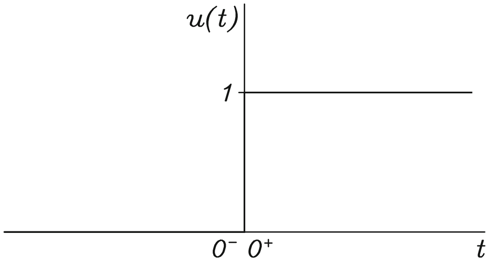


Fig. 2.17 Unit-step function applied at $t = 0$

Note that the unit-step function is undefined in the interval $0^- < t < 0^+$. This does not bother us, because the values of this function in that interval are never needed, except for the basic assumption that this function remains bounded everywhere, including that interval. The unit-step function is needed to represent *abrupt* changes of variables upon which a function jumps instantaneously from one value to another by a *finite* amount. This corresponds to physical situations such as a constant, finite force applied suddenly onto a mass, the sudden closing of a switch in a circuit driven by a battery, the sudden exposure of a body at a given temperature to a constant, finite temperature, different from that of the body, and so on.

The relationship between the unit-step and the unit-impulse functions is obviously

$$u(t) = \int_{-\infty}^{t} \delta(\theta) d\theta \tag{2.34}$$

where θ is a dummy variable of integration. Hence, the unit-step function is dimensionless. Furthermore,

$$\delta(t) = \frac{du(t)}{dt} \tag{2.35}$$

2.4.4 The Unit Ramp

One more function of interest is the *unit-ramp function*, $r(t)$, defined as

$$r(t) \equiv \begin{cases} 0, & \text{for} \quad t \leq 0^- \\ t, & \text{for} \quad t \geq 0^+. \end{cases} \tag{2.36}$$

The unit-ramp function is sketched in Fig. 2.18.

Note that $r(t)$ can be interpreted as the integral of $u(t)$, i.e.,

$$r(t) \equiv \int_{-\infty}^{t} u(\theta) d\theta \tag{2.37}$$

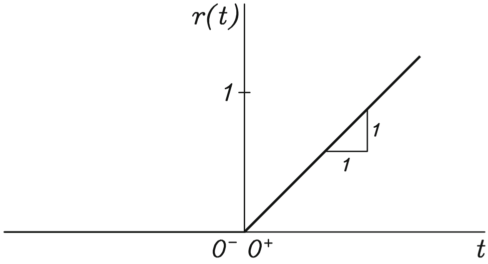


Fig. 2.18 Unit-ramp function applied at $t = 0$

with θ, again, being a dummy variable of integration. Moreover, we have the relations:

$$u(t) = \frac{dr(t)}{dt}, \quad \delta(t) = \frac{d^2 r(t)}{dt^2} \tag{2.38}$$

Obviously, the unit-ramp function has units of time, i.e., second.

We will now derive explicit expressions for the impulse responses of first- and second-order LTI dynamical systems.

2.4.5 The Impulse Response

In this subsection we consider the response of linear time-invariant dynamical systems to a unit impulse, the response at hand being termed the *impulse response*—of the system at hand, of course. The importance of the impulse response cannot be overstated. For example, knowing the impulse response of a LTIS we can obtain, by superposition, the response of the same system to *any* input, provided that the initial conditions are zero in all cases. On the other hand, the unit impulse models quite effectively inputs that have a very short duration but a very large amplitude and hence, their effect on the behavior of the system under study is not negligible. Examples of such inputs appear whenever collisions occur. For example, when hitting a ball with a tennis racket, the ball is acted upon by an infinitely large force during an infinitesimally small interval of time. As a consequence, the ball undergoes a finite change in its velocity that implies a finite change in its momentum. This finite change can be explained as the result of a finite impulse acting on the ball.

The impulse responses of first- and second-order LTI systems are derived in this order. After this, we show how, by exploiting the properties of *linearity* and *time-invariance* of the systems at hand, the impulse response of such systems can be used to determine the response of the same systems to *any* type of input by means of the *convolution integral*. The responses of first- and second-order LTI systems for any arbitrary input will then be derived.

2.4.5.1 First-order Systems

In this case, the associated differential equation is

$$\dot{x} = -ax + \delta(t), \quad t > 0^-, \quad x(0^-) = 0 \tag{2.39}$$

where we have assumed that the system is at rest prior to the application of the impulse. A simple way of computing the response of the foregoing system is by calculating its state $x(t)$ at $t = 0^+$ and rewriting Eq. 2.39 for $t \geq 0^+$, when the impulse function is zero, thereby reducing the response of the system to the zero-input response, which was already found. In order to find $x(0^+)$, Eq. 2.39 is rewritten in the form:

$$\dot{x} + ax = \delta(t), \quad t > 0^-, \quad x(0^-) = 0$$

The presence of a delta function on the right-hand side of the previous equation requires a delta function on the left-hand side. Now, let us look for this function there. It can be either in x or in $\dot{x}$. If it were in x, then $\dot{x}$ would be proportional to a doublet, which should be balanced by a doublet on the right-hand side. Since no such doublet is present there, this possibility is discarded. Then, the impulsive function must be in the $\dot{x}$ term, which implies that x is a multiple of the unit-step function, and hence, at the origin, x undergoes a finite jump. This means that, in the neighborhood of $t = 0$, x remains bounded. Now we have enough information on all terms appearing in Eq. 2.39 to perform their integral from $t = 0^-$ to $t = 0^+$, which is done below:

$$\int_{0^-}^{0^+} \dot{x}dt = -a\int_{0^-}^{0^+} xdt + \int_{0^-}^{0^+} \delta(t)dt \tag{2.40}$$

The integral of the left-hand side is readily identified as $x(0^+)$; the first integral appearing on the right-hand side vanishes, for $x(t)$, not containing any impulsive component, is finite in the neighborhood of $t = 0$. Finally, the last integral of that equation is simply $+1$ by definition, and hence, one has

$$x(0^+) = 1$$

Now, Eq. 2.39 can be rewritten as:

$$\dot{x} = -ax, \quad t \geq 0^+, \quad x(0^+) = 1 \tag{2.41}$$

with zero-input term and nonzero initial conditions. The response of the system appearing in Eq. 2.41 was already found in Sect. 2.2. If we now just replace 0 of Eq. 2.8 with 0^+, and recall the initial condition of the system at hand, we readily derive

$$x(t) = \begin{cases} 0, & \text{for} \quad t \leq 0^-; \\ e^{-at}, & \text{for} \quad t \geq 0^+ \end{cases} \tag{2.42a}$$

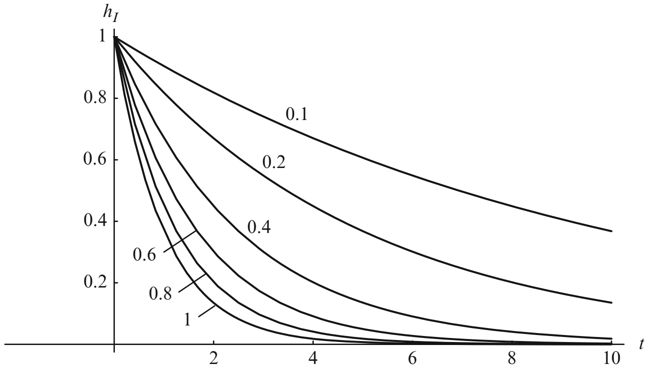


Fig. 2.19 Impulse response of a first-order system for various values of a

The response given in Eq. 2.42a is the impulse response of a first-order system; we shall denote this response by $h_I(t)$, and thus, in compact form,

$$h_I(t) \equiv e^{-at}\, u(t) \tag{2.42b}$$

which is sketched in Fig. 2.19, for various values of parameter a. Notice that, in the figure, $h_I(t) = 0$, for $t < 0^-$, an important feature of the response at hand that escapes to methods relying on math-book solutions to ODEs.

Note that the integral of the differential equation (2.39) given in Eq. 2.42a should verify (1) the differential equation and (2) the initial conditions. Below we show that it indeed verifies both. In fact, upon differentiation of $x(t)$ as given by Eq. 2.42b, we obtain

$$\dot{h}_I(t) = -ae^{-at}\, u(t) + e^{-at}\, \delta(t)$$

where relation (2.35) has been recalled. Now, since we are considering the response of the system for $t \geq 0^+$ only, at which $\delta(t) = 0$ and $u(t) = 1$, the foregoing expression readily reduces to

$$\dot{h}_I(t) = -ae^{-at} = -ah_I(t), \quad t \geq 0^+$$

thereby showing that the solution found verifies indeed the differential equation. Moreover,

$$h_I(0^+) = \underbrace{e^{-a0^+}}_{1}\underbrace{u(0^+)}_{1} = 1$$

and hence, h_I also verifies the initial conditions, the proposed response, Eq. 2.42b, thus being a valid solution to the *initial-value problem* (2.41), and hence to the original problem (2.39).

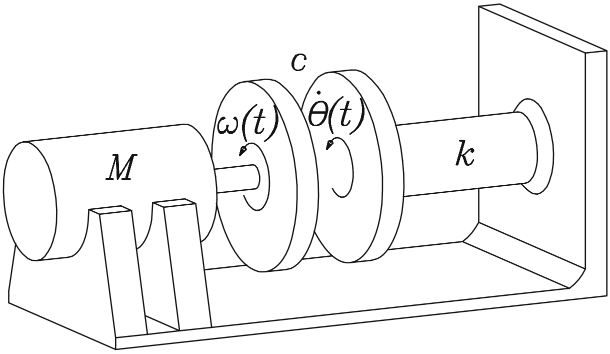


Fig. 2.20 Fluid clutch

Example 2.4.1 (Fluid-clutch Tests). Consider the fluid clutch of Example 1.6.10. The system is reproduced in Fig. 2.20 for quick reference. If the left-end disk is displaced suddenly through an angle ϕ_0, determine the ensuing motion of the right-end plate, $\theta(t)$. If we let $\phi(t)$ denote the angular displacement of the left-end plate, then $\omega(t) = \dot{\phi}(t)$.

Solution: The mathematical model of this system can be readily shown to be

$$\dot{\theta} + \frac{1}{\tau}\theta = \omega(t), \quad \theta(0^-) = 0, \quad \omega(t) = \dot{\phi}(t), \quad \phi(t) = \phi_0 u(t), \quad t > 0^-$$

where the time constant τ is defined as $\tau \equiv c/k$ and the input angle $\phi(t)$ has been modeled as a unit-step function. Hence,

$$\omega = \dot{\phi}(t) = \phi_0 \delta(t)$$

and the model now takes on the form

$$\dot{\theta} + \frac{1}{\tau}\theta = \phi_0 \delta(t), \quad \theta(0) = 0, \quad t > 0$$

In the next step, we transform the above model into zero-input form by expressing it as a homogeneous ODE with non-zero initial condition. To this end, we integrate both sides of the equation between 0^- and 0^+, while taking into account that $\theta(t)$ is not impulsive. We realize this by resorting to the arguments introduced above. From this integration we obtain

$$\theta(0^+) - \theta(0^-) = \phi_0$$

and, by virtue of the original initial condition, $\theta(0^-) = 0$ and hence, $\theta(0^+) = \phi_0$, the homogeneous ODE thus taking the form

$$\dot{\theta} + \frac{1}{\tau}\theta = 0, \quad \theta(0^+) = \phi_0, \quad t > 0^+$$

Therefore, the time response sought is

$$\theta(t) = \phi_0 e^{-t/\tau}, \quad t > 0^+$$

2.4.5.2 Second-order Undamped Systems

In this case, the governing equation is

$$\ddot{x} + \omega_n^2 x = \delta(t), \quad x(0^-) = 0, \quad \dot{x}(0^-) = 0, \quad t > 0^- \tag{2.42c}$$

As in the case of first-order systems, the impulse on the right-hand side must be balanced with an impulse on the left-hand side. Moreover, this impulse cannot lie in the x term. If it did, then $\dot{x}$ would necessarily contain a doublet and $\ddot{x}$ a *triplet*, which should be balanced in the right-hand side. Since neither triplet nor doublet appear in that side, x is not impulsive, and hence, is bounded at the origin. The only impulsive term of the LHS[7] of Eq. 2.42c is, then, the first term. Next, we integrate both sides of the same equation between 0^- and 0^+, thereby obtaining

$$\int_{0^-}^{0^+} \ddot{x} dt + \omega_n^2 \int_{0^-}^{0^+} x dt = \int_{0^-}^{0^+} \delta(t) dt \tag{2.42d}$$

The first integral of the left-hand side of Eq. 2.42d yields, as in the first-order case,

$$\int_{0^-}^{0^+} \ddot{x} dt = \dot{x}(0^+) - \dot{x}(0^-) = \dot{x}(0^+) \tag{2.42e}$$

the second integral vanishing because its integrand is finite at the origin and the integration is performed over an infinitesimally small time-interval around the origin. Finally, the integral of the right-hand side yields 1. Thus, Eq. 2.42d reduces to

$$\dot{x}(0^+) = 1$$

Moreover, since $\dot{x}(t)$ is bounded at the origin, $x(t)$ does not undergo any finite jump at the origin, and hence,

$$x(0^+) = x(0^-) = 0$$

Thus, Eq. 2.42c can be rewritten as

$$\ddot{x} + \omega_n^2 x = 0, \quad x(0^+) = 0, \quad \dot{x}(0^+) = 1, \quad t \geq 0^+$$

[7] Abbreviation of *left-hand side*.

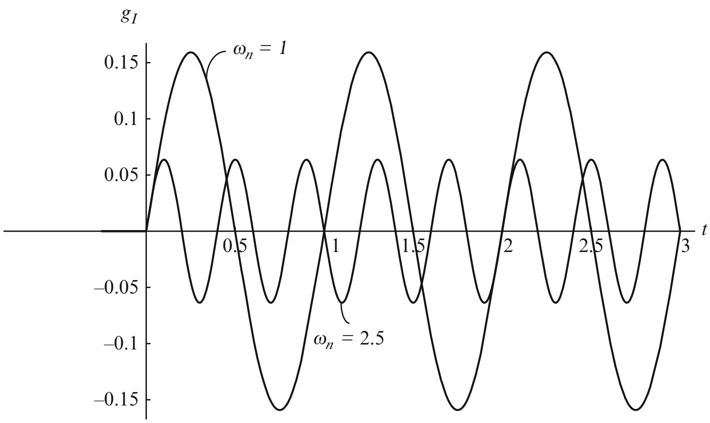


Fig. 2.21 Impulse response of a second-order undamped system

Therefore, the time response of the system under study can be obtained from the general expression, Eq. 2.13, as

$$x(t) = \begin{cases} 0, & \text{for} \quad t \leq 0^-; \\ (\sin \omega_n t)/\omega_n, & \text{for} \quad t \geq 0^+ \end{cases}$$

The impulse response of the given undamped second-order system is represented henceforth by $g_I(t)$, i.e., in compact form,

$$g_I(t) \equiv \frac{1}{\omega_n}(\sin \omega_n t)u(t) \tag{2.42f}$$

which is plotted in Fig. 2.21, for $\omega_n = 1\ \text{s}^{-1}$ and $\omega_n = 2.5\ \text{s}^{-1}$. Again, notice that the foregoing time response vanishes for values of $t \leq 0^-$.

2.4.5.3 Second-order Damped Systems

In this case, the governing equation is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \delta(t), \quad x(0^-) = 0, \quad \dot{x}(0^-) = 0, \quad t > 0^- \tag{2.42g}$$

As in the previous case, the impulse on the right-hand side must be balanced with an impulse on the left-hand side. Moreover, this impulse can neither lie in the x nor in the $\dot{x}$ term. If it lied, say, in the latter, then $\ddot{x}$ would necessarily contain a doublet, which should be balanced in the right-hand side. Since no doublet appears in that side, $\dot{x}$ and, consequently, x do not contain impulsive functions, and are, hence, bounded at the origin. Next, we integrate both sides of Eq. 2.42g between 0^- and 0^+, thereby obtaining

$$\int_{0^-}^{0^+} \ddot{x}dt + 2\zeta\omega_n \int_{0^-}^{0^+} \dot{x}dt + \omega_n^2 \int_{0^-}^{0^+} xdt = \int_{0^-}^{0^+} \delta(t)dt \tag{2.42h}$$

The first integral of the left-hand side of Eq. 2.42h is readily evaluated, namely,

$$\int_{0^-}^{0^+} \ddot{x}dt = \dot{x}(0^+) - \dot{x}(0^-) = \dot{x}(0^+)$$

where the given initial conditions of Eq. 2.42g were taken into account.

Moreover, the second and the third integrals of the left-hand side of Eq. 2.42h vanish because their integrands are finite at the origin and the integration is performed over an infinitesimally small time-interval around the origin. Finally, the integral of the right-hand side yields 1, Eq. 2.42h thus reducing to

$$\dot{x}(0^+) = 1$$

Furthermore, since $\dot{x}(t)$ is bounded at the origin, $x(t)$ does not undergo any finite jump at the origin, and hence,

$$x(0^+) = x(0^-) = 0$$

Thus, Eq. 2.42g can be rewritten as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \quad x(0^+) = 0, \quad \dot{x}(0^+) = 1, \quad t \geq 0^+$$

Therefore, the time response of the system under study can be obtained from the general expressions derived in Sect. 2.3.2. For *underdamped systems*, for example, we have, from Eq. 2.23a,

$$x(t) = \begin{cases} 0, & \text{for} \quad t \leq 0^-; \\ (e^{-\zeta\omega_n t} \sin\omega_d t)/\omega_d, & \text{for} \quad t \geq 0^+ \end{cases}$$

By analogy with the previous cases, the response of the underdamped second-order system to a unit impulse is termed the *impulse response* of that system, and is denoted by $j_I(t)$, i.e., in compact form,

$$j_I(t) \equiv \frac{e^{-\zeta\omega_n t}}{\omega_d}(\sin\omega_d t)u(t) \tag{2.42i}$$

The impulse response derived above is represented graphically in Fig. 2.22 for various values of ζ, with ω_n fixed. Needless to say, $j_I(t) = 0$, for $t \leq 0^-$.

The impulse responses $k_I(t)$ and $l_I(t)$ of critically damped and overdamped systems, respectively, are derived likewise, the details of their derivation being left

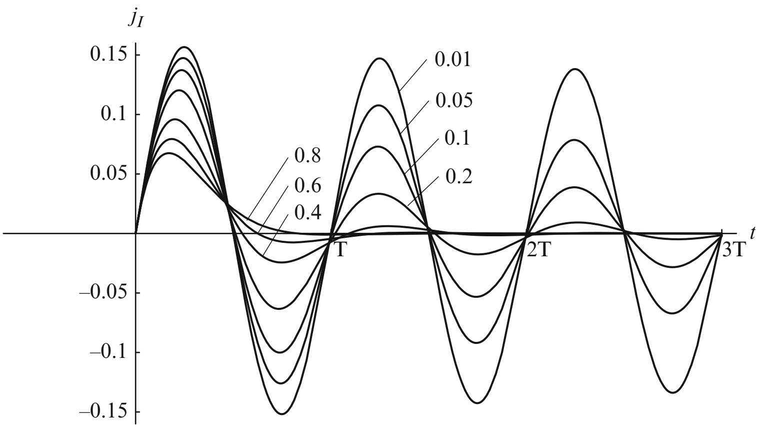


Fig. 2.22 Impulse response of a second-order underdamped system for various values of ζ

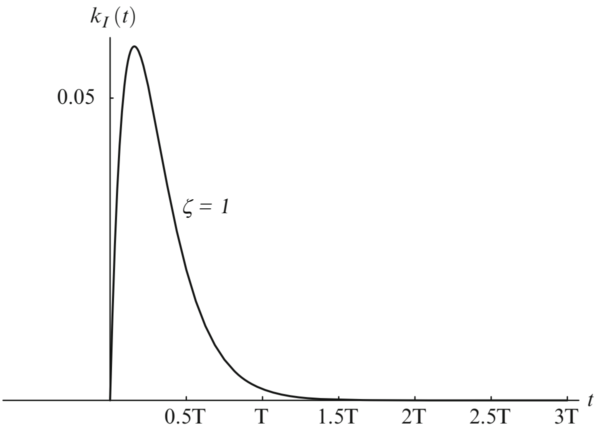


Fig. 2.23 Impulse response of a second-order critically damped system

as an exercise to the reader. These responses are displayed in Figs. 2.23 and 2.24, respectively, the latter for various values of ζ, expressions for these being included in Eqs. 2.91 and 2.92a. It goes without saying that $k_I(t) = 0$ and $l_I(t) = 0$, for $t \le 0^-$.

Example 2.4.2 (Damping Identification from the Impulse Response). A test pad[8] is modeled as shown in Fig. 2.25. In this model, the steel pad of mass m_1 is mounted on relatively soft springs. The light damping of the springs, as yet unknown but assumed to be linear, is represented in the model by the dashpot. In order to determine the physical parameters of the pad-suspension system, it has been

[8]Taken, with some modifications, from [3].

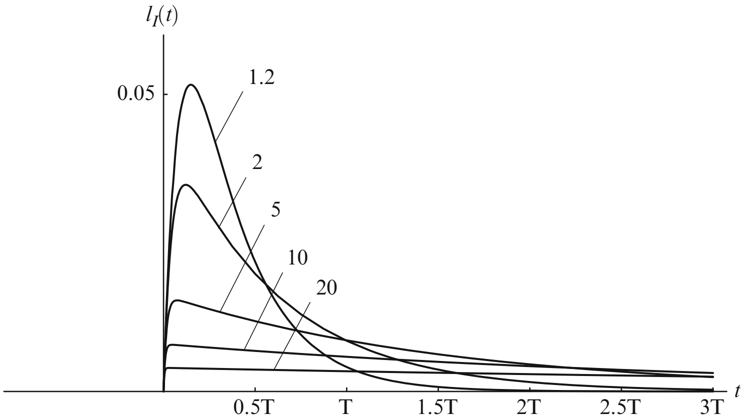


Fig. 2.24 Impulse response of a second-order overdamped system

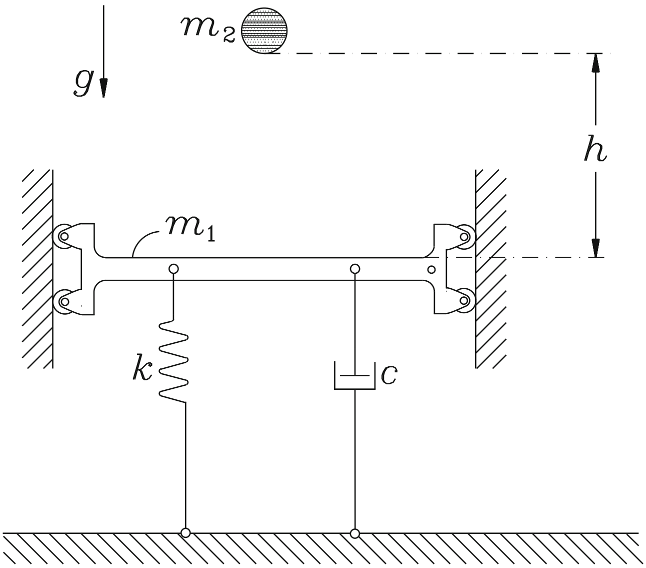


Fig. 2.25 Test pad

proposed to record its impulse response. To this end, a steel ball of mass m_2 is dropped onto the center of the pad from a height h, and then caught by an observer on the first bounce. The ensuing motion of the pad is then recorded, as appearing in Fig. 2.26. Under the assumption that the collision of the ball with the pad is perfectly elastic, (a) estimate, from the plot of this figure, the natural frequency and the damping ratio of the system, and (b) calculate the maximum excursion of the pad in its first oscillation. It is known that the ratio m_2/m_1 is 0.01 and $h = 1$ m.

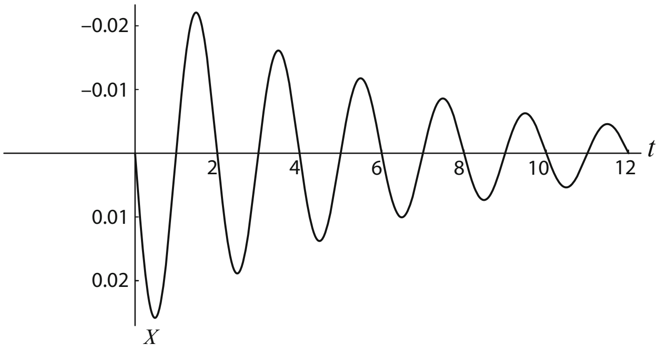


Fig. 2.26 Impulse response of the test pad

Solution:

(a) Let

$$v_{\mathrm{rel}}(t) \equiv v_2(t) - v_1(t)$$

be the relative velocity of the ball with respect to the pad, for every time $t \geq 0$. Moreover, we recall that, for a perfectly elastic shock, we have

$$v_{\mathrm{rel}}(0^+) = -v_{\mathrm{rel}}(0^-)$$

The relative velocity just before the impact is equal to the velocity of the ball, i.e.,

$$v_{\mathrm{rel}}(0^-) = v_2(0^-) - 0 = \sqrt{2gh}$$

with the positive direction defined downward. Thus,

$$v_{\mathrm{rel}}(0^+) \equiv v_2(0^+) - v_1(0^+) = -\sqrt{2gh} \tag{2.43}$$

Now, we note that the velocities of the ball and the pad undergo jumps upon colliding, but both remain finite. If we let j_d denote the impulse developed by the dashpot and j_s denote that developed by the spring, then

$$j_d = \int_{0^-}^{0^+} cv_1(t)dt = 0 \quad \text{because } v_1 \text{ is finite} \tag{2.44}$$

$$j_s = \int_{0^-}^{0^+} kx_1(t)dt = 0 \quad \text{because } x_1 \text{ is finite} x \tag{2.45}$$

so that the total impulse is zero. Thus, the total change in linear momentum, Δp, is zero as well, which means

$$\Delta p = m_1 v_1(0^+) + m_2 v_2(0^+) - m_2 v_2(0^-) = 0$$

from which we obtain, upon dividing all terms of the above equation by m_1,

$$v_1(0^+) + \alpha v_2(0^+) = \alpha\sqrt{2gh} \tag{2.46}$$

where $\alpha \equiv m_2/m_1 = 0.01$. Now, solving Eqs. 2.43 and 2.46 for $v_1(0^+)$ and $v_2(0^+)$ gives

$$v_1(0^+) = \frac{2\alpha}{1+\alpha}\sqrt{2gh}, \qquad v_2(0^+) = -\frac{1-\alpha}{1+\alpha}\sqrt{2gh}$$

Noting that $x_1(0^+) = 0$, the time response of Eq. 2.23a takes the form

$$x_1(t) = \frac{e^{-\zeta\omega_n t}}{\omega_d}(\sin\omega_d t)\, v_1(0^+)$$

Moreover,

$$v_1(0^+) = \frac{0.02}{1.01}\sqrt{2 \times 9.81 \times 1.0} = 0.08771 \text{ m/s}$$

From the plot of Fig. 2.26, we can readily find that the damped frequency ω_d of the system is 0.5 Hz. Now, the damping ratio ζ of the system can be found via the logarithmic decrement δ, which can be estimated from the same plot. Here, it is apparent that the system is, in fact, lightly damped, and so, the decrement of the amplitude of the oscillations from one cycle to the next one is very small. Measuring it from the plot, then, can lead to substantial error. In order to cope with this uncertainty, we measure the decrement through various cycles. From the figure, it is apparent that we have information of up to six cycles, and so, if we measure the first and the sixth negative peaks, we obtain $x_1 = 0.022$ m and $x_6 = 0.005$ m, which, when plugged into Eq. 2.29, with $k = 1$ and $N = 5$, yield

$$\delta = \frac{1}{5}\ln\left(\frac{x_1}{x_6}\right) = \frac{1}{5}\ln(4.4) = 0.296$$

For light damping, we have

$$\zeta \approx \frac{\delta}{2\pi} = 0.04711$$

Now, the natural frequency ω_n is simply obtained as

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{2\pi/T}{\sqrt{1-\zeta^2}} = \frac{2\pi/2}{\sqrt{1-0.04711^2}} = 3.1451 \text{ rad/s}$$

where, from inspection of the plot of Fig. 2.26, $T = 2$ s, thereby completing the identification of the parameters involved.

(b) Now, in order to calculate the maximum excursion of the pad at the first oscillation, all we need is calculate $x_1(t)$ at the instant t_M at which $\dot{x}_1$ first vanishes. From the above expression for $x_1(t)$, we can readily derive

$$\dot{x}_1(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t - \zeta\sin\omega_d t\right)v_1(0^+)$$

and hence, t_1 can be found as the instant at which the term inside the parentheses of the above expression vanishes, i.e., from

$$\sqrt{1-\zeta^2}\cos\omega_d t_1 - \zeta\sin\omega_d t_1 = 0$$

whence,

$$\tan\omega_d t_1 = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Upon substituting the foregoing value of ζ, 0.04711, into the above expression, we obtain

$$\tan\omega_d t_1 = 21.1843 \quad \Rightarrow \quad \omega_d t_1 = 87.2974^\circ = 1.5236 \text{ rad}$$

and hence, $\sin\omega_d t_1 = 0.9989$, which yields

$$\omega_n t_1 = \frac{\omega_d t_1}{\sqrt{1-\zeta^2}} = \frac{1.5236}{0.9989} = 1.5253 \text{ rad}$$

Therefore,

$$x_1(t_1) = x_{\max} = \frac{e^{-0.04711\times 1.5253}}{2\pi\times 0.5}\times 0.9989\times 0.08771 = 0.02595$$

with an accuracy that could not have been obtained from the plot, thereby completing the solution.

2.4.6 The Convolution (Duhamel) Integral

The *unit impulse* was shown above to be useful in the modeling of impulsive inputs of extremely short duration, but the usefulness of this concept goes far beyond that.

Specifically, we can use the impulse response, along with the properties of *linearity* and *time invariance* of LTI systems, to determine the response of these systems to *any* input *under zero initial conditions*. The time response thus obtained is what is known as the *zero-state response* of the system under analysis, also known as the forced response, as pointed out already in the Preamble.

As a matter of fact, if we have the impulse response of a LTIS, then we do not even have to know the system in detail in order to determine its zero-state response to any input. To illustrate this statement, we take a black-box approach; we shall thus see that the zero-state response of the system under study can be obtained as the *convolution*, to be defined presently, of the impulse response of the system with the input.

We shall resort below to a fundamental identity, that we shall prove in detail. Prior to this, we need the relation

$$f(t)\delta(t) \equiv f(0)\delta(t) \tag{2.47a}$$

which follows because the delta function is zero everywhere, except at the origin, where it attains an infinite value. Note, moreover, that, if the impulse is applied at $t = \tau > 0$, rather than at $t = 0$, then, Eq. 2.47 takes the form

$$f(t)\delta(t-\tau) \equiv f(\tau)\delta(t-\tau) \equiv f_\tau\delta(t-\tau) \tag{2.47b}$$

where f_τ is a constant, for fixed τ. Thus,

$$\int_0^t f(\tau)\delta(t-\tau)d\tau \equiv \int_0^t f(t)\delta(t-\tau)d\tau \equiv f(t)\int_0^t \delta(t-\tau)d\tau$$

the last identity following because $f(t)$ is independent from the integration variable, τ. Now we evaluate the integral appearing in the rightmost-hand side of the above equation. To do this, we let $\theta \equiv t-\tau$ and regard t as fixed, which thus leads to $d\theta = -d\tau$, and hence,

$$\int_0^t \delta(t-\tau)d\tau = -\int_t^0 \delta(\theta)d\theta \equiv \int_0^t \delta(\theta)d\theta$$

Furthermore, we recall relation (2.34) and the definition of $\delta(t)$, Eq. 2.30, the last integral thus yielding

$$\int_0^t \delta(\theta)d\theta \equiv \int_{-\infty}^t \delta(\theta)d\theta \equiv u(t)$$

But we are interested in positive values of t, for we are studying the response of the system to an excitation applied at $t = 0$ and observing its behavior thereafter. Thus, the unit-step function becomes 1, thereby proving that

$$f(t) \equiv \int_0^t f(\tau)\delta(t-\tau)d\tau \tag{2.48}$$

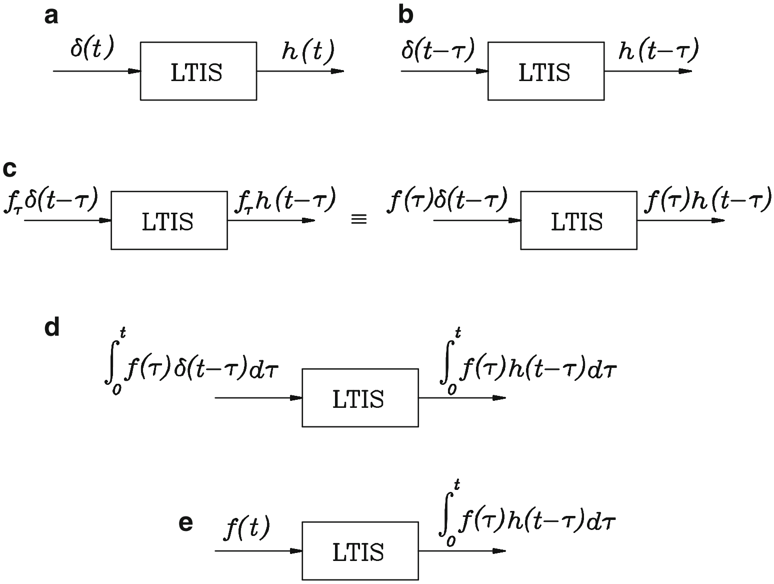


Fig. 2.27 (**a**) The impulse response of a LTIS; (**b**) the same response to a delayed impulse; (**c**) the response of the system to a modulated, delayed impulse; (**d**) the response of the system to a continuous train of modulated, delayed impulses; (**e**) the zero-state response as the convolution of the input with the impulse response of the system

which is the identity sought. i.e., $f(t)$ *can be regarded as a continuous train of impulses applied at instants* τ*, of amplitude* $f_\tau \equiv f(\tau)$*, for* $0 \le \tau \le t$.

Now, in order to obtain the response $x(t)$ of the system to an arbitrary input $f(t)$, we regard the latter as the integral appearing in the right-hand side of Eq. 2.48. Moreover, we denote the impulse response of the same system, which can be *any* system, as long as it is linear and time-invariant, by $h(t)$, as indicated with the block diagram of Fig. 2.27a. Now, by time-invariance, the response of the system to a delayed input $\delta(t-\tau)$ is a delayed impulse response $h(t-\tau)$, as shown in Fig. 2.27b. If, now, the delayed impulse is modulated with an amplitude f_τ, the corresponding response is, by homogeneity, $f_\tau h(t-\tau)$, as illustrated in Fig. 2.27c, with f_τ defined as in Eq. 2.47b. Furthermore, by additivity, the response of the system to a *continuous* train of delayed, modulated impulses, is, in turn, a *continuous* train of delayed, modulated impulse responses, both trains being represented as integrals between $\tau = 0$ and $\tau = t$, because of the continuity of the train, as illustrated in Fig. 2.27d. Finally, by virtue of the identity shown in Eq. 2.48, the train of inputs of Fig. 2.27d is readily identified as $f(t)$, thereby obtaining the input–output relation of Fig. 2.27e, namely,

$$x(t) = \int_0^t f(\tau)h(t-\tau)d\tau \tag{2.49}$$

The integral appearing in Eq. 2.49 is called the *convolution*, or the *Duhammel integral*, of the two functions, $f(t)$ and $h(t)$. More generally, given any two functions $\phi(t)$ and $\psi(t)$, their convolution, represented as $\phi(t) * \psi(t)$, is defined as

$$\phi(t) * \psi(t) \equiv \int_0^t \phi(\tau)\psi(t-\tau)d\tau \tag{2.50}$$

Important properties of the convolution are given below:

1. Commutativity:

$$\phi(t) * \psi(t) = \psi(t) * \phi(t) \tag{2.51a}$$

2. Distributivity:

$$\phi(t) * [g_1(t) + g_2(t)] = \phi(t) * g_1(t) + \phi(t) * g_2(t) \tag{2.51b}$$

3. Linear homogeneity:

$$\phi(t) * [\alpha\psi(t)] = \alpha\phi(t) * \psi(t) \tag{2.51c}$$

Note that commutativity implies the identity shown below, whose **proof** is left to the reader:

$$\int_0^t \phi(\tau)\psi(t-\tau)d\tau \equiv \int_0^t \phi(t-\tau)\psi(\tau)d\tau \tag{2.52}$$

distributivity and homogeneity following from the properties of the Riemann integral.

From the above equation, we can rewrite Eq. 2.49 alternatively as

$$x(t) = \int_0^t f(t-\tau)h(\tau)d\tau \tag{2.53}$$

In summary, we have seen that the response of a linear time-invariant system to an arbitrary input $f(t)$ is equal to the convolution of the input with the impulse response of the system. Thus, the response of a LTIS to *any input* can be determined from its impulse response. We can then say that the behavior of a LTI system is characterized completely by its impulse response.

Given the convolution theorem for a general LTIS, the responses of first- and second-order systems to an arbitrary input can be readily derived, as shown below.

2.4.6.1 First-order Systems

Inserting the expression for the impulse response $h_I(t)$ given in Eq. 2.42b into Eq. 2.53, we obtain

$$x(t) = \int_0^t f(t-\tau)e^{-a\tau}d\tau \tag{2.54}$$

2.4.6.2 Second-order Undamped Systems

Likewise, if we recall the expression for the impulse response $g_I(t)$ given in Eq. 2.42f, and substitute it into Eq. 2.53, we obtain

$$x(t) = \int_0^t f(t-\tau)\frac{1}{\omega_n}(\sin\omega_n\tau)\,d\tau \tag{2.55}$$

2.4.6.3 Second-order Damped Systems

We consider first the convolution of underdamped systems. To derive this convolution, we recall the expression for the pertinent impulse response $j_I(t)$, as given in Eq. 2.42i, and substitute it into Eq. 2.53, thus obtaining

$$x(t) = \int_0^t f(t-\tau)\frac{e^{-\zeta\omega_n\tau}}{\omega_d}(\sin\omega_d\tau)\,d\tau \tag{2.56}$$

the convolution for critically damped systems being derived likewise, i.e.,

$$x(t) = \int_0^t f(t-\tau)\tau e^{-\omega_n\tau}d\tau \tag{2.57}$$

Finally, the convolution for overdamped systems takes the form

$$x(t) = \int_0^t f(t-\tau)\frac{e^{-\zeta\omega_n\tau}}{\sqrt{1-\zeta^2}}\sinh\left(\sqrt{1-\zeta^2}\omega_n\tau\right)d\tau \tag{2.58}$$

Note that the foregoing results could have been obtained alternatively by inserting the appropriate impulse response function in Eq. 2.49.

2.5 Response to Abrupt and Impulsive Inputs

The step function models applications of *abrupt* inputs, such as the sudden closing of a circuit or the sudden arrest of a body undergoing a perfectly plastic collision with a stationary body of infinite mass, e.g., a speeding car hitting a wall.[9] The impulse function, on the other hand, models forces of a very short duration that nevertheless affect the state of the system due to the finite amount of energy transferred to or from the system in infinitesimal amounts of time. While we have already studied the response of first- and second-order LTI dynamical systems to

[9]Note that an abrupt change in the velocity implies an impulse in the acceleration.

unit impulses, the response to step inputs is yet to be studied. In addition, other situations occurring in real life, such as the sudden relocation of a heavy body, without a noticeable change in its velocity, such as a table driven by an indexing mechanism—a Geneva mechanism, for example—have not yet been studied. In this section, we shall determine the responses of the system of interest to various types of impulsive and abrupt inputs. These inputs can be used to model certain types of physical phenomena, as discussed earlier. The responses can be found using three different approaches:

- By using the convolution theorem
- By direct integration of the differential equation, with the appropriate input function inserted
- By making use of the properties illustrated in Figs. 2.4 and 2.5

We determine below the responses of first- and second-order systems to unit-doublet, unit-step, and unit-ramp inputs using either or both of the last two methods. The use of the convolution will be discussed in Sect. 2.7.

2.5.1 First-order Systems

2.5.1.1 Step Response

If the system under study is acted upon by a unit-step input, then Eq. 2.39 becomes

$$\dot{x} = -ax + u(t), \quad t \geq 0^-, \quad x(0^-) = 0 \tag{2.59}$$

By linearity, the response of the system under study to the unit step is the integral of the response of the same system to the unit impulse, since the former is the integral of the latter. We next denote the response to the unit step by $h_S(t)$, i.e.,

$$h_S(t) = \int_0^t h_I(\theta)d\theta \tag{2.60a}$$

where θ is a dummy variable of integration. Thus,

$$h_S(t) \equiv \begin{cases} 0, & t \leq 0^-; \\ (1 - e^{-at})/a, & t \geq 0^+ \end{cases} \tag{2.60b}$$

or, in a more compact form,

$$h_S(t) \equiv \frac{1 - e^{-at}}{a} u(t) \tag{2.60c}$$

which can be called the *step response* of the first-order system under study. This response is plotted in Fig. 2.28 for various values of positive a. Note that systems

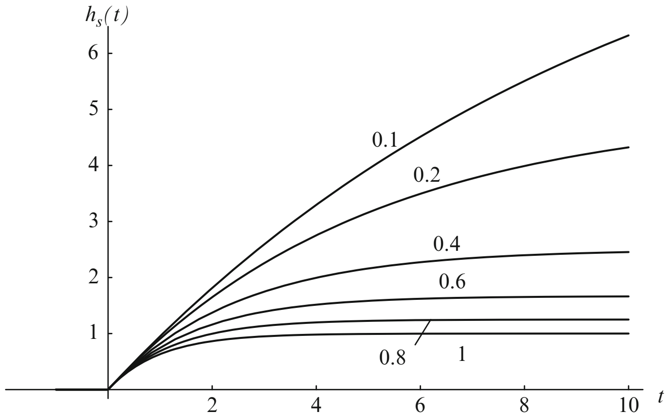


Fig. 2.28 Unit-step response of a first-order system for various positive values of a

with a small time constant are capable of following an input faster than those with a large time constant. That is, systems with a large time constant are slower than systems with a small one.

2.5.1.2 Ramp Response

The mathematical model now takes the form

$$\dot{x} = -ax + r(t), \quad t > 0^-, \quad x(0^-) = 0 \tag{2.61}$$

In Exercise 2.30 the reader is asked to derive the *ramp response* of the system at hand, denoted by $h_R(t)$, as shown below:

$$h_R(t) \equiv \frac{1}{a}\left[r(t) - \frac{1}{a}(1 - e^{-at})u(t)\right] \tag{2.62}$$

2.5.2 Second-order Undamped Systems

2.5.2.1 Doublet Response

In this case, the governing equation is

$$\ddot{x} + \omega_n^2 x = \dot{\delta}(t), \quad x(0^-) = 0, \quad \dot{x}(0^-) = 0, \quad t > 0^- \tag{2.63}$$

The doublet appearing in the right-hand side of Eq. 2.63 must be balanced with a corresponding term in the left-hand side, which cannot appear in the x-term. Indeed, if it did, then the $\dddot{x}$ term would comprise a quadruplet, which does not appear in the right-hand side. Thus, the doublet appears only in the $\ddot{x}$ term, and hence, the x term comprises a step function and is thus finite in a neighborhood around the origin. Now, both sides of Eq. 2.63 are integrated between $t = 0^-$ and $t > 0^-$, thus obtaining

$$\dot{x} + \omega_n^2 \int_{0^-}^{t} x d\tau = \delta(t), \quad x(0^-) = 0, \quad t > 0^- \tag{2.64a}$$

where the initial condition on $\dot{x}(t)$ is no longer needed, for the highest-order derivative appearing in the foregoing equation is $\dot{x}$. Next, the two sides of Eq. 2.64a are integrated between 0^- and 0^+, thereby obtaining

$$\int_{0^-}^{0^+} \dot{x} dt + \omega_n^2 \int_{0^-}^{0^+} \int_{0^-}^{t} x d\tau dt = \int_{0^-}^{0^+} \delta(t) dt \tag{2.64b}$$

The first integral of the left-hand side of Eq. 2.64b is readily recognized as $x(0^+) - x(0^-)$, the second integral vanishing because x is bounded at the origin, as proven above, and hence, its integral is bounded in a neighborhood around the origin as well. Furthermore, the integral of the right-hand side reduces to 1, and hence, Eq. 2.64b takes the form

$$x(0^+) - x(0^-) = 1$$

Moreover, if we take into account the initial conditions of Eq. 2.63, then

$$x(0^+) = 1$$

The above discussion shows that a unit doublet input indeed produces an instantaneous change in position of unit magnitude. A doublet of magnitude $A\dot{\delta}(t)$ would produce an instantaneous position change of magnitude A, without affecting its velocity. Furthermore, since $x(t)$ is a multiple of the unit-step function, $\dot{x}(t)$ is a multiple of the unit-impulse function, and hence, $\dot{x}(0^+) = \dot{x}(0^-) = 0$, i.e., it attains infinite values inside the interval $0^- < t < 0^+$, but is zero elsewhere. Then, Eq. 2.63 reduces to the form:

$$\ddot{x} + \omega_n^2 x = 0, \quad x(0^+) = 1, \quad \dot{x}(0^+) = 0, \quad t \geq 0^+$$

whose time response was already found in Eq. 2.13 for a more general case. Taking the foregoing equation into account, we can write

$$x(t) = \begin{cases} 0, & \text{for} \quad t \leq 0^-; \\ \cos \omega_n t, & \text{for} \quad t \geq 0^+ \end{cases} \tag{2.65a}$$

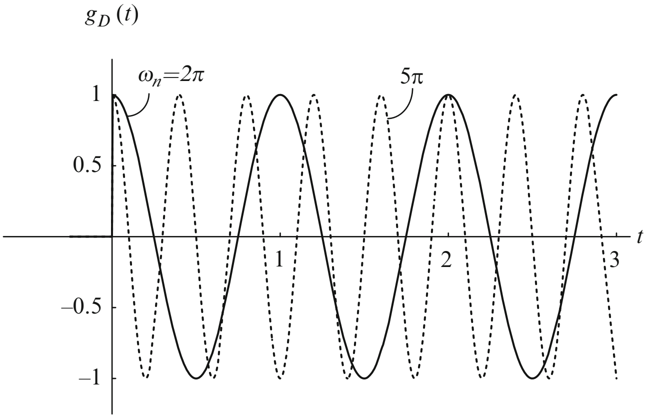


Fig. 2.29 Doublet response of an undamped second-order system

By analogy with Eq. 2.60c, the response of Eq. 2.65a can be called the *doublet response* of the given second-order system, and is denoted by $g_D(t)$, i.e., in compact form,

$$g_D(t) \equiv (\cos \omega_n t)\, u(t) \tag{2.65b}$$

its plot being included in Fig. 2.29, for two different values of ω_n.

Alternatively, since the unit doublet is the first derivative of the unit impulse, by Eq. 2.2 we can obtain the doublet response by differentiating the impulse response, namely,

$$g_D(t) = \frac{d}{dt} g_I(t)$$

i.e.,

$$g_D(t) = \frac{d}{dt}\left(\frac{1}{\omega_n}(\sin \omega_n t)\, u(t)\right) = (\cos \omega_n t)\, u(t) + \frac{1}{\omega_n}(\sin \omega_n t)\, \delta(t) \tag{2.66}$$

Recalling the definitions of $\delta(t)$ and $u(t)$, we see that the last term in the rightmost-hand side is zero for $t \geq 0^+$ and that $g_D(t) = 0$ for $t \leq 0^-$. Thus, $g_D(t)$, as obtained in Eq. 2.66, reduces to Eq. 2.65b.

A mass-spring system is acted upon by a doublet when its mass is given a sudden relocation, without affecting its velocity. For example, when a crank turning at a very high speed performs an indexing operation on a Geneva[10] wheel, the latter undergoes a finite rotation in no time at all. This situation can be modeled as the system appearing in Eq. 2.63, when the elasticity of the shaft attached to the crank is taken into account.

[10]See Fig. 3.2.

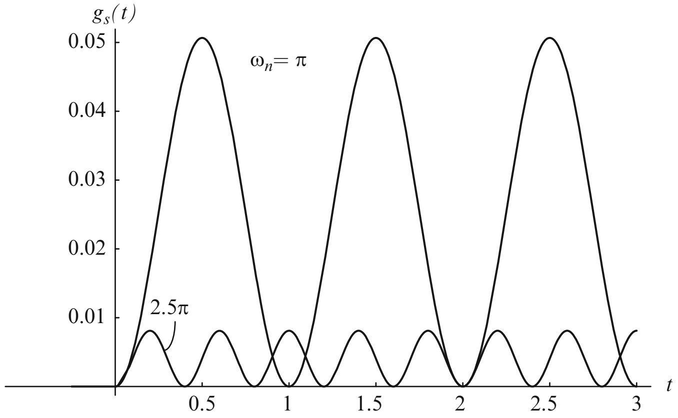


Fig. 2.30 Unit-step response of a second-order undamped system

2.5.2.2 Step Response

If now the system is acted upon by a unit step, we have

$$\ddot{x}+\omega_n^2 x = u(t), \quad t>0^-, \quad x(0^-)=0, \quad \dot{x}(0^-)=0 \tag{2.67}$$

Again, by linearity, we can express the response of the system under study as the integral of the impulse response of the same system, i.e.,

$$x(t)=\int_0^t g_I(\theta)d\theta = \frac{1-\cos\omega_n t}{\omega_n^2}u(t) \tag{2.68}$$

Expression (2.68) can be called the *step response* of the undamped second-order system under study. We will represent this response as $g_S(t)$, i.e.,

$$g_S(t) \equiv \frac{1-\cos\omega_n t}{\omega_n^2}u(t) \tag{2.69}$$

This response is plotted in Fig. 2.30.

2.5.2.3 Ramp Response

We now have

$$\ddot{x}+\omega_n^2 x = r(t), \quad t>0^-, \quad x(0^-)=0, \quad \dot{x}(0^-)=0 \tag{2.70}$$

The reader is invited in Exercise 2.30 to derive the *ramp response* of the system at hand, denoted by $g_R(t)$, as shown below:

$$g_R(t) \equiv \frac{1}{\omega_n^2}\left[r(t) - \frac{1}{\omega_n}(\sin\omega_n t)u(t)\right] \tag{2.71}$$

2.5.3 *Second-order Damped Systems*

2.5.3.1 Doublet Response

In this case, the governing equation is

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \dot{\delta}(t), \quad x(0^-) = 0, \quad \dot{x}(0^-) = 0, \quad t > 0^- \tag{2.72}$$

The doublet appearing in the right-hand side of Eq. 2.72 must be balanced with a corresponding term in the left-hand side, which can neither appear in the second nor in the third term of that side. Indeed, if it appeared in the $\dot{x}$ term, then the $\ddot{x}$ term would comprise a triplet, which does not appear in the right-hand side. Likewise, if the doublet appeared in the x term, then the $\ddot{x}$ term would have a quadruplet, which should be balanced in the right-hand side. Thus, the doublet appears only in the $\ddot{x}$ term, and hence, the $\dot{x}$ term comprises an impulse and the x term, a step function. Now, both sides of Eq. 2.72 are integrated between $t = 0^-$ and $t > 0^-$, thus obtaining

$$\dot{x} + 2\zeta\omega_n x + \omega_n^2\int_{0^-}^{t} x d\tau = \delta(t), \quad x(0^-) = 0, \quad t > 0^- \tag{2.73}$$

where, again the initial condition on $\dot{x}(t)$ is not needed, for the highest-order derivative appearing in the foregoing equation is $\dot{x}$.

Further, the two sides of Eq. 2.73 are integrated between 0^- and 0^+, which yields

$$\int_{0^-}^{0^+} \dot{x}dt + 2\zeta\omega_n\int_{0^-}^{0^+} xdt + \omega_n^2\int_{0^-}^{0^+}\int_{0^-}^{t} xd\tau = \int_{0^-}^{0^+} \delta(t)dt \tag{2.74}$$

The first integral of the left-hand side of Eq. 2.74 is readily recognized as $x(0^+) - x(0^-)$, the second and the third integrals of the same side vanishing because x and its integral were found to be bounded at the origin. The integral of the right-hand side reduces to 1, and hence,

$$x(0^+) - x(0^-) = 1$$

which, in light of the initial conditions of Eq. 2.72, leads to

$$x(0^+) = 1$$

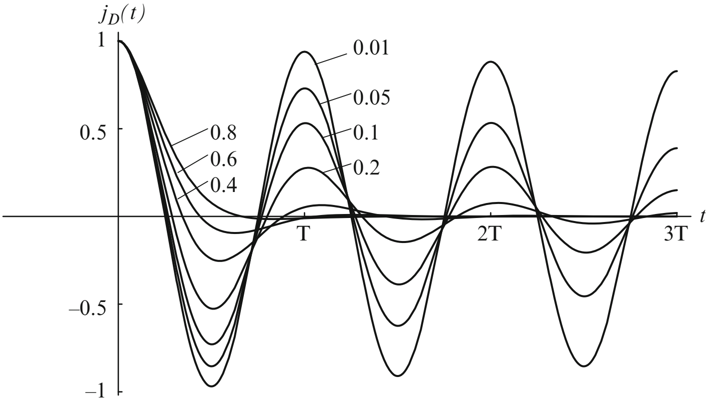


Fig. 2.31 Doublet response of an underdamped second-order system for various values of ω_n

Since $x(t)$ is a multiple of the unit-step function, $\dot{x}(t)$ is a multiple of the unit-impulse function, and hence, $\dot{x}(0^+) = \dot{x}(0^-) = 0$, i.e., it attains infinite values inside the interval $0^- < t < 0^+$, but is zero everywhere else. In summary, Eq. 2.72 reduces to a homogeneous equation with nonzero initial conditions, namely,

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \quad x(0^+) = 1, \quad \dot{x}(0^+) = 0, \quad t \geq 0^+ \tag{2.75}$$

whose time response was already found in Eq. 2.23a for a more general case, and hence, for *underdamped* systems,

$$x(t) = \begin{cases} 0, & \text{for} \quad t \leq 0^-; \\ \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\zeta\sin\omega_d t + \sqrt{1-\zeta^2}\cos\omega_d t\right), & \text{for} \quad t \geq 0^+ \end{cases}$$

The response of a damped, second-order system to a unit doublet can be called the *doublet response* of this system, and is, henceforth, represented by $j_D(t)$, i.e., in compact form,

$$j_D(t) \equiv \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\zeta\sin\omega_d t + \sqrt{1-\zeta^2}\cos\omega_d t\right)u(t) \tag{2.76a}$$

The doublet response of the system under study is plotted in Fig. 2.31, for various values of ζ between 0 and 1.

Alternatively, as done with the undamped second-order system, we can also obtain the doublet response by taking the first derivative of the impulse response, namely,

$$j_D(t) = \frac{d}{dt}j_I(t) = \frac{d}{dt}\left[\frac{e^{-\zeta\omega_n t}}{\omega_d}(\sin\omega_d t)u(t)\right]$$

i.e.,

$$j_D(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right]u(t) + \left[\frac{e^{-\zeta\omega_n t}}{\omega_d}(\sin\omega_d t)\right]\delta(t)$$

However, in light of Eq. 2.47a, the second term of the right-hand side of the above expression vanishes, $j_D(t)$ thus reducing to the form given in Eq. 2.76a.

The doublet responses of critically damped and overdamped systems are left as exercises.

2.5.3.2 Step Response

If the same system is acted upon by a unit-step function, we have

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u(t), \quad t > 0^-, \quad x(0^-) = 0, \quad \dot{x}(0^-) = 0 \tag{2.77}$$

whose response can be found, again by superposition, as the integral of the response of the same system to a unit impulse. Thus, for *underdamped* systems we have

$$x(t) = \int_0^t j_I(\theta)d\theta = \int_0^t \frac{e^{-\zeta\omega_n\theta}}{\omega_d}(\sin\omega_d\theta)\,d\theta \tag{2.78}$$

The integral in Eq. 2.78 can be readily evaluated if $\sin\omega_d\theta$ is expressed as

$$\sin\omega_d\theta \equiv \frac{e^{j\omega_d\theta} - e^{-j\omega_d\theta}}{2j}, \quad j = \sqrt{-1}$$

thereby obtaining

$$x(t) = \frac{1}{2j\omega_d}\int_0^t \left[e^{-(\zeta\omega_n - j\omega_d)\theta} - e^{-(\zeta\omega_n + j\omega_d)\theta}\right]d\theta$$

and hence

$$x(t) = \begin{cases} 0, & t \le 0^-; \\ \left[1 - e^{-\zeta\omega_n t}(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t)\right]/\omega_n^2, & t \ge 0^+ \end{cases}$$

Again, we call the response displayed above the *step response* of the second-order underdamped system, and denote it by $j_S(t)$, i.e., in compact form,

$$j_S(t) \equiv \frac{1}{\omega_n^2}\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\right]u(t) \tag{2.79}$$

which is plotted in Fig. 2.32, for fixed ω_n and various values of ζ.

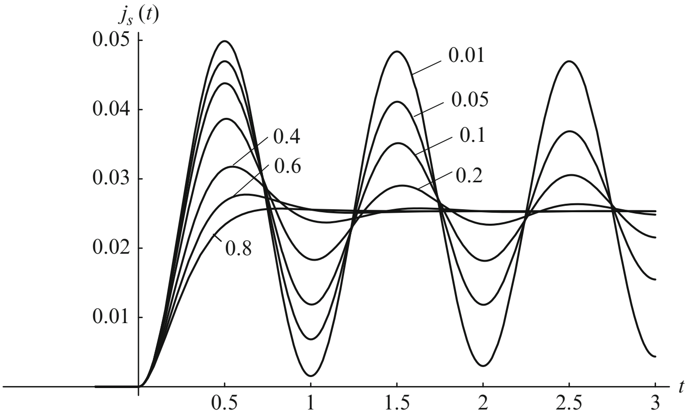


Fig. 2.32 Step response of a second-order underdamped system for various values of ζ

The step responses of critically damped and overdamped systems are left, again, as **exercises**.

2.5.3.3 Ramp Response

If now the system is acted upon by a ramp function, we have

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = r(t), \quad t > 0^-, \quad x(0^-) = 0, \quad \dot{x}(0^-) = 0 \tag{2.80}$$

As the reader can verify, the response of a second-order underdamped system to a ramp, denoted by $j_R(t)$, is given by

$$j_R(t) \equiv \frac{1}{\omega_n^2}\left[r(t) - \frac{2\zeta}{\omega_n}u(t)\right] + \frac{e^{-\zeta\omega_n t}}{\omega_n^3}\left(2\zeta\cos\omega_d t + \frac{2\zeta^2 - 1}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)u(t) \tag{2.81}$$

Likewise, the response of a second-order, critically damped system to a ramp, denoted by $k_R(t)$, is given by

$$k_R(t) \equiv \frac{1}{\omega_n^2}\left(1 - e^{-\omega_n t}\right)r(t) - \frac{2}{\omega_n^3}\left(1 - e^{-\omega_n t}\right)u(t) \tag{2.82}$$

Finally, the ramp response of an overdamped second-order system, denoted by $l_R(t)$, is given by

$$l_R(t) \equiv \frac{1}{\omega_n^2}\left(r(t) - \frac{2\zeta}{\omega_n}u(t)\right) + \frac{e^{-\zeta\omega_n t}}{\omega_n^3}\left[2\zeta\cosh(\rho\omega_n t) + \frac{2\zeta^2 - 1}{\rho}\sinh(\rho\omega_n t)\right]u(t) \tag{2.83a}$$

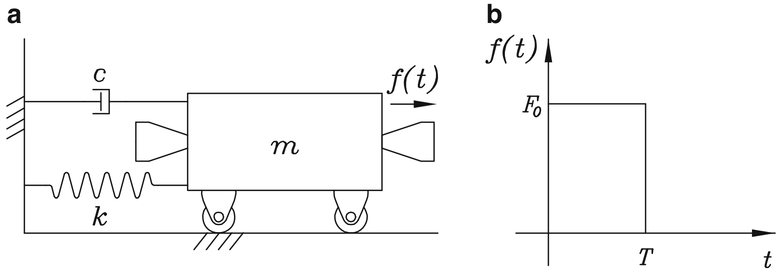


Fig. 2.33 Testbed for thrusters (**a**) its iconic model, and (**b**) its input under a firing thruster

where

$$\rho \equiv \sqrt{\zeta^2 - 1} \tag{2.83b}$$

An example of input that can be represented as a superposition of inputs studied here is the *pulse*, introduced below: A *pulse* $p(t)$ is a signal, i.e., a function of time, that is characterized by its *short duration*. Formally,

$$p(t) = \begin{cases} 0, & t < 0^- \\ \Pi(t), & 0^+ \leq t \leq T^- \\ 0, & t > T^+ \end{cases} \tag{2.84}$$

where provisions here have been made to allow for possible discontinuities in the function $\Pi(t)$. As well, T is a "small" time-interval with respect to the observation time.

2.5.4 Superposition

Sometimes, a LTIS is acted upon by an input that is abrupt, impulsive, or a combination of both, but is none of the above functions. Nevertheless, this input can be decomposed into a linear combination of doublets, impulses, steps, and ramps. By homogeneity, additivity, and time-invariance, the response can then be readily obtained by *superposition* as the corresponding linear combination of the individual responses to the inputs studied in this section.

Example 2.5.1 (Response of a Flexible System Under Thruster-firing). The iconic model of a mechanical system used to test on-off thrusters for space applications is shown in Fig. 2.33a. Such a thruster fires for a given finite time-interval T, thus providing a constant force F_0 to the mass, during that period, as shown in Fig. 2.33b, the force thus being a pulse with $\Pi(t) = F_0$. Find the time response of the underdamped system to this pulse.

Solution: The pulse of Fig. 2.33b can be represented as the sum of two step functions of the same amplitude F_0 and opposite signs, the positive step being applied at $t = 0$, the negative at $t = T$, i.e.,

$$f(t) = F_0 u(t) - F_0 u(t - T)$$

the mathematical model of the system being

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

As the system is underdamped, its step response is $j_S(t)$, as given by Eq. 2.79. We can, therefore, write the time response sought in the form

$$x(t) = \frac{F_0}{m} j_S(t) - \frac{F_0}{m} j_S(t - T)$$

Since we have already found an expression for $j_S(t)$, the foregoing expression would suffice for our purposes. For completeness, we expand this response as

$$\begin{aligned} x(t) = \frac{F_0}{m\omega_n^2} \Bigg\{ & \left[1 - e^{-\zeta\omega_n t} \left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t \right) \right] u(t) \\ & - \left[1 - e^{-\zeta\omega_n (t-T)} \left(\cos\omega_d (t-T) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d (t-T) \right) \right] u(t-T) \Bigg\} \end{aligned}$$

This total time response is sketched in Fig. 2.34 as the sum of two step responses.

2.6 The Total Time Response

The *total time response* of dynamical systems is the time response under non-zero input and non-zero initial conditions. By superposition, the total time response of linear dynamical systems and, in particular, of LTI dynamical systems, is simply the sum of the zero-input and the zero-state responses. We summarize below these results for each of the systems under study.

2.6.1 First-order Systems

Here the system of interest is modeled by

$$\dot{x} = -ax + f(t), \quad t > 0^-, \quad x(0^-) = x_0 \tag{2.85}$$

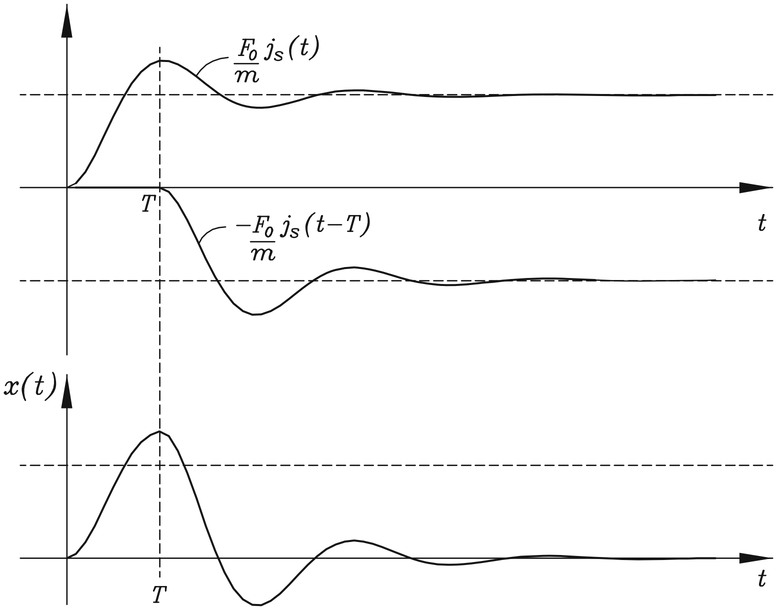


Fig. 2.34 Time response of the testbed to a thruster pulse

If we recall the zero-input response of Eq. 2.8 and the zero-state response given by the convolution of Eq. 2.54, the required time response is

$$x(t) = e^{-at}x_0 + \int_0^t f(t-\tau)e^{-a\tau}d\tau \tag{2.86}$$

2.6.2 Second-order Systems

As in previous sections, here we distinguish among the usual cases, as described below.

2.6.2.1 Undamped Systems

For second-order undamped systems, we have

$$\ddot{x} + \omega_n^2 x = f(t), \quad x(0^-) = x_0, \quad \dot{x}(0^-) = v_0, \quad t > 0^- \tag{2.87}$$

The time response of this system is then the sum of the zero-input response of Eq. 2.13 and the zero-state response given by the corresponding convolution, as appearing in Eq. 2.55, namely,

$$x(t) = (\cos\omega_n t)x_0 + \frac{1}{\omega_n}(\sin\omega_n t)v_0 + \int_0^t f(t-\tau)\frac{1}{\omega_n}(\sin\omega_n\tau)\,d\tau \tag{2.88}$$

2.6.2.2 Damped Systems

A second-order damped system acted upon by a non-zero input and non-zero initial conditions is represented below:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t), \quad x(0^-) = x_0, \quad \dot{x}(0^-) = v_0, \quad t > 0^- \tag{2.89}$$

For underdamped systems, the time response is derived as the sum of the zero-input response of Eq. 2.23a and the zero-state response given by the convolution of Eq. 2.56, namely,

$$\begin{aligned} x(t) = {} & \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right)x_0 + \frac{e^{-\zeta\omega_n t}}{\omega_d}(\sin\omega_d t)v_0 \\ & + \int_0^t f(t-\tau)\frac{e^{-\zeta\omega_n\tau}}{\omega_d}(\sin\omega_d\tau)\,d\tau \end{aligned} \tag{2.90}$$

Correspondingly, the total responses of critically damped and overdamped systems are shown below:

$$x(t) = e^{-\omega_n t}\left[(1+\omega_n t)x_0 + t v_0\right] + \int_0^t f(t-\tau)\tau e^{-\omega_n\tau}d\tau \tag{2.91}$$

and

$$\begin{aligned} x(t) = {} & \frac{e^{-\zeta\omega_n t}}{r}\left[\left[r\cosh(r\omega_n t) + \zeta\sinh(r\omega_n t)\right]x_0 + \frac{1}{\omega_n}\sinh(r\omega_n t)v_0\right] \\ & + \int_0^t f(t-\tau)\frac{e^{-\zeta\omega_n\tau}}{r\omega_n}\sinh(r\omega_n\tau)\,d\tau \end{aligned} \tag{2.92a}$$

$$r \equiv \sqrt{\zeta^2 - 1} \tag{2.92b}$$

In simulation studies it is useful to have the foregoing total responses for all three cases of damped systems in a single expression, which can be done with the aid of the concept of state variable introduced in Sect. 2.3.2. The zero-input response of damped systems in state-variable form is given in Eq. 2.21. The zero-state response in state-variable form can be derived from the convolution, but we have three different formulas for this, namely, for each of the three associated cases. A generally applicable convolution expression in terms of state variables is derived below. To this end, we first cast the governing equation, Eq. 2.89, in state-variable form, namely,

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}f(t), \quad t > 0, \quad \mathbf{z}(0) = \mathbf{z}_0 \tag{2.93}$$

with $\mathbf{z}$ and $\mathbf{A}$ defined in Eqs. 2.15, 2.16, 2.18a, b, while $\mathbf{b}$ as defined as

$$\mathbf{b} \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.94}$$

Thus, the impulse response of the system in state-variable form is defined as the response of the system under $\mathbf{z}_0 = \mathbf{0}$ and $f(t) = \delta(t)$. We can determine, then, the impulse response of the system at hand in exactly the same manner as we did for the scalar first-order system in Sect. 2.4.5. That is, we transform Eq. 2.93 with zero initial condition and acted upon by a unit impulse into a system with zero input and non-zero initial condition at time 0^+. We do this by integration of both sides of the aforementioned equation, with $f(t) = \delta(t)$, between $t = 0^-$ and $t = 0^+$, which yields

$$\mathbf{z}(0^+) - \mathbf{z}(0^-) = \int_{0^-}^{0^+} \mathbf{A}\mathbf{z}(t)dt + \int_{0^-}^{0^+} \mathbf{b}\delta(t)dt \tag{2.95}$$

In the foregoing equation, the left-hand side reduces to $\mathbf{z}(0^+)$, while the first integral of the right-hand side vanishes because $\mathbf{z}(t)$ remains bounded at the origin. Moreover, the second integral reduces to $\mathbf{b}$ by virtue of the definition of the impulse function, and hence,

$$\mathbf{z}(0^+) = \mathbf{b}$$

Thus, the system under study takes the form

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}, \quad \mathbf{z}(0^+) = \mathbf{b}, \quad t > 0^+ \tag{2.96}$$

whose time response, henceforth represented by $\mathbf{z}_I(t)$, can be readily derived from Eq. 2.21, namely,

$$\mathbf{z}_I(t) = e^{\mathbf{A}t}\mathbf{b}u(t) \tag{2.97}$$

which is the impulse response of the system under study. Now, obtaining the zero-state response for an arbitrary input is straightforward if we *convolve* the foregoing impulse response with a given input $f(t)$, thus deriving

$$\mathbf{z}(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{b}f(t-\tau)d\tau \tag{2.98}$$

and hence, the total response of the system takes the form

$$\mathbf{z}(t) = e^{\mathbf{A}t}\mathbf{z}_0 + \int_0^t e^{\mathbf{A}\tau}\mathbf{b}f(t-\tau)d\tau \tag{2.99a}$$

or, by virtue of the commutativity of the convolution,

$$\mathbf{z}(t) = e^{\mathbf{A}t}\mathbf{z}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b}f(\tau)d\tau \tag{2.99b}$$

The two above expressions for the total response of damped systems are valid, in fact, for arbitrary n-degree-of-freedom (n-dof) systems, in which the state-variable vector $\mathbf{z}$ becomes $2n$-dimensional, while vector $\mathbf{b}$ $2n$-dimensional, and matrix $\mathbf{A}$ of $2n \times 2n$.

2.7 The Harmonic Response

Besides impulsive and abrupt inputs, dynamical systems can be subjected to persistent, time-varying inputs that behave in an unpredictable manner. This is the case in aircraft subjected to gust winds and turbulence or terrestrial vehicles traveling on imperfect roads. While these systems are subjected to time-varying inputs that are difficult, if not impossible, to describe in the form of an *explicit* function $f(t)$, these inputs can be regarded as the summation of infinitely-many inputs of a much simpler structure. In fact, as the French mathematician Joseph Fourier (1768–1830) showed in his famous work *Théorie analytique de la chaleur*, published in 1812, any periodic function can be decomposed into an infinite sum of sines and cosines of multiples of a fundamental frequency. Such an expansion of a periodic function is known as a *Fourier series*. By extension, the same decomposition can be applied to arbitrary functions using a continuous distribution of frequencies that can be determined using what is known as the *Fourier transform*. Thus, given any input, not necessarily periodic, it is always possible to find its frequency content by a study that is known as *spectral analysis*. For linear systems, such a decomposition, whether in a continuum of frequencies or in a discrete, although infinite set of multiples of a fundamental frequency, is extremely useful because it allows the analyst, by superposition, to find the response of the system to that arbitrary input. Indeed, by means of superposition, the said response can be found as an infinite sum, i.e., a series of responses to harmonic (sinusoidal and cosinusoidal) inputs. These facts allow us to understand the importance of the response of dynamical systems to simple harmonic inputs. We will term, henceforth, such a response the *harmonic response* of the system at hand, which also goes by the name of *frequency response*.

Below we study the response of mechanical first- and second-order systems to harmonic inputs and then, using a Fourier expansion, we derive the response of these systems to any periodic function. While the Fourier transform of arbitrary, possibly random functions allows the study of the response of dynamical systems to unpredictable or *stochastic* inputs, we will skip in this book the Fourier transform and focus on the Fourier series.

The philosophy behind the analysis that follows is that we are interested in the behavior of systems *after a very long time* has elapsed since an initial time, i.e., since the system started being perturbed or excited with an input of the nature mentioned above. Hence, initial conditions become immaterial and can be set arbitrarily. Now, when we speak of *long periods*, we must realize that we are speaking of a time span that is large with respect to the *natural time scale* of the system under study. In fact, every single-dof system studied in Chap. 1 has one distinct time scale, namely, the *time constant* for first-order systems or the *natural period*, i.e., the inverse of the natural frequency, for second-order systems. Moreover, all physical systems always contain a certain amount of damping, that takes care of any nonzero initial conditions whose effect on the system response becomes negligible after a certain finite time. Thus, we begin by deriving the zero-state response of the systems at hand to harmonic excitations. For example, when we study the vibrations in the fuselage of aircraft under turbulence occurring at cruising conditions, the initial conditions become irrelevant. We might as well consider, then, that the system, in this case the fuselage, started its motion at time $-\infty$.

We will assume a harmonic input, i.e., the forcing term in the governing equations is assumed to have the form

$$f(t) = A\cos\omega t + B\sin\omega t \tag{2.100}$$

We can study the response of the systems at hand to the input of Eq. 2.100 by superposition, i.e., by finding first the response to an input of the form $\cos\omega t$, then that to one of the form $\sin\omega t$ and then express the total input as a suitable linear combination of the two foregoing responses. Of course, by linearity, the coefficients of that linear combination are the corresponding coefficients A and B of Eq. 2.100. Coefficients A and B are termed the *amplitude* of the cosine and the sine signals, respectively.

Below we study the time response of first- and second-order systems and show that this response can be decomposed into a *transient part* and a *steady-state part*. The transient part decays with time and, after a finite time T has elapsed, it becomes negligible. What remains is a harmonic response that constitutes the *steady-state response*. In particular, for a harmonic excitation of the form

$$f(t) = A\cos\omega t \tag{2.101}$$

the steady-state response, represented by $x_{CS}(t)$, takes the form

$$x_{CS}(t) = M\cos(\omega t + \phi) \tag{2.102}$$

where M and ϕ are the *magnitude* and *phase* of the response under study. Thus, the harmonic response is fully characterized by these two parameters, as shown in Fig. 2.35. It will become apparent that both the magnitude and the phase are functions of: (a) the amplitude and the frequency of the input and (b) the parameters—τ for first-order, ζ and ω_n for second-order systems. Below we derive

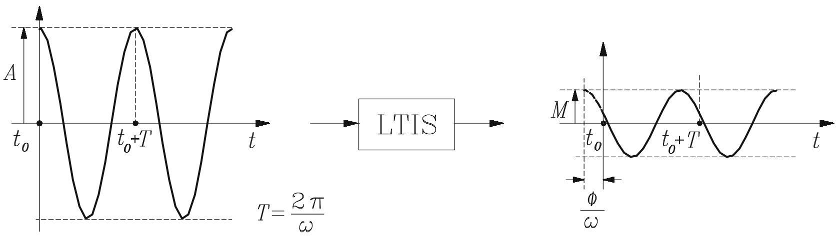


Fig. 2.35 Harmonic response of a linear, time-invariant dynamical system

expressions for the magnitude and phase of the harmonic response of the systems of interest and plot what is known as their *frequency response plots*. These plots are also known as the *Bode plots* of the system at hand.

Prior to our study, we recall some useful trigonometric identities, namely,

$$C_C \cos\omega t + C_S \sin\omega t = K\cos(\omega t + \alpha_C) \tag{2.103a}$$

where

$$K = \sqrt{C_C^2 + C_S^2}, \qquad \alpha_C = -\arctan\left(\frac{C_S}{C_C}\right) \tag{2.103b}$$

and

$$C_C \cos\omega t + C_S \sin\omega t = K\sin(\omega t + \alpha_S) \tag{2.104a}$$

where

$$\alpha_S = \arctan\left(\frac{C_C}{C_S}\right) \tag{2.104b}$$

with K as defined in Eq. 2.103b.

Note that if the harmonic functions of the left-hand side of Eqs. 2.103a and 2.104a involve themselve a phase angle β, then the phase angle of the harmonic functions in the right-hand side changes correspondingly, i.e.,

$$C_C \cos(\omega t + \beta) + C_S \sin(\omega t + \beta) = K\cos(\omega t + \alpha_C') \quad \text{or} \quad K\sin(\omega t + \alpha_S') \tag{2.105a}$$

where the coefficient K is the same as that of Eq. 2.103b, but the phase angles are now

$$\alpha_C' = \alpha_C + \beta, \quad \alpha_S' = \alpha_S + \beta \tag{2.105b}$$

2.7.1 The Unilateral Harmonic Functions

It will be useful to derive a relationship between the zero-state response of a system to inputs of the forms $(\cos\omega t)u(t)$ and $(\sin\omega t)u(t)$. These two functions are zero up until $t = 0^-$; they are identical to the harmonic functions afterwards, and henceforth termed the *unilateral harmonic functions*. Note that, if observed at instants $t \gg 0$, the unilateral harmonic functions become the usual harmonic functions.

Let $x_C(t)$ be the zero-state response (ZSR) of any linear system to an input of the form $(\cos \omega t)u(t)$; likewise, let $x_S(t)$ be the ZSR of the same system to an input of the form $(\sin \omega t)u(t)$; and let $x_I(t)$ be the impulse response of the same system. Note that here we do not distinguish among undamped, underdamped, critically damped, and overdamped systems; neither do we distinguish between first- and second-order systems. In the process, we will need the time-derivatives of the unilateral harmonic functions, which we readily derive:

$$\frac{d}{dt}[(\cos \omega t)u(t)] = -\omega(\sin \omega t)u(t) + (\cos \omega t)\delta(t)$$

But, from Eq. 2.47a,

$$(\cos \omega t)\delta(t) = (\cos(0))\delta(t) \equiv \delta(t)$$

and so,

$$\frac{d}{dt}[(\cos \omega t)u(t)] = -\omega(\sin \omega t)u(t) + \delta(t) \tag{2.106}$$

or, solving for $(\sin \omega t)u(t)$,

$$(\sin \omega t)u(t) = -\frac{1}{\omega}[(\cos \omega t)u(t)] + \frac{1}{\omega}\delta(t) \tag{2.107}$$

Thus, the response of the system to the unilateral sine function can be derived as a linear combination of the response to the unilateral cosine function and the impulse response, by merely mimicking the foregoing relation among inputs in terms of the corresponding relation among responses. This we can do by exploiting the properties of LTIS, namely,

$$x_S(t) = -\frac{1}{\omega}\frac{d}{dt}x_C(t) + \frac{1}{\omega}x_I(t) \tag{2.108}$$

Likewise,

$$\frac{d}{dt}[(\sin \omega t)u(t)] = \omega(\cos \omega t)u(t) + (\sin \omega t)\delta(t)$$

Again, from Eq. 2.47,

$$(\sin \omega t)\delta(t) \equiv (\sin(0))\delta(t) \equiv 0$$

and so,

$$\frac{d}{dt}[(\sin \omega t)u(t)] = \omega(\cos \omega t)u(t) \tag{2.109}$$

whence,

$$(\cos \omega t)u(t) = \frac{1}{\omega}\frac{d}{dt}[(\sin \omega t)u(t)] \tag{2.110}$$

Thus, we can find the response of a LTIS to the unilateral sine function in terms of its response to the unilateral cosine function by simply mimicking the foregoing expression, in exactly the same way as done previously, i.e.,

$$x_C(t) = \frac{1}{\omega}\frac{d}{dt}x_S(t) \tag{2.111}$$

The presence of the impulse response in Eq. 2.108 is to be highlighted. This term appears in that equation by virtue of the corresponding impulsive term in Eq. 2.106.

2.7.2 First-order Systems

The equation of motion of a first-order system is recalled below:

$$\dot{x} + ax = f(t) \tag{2.112}$$

In the subsequent discussion, we shall assume that the parameter a characterizing the system is positive. Moreover, the zero-state response of a first-order system to the $(\cos \omega t)u(t)$ input, denoted, as above, by $x_C(t)$, will be derived. Now, since the unit step function is dimensionless, the input function in Eq. 2.112 is dimensionless, which means that its left-hand side is also dimensionless, and hence, $x(t)$ **is expressed in units of time**. This should not bother the reader, for actual physical excitations always carry physical units that are accounted for by the units of the *amplitudes* A and B of Eq. 2.100.

The response sought can be expressed in terms of the convolution, namely,

$$x_C(t) = \int_0^t \cos \omega(t-\tau)h_I(\tau)d\tau \tag{2.113a}$$

where $h_I(t)$ is the impulse response of the system under study, i.e.,

$$h_I(t) = e^{-at}u(t) \tag{2.113b}$$

which is recalled to be dimensionless.

Now, the integral of Eq. 2.113a can be obtained in many ways. If a straightforward approach is used, we would substitute the cosine function of the convolution by its exponential representation, namely,

$$\cos \omega t \equiv \frac{e^{j\omega t} + e^{-j\omega t}}{2} \tag{2.114}$$

with j defined as the imaginary unity, i.e., as $j \equiv \sqrt{-1}$. Alternatively, one can simply search in a table of integrals or resort to *symbolic computations*, using scientific software—e.g., *Maple*, *Mathematica* or *Macsyma*. The result is, in any case, as indicated below:

$$x_C(t) = \frac{1}{a^2+\omega^2}\left(-ae^{-at} + a\cos\omega t + \omega\sin\omega t\right)u(t) \tag{2.115}$$

from which we can readily identify the transient and the steady-state components of the time response. Indeed, since we assumed at the outset that $a > 0$, the exponential term in the above expression decays with time, and hence, the transient part of the response, $x_{CT}(t)$, is readily identified, namely,

$$x_{CT}(t) = \frac{-a}{a^2+\omega^2}e^{-at}u(t) \tag{2.116}$$

while the steady-state component, $x_{CS}(t)$, is given by

$$x_{CS}(t) = \frac{1}{a^2+\omega^2}(a\cos\omega t + \omega\sin\omega t)u(t)$$

Since we are interested in this section in studying the behavior of the systems at hand after a long time has elapsed, we can assume that the system started being excited in the distant past. For the time frame of observation in which we are interested, we can write $u(t) = 1$, and the above equation can then be rewritten in the form

$$x_{CS}(t) = \left(\frac{a}{a^2+\omega^2}\right)\cos\omega t + \left(\frac{\omega}{a^2+\omega^2}\right)\sin\omega t \tag{2.117}$$

Using Eqs. 2.103a, b, we may rewrite this expression in the form

$$x_{CS}(t) = M\cos(\omega t + \phi) \tag{2.118a}$$

where

$$M = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \phi = -\arctan\left(\frac{\omega}{a}\right) \tag{2.118b}$$

For completeness, we derive now the time response of the same system to an input of the form $(\sin\omega t)u(t)$ using Eq. 2.108. First, we calculate

$$\begin{aligned}\frac{d}{dt}(x_C(t)) = {} & \frac{1}{a^2+\omega^2}\left(a^2e^{-at} - a\omega\sin\omega t + \omega^2\cos\omega t\right)u(t) \\ & + \frac{1}{a^2+\omega^2}\left(-ae^{-at} + a\cos\omega t + \omega^2\sin\omega t\right)\delta(t)\end{aligned} \tag{2.119}$$

which, by virtue of Eq. 2.47, simplifies to

$$\frac{d}{dt}\left(x_C(t)\right) = \frac{1}{a^2+\omega^2}\left(a^2 e^{-at} - a\omega \sin \omega t + \omega^2 \cos \omega t\right) u(t)$$

Moreover, we recall that

$$x_I(t) = e^{-at} u(t)$$

Thus, from Eq. 2.108, we have

$$\begin{aligned} x_S(t) &= -\frac{1}{\omega}\frac{1}{a^2+\omega^2}\left(a^2 e^{-at} - a\omega \sin \omega t + \omega^2 \cos \omega t\right) u(t) + \frac{1}{\omega} e^{-at} u(t) \\ &= -\frac{1}{a^2+\omega^2}\left[-\omega e^{-at} - a \sin \omega t + \omega \cos \omega t\right] u(t) \end{aligned} \tag{2.120}$$

The transient component of the response $x_S(t)$, denoted by $x_{ST}(t)$, is readily singled out from Eq. 2.120, namely,

$$x_{ST}(t) = \frac{\omega}{a^2+\omega^2} e^{-at} u(t) \tag{2.121}$$

Likewise, the steady-state component of the same response, denoted by $x_{SS}(t)$, is identified from Eq. 2.120 as

$$x_{SS}(t) = \frac{1}{a^2+\omega^2}\left(a \sin \omega t - \omega \cos \omega t\right) u(t)$$

After a long time t has elapsed, the unit-step function is identical to unity, and hence, the above expression becomes

$$x_{SS}(t) = \frac{1}{a^2+\omega^2}\left(a \sin \omega t - \omega \cos \omega t\right) \tag{2.122}$$

Correspondingly, the magnitude and the phase of the $x_{SS}(t)$ response can be readily identified from the above expression for this signal, thereby obtaining

$$x_{SS}(t) = \left(\frac{a}{a^2+\omega^2}\right) \sin \omega t - \left(\frac{\omega}{a^2+\omega^2}\right) \cos \omega t \tag{2.123}$$

Using Eq. 2.104a, we may write this equation as

$$x_{SS}(t) = M \sin(\omega t + \phi) \tag{2.124a}$$

where

$$M = \frac{1}{\sqrt{a^2 + \omega^2}} \qquad \phi = -\arctan\left(\frac{\omega}{a}\right) \tag{2.124b}$$

which are essentially the same as those derived for the cosine input, as they should.

2.7.3 Second-order Systems

We attempt first to find the harmonic response of undamped systems using the approach introduced for first-order systems.

2.7.3.1 The Response to the Unilateral Cosine Function

For a unilateral cosine input, we have

$$\ddot{x} + \omega_n^2 x = (\cos\omega t)u(t) \tag{2.125}$$

In the above equation, the right-hand side is, again, dimensionless, and so is its left-hand side, which implies that $x(t)$ **is represented in units of time-squared**.

Further, we recall the impulse response of second-order undamped systems, namely,

$$g_I(t) = \frac{1}{\omega_n}(\sin\omega_n t)u(t)$$

which is recalled to bear units of time.

Upon calculation of the convolution of the $(\cos\omega t)u(t)$ input with the foregoing impulse response, we obtain

$$x(t) = \frac{1}{\omega_n^2 - \omega^2}\left(\cos\omega t - \cos\omega_n t\right)u(t) \tag{2.126}$$

from which we cannot identify a decaying term, as we did for first-order systems, and our attempt to follow the previous approach stops here.

An alternative approach to finding the harmonic response of undamped second-order systems is now introduced. We first derive the harmonic response of underdamped systems, the response associated with undamped systems being obtained as the limiting case, when $\zeta \to 0$, of the harmonic response thus found.

Let us then consider the second-order damped system excited by a unilateral cosine function:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = (\cos\omega t)u(t) \tag{2.127}$$

Moreover, the impulse response of the above system for the underdamped case is

$$j_I(t) = \frac{e^{-\zeta\omega_n t}}{\omega_d}(\sin\omega_d t)u(t) \tag{2.128}$$

In order to obtain the response of the system of Eq. 2.127, we resort to the convolution of the $(\cos\omega t)u(t)$ input with the foregoing impulse response. This convolution is calculated in exactly the same manner as done for first-order systems, thereby obtaining the desired expression $x_C(t)$ for the given input, namely,

$$x_C(t) = \frac{N(t)}{D}u(t) \tag{2.129a}$$

with $N(t)$ and D defined as

$$N(t) \equiv \left[\frac{-\zeta(\omega^2+\omega_n^2)}{\sqrt{1-\zeta^2}}\sin\omega_d t + (\omega^2-\omega_n^2)\cos\omega_d t\right]e^{-\zeta\omega_n t} - \left[(\omega^2-\omega_n^2)\cos\omega t - 2\zeta\omega\omega_n\sin\omega t\right] \tag{2.129b}$$

$$D \equiv \omega^4 - 2\omega^2\omega_n^2 + \omega_n^4 + 4\zeta^2\omega^2\omega_n^2 \equiv (\omega^2-\omega_n^2)^2 + 4\zeta^2\omega^2\omega_n^2 \tag{2.129c}$$

We thus have

$$x_C(t) = \frac{1/\omega_n^2}{(1-r_f^2)^2+4\zeta^2 r_f^2}\left\{e^{-\zeta\omega_n t}\left[\frac{-\zeta(1+r_f^2)}{\sqrt{1-\zeta^2}}\sin\omega_d t - (1-r_f^2)\cos\omega_d t\right] + \left[2\zeta r_f\sin\omega t + (1-r_f^2)\cos\omega t\right]\right\}u(t) \tag{2.130a}$$

with the *frequency ratio* r_f defined as

$$r_f \equiv \frac{\omega}{\omega_n} \tag{2.130b}$$

Since $\zeta > 0$ and $\omega_n > 0$, we can see that the transient term is that comprising the exponential in Eq. 2.130a, namely

$$x_{CT}(t) = \frac{e^{-\zeta\omega_n t}}{\omega_n^2\left[(1-r_f^2)^2+(2\zeta r_f)^2\right]}\left[\frac{-\zeta(1+r_f^2)}{\sqrt{1-\zeta^2}}\sin\omega_d t - (1-r_f^2)\cos\omega_d t\right]u(t)$$

The steady-state component x_{CS} of the response given in Eq. 2.130a can now be recognized as

$$x_{CS}(t) = \frac{1}{\omega_n^2\left[(1-r_f^2)^2+(2\zeta r_f)^2\right]}\left[2\zeta r_f\sin\omega t + (1-r_f^2)\cos\omega t\right]u(t)$$

Again, we are interested only in the response after a long time has elapsed since the system was first excited, and so, the unit-step function can be replaced by unity in the above expression, thus obtaining

$$x_{CS}(t) = \frac{2\zeta r_f}{\omega_n^2\left[(1-r_f^2)^2+(2\zeta r_f)^2\right]}\sin\omega t + \frac{1-r_f^2}{\omega_n^2\left[(1-r_f^2)^2+(2\zeta r_f)^2\right]}\cos\omega t \tag{2.131}$$

Using Eq. 2.103a we may rewrite this expression in the form

$$x_{CS} = M\cos(\omega t+\phi) \tag{2.132a}$$

where

$$M = \frac{1/\omega_n^2}{\sqrt{(1-r_f^2)^2+(2\zeta r_f)^2}} \qquad \phi = -\arctan\left(\frac{2\zeta r_f}{1-r_f^2}\right) \tag{2.132b}$$

Now the total response of an undamped second-order system to a unilateral cosine input $(\cos\omega t)u(t)$ can be derived by setting $\zeta = 0$ in Eq. 2.130a, thereby obtaining

$$x_C(t) = \frac{1/\omega_n^2}{1-r_f^2}(\cos\omega t - \cos\omega_n t)\,u(t) \tag{2.133}$$

which is identical to the expression derived directly from the convolution theorem and displayed in Eq. 2.126.

The corresponding harmonic response of undamped systems can now be derived by setting $\zeta = 0$ in Eq. 2.131, which yields the associated steady-state response as

$$x_{CS}(t) = \frac{1/\omega_n^2}{1-r_f^2}\cos\omega t \tag{2.134}$$

From the above expression, it is clear that the magnitude of the response is the absolute value of the coefficient of $\cos\omega t$. For the phase, we must consider two cases. First, when $r_f < 1$, the denominator of the said coefficient is positive and, hence, the coefficient is also positive. Thus, the response to the input $(\cos\omega t)u(t)$ is exactly the same signal, except for a difference in magnitude; the response is therefore in phase with the input, i.e., $\phi = 0$. On the other hand, when $r_f > 1$, the denominator and, hence, the whole coefficient of $\cos\omega t$, are negative. Thus, the response of the system, apart from a difference in amplitude, is opposite in sign to the input, which corresponds to $\phi = \pm 180°$. However, the positive sign does not make physical sense, since it implies that the response "leads" the input, i.e., the system responds to the input before it is applied. The negative sign implies a lag in phase, which is consistent with the actual behavior of dynamical systems. The response of the undamped system to the given input is, then,

$$x_{CS}(t) = M\cos(\omega t + \phi) \tag{2.135a}$$

where

$$M = \frac{1/\omega_n^2}{|1 - r_f^2|}, \qquad \phi = \begin{cases} 0^\circ & \text{if } r_f < 1; \\ -180^\circ & \text{if } r_f > 1. \end{cases} \tag{2.135b}$$

The foregoing expressions for the amplitude and phase of the harmonic response of second-order systems were derived from the steady-state response of underdamped systems. However, a close look at those derivations reveals that nothing prevents us from applying them to critically damped and overdamped systems as well. In fact, $M\cos(\omega t + \phi)$ is a particular solution of the second-order damped system excited by the input $\cos(\omega t)$.

As a matter of fact, the steady-state responses derived above are nothing but *particular solutions* of the associated ODEs, as the reader can readily verify. Upon substituting the foregoing expressions for the harmonic response of second-order systems into the corresponding mathematical model, it will become apparent that those expressions verify indeed the model, if with particular initial conditions. In the realm of the harmonic response, we are interested in the behavior of the system at arbitrarily large values of t, and so, the initial conditions in this case become irrelevant.

2.7.3.2 Resonance

One special situation occurs when $r_f = 1$, or $\omega = \omega_n$. From Eq. 2.135a, we see that M attains unbounded values when this occurs. Thus, when a harmonic input is applied to a second-order undamped system at a frequency exactly equal to its natural frequency, the magnitude of its response is infinite. This phenomenon is called *resonance* and does not occur in first-order systems. Of course, such a situation could never occur in a real-life system since, first, damping is always present to some extent in actual systems, and, more importantly, the assumption of the system being linear would no longer apply, as large amplitudes in the response would occur, thus making the underlying model no longer valid. Still, the concept is of extreme practical importance since, in actual systems, a very large and violent response is produced whenever an exciting force is near the natural frequency of the system. Thus, great care must go into the design of mechanical systems that are required to remain as unperturbed as possible in the presence of harmonic disturbances. What is needed in these cases is, apparently, a design that will ensure that the system at hand is not subjected to excitations bearing frequencies near its natural frequency at any time during their normal operating conditions.

2.7.3.3 The Response to the Unilateral Sine Function

For completeness, and for further reference, we derive below the response of underdamped systems to a unilateral sine input, $(\sin\omega t)u(t)$; the desired response is denoted here by $x_S(t)$. We do this by resorting to the same idea introduced earlier for first-order systems, namely, by recalling Eq. 2.108. To this end, we must first differentiate $x_C(t)$ as given by Eqs. 2.129a, b, i.e.,

$$\frac{d}{dt}(x_C(t)) = \frac{\dot{N}(t)}{D}u(t) + \frac{N(t)}{D}\delta(t)$$

where $N(t)$ and D are given in Eqs. 2.129b, d; it will become more convenient to write the latter in the form

$$D = \omega_n^4\left[(1-r_f^2)^2 + 4\zeta^2 r_f^2\right] \tag{2.136}$$

The impulsive term in the above expression for the time-derivative of $x_C(t)$ reduces to

$$\frac{N(t)}{D}\delta(t) = \frac{N(0)}{D}\delta(t)$$

but $N(0)$ vanishes, and hence,

$$\frac{d}{dt}(x_C(t)) = \frac{\dot{N}(t)}{D}u(t) \tag{2.137}$$

The time derivative $\dot{N}(t)$ is readily found to be, with the aid of computer algebra,

$$\begin{aligned}
\dot{N}(t) = {} & -\zeta\omega_n e^{-\zeta\omega_n t}\left[-\frac{\zeta(\omega^2+\omega_n^2)}{\sqrt{1-\zeta^2}}\sin\omega_d t + (\omega^2-\omega_n^2)\cos\omega_d t\right] \\
& + e^{-\zeta\omega_n t}\left[-\frac{\zeta\omega_d(\omega^2+\omega_n^2)}{\sqrt{1-\zeta^2}}\cos\omega_d t - \omega_d(\omega^2-\omega_n^2)\sin\omega_d t\right] \\
& + \left[2\zeta\omega^2\omega_n\cos\omega t + \omega(\omega^2-\omega_n^2)\sin\omega t\right]
\end{aligned}$$

or, after rearrangement of terms,

$$\begin{aligned}
\dot{N}(t) = {} & e^{-\zeta\omega_n t}\left\{\left[\frac{\zeta^2\omega_n(\omega^2+\omega_n^2)}{\sqrt{1-\zeta^2}} - \sqrt{1-\zeta^2}\,\omega_n(\omega^2-\omega_n^2)\right]\sin\omega_d t\right. \\
& \left. -\zeta\omega_n(\omega^2-\omega_n^2+\omega^2+\omega_n^2)\cos\omega_d t\right\} \\
& + A\omega_n^3\left[2\zeta r_f^2\cos\omega t + r_f(r_f^2-1)\sin\omega t\right]
\end{aligned}$$

Upon further rearranging of terms, the above expression reduces to

$$\dot{N}(t) = \omega_n^3 e^{-\zeta\omega_n t}\left(\frac{1-r_f^2+2\zeta^2 r_f^2}{\sqrt{1-\zeta^2}}\sin\omega_d t - 2\zeta r_f^2\cos\omega_d t\right) + \omega_n^3\left[2\zeta r_f^2\cos\omega t + r_f(r_f^2-1)\sin\omega t\right] \tag{2.138}$$

We can now substitute Eq. 2.137 and the corresponding impulse response, $j_I(t)$, into Eq. 2.108. For quick reference, we recall below the impulse response of underdamped systems:

$$j_I(t) = \frac{e^{-\zeta\omega_n t}}{\omega_d}(\sin\omega_d t)$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$. After substituting the previous results into Eq. 2.108, and performing some algebraic manipulations, we obtain

$$x_S(t) = -\frac{1/\omega_n^2}{(1-r_f^2)^2+(2\zeta r_f)^2}H(t)u(t) \tag{2.139a}$$

where

$$H(t) \equiv e^{-\zeta\omega_n t} r_f\left[\frac{1-r_f^2-2\zeta^2}{\sqrt{1-\zeta^2}}\sin\omega_d t - 2\zeta\cos\omega_d t\right] + 2\zeta r_f\cos\omega t + (r_f^2-1)\sin\omega t \tag{2.139b}$$

Now, the transient response $x_{ST}(t)$ of the system at hand to the unilateral sine input is readily identified from the above expression, namely,

$$x_{ST}(t) = \frac{-(1/\omega_n^2)}{(1-r_f^2)^2+(2\zeta r_f)^2}e^{-\zeta\omega_n t} r_f\left[\frac{1-r_f^2-2\zeta^2}{\sqrt{1-\zeta^2}}\sin\omega_d t - 2\zeta\cos\omega_d t\right]u(t) \tag{2.140}$$

while the corresponding steady-state response is

$$x_{SS}(t) = \frac{-(1/\omega_n^2)}{(1-r_f^2)^2+(2\zeta r_f)^2}\left[2\zeta r_f\cos\omega t - (1-r_f^2)\sin\omega t\right]u(t)$$

Again, we are interested, in the steady state, at times t very large, at which $u(t)$ can be replaced by unity, the above response then becoming

$$x_{SS}(t) = \frac{-(1/\omega_n^2)}{(1-r_f^2)^2+(2\zeta r_f)^2}\left[2\zeta r_f\cos\omega t - (1-r_f^2)\sin\omega t\right] \tag{2.141}$$

and hence, $x_{SS}(t)$ has the form

$$x_{SS}(t) = M\sin(\omega t + \phi) \tag{2.142a}$$

where

$$M = \frac{1/\omega_n^2}{\sqrt{(1-r_f^2)^2 + (2\zeta r_f)^2}}, \quad \phi = -\arctan\left(\frac{2\zeta r_f}{1-r_f^2}\right) \tag{2.142b}$$

The total response of an undamped system to a unilateral sine input can be derived by setting $\zeta = 0$ in the expression for $x_S(t)$ given by Eq. 2.139 to obtain

$$x_S(t) = \frac{1/\omega_n^2}{1-r_f^2}\left(\sin\omega t - r_f \sin\omega_n t\right) u(t) \tag{2.143}$$

Likewise, the steady-state response of an undamped system to a sinusoidal input is now derived by setting $\zeta = 0$ in the expression derived for $x_{SS}(t)$, thereby obtaining

$$x_{SS}(t) = M_S \sin(\omega t + \phi) \tag{2.144a}$$

where

$$M_S = \frac{(1/\omega_n^2)}{|1-r_f^2|}, \qquad \phi = \begin{cases} 0^\circ & \text{if } r_f < 1; \\ -180^\circ & \text{if } r_f > 1 \end{cases} \tag{2.144b}$$

It is instructive to analyze the total response of an undamped system when excited by a harmonic input with a frequency identical to that of the system, i.e., when $\omega = \omega_n$ or, equivalently, when $r_f = 1$. For example, if we set $\omega = \omega_n$ in Eq. 2.133 or Eq. 2.143, we obtain an indeterminacy, i.e., $0/0$. In order to resolve this indeterminacy, we apply L'Hospital's rule to Eq. 2.133, that we rewrite in the form

$$x_C(t) = \frac{1/\omega_n^2}{1-r_f^2}(\cos\omega_n r_f t - \cos\omega_n t) \equiv F(r_f) \equiv \frac{N(r_f)}{D(r_f)}$$

and we regard this expression as a function of r_f only. By L'Hospital's rule, then,

$$F(1) = \lim_{r_f \to 1} \frac{N'(r_f)}{D'(r_f)} = \left.\frac{-(1/\omega_n^2)\omega_n t \sin\omega_n r_f t}{-2r_f}\right|_{r_f=1}$$

which yields

$$F(1) = \frac{1}{2\omega_n} t \sin\omega_n t$$

The function $2\omega_n F(t)$ is plotted in Fig. 2.36 for $\omega_n = 1\,\text{Hz} = 2\pi\,\text{rad/s}$, which shows a dramatic case of instability. Note that the amplitude of the above response grows linearly with time.

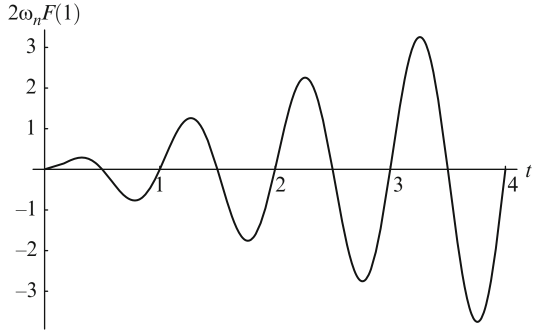


Fig. 2.36 Resonant response of an undamped second-order system

2.7.4 The Response to Constant and Linear Inputs

We consider a load that has been suspending from a crane by means of a cable during a time long enough as to have allowed all transient components of its time response to have faded away. If the crane is subjected to the sole weight of the load, we have an example of a mass-spring system acted upon by a constant input, the weight of the load. Likewise, a terrestrial vehicle, like that of Example 2.3.3, that has been climbing up a slope for a time long enough as to have allowed for the decay of the transient components of its time response, is an instance of a mass-spring-dashpot system acted upon by a linear input. In this subsection we study the time response of these systems, which is quite straightforward to obtain.

For the sake of brevity, we skip first-order and undamped second-order systems here, and focus on second-order underdamped systems. Moreover, in all cases considered here we assume zero initial conditions, with the purpose of being able to apply superposition. We thus have

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = A, \quad x(0) = 0, \quad \dot{x}(0) = 0 \tag{2.145}$$

where A is a constant, and initial conditions are given at $t = 0$, without mentioning explicitly whether at 0^+ or at 0^-, a difference that now becomes irrelevant because the excitation is smooth, and hence, does not contain any impulsive or abrupt components. The simplest way of obtaining the time response of the system at hand is by rewriting the foregoing equation in the form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2\left(x - \frac{1}{\omega_n^2}A\right) = 0, \quad x(0) = 0, \quad \dot{x}(0) = 0$$

It is thus apparent that we can cast the foregoing system in a zero-input form with a simple change of variable, namely,

$$\xi = x - \frac{1}{\omega_n^2}A, \quad \text{or} \quad x = \xi + \frac{1}{\omega_n^2}A \tag{2.146}$$

which implies that the initial conditions will have to be given in terms of $\xi(0)$ and $\dot{\xi}(0)$, i.e.,

$$\ddot{\xi} + 2\zeta\omega_n\dot{\xi} + \omega_n^2\xi = 0, \quad \xi(0) = -\frac{1}{\omega_n^2}A, \quad \dot{\xi}(0) = 0$$

Now it is a simple matter to derive the time response of the above system, which turns out to be

$$\xi(t) = -\frac{A}{\omega_n^2}\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t)$$

Therefore, upon returning to the original variable $x(t)$, the foregoing expression leads to the time response of an underdamped second-order system to a constant input, namely,

$$x_{\text{const}}(t) = \frac{A}{\omega_n^2}\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t)\right] \tag{2.147}$$

which the reader is invited to compare with the corresponding step response.

Now we derive the time response of the same system to an input that is linear with time, under zero initial conditions, i.e.,

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = At, \quad x(0) = 0, \quad \dot{x}(0) = 0 \tag{2.148}$$

Let us call $x_{\text{lin}}(t)$ the time response of the system under study to a linear input, with zero initial conditions. Since the linear input is the integral of the constant input, by linearity we have

$$x_{\text{lin}}(t) = \int_0^t x_{\text{const}}(\theta)d\theta$$

Upon performing the above integration with the aid of computer algebra, we have

$$x_{\text{lin}}(t) = \frac{A}{\omega_n^3}\left\{\omega_n t + e^{-\zeta\omega_n t}\left[2\zeta\cos\omega_d t + \frac{(2\zeta^2-1)}{\sqrt{1-\zeta^2}}\sin\omega_d t\right] - 2\zeta\right\} \tag{2.149}$$

whence it is apparent that the initial condition $x_{\text{lin}}(0) = 0$ is satisfied. The time-derivative of the foregoing expression need not be computed, for it is identical to $x_{\text{const}}(t)$, by definition. Since this function verifies the initial condition $x_{\text{const}}(0) = 0$, we have that the expression for $x_{\text{lin}}(t)$ indeed satisfies the given zero initial conditions, thereby completing the computation of the time responses of second-order damped systems to constant and linear inputs.

2.7.5 The Power Dissipated By a Damped Second-order System

Damping can be determined experimentally if the energy dissipated by a damped second-order system and the motion undergone by the system are both known. In particular, when the system is excited by a harmonic force, we know that the system undergoes harmonic motion as well. Below we derive the relation that allows us to determine the damping coefficient c when we know the amount of energy E_C dissipated by a damped second-order system per cycle as well as its amplitude M and the frequency ω.

The system at hand is assumed to undergo a motion of the form

$$x(t) = M\cos(\omega t + \phi) \tag{2.150}$$

and hence,

$$\dot{x}(t) = -\omega M\sin(\omega t + \phi) \tag{2.151}$$

Thus, from Chap. 1, the power Π_d dissipated by the dashpot takes the form

$$\Pi_d(t) = c\dot{x}^2(t) = c\,\omega^2 M^2\sin^2(\omega t + \phi) \tag{2.152}$$

which can be further expressed as

$$\Pi_d(t) = \frac{1}{2}c\,\omega^2 M^2\,[1 - \cos 2(\omega t + \phi)] \tag{2.153}$$

Now, E_C is obtained simply by integration of the power $\Pi_d(t)$ dissipated throughout a complete cycle, i.e.,

$$E_C = \int_0^{2\pi/\omega} \Pi_d(t)dt = \frac{1}{2}c\,\omega M^2 \int_0^{2\pi} [1 - \cos 2(\omega t + \phi)]\,\omega dt \tag{2.154}$$

the integral thus reducing to 2π, and hence,

$$E_C = \pi c\,\omega M^2$$

Therefore, if all parameters E_C, ω and M are known, the damping coefficient can be readily found as

$$c = \frac{E_C}{\pi\omega M^2} \tag{2.155}$$

2.7.6 The Bode Plots of First- and Second-order Systems

In the preceding section we derived expressions for the magnitude of the response of first- and second-order systems to a harmonic input. This magnitude has a unit associated with it, which is the same as that of the variable x. In engineering practice, it is often advantageous to work with dimensionless quantities. Thus, we will define a dimensionless quantity μ called the *magnification factor*. For first-order systems, M has units of a^{-1}, and hence, Ma or, equivalently, M/τ, is dimensionless and plays the role of the magnification factor μ, i.e.,

$$\mu_f \equiv Ma \equiv \frac{M}{\tau} \tag{2.156}$$

Likewise, for second-order systems, M has units of time squared, and hence, we can render it dimensionless by multiplying it by a quantity with units of frequency-squared. The obvious candidate is ω_n^2, i.e., the square of the natural frequency of the system at hand. Thus, for second-order systems, the magnification factor is defined as

$$\mu_s = M\omega_n^2 \tag{2.157}$$

With the preceding definitions, we can readily derive the magnification factors for first- and second-order systems, using the expressions derived for M previously. These are listed below, along with the corresponding expression for the phase angle. As before, for second-order systems, only the undamped and underdamped cases are considered, the critically damped and overdamped cases being left as exercises.

For first-order systems, we have

$$\mu_f = \frac{1}{\sqrt{1+\left(\frac{\omega}{a}\right)^2}}, \qquad \phi_f = -\arctan\left(\frac{\omega}{a}\right) \tag{2.158}$$

while, for second-order undamped systems,

$$\mu_{su} = \frac{1}{|1-r_f^2|}, \qquad \phi_{su} = \begin{cases} 0^\circ & \text{if } r_f < 1; \\ -180^\circ & \text{if } r_f > 1 \end{cases} \tag{2.159}$$

Finally, for second-order underdamped, critically damped and overdamped systems,

$$\mu_{sd} = \frac{1}{\sqrt{(1-r_f^2)^2 + (2\zeta r_f)^2}}, \qquad \phi_{sd} = -\arctan\left(\frac{2\zeta r_f}{1-r_f^2}\right) \tag{2.160}$$

Plots of μ and ϕ versus $\log \omega$ are known as *Bode plots* of the system and provide all the information of the harmonic response of the system under study. However, the $\log(\cdot)$, like the exponential $e^{(\cdot)}$, of an argument with physical units is meaningless, which calls for a *normalization* of ω in the Bode plots, as explained below. Figures 2.37 and 2.38 show the Bode plots of first- and second-order systems. Note that, for first-order systems, we have plotted μ and ϕ versus ω/a, the latter being a dimensionless quantity similar to the frequency ratio of second-order systems.

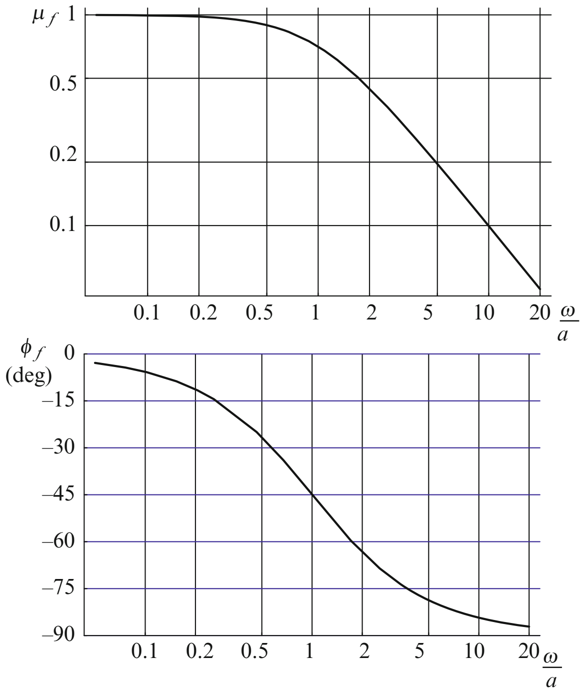


Fig. 2.37 Magnification factor and phase plots for first-order system

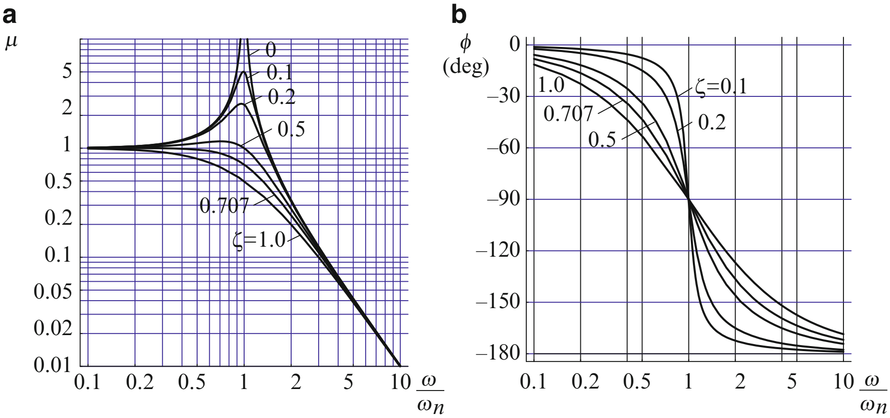


Fig. 2.38 Bode plots of second-order system for different values of damping ratio ζ: (**a**) magnification factor; and (**b**) phase

Similarly, for second-order systems, μ and ϕ are plotted versus the frequency ratio $\omega/\omega_n \equiv r_f$, which is also a dimensionless quantity. For both systems, the scales for μ, ω/a and ω/ω_n are logarithmic to a base of 10. Moreover, the Bode plots of second-order systems are shown for undamped, underdamped and critically damped systems; for underdamped systems, plots for different values of the damping ratio ζ are displayed.

Magnification factors are usually given in *decibels*, abbreviated db. A nondimensional variable x is given in a decibel scale according with the definition

$$x_{\mathrm{db}} = 20\log_{10}(x) \quad \mathrm{db} \tag{2.161}$$

That is, x increases by an order of magnitude every 20 db. Likewise, the interval in which the frequency ratio increases by one order of magnitude is called a *decade*, abbreviated dec; sometimes, the *octave* is used instead, which is defined as the interval over which the frequency ratio is doubled, and abbreviated oct. For most engineering applications, the decade is the most useful unit, for which reason, this is what we will use here.

Note that, from the Bode plots displayed in Figs. 2.37 and 2.38, it is apparent that for very low values of the frequency ratio, the plot of the magnification ratio flattens out at a constant value of unity. Likewise, for very high values of the same ratio, the same plot flattens out at a constant slope. Moreover, while this slope is of -20 db/dec for first-order systems, it is of -40 db/dec for second-order systems. Note, for example, the similarity between the magnification Bode plots for first-order and critically damped second-order systems. The only difference in these plots is their slope at high frequency ratios. Instruments with Bode plots of the forms of Figs. 2.37 and 2.38 are said to be *low-pass filters*, as they reject frequencies above the reference frequency, a or ω_n.

In general, n-order systems, which occur in multi-dof systems, have Bode plots that flatten out at high frequency ratios at values of $-20n$ db/dec.

2.7.7 Applications of the Harmonic Response

From the Bode plots of damped second-order systems, it is apparent that, at excitation frequencies ω that are very low with respect to the natural frequency ω_n of the system, the magnitude of the harmonic response of these systems is virtually the same as that of the excitation. By the same token, at very high frequencies, the magnitude of the harmonic response becomes negligibly low. In the subsections below we show how we can apply these observations to practical design problems. We may, for example, want to design a pneumatic hammer so that, at the tool-end, it will transmit a very high force, capable of breaking hard rock while, at the handle end, it will transmit a gentle force to the operator. Alternatively, we may want to design instruments to measure accurately displacement, velocity or acceleration. In these cases, we want to obtain a signal transmission with the highest fidelity, and so, we should make sure that the natural frequency of our instrument is tuned with the frequency range of the signal that we want to measure.

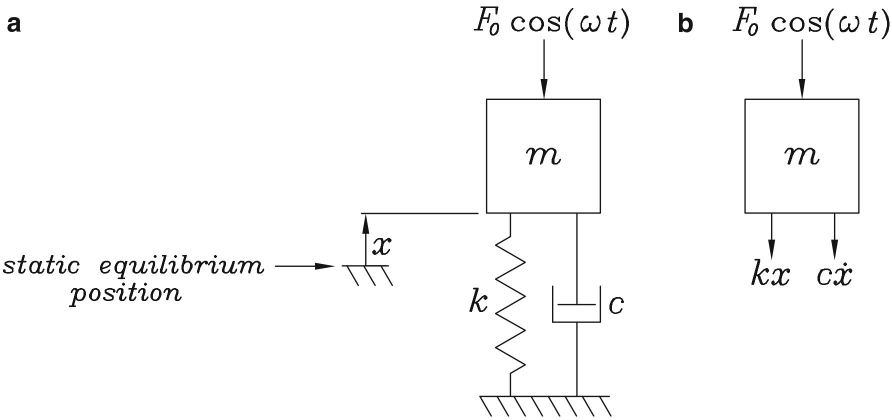


Fig. 2.39 A pneumatic hammer under harmonic excitation: (**a**) Iconic model and (**b**) its tool FBD

2.7.7.1 Vibration Isolation

Consider the model of a pneumatic rock-breaking hammer depicted in Fig. 2.39a, in which the force $F_0 \cos \omega t$ is applied by the rock on the cutting tool and the support, depicted as an inertial frame, is the handle, which is firmly held by the operator. The latter is now regarded as an inertial frame. We want to determine the relation between the force transmitted to the handle and the force applied onto the tool, for design purposes. Moreover, **the static force due to gravity in this and the examples below is irrelevant, as it does not vary harmonically, and hence, does not appear in our analysis**.

First we need to derive an expression for the transmitted force $F(t)$. From the free-body diagram of Fig. 2.39b, it is apparent that this force is the resultant of the forces acting at the ends of the spring and the dashpot, respectively. Hence,

$$F(t) = c\dot{x}_{CS} + kx_{CS}$$

where x_{CS} is the steady-state component of the displacement under a cosine excitation. Thus, the transmitted force is a linear combination of the steady-state displacement and velocity of the mass. Now, since the input is a harmonic function, so is the steady-state displacement, and hence, the velocity. The transmitted force is, then, harmonic as well, and we can fully determine it by its magnitude F_T and its phase ψ. For ease of manipulation, we represent the steady-state displacement in the form given by Eq. 2.102, and hence, the steady-state velocity is obtained by simple differentiation of the foregoing expression. For quick reference, we include expressions for these two items below:

$$x_{CS}(t) = M\cos(\omega t + \phi), \quad \dot{x}_{CS}(t) = -M\omega\sin(\omega t + \phi)$$

Therefore,

$$F(t) = -cM\omega\sin(\omega t + \phi) + kM\cos(\omega t + \phi) \tag{2.162}$$

and recalling that $c = 2\zeta\omega_n m$ and $k = \omega_n^2 m$, we have

$$\begin{aligned} F(t) &= -2\zeta\omega_n mM\omega\sin(\omega t + \phi) + \omega_n^2 mM\cos(\omega t + \phi) \\ &= \omega_n^2 mM\left[-2\zeta r_f\sin(\omega t + \phi) + \cos(\omega t + \phi)\right] \end{aligned}$$

Using Eqs. 2.103a and 2.103b, we may rewrite $F(t)$ as

$$F(t) = F_T\cos(\omega t + \psi) \tag{2.163}$$

where

$$F_T \equiv \omega_n^2 Mm\sqrt{1 + (2\zeta r_f)^2} \tag{2.164}$$

and, as the reader is invited to verify, M is given in this case as

$$M = \frac{F_0/m}{\omega_n^2\sqrt{(1 - r_f^2)^2 + (2\zeta r_f)^2}} \tag{2.165}$$

We have, therefore,

$$F_T \equiv \frac{F_0\sqrt{1 + (2\zeta r_f)^2}}{\sqrt{(1 - r_f^2)^2 + (2\zeta r_f)^2}} \tag{2.166}$$

Moreover, using Eq. 2.105b, we have

$$\psi \equiv \tan^{-1}(2\zeta r_f) + \phi = \tan^{-1}(2\zeta r_f) - \tan^{-1}\left(\frac{2\zeta r_f}{1 - r_f^2}\right) \tag{2.167}$$

The magnification factor μ_F of the transmitted force is thus defined as the ratio

$$\mu_F \equiv \frac{F_T}{F_0} = \sqrt{\frac{1 + (2\zeta r_f)^2}{(1 - r_f^2)^2 + (2\zeta r_f)^2}} \tag{2.168}$$

Plots of both the magnification factor μ_F and the phase angle ψ are shown in Fig. 2.40.

The similarity between the Bode plots of Figs. 2.38 and 2.40 is to be highlighted. Note that both magnification plots converge to the same asymptotes for very low frequency ratios, namely, at a constant value of unity. Furthermore, at very high frequency ratios, the Bode plots of Fig. 2.40 flatten out at a constant slope of -40 db/dec, as $\mu_F \to 0$. Moreover, the magnification plots of the same figure, for

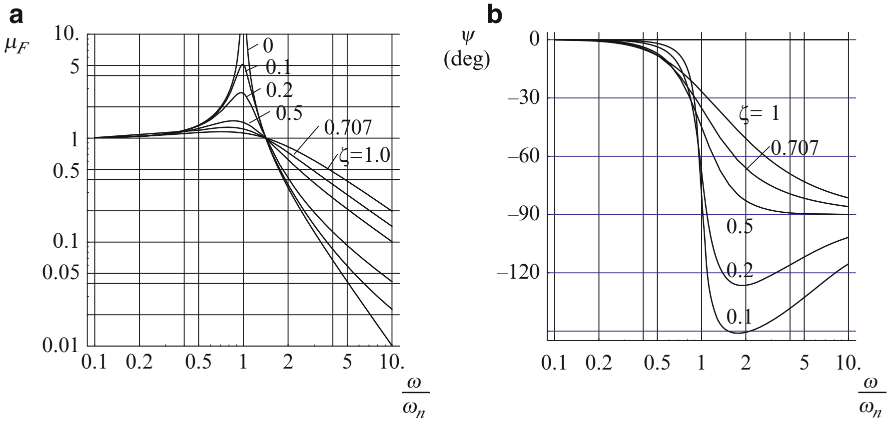


Fig. 2.40 The Bode plots of the force transmitted by a damped second-order system to its base: (**a**) magnification factor; and (**b**) phase

all values of damping coefficient ζ, cross the $\mu_F = 1$ axis at the same point. A quick calculation shows that the crossing point is located at $r_f = \sqrt{2}$, a fact that the reader is encouraged to verify.

With regard to the design of pneumatic hammers as described above, we can use the Bode plots of Fig. 2.40. From these plots it is apparent that the force transmitted to the operator will be negligibly small as long as the operation frequency ω of the rock-breaking force is substantially above $\sqrt{2}$ times the natural frequency of the tool-pneumatic cylinder of the hammer, when the cylinder is modeled as a spring-dashpot array in parallel, as shown in Fig. 2.39.

An alternative application of the above-derived Bode plots is in the design of foundations for equipment that is to remain stationary in the presence of parasitical vibration of the ground. Such vibration is usually present in environments where various machines are in operation. Equipment that should remain undisturbed by this vibration include instruments and precision machine tools. To illustrate this idea, let us assume that the equipment to be isolated from parasitical vibration can be safely modeled as a rigid body of mass m and the foundation as a spring-dashpot array in parallel, exactly as the model shown in Fig. 2.39a. Furthermore, we assume that the ground undergoes a vertical displacement $y(t) = Y\cos\omega t$. We want to relate the harmonic response of the system, $x(t)$, to this input. To this end, we sketch first the iconic model of the system in Fig. 2.41, and then derive its mathematical model.

The free-body diagram of the mass of Fig. 2.41 leads to

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

and hence, in normal form,

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{y} + \omega_n^2 y$$

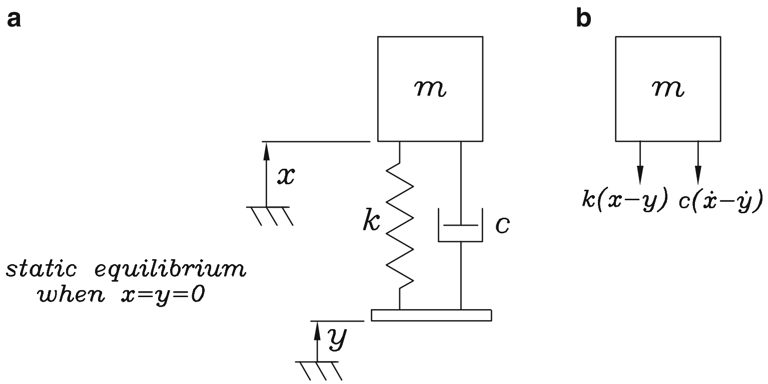


Fig. 2.41 The iconic model of a mass suspended on a viscoelastic foundation that is subjected to vibration

Further, by virtue of the harmonic form of the signal $y(t)$, we have

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x = -2\zeta\omega_n\omega Y\sin\omega t+\omega_n^2 Y\cos\omega t$$

That is, the system is subject to a linear combination of a sine and a cosine signal. Using Eq. 2.103a, we rewrite the right-hand side of the above expression in cosine form, thereby obtaining

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x = A\cos(\omega t+\sigma) \tag{2.169}$$

where

$$A \equiv \omega_n^2\sqrt{1+(2\zeta r_f)^2}\,Y, \qquad \sigma \equiv \arctan(2\zeta r_f)$$

Using Eq. 2.132a, the steady–state response of the system is given by

$$x_{CS}(t) = M\cos(\omega t+\sigma+\phi) \tag{2.170a}$$

where

$$M = \frac{A/\omega_n^2}{\sqrt{(1-r_f^2)^2+(2\zeta r_f)^2}} \tag{2.170b}$$

or

$$M = \frac{Y\sqrt{1+(2\zeta r_f)^2}}{\sqrt{(1-r_f^2)^2+(2\zeta r_f)^2}} \tag{2.170c}$$

and

$$\phi = -\tan^{-1}\left(\frac{2\zeta r_f}{1-r_f^2}\right) \tag{2.170d}$$

Hence, the magnification factor μ_Y of the transmitted motion is given by

$$\mu_Y \equiv \frac{M}{Y} \equiv \sqrt{\frac{1+(2\zeta r_f)^2}{(1-r_f^2)^2+(2\zeta r_f)^2}} \tag{2.171}$$

which is identical to the magnification factor found above for the transmitted force μ_F, and so, a Bode plot for this item is not needed.

For design purposes, then, we aim to design a foundation so that it will isolate the vibration $y(t)$ by giving the foundation a natural frequency that will be substantially below all parasitical frequencies. In this way, the parasitical frequency ratios will be substantially to the right of the value $r_f = \sqrt{2}$ of the Bode plot of Fig. 2.40.

One more application of the Bode plots of Fig. 2.40 is in the explanation of a phenomenon worrying the civil engineers in charge of road maintenance. Transport-regulating bodies specify the maximum allowable static load (mass) per axle in terrestrial vehicles, whether these transport people or goods. Thus, buses and trucks are subjected to the same regulations; transportation companies, with the aim of maximizing their profit per trip, load their units to the limit. However, as it turns out, buses produce much less damage to the road than trucks. The explanation can be found in the difference in the suspension of the two types of vehicles: those of buses are designed to give the passengers a comfortable ride; those of trucks are designed to require the least maintenance, and hence, the only damping that the latter have is the one that comes within the leaf springs that they use. In order to explain the difference that the damping makes, let us model the load-suspension system pertaining to one axle with the iconic model of Fig. 2.13a. Furthermore, we model the irregular road profile as a sinusoidal track, thereby ending up with the iconic model of Fig. 2.41, where $y = Y\cos\omega t$, and the Bode plots of Fig. 2.40 apply. Moreover, we can safely assume that trucks and buses, like any other terrestrial vehicle, have suspensions that produce vertical vibrations with a natural frequency of $\omega_n = 1$ Hz, the only difference in the two types of suspension then being in the damping. Now, the damping in trucks is very low, say, around 0.1, while that of buses can be safely assumed to be around 0.5. From Fig. 2.40a, it is apparent that μ for $\zeta = 0.1$ reaches a peak that is around four times that occurring at $\zeta = 0.5$. As a consequence, the truck transmits to the road a dynamic force that is around four times as big as that transmitted by the bus.

2.7.7.2 Velocity Meter Design

Contrary to the second example of the preceding section, we may want in some instances, such as when designing an instrument to measure the velocity of a moving body, to be able to pick up the desired velocity within its frequency range with the highest fidelity. What we mean by the latter is that we want to be able to subject a mass m to a harmonic displacement that will follow the velocity signal to be measured with the highest possible magnitude and the smallest possible phase angle.

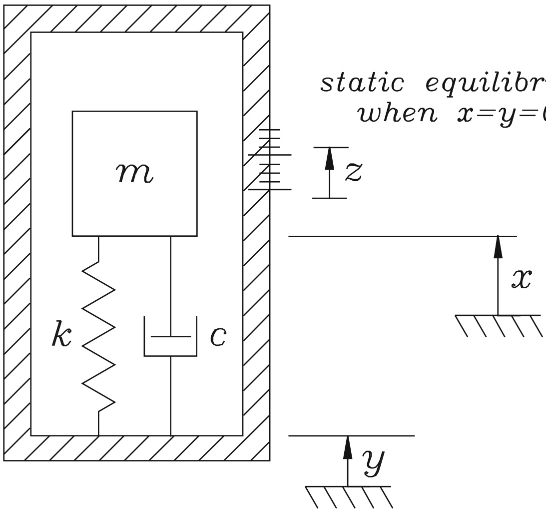


Fig. 2.42 An iconic model of the velocity meter

In order to attain this goal, obviously, the two signals, displacement of m and measured velocity, must have the same frequency. The purpose of this design task is to determine the physical parameters m, c and k of a velocity meter, modeled as a mass-spring-dashpot system, so that it will exhibit the harmonic response described above. Our iconic model in this case is shown in Fig. 2.42, where $y(t)$ is the displacement of the object whose velocity is to be measured, onto which the base of the instrument is firmly attached; moreover, x is the displacement of the mass m and the pick-up signal $z(t)$ is defined as $z \equiv y - x$. The mathematical model is now readily derived from Fig. 2.42 in the form

$$m\ddot{x} = -c(\dot{x} - \dot{y}) - k(x - y) \tag{2.172}$$

or

$$\ddot{x} + 2\zeta\omega_n(\dot{x} - \dot{y}) + \omega_n^2(x - y) = 0 \tag{2.173}$$

or, in terms of z, the measured signal,

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \ddot{y} \tag{2.174}$$

We now assume that the velocity signal $\dot{y}$ of interest is harmonic, of the form

$$\dot{y} = V\cos\omega t \tag{2.175}$$

and hence, the foregoing model takes the form

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\omega V\sin\omega t$$

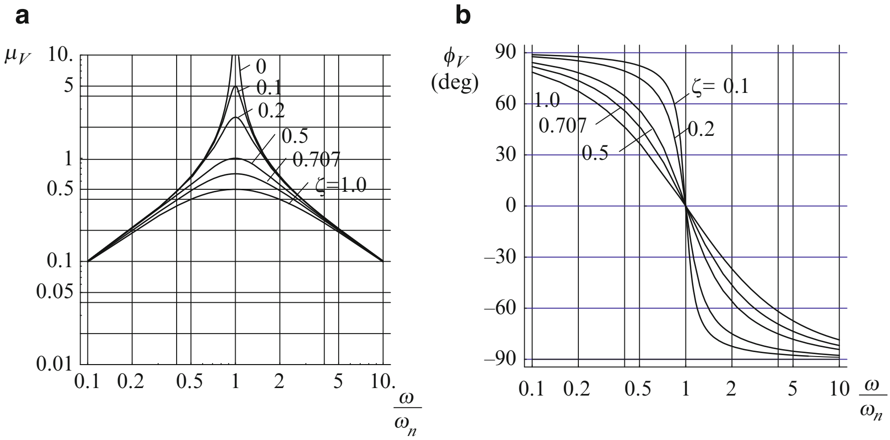


Fig. 2.43 The Bode plots of the velocity transmission of a velocity meter

However, since we want our instrument to follow the velocity signal, which is cosinusoidal, we have to express the foregoing model with a cosinusoidal excitation as well, which is done by simply adding a phase of 90° to the above sinusoidal signal, thus obtaining

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \omega V \cos\left(\omega t + \frac{\pi}{2}\right) \tag{2.176}$$

The steady-state response of the system, then, can be written as

$$z = \omega V M \cos(\omega t + \phi_V) \tag{2.177}$$

where M and ϕ_V take the forms of Eqs. 2.142a, b, except that the latter is augmented by a phase angle of $\pi/2$. Now, the magnification factor that we need is one relating $\omega V M$ with the amplitude of the excitation signal V, as a dimensionless quantity. We thus define the magnification factor of interest, along with the corresponding phase angle, as shown below:

$$\mu_V \equiv \frac{\omega_n \omega V M}{V} = \frac{r_f}{\sqrt{(1-r_f^2)^2 + (2\zeta r_f)^2}}, \qquad \phi_V = \frac{\pi}{2} - \arctan\left(\frac{2\zeta r_f}{1-r_f^2}\right) \tag{2.178}$$

which are plotted in Fig. 2.43.

An inspection of the Bode plots of Fig. 2.43 reveals that the frequency range in which the magnification factor of this instrument attains its peak value is remarkably narrow. Such an instrument is said to have a narrow *bandwidth*, which makes it useless for our envisioned application. A narrow bandwidth means that the magnification factor attains values above unity in a correspondingly narrow range

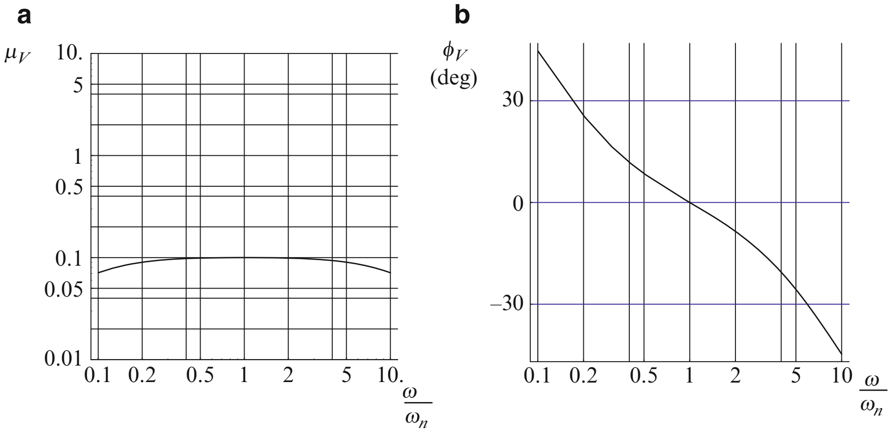


Fig. 2.44 The Bode plots of the velocity transmission of a velocity meter with a high damping ratio $\zeta = 5$

of values of r_f. A means of enhancing the bandwidth of this instrument is to design it with a rather high damping ratio, say $\zeta = 5$, as made apparent by the plot of Fig. 2.44a. From this plot, it is apparent that the velocity meter under study is useful for the measurement of velocity signals comprising a frequency ranging from $0.2\omega_n$ to about $10\omega_n$. The corresponding phase angle is shown in Fig. 2.44b.

2.7.7.3 Accelerometer Design

Accelerometers are designed to provide a displacement signal that follows as closely as possible a harmonic acceleration signal with a high amplitude and a low phase angle. We thus assume that the accelerometer is modeled as the system of Fig. 2.41, the acceleration signal $\ddot{y}(t)$ that we want to measure being harmonic, namely,

$$\ddot{y}(t) = A\cos\omega t$$

the underlying mathematical model thus taking the form

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \ddot{y}$$

with z defined as

$$z \equiv y - x$$

The steady-state response thus becoming

$$z(t) = M_A\cos(\omega t + \phi_A)$$

with M_A given as

$$M_A = \frac{A}{\omega_n^2\sqrt{(1-r_f^2)^2+(2\zeta r_f)^2}}$$

the phase being given as in Eq. 2.160, namely, as

$$\phi_A = -\arctan\left(\frac{2\zeta r_f}{1-r_f^2}\right)$$

Now, we define the magnification factor μ_A as

$$\mu_A \equiv \frac{\omega_n^2 M_A}{A}$$

thereby obtaining

$$\mu_A = \frac{1}{\sqrt{(1-r_f^2)^2+(2\zeta r_f)^2}}$$

which is identical to the magnification factor of Eq. 2.160 and plotted in Fig. 2.38.

2.7.7.4 Seismograph Design

The instrument of Fig. 2.42 can be used as a *seismogragh* if we want to measure a harmonic displacement $y(t)$ of the base with the pick-up signal $z \equiv y - x$. The mathematical model we have is identical to that derived for the accelerometer, i.e.,

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \ddot{y}$$

Upon the assumption that the displacement $y(t)$ is harmonic, we can write

$$y(t) = Y\cos\omega t$$

and hence,

$$\ddot{y} = -\omega^2 Y\cos\omega t$$

the mathematical model thus becoming

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = -\omega^2 Y\cos\omega t \equiv \omega^2 Y\cos(\omega t + \pi)$$

The steady-state response of the system is then of the form

$$z(t) = M_Y\cos(\omega t + \phi_Y)$$

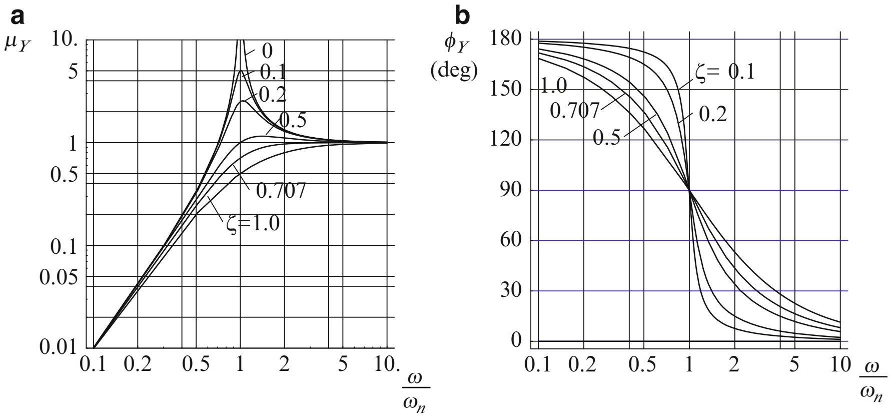


Fig. 2.45 The Bode plots of a seismograph

with M_Y and ϕ_Y given as

$$M_Y = \frac{r_f^2 Y}{\sqrt{(1-r_f^2)^2 + (2\zeta r_f)^2}}, \qquad \phi_Y = \pi - \arctan\left(\frac{2\zeta r_f}{1-r_f^2}\right)$$

The magnification factor μ_Y is thus defined as

$$\mu_Y \equiv \frac{M_Y}{Y} = \frac{r_f^2}{\sqrt{(1-r_f^2)^2 + (2\zeta r_f)^2}}$$

Plots of the magnification factor and the phase angle are shown in Fig. 2.45.

Note that the Bode plots of this instrument reveal that the seismograph is insensitive to low frequencies but very sensitive to high frequencies. The instrument is thus said to be a *high-pass filter*.

In summary, the Bode plots obtained here have a common feature: the difference between the high-frequency asymptote slope and the low-frequency asymptote slope is always +20 db/dec. This feature is common to all second-order systems.

2.7.8 Further Applications of Superposition

In the two examples below, we include further applications of superposition, while resorting to the time responses derived for harmonic inputs. We do this to analyze the response of a second-order system to a pulse-like input, which is not periodic.

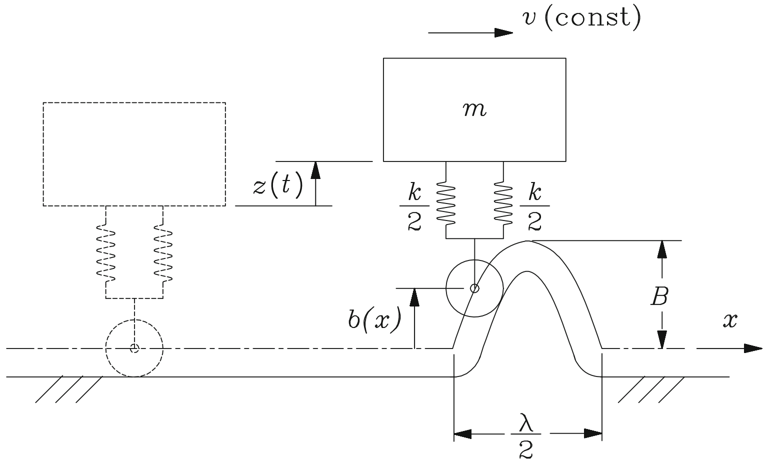


Fig. 2.46 The iconic model of a terrestrial vehicle upon encountering a bump

Example 2.7.1 (An Undamped Terrestrial Vehicle Hitting a Bump). A highly simplified iconic model of a terrestrial vehicle traveling at a constant speed v, upon hitting a bump of height B, is shown in Fig. 2.46.

In this model, we neglect the damping in the suspension and consider only its stiffness. Moreover, we assume that the bump does not affect the horizontal, uniform motion of the vehicle, the bump being modeled via the function $b(x)$, defined below:

$$b(x) \equiv \begin{cases} B\sin(2\pi x/\lambda), & \text{for} \quad 0 \le x \le \lambda/2; \\ 0 & \text{otherwise} \end{cases}$$

Here, we have $x = vt$ and hence, the period of the sine function, in terms of the wavelength λ, is $T = \lambda/v$. Determine the motion of the suspension-mass system after the vehicle has hit the bump.

Solution: In order to ease our task, we start by realizing that $b(x)$ can be synthesized as

$$b(x) = B\left(\sin\frac{2\pi x}{\lambda}\right)u(x) + B\left(\sin\frac{2\pi(x-\lambda/2)}{\lambda}\right)u\left(x-\frac{\lambda}{2}\right)$$

and so, we can obtain the desired response by superposition, i.e., as the sum of the responses to each of the two foregoing unilateral sinusoidal functions. The mathematical model of the system at hand, then, takes the form

$$\ddot{z} + \omega_n^2 z = \omega_n^2 B\left[\left(\sin\frac{2\pi vt}{\lambda}\right)u(t) + \left(\sin\frac{2\pi v(t-\lambda/(2v))}{\lambda}\right)u\left(t-\frac{\lambda}{2v}\right)\right]$$

where we have replaced $b(x)$ by $b(t)$ for consistency with the model, since the independent variable of the latter is time. Moreover, we assume that the vehicle travels undisturbed up until it hits the bump, at which time we start observing the system and set arbitrarily $t = 0$. The time response of interest can now be derived by superimposing the time responses of each of the two sine inputs of the above model. We start, then, by deriving the response to the first input. To this end, we use the total response $x_S(t)$ derived in Eq. 2.143 for an undamped second-order system acted upon by a harmonic input of the form $(\sin \omega t)u(t)$, with zero initial conditions, namely,

$$x_S(t) = \frac{1/\omega_n^2}{1-r_f^2}\left(\sin \omega t - r_f \sin \omega_n t\right), \qquad \omega \equiv \frac{2\pi v}{\lambda} \equiv \frac{2\pi}{T}, \qquad r_f = \frac{\omega}{\omega_n} = \frac{2\pi v}{\lambda}\sqrt{\frac{m}{k}}$$

That is, the time it takes the vehicle to traverse the bump is one-half of the period T associated with the wavelength λ and the speed v. The total response $z(t)$ of the system at hand is now derived by linearity and time-invariance as

$$\begin{aligned} z(t) = \tfrac{B}{1-r_f^2} \Big\{ & \left[\sin \omega t - r_f \sin \omega_n t\right] u(t) \\ & + \left[\sin \omega \left(t - \frac{T}{2}\right) - r_f \sin \omega_n \left(t - \frac{T}{2}\right)\right] u\left(t - \frac{T}{2}\right) \Big\} \end{aligned}$$

or, after some rearrangement of terms,

$$\begin{aligned} z(t) = & \frac{B}{1-r_f^2}\left\{(\sin \omega t)u(t) + \left[\sin \omega \left(t - \frac{T}{2}\right)\right] u\left(t - \frac{T}{2}\right)\right\} \\ & - \frac{Br_f}{1-r_f^2}\left\{(\sin \omega_n t)u(t) + \left[\sin \omega_n \left(t - \frac{T}{2}\right)\right] u\left(t - \frac{T}{2}\right)\right\} \end{aligned}$$

Note that the first term in the foregoing expression is nothing but the bump function $b(t)$ divided by $(1-r_f^2)$. Now, as to the second term, this looks like a bump as well, but it is not. Indeed, this term is a linear combination of two harmonics of the same kind, i.e., of sines, with one delayed with respect to the other by half a period of the bump signal. Since the natural frequency ω_n of the system is, in general, different from the bump frequency ω, the two sinusoidal terms of the above expression do no cancel each other. We thus have, in summary,

$$z(t) = \frac{1}{1-r_f^2}\left[b(t) - r_f \beta(t)\right]$$

where

$$\beta(t) \equiv B\left\{(\sin \omega_n t)u(t) + \left[\sin \omega_n \left(t - \frac{T}{2}\right)\right] u\left(t - \frac{T}{2}\right)\right\}$$

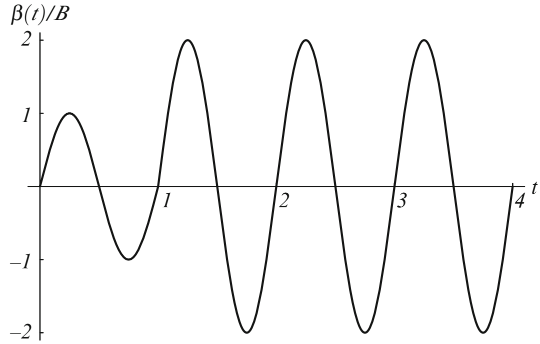


Fig. 2.47 The plot of $\beta(t)/B$ appearing in the response of the vehicle with an undamped suspension upon hitting a bump, for $\omega_n = 1\,\text{Hz} = 2\pi\,\text{rad/s}$ and $T = 2$ s

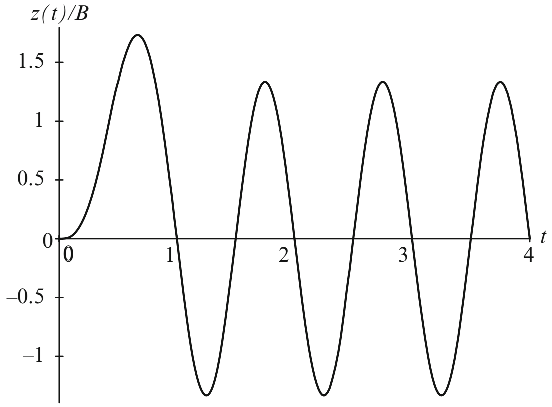


Fig. 2.48 Nondimensional time response of the undamped suspension to a bump with $\omega_n = 1$ Hz $= 2\pi$ rad/s and $T = 2$ s

The ratio $\beta(t)/B$ is plotted in Fig. 2.47 for $\omega_n = 1\,\text{Hz} = 2\pi\,\text{rad/s}$ and $T = 2$ s, the total time response $z(t)$ being plotted in Fig. 2.48 in nondimensional form.

That is, the response of the undamped system to a bump is a linear combination of the bump and the function $\beta(t)$, the amplitude of this response increasing as the difference $1 - r_f^2$ decreases. Note that this difference vanishes when the frequency of the bump equals that of the spring-mass system, which occurs, in terms of the given parameters, when

$$\frac{\lambda}{v} = \frac{2\pi}{\omega_n}$$

Most terrestrial vehicles have a natural frequency of about 1 Hz, i.e., of $\omega_n = 2\pi\ \text{s}^{-1}$. If we were going to encounter a sinusoidal bump with a wavelength λ obeying the foregoing relation, for a given traveling speed v, we would like to have an idea of the order of magnitude of the parameters involved, for realistic situations. For example, if the traveling speed is around 72 km/h, then $v = 20$ m/s, which thus would yield a value $\lambda = 20$ m, a rather huge wavelength, and hence, not realistic. A more realistic value of λ is 1 m, which, for the same natural frequency of the vehicle, gives a velocity $v = 1$ m/s or 3.6 km/h, a rather unusually small value.

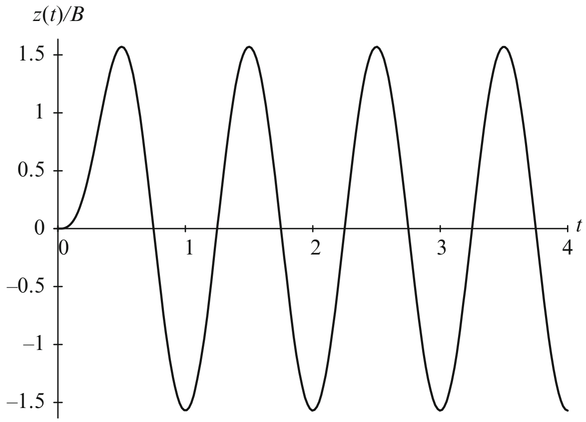


Fig. 2.49 Nondimensional time response of the undamped suspension with $\omega_n = 1$ Hz and $T = 2\pi/\omega_n = 1$ s

While a value of $r_f = 1$ is, therefore, unlikely to happen in real-life situations, it is instructive to find the response of the vehicle under this condition. To this end, we regard function $z(t)$, as given above, as a function of r_f, i.e.,

$$z(r_f) \equiv \frac{N(r_f)}{D(r_f)} \equiv F(r_f)$$

with $N(r_f)$ and $D(r_f)$ given by

$$N(r_f) \equiv b(t; r_f) - r_f \beta(t), \quad D(r_f) \equiv 1 - r_f^2 \tag{2.179}$$

By application of L'Hospital's rule, we have

$$F(1) = \lim_{r_f \to 1} \frac{N'(r_f)}{D'(r_f)}$$

and hence, as the reader is invited to verify,

$$z(t) = -\frac{1}{2}\left\{t\dot{b}(t) - b(t) - \pi B[\cos(\omega_n t - \pi)]u(t - \frac{\pi}{\omega_n})\right\}, \text{ for } \quad r_f = 1 \text{ or } \omega_n \lambda = 2\pi v$$

Therefore, the vertical oscillations of the vehicle under study remain bounded, even under these apparently resonant conditions. The explanation is simple: r_f refers to the ratio ω/ω_n, where $\omega = 2\pi r/\lambda$ for the system of Fig. 2.46, with r denoting the radius of the wheel. However, notice that the input to this system is not harmonic, but a bump. The foregoing response is plotted in Fig. 2.49 for $\omega_n = 1$ Hz and $r_f = 1$.

It is pointed out here that, the bump being zero for $t > \lambda/v$, it disappears from the time response after the vehicle has traversed it. Hence, the remainder of $z(t)$, namely, the term containing the function $\beta(t)$, is the steady-state response of the system.

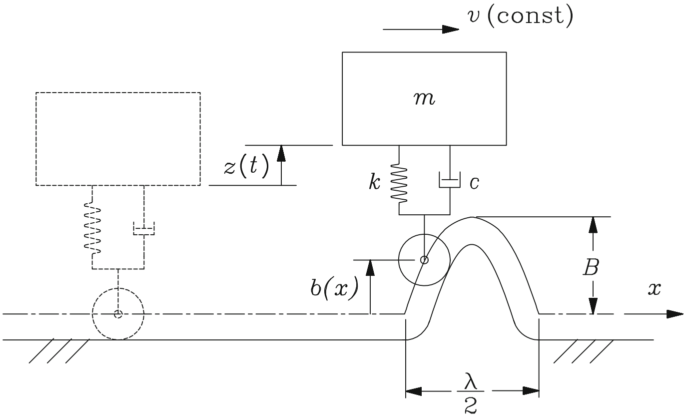


Fig. 2.50 The iconic model of a terrestrial vehicle upon encountering a bump

Example 2.7.2 (A Damped Terrestrial Vehicle Encountering a Bump). Find the time response of the vehicle introduced in the foregoing example, if we add "shocks" to its suspension, as shown in Fig. 2.50, under the assumption that the system at hand becomes underdamped.

Solution: The model now takes the form

$$\ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = 2\zeta\omega_n\dot{b}(t) + \omega_n^2 b(t)$$

and hence, the response can be synthesized as the sum of the two responses $z_b(t)$ and $z_{\dot{b}}(t)$ to the first and the second terms of the right-hand side, respectively. We start by calculating $\dot{b}(t)$, namely,

$$\dot{b}(t) = \frac{db(x)}{dx}\frac{dx}{dt} \equiv \frac{db(x)}{dx}v = b'(x)v$$

and then find $b'(x)$. From the expression derived in the previous example for $b(x)$, it is now a simple matter to show that

$$b'(x) = \frac{2\pi B}{\lambda}\left[\left(\cos\frac{2\pi x}{\lambda}\right)u(x) + \left(\cos\frac{2\pi(x-\lambda/2)}{\lambda}\right)u\left(x-\frac{\lambda}{2}\right)\right]$$

2.7.9 Derivation of $z_b(t)$

To derive $z_b(t)$ we use the $x_S(t)$ response for the second-order underdamped system derived in Sect. 2.7.3, and invoke linearity and time-invariance, thereby obtaining

$$z_b(t) = \omega_n^2 B x_S(t) + \omega_n^2 B x_S\left(t - \frac{T}{2}\right)$$

and hence,

$$z_b(t) = -\frac{B}{(1-r_f^2)^2+4\zeta^2 r_f^2}\left[r_f e^{-\zeta\omega_n t}\left(\frac{1-2\zeta^2-r_f^2}{\sqrt{1-\zeta^2}}\sin\omega_d t - 2\zeta\cos\omega_d t\right)\right.$$
$$\left. + 2\zeta r_f\cos\omega t + (r_f^2-1)\sin\omega t\right]u(t)$$
$$-\frac{B}{\left(1-r_f^2\right)^2+4\zeta^2 r_f^2}\left\{r_f e^{-\zeta\omega_n\left(t-\frac{T}{2}\right)}\left[\frac{1-2\zeta^2-r_f^2}{\sqrt{1-\zeta^2}}\sin\omega_d\left(t-\frac{T}{2}\right)\right.\right.$$
$$\left.\left.-2\zeta\cos\omega_d\left(t-\frac{T}{2}\right)\right]+2\zeta r_f\cos\omega\left(t-\frac{T}{2}\right)+(r_f^2-1)\sin\omega\left(t-\frac{T}{2}\right)\right\}u\left(t-\frac{T}{2}\right)$$

which can be recast in the form

$$z_b(t) = \frac{-1}{(1-r_f^2)^2+4\zeta^2 r_f^2}\left\{B r_f e^{-\zeta\omega_n t}\left(\frac{1-2\zeta^2-r_f^2}{\sqrt{1-\zeta^2}}\sin\omega_d t - 2\zeta\cos\omega_d t\right)u(t)\right.$$
$$+B r_f e^{-\zeta\omega_n\left(t-\frac{T}{2}\right)}\left[\frac{1-2\zeta^2-r_f^2}{\sqrt{1-\zeta^2}}\sin\omega_d\left(t-\frac{T}{2}\right)-2\zeta\cos\omega_d\left(t-\frac{T}{2}\right)\right]u\left(t-\frac{T}{2}\right)$$
$$+2\zeta r_f B\left[(\cos\omega t)\,u(t)+\cos\omega\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)\right]$$
$$\left.+(r_f^2-1)B\left[(\sin\omega t)\,u(t)+\sin\omega\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)\right]\right\}$$

2.7.9.1 Derivation of $z_{\dot{b}}(t)$

This response is derived likewise, i.e., by means of the response $x_C(t)$ of an underdamped system to a unilateral cosine input, namely,

$$z_{\dot{b}}(t) = (2\zeta\omega_n)\omega B x_C(t)u(t) + (2\zeta\omega_n)\omega B x_C\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)$$

Hence, upon substituting both $x_C(t)$ and $x_C(t-T/2)$ into the foregoing expression, we obtain

$$z_{\dot{b}}(t) = \frac{2\zeta r_f B}{(1-r_f^2)^2+4\zeta^2 r_f^2}\left\{e^{-\zeta\omega_n t}\left[\frac{-\zeta(1+r_f^2)}{\sqrt{1-\zeta^2}}\sin\omega_d t-(1-r_f^2)\cos\omega_d t\right]\right.$$
$$\left.+2\zeta r_f\sin\omega t+(1-r_f^2)\cos\omega t\right\}u(t)$$

$$+\frac{2\zeta r_f B}{\left(1-r_f^2\right)^2+4\zeta^2 r_f^2}\left\{e^{-\zeta\omega_n\left(t-\frac{T}{2}\right)}\left[\frac{-\zeta(1+r_f^2)}{\sqrt{1-\zeta^2}}\sin\omega_d\left(t-\frac{T}{2}\right)\right.\right.$$
$$\left.-(1-r_f^2)\cos\omega_d\left(t-\frac{T}{2}\right)\right]+2\zeta r_f\sin\omega\left(t-\frac{T}{2}\right)$$
$$\left.+(1-r_f^2)\cos\omega\left(t-\frac{T}{2}\right)\right\}u\left(t-\frac{T}{2}\right)$$

which can be rewritten as

$$z_{\dot{b}}(t)=\frac{2\zeta r_f}{(1-r_f^2)^2+4\zeta^2 r_f^2}\left\{Be^{-\zeta\omega_n t}\left[\frac{-\zeta(1+r_f^2)}{\sqrt{1-\zeta^2}}\sin\omega_d t-(1-r_f^2)\cos\omega_d t\right]u(t)\right.$$
$$+Be^{-\zeta\omega_n\left(t-\frac{T}{2}\right)}\left[\frac{-\zeta(1+r_f^2)}{\sqrt{1-\zeta^2}}\sin\omega_d\left(t-\frac{T}{2}\right)\right.$$
$$\left.-(1-r_f^2)\cos\omega_d\left(t-\frac{T}{2}\right)\right]u\left(t-\frac{T}{2}\right)$$
$$+2\zeta r_f B\left[(\sin\omega t)u(t)+\left(\sin\omega\left(t-\frac{T}{2}\right)\right)u\left(t-\frac{T}{2}\right)\right]$$
$$\left.+(1-r_f^2)B\left[(\cos\omega t)u(t)+\left(\cos\omega\left(t-\frac{T}{2}\right)\right)u\left(t-\frac{T}{2}\right)\right]\right\}$$

The total response $z(t)$ of the vehicle is, then,

$$z(t)=z_b(t)+z_{\dot{b}}(t)$$

Shown in Fig. 2.51 are the individual responses $z_b(t)$, $z_{\dot{b}}(t)$ and their sum $z(t)$, for $\omega_n = 1\,\mathrm{Hz} = 2\pi\,\mathrm{rad/s}$, $\zeta = 0.707$ and $T = 2$ s.
Animation of this time response is available in `3-DampedBump1dof.mw`.

2.8 The Periodic Response

In this section, we will assume that the systems under study are subjected to an arbitrary, though periodic, input; otherwise, conditions are similar to those of the harmonic response. Thus, we assume that a very long time has elapsed since the system under analysis was first perturbed. This is the case in many real-life situations, like aircraft fuselage under vibrations produced by the turbines at cruising speed or bodies of terrestrial vehicles under vibrations produced by the engine running at constant r.p.m., and so forth. In these examples, the source of vibration is either a turbine or an IC engine, both running at uniform angular velocity, thus producing a periodic, although non-harmonic, input.

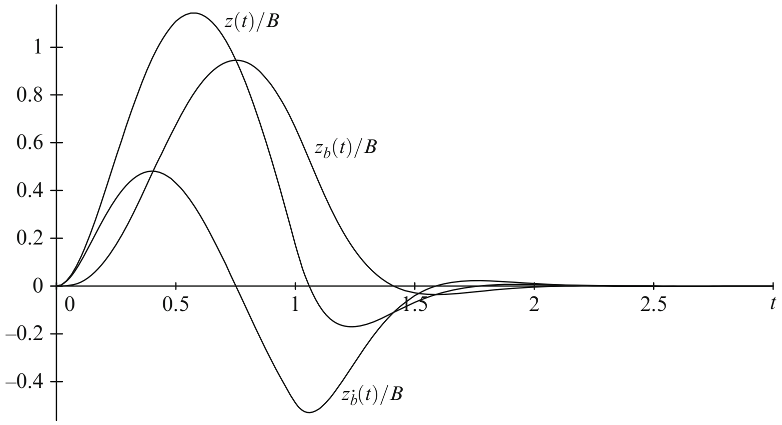


Fig. 2.51 Time response of an underdamped suspension encountering a bump

By means of *Fourier analysis*, we will decompose any periodic input into an infinite summation, i.e., a series, of harmonics of multiples of a fundamental frequency. To do this, we need some background, as described below:

2.8.1 Background on Fourier Analysis

We begin with a few definitions that will prove useful in the ensuing analysis. A function $f(t)$ is *periodic* of period T if, for any real value of t,

$$f(t) = f(t+T) \tag{2.180}$$

An *even function* $f(t)$ is a function with the property

$$f(t) = f(-t) \tag{2.181}$$

while an *odd function* $f(t)$ obeys

$$f(t) = -f(-t) \tag{2.182}$$

Now, a function need not be even or odd *everywhere*, i.e., for any value of t, to be of interest to us. Since we are interested in periodic functions, we will look at the functions under study only within a period of length T, and so, we may add that a certain function is even or odd *within the interval* $[-T/2, +T/2]$ if the above definitions hold not necessarily everywhere, but within the said interval. Moreover, we recall below a few facts that will prove to be useful in this discussion. The pertinent proofs are available in any book on Fourier analysis or partial differential equations.

Fact 1: Let a function $f(t)$ be even in the interval $[-T/2,+T/2]$ and a function $g(t)$ be odd in the same interval. The integral of the product of these functions over the same interval vanishes.

An example of even function is the cosine function, one of odd function is the sine function. Note that the cosine and the sine functions are even or, correspondingly, odd, everywhere.

It is now apparent that the sum of an arbitrary number of even functions is even as well, a similar statement holding for odd functions. Here, we must realize that, if a function is not even, it need not be odd, while keeping in mind the result below:

Fact 2: Any function $f(t)$ can always be decomposed into the sum of an even and an odd component, $f_E(t)$ and $f_O(t)$, respectively, which are given by

$$f_E(t) \equiv \frac{1}{2}[f(t)+f(-t)], \quad f_O(t) \equiv \frac{1}{2}[f(t)-f(-t)] \tag{2.183}$$

Given *any periodic* function $f(t)$ of period T, its *Fourier expansion* is given below:

$$f(t) = a_0 + \sum_1^\infty a_k \cos\left(\frac{2\pi k}{T}t\right) + \sum_1^\infty b_k \sin\left(\frac{2\pi k}{T}t\right) \tag{2.184a}$$

where

$$a_0 = \frac{1}{T}\int_{\mathscr{I}} f(t)dt \tag{2.184b}$$

$$a_k = \frac{2}{T}\int_{\mathscr{I}} f(t)\cos\left(\frac{2\pi kt}{T}\right)dt, \quad k = 1,2,\ldots,\text{etc.} \tag{2.184c}$$

$$b_k = \frac{2}{T}\int_{\mathscr{I}} f(t)\sin\left(\frac{2\pi kt}{T}\right)dt, \quad k = 1,2,\ldots,\text{etc.} \tag{2.184d}$$

with $\mathscr{I}$ denoting *any* interval of length T, such as $[-T/2,T/2]$ or $[0,T]$.

We note that a_0 is nothing but the *mean value* of $f(t)$ in the interval of integration. Moreover, $\omega_0 = 2\pi/T$ is known as the *fundamental frequency* of $f(t)$.

The process under which the foregoing coefficients a_k and b_k are determined, for a given periodic function $f(t)$, is known as *Fourier analysis* or *spectral analysis*. Needless to say, the Fourier analysis of a given function can be conducted with paper and pencil by quadrature of the underlying integrals or by table-lookup only in special textbook-type of cases. Otherwise, the analyst has to resort to numerical methods. Note that the table-lookup procedure can be greatly eased and broadened if symbolic computation software is used. In many instances, integrals that are not available in tables can be found with the aid of this type of software.

Example 2.8.1 (A Simple Example). Find the Fourier expansion of the function

$$f(t) = \sin^2 \omega_0 t$$

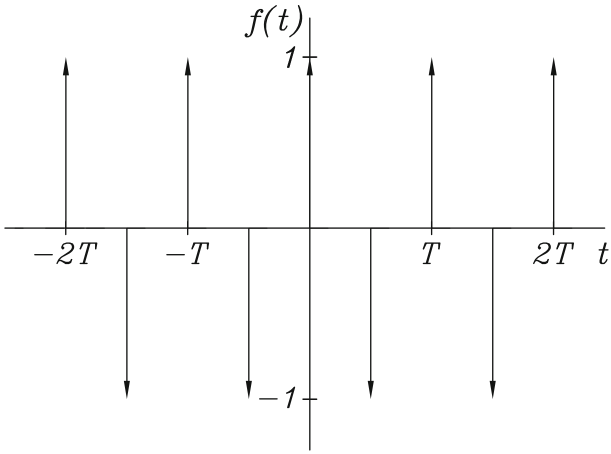


Fig. 2.52 A train of impulses

Solution: This is a periodic function of fundamental period $T = 2\pi/\omega_0$. Its Fourier expansion is particularly simple because we do not have to go through the calculation of its Fourier coefficients from integrals. Indeed, our task will be greatly eased if we recall the identity below:

$$\sin^2 \omega_0 t \equiv \frac{1}{2} - \frac{1}{2}\cos 2\omega_0 t$$

which is already the Fourier expansion of the given function, and hence,

$$a_0 = \frac{1}{2}, \quad a_2 = -\frac{1}{2}$$

all other coefficients vanishing.

Example 2.8.2 (Fourier Analysis of a Train of Impulses). Obtain the Fourier expansion of the train of impulses given below:

$$f(t) = \sum_{i=-\infty}^{\infty} (-1)^i \delta\left(t - \frac{iT}{2}\right)$$

which is plotted in Fig. 2.52.

Solution: From symmetry considerations it is apparent that the mean value of the function at hand is zero. It is also apparent that this function is even, and hence,

$$a_0 = 0, \quad b_k = 0, \quad k = 1, 2, \ldots$$

The remaining Fourier coefficients are calculated below:

$$\begin{aligned}
a_k &= \frac{2}{T}\int_{-T^-/2}^{+T^-/2} f(t)[\cos(k\omega_0 t)]dt \\
&= \frac{2}{T}\int_{-T^-/2}^{+T^-/2} \left[\sum_{i=-\infty}^{\infty}(-1)^i\delta\left(t-\frac{iT}{2}\right)\right]\cos\left(\frac{2\pi kt}{T}\right)dt \\
&= \frac{2}{T}\sum_{i=-\infty}^{\infty}(-1)^i\int_{-T^-/2}^{+T^-/2}\delta\left(t-\frac{iT}{2}\right)\cos\left(\frac{2\pi kt}{T}\right)dt
\end{aligned}$$

where the integration extremes have been set so as to take into account the discontinuities in the impulse functions in the integrand.

It is apparent that the interval of integration comprises only two impulses, those at $t=-T/2$ and at $t=0$, the impulse at $t=+T/2$ being left out of the interval. Hence, the expression for a_k reduces to

$$\begin{aligned}
a_k &= \frac{2}{T}\int_{-T^-/2}^{+T^-/2}\left[-\delta\left(t+\frac{T}{2}\right)+\delta(t)\right]\cos\left(\frac{2\pi kt}{T}\right)dt \\
&= \frac{2}{T}\left[-\int_{-T^-/2}^{+T^-/2}\delta\left(t+\frac{T}{2}\right)\cos\left(\frac{2\pi kt}{T}\right)dt+\int_{-T^-/2}^{+T^-/2}\delta(t)\cos\left(\frac{2\pi kt}{T}\right)dt\right]
\end{aligned}$$

In light of the identity appearing in Eq. 2.47, the above integrands are evaluated as

$$\begin{aligned}
\delta\left(t+\frac{T}{2}\right)\cos\left(\frac{2\pi kt}{T}\right) &= \delta\left(t+\frac{T}{2}\right)\cos\left[\frac{2\pi k}{T}\left(-\frac{T}{2}\right)\right] \\
&= \delta\left(t+\frac{T}{2}\right)\cos(k\pi) = \delta\left(t+\frac{T}{2}\right)(-1)^k \\
\delta(t)\cos\left(\frac{2\pi kt}{T}\right) &= \delta(t)\cos\left[\frac{2\pi k}{T}(0)\right] = \delta(t)
\end{aligned}$$

Therefore,

$$a_k = \frac{2}{T}\left[-\int_{-T^-/2}^{+T^-/2}\delta\left(t+\frac{T}{2}\right)(-1)^k dt+\int_{-T^-/2}^{+T^-/2}\delta(t)dt\right] = \frac{2}{T}\left[-(-1)^k+1\right]$$

and hence,

$$a_{2k-1} = \frac{4}{T}, \quad a_{2k} = 0$$

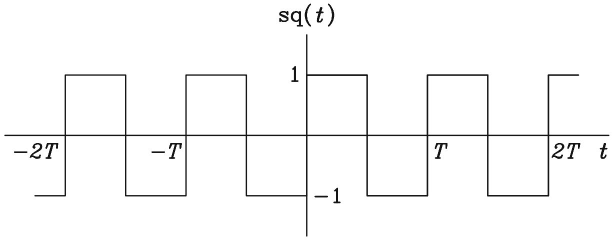


Fig. 2.53 A square wave

i.e., the Fourier expansion of the train of impulses contains only cosine terms of odd-numbered frequencies, all these terms with identical coefficients, namely,

$$f(t) \equiv \sum_{i=0}^{\infty}(-1)^i\delta\left(t-\frac{iT}{2}\right) = \frac{4}{T}\sum_{k=1}^{\infty}\cos\left[\frac{2(2k-1)\pi}{T}t\right]$$

Example 2.8.3 (Fourier Analysis of a Square Wave). Determine the Fourier expansion of the square wave $\mathrm{sq}(t)$ displayed in Fig. 2.53.

Solution: It is not very difficult to realize that the integral of the train of impulses yields a train of pulses, a pulse having the shape depicted in Fig. 2.33b. Indeed, every negative impulse produces a sudden decrease of one unity, while every positive impulse produces an increase of one unity in the above integral. We can therefore realize that the integral of $f(t)$ is a signal that oscillates, about a given mean value $\overline{f}$, between $-1/2$ and $+1/2$, the jumps occurring every time an impulse fires. Therefore, the above integral undergoes oscillations of amplitude equal to $1/2$, while the given square wave undergoes oscillations of unit amplitude. As a consequence, we have

$$\mathrm{sq}(t) = 2\int_t f(\theta)d\theta$$

where the integral has been left indefinite and the integration takes place in the domain of the dummy variable θ. Moreover, because of the linearity of the integral operation,[11] the Fourier series of the integral of the train of impulses is the integral of the Fourier series of the train, i.e.,

$$\mathrm{sq}(t) = \frac{8}{T}\sum_{k=1}^{\infty}\int_t \cos\left[\frac{2(2k-1)\pi}{T}\theta\right]d\theta + C$$

[11]The integral operation is both additive and homogeneous, and hence, linear.

where the integration constant is found from the condition that the mean value of sq(t) is zero, which is apparent from Fig. 2.53. Moreover, upon evaluation of the integral,

$$\mathrm{sq}(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\left[\frac{2(2k-1)\pi}{T}t\right] + C$$

Since the mean value of each term of the series is zero, that of the whole series is zero as well, and hence, $C = 0$, which thus leads to

$$\mathrm{sq}(t) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\left[\frac{2(2k-1)\pi}{T}t\right]$$

Alternatively, we can proceed to evaluate the Fourier coefficients by direct integration, as we show below. To this end, we observe first that the square wave is an odd function with zero mean value and hence,

$$a_k = 0, \quad k = 0, 1, 2, \ldots$$

the remaining coefficients being calculated from

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2}\mathrm{sq}(t)\sin\left(\frac{2\pi kt}{T}\right)dt$$

which, after evaluation and simplification, yields

$$b_k = \frac{2}{\pi k}\left[1-\cos(\pi k)\right]$$

and hence,

$$b_k = \begin{cases} 4/(\pi k), & \text{for} \quad k \quad \text{odd;} \\ 0, & \text{for} \quad k \quad \text{even} \end{cases}$$

Therefore,

$$\begin{aligned} \mathrm{sq}(t) &= \frac{4}{\pi}\sum_{1}^{\infty}\frac{1}{2k-1}\sin\left[\frac{2\pi(2k-1)t}{T}\right] \equiv \frac{4}{\pi}\sum_{0}^{\infty}\frac{1}{2k+1}\sin\left[\frac{2\pi(2k+1)t}{T}\right] \\ &= \frac{4}{\pi}\left[\sin\left(\frac{2\pi t}{T}\right) + \frac{1}{3}\sin\left(\frac{6\pi t}{T}\right) + \frac{1}{5}\sin\left(\frac{10\pi t}{T}\right) + \ldots\right] \end{aligned}$$

which thus confirms the result obtained above.

Example 2.8.4 (Fourier Analysis of a Monotonic Function). Shown in Fig. 2.54 is the profile of a pre-Columbian pyramid, of those found by tourists when visiting archeological sites in Mexico. This profile is labelled pyr(x) in that figure, in which the *wave length* λ now plays the role of the period T in the previous examples. In

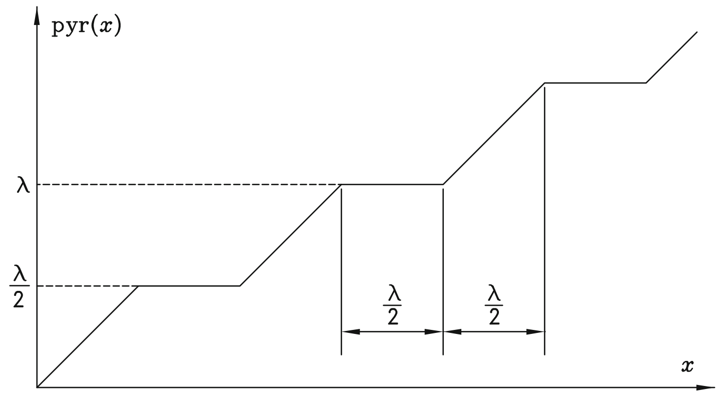


Fig. 2.54 A Pre-Columbian pyramid

thinking of tourists using wheelchairs, we would like to design the suspension of a motor-driven wheelchair to help these tourists climb up these pyramids comfortably. In order to do this, let us find the Fourier expansion of the profile.

Solution: We first note that the given profile can be obtained from the integral of the square wave of Fig. 2.53 when shifted 1 upwards. Thus, all we need to obtain the Fourier series of the given profile is (a) add 1 to the square-wave Fourier expansion derived above, (b) integrate each term of the Fourier series thus resulting, and (c) divide each integral by 2, i.e.,

$$\mathrm{pyr}(x) = \frac{1}{2}\left[\int_x \left\{1 + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\left[\frac{2\pi(2k-1)\xi}{\lambda}\right]\right\} d\xi\right]$$

in which ξ is a dummy variable of integration. Hence,

$$\mathrm{pyr}(x) = \frac{1}{2}x - \frac{\lambda}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\left[\frac{2\pi(2k-1)x}{\lambda}\right] + C$$

where C is a constant of integration that can be obtained by noting that $\mathrm{pyr}(0) = 0$ in Fig. 2.54. Thus,

$$0 = C - \frac{\lambda}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

But, as the reader can verify, e.g., using computer algebra,

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

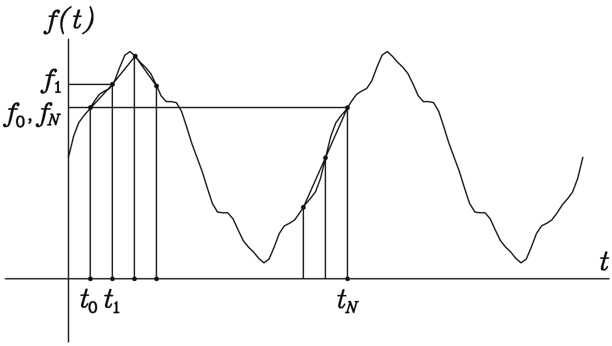


Fig. 2.55 Polygonal approximation of function $f(t)$

and hence,

$$C = \frac{\lambda}{\pi^2}\frac{\pi^2}{8} = \frac{\lambda}{8}$$

Therefore,

$$\begin{aligned}
\mathrm{pyr}(x) &= \frac{\lambda}{8} + \frac{1}{2}x - \frac{\lambda}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos\left[\frac{2\pi(2k-1)x}{\lambda}\right] \\
&= \frac{\lambda}{8} + \frac{1}{2}x - \frac{\lambda}{\pi^2}\left[\cos\left(\frac{2\pi x}{\lambda}\right) + \frac{1}{9}\cos\left(\frac{6\pi x}{\lambda}\right) + \frac{1}{25}\cos\left(\frac{10\pi x}{\lambda}\right) + \ldots\right]
\end{aligned}$$

Notice that the above expansion includes a linear term, which accounts for the monotonicity of $\mathrm{pyr}(x)$.

2.8.2 The Computation of the Fourier Coefficients

Here we study the procedure under which the Fourier coefficients of a periodic function are computed using a numerical approach. We aim to compute the coefficients $\{a_k\}_0^\infty$ and $\{b_k\}_1^\infty$ of the Fourier expansion of a periodic function $f(t)$. We shall do this by simple *numerical quadrature*, as explained by Kahaner et al. [4]. Various schemes of numerical quadrature are available, the one that lends itself best to our purposes is that based on the *trapezoidal rule*. In this scheme, the interval of integration is divided into a number N of subintervals $t_{i-1} \le t \le t_i$, for $i = 1, \ldots, N$, and a set of $N+1$ points of coordinates (t_i, f_i), with $f_i \equiv f(t_i)$, for $i = 0, 1, \ldots, N$, is defined in the $f(t)$-vs.-t plane. In the next step, every pair of neighboring points is joined with a straight line, thereby obtaining a *polygonal approximation* of the given function, as illustrated in Fig. 2.55.

The desired integral is then approximated as the area below this polygon. Note that each side of the polygon, along with its two vertical lines and the segment joining points t_{i-1} and t_i in the t axis forms a trapezoid, which is the reason why this scheme is called the trapezoidal rule. Moreover, we shall refer to this trapezoid as the ith trapezoid, its area A_i being given by

$$A_i = \frac{1}{2}\Delta t_i(f_{i-1} + f_i), \quad \Delta t_i \equiv t_i - t_{i-1}, \quad f_i \equiv f(t_i) \tag{2.185}$$

Within our scheme, we shall divide the interval of integration *uniformly*, i.e.,

$$\Delta t_i \equiv \frac{T}{N}, \quad i = 1, \ldots, N \tag{2.186}$$

Now, a_0 is readily approximated as

$$\begin{aligned} a_0 &\approx \frac{1}{T}\frac{T}{N}\left[\frac{1}{2}(f_0 + f_1) + \frac{1}{2}(f_1 + f_2) + \ldots + \frac{1}{2}(f_{N-2} + f_{N-1}) + \frac{1}{2}(f_{N-1} + f_N)\right] \\ &= \frac{1}{N}\left[\frac{1}{2}(f_0 + f_N) + f_1 + f_2 + \ldots + f_{N-2} + f_{N-1}\right] \end{aligned}$$

But, since $f(t)$ is periodic, $f_N = f_0$, and hence,

$$a_0 \approx \frac{1}{N}(f_0 + f_1 + f_2 + \ldots + f_{N-2} + f_{N-1}) \tag{2.187}$$

i.e., a_0 becomes the *mean value* of the set of sampled function values $\{f_i\}_0^{N-1}$. Moreover,

$$\begin{aligned} a_k \approx \frac{2}{T}\frac{T}{N}\Big\{&\frac{1}{2}[f_0\cos(k\omega_0 t_0) + f_1\cos(k\omega_0 t_1)] \\ &+ \frac{1}{2}[f_1\cos(k\omega_0 t_1) + f_2\cos(k\omega_0 t_2)] + \ldots \\ &+ \frac{1}{2}[f_{N-2}\cos(k\omega_0 t_{N-2}) + f_{N-1}\cos(k\omega_0 t_{N-1})] \\ &+ \frac{1}{2}[f_{N-1}\cos(k\omega_0 t_{N-1}) + f_N\cos(k\omega_0 t_N)]\Big\} \end{aligned}$$

But, for $j = 1, \ldots, N$,

$$t_j \equiv t_0 + \frac{jT}{N} \quad \text{and} \quad \cos(k\omega_0 t_N) = \cos(k\omega_0 t_0)$$

and, hence, the foregoing expression becomes

$$a_k \approx \frac{2}{N}\left\{ f_0 \cos(k\omega_0 t_0) + f_1 \cos\left[k\omega_0\left(t_0 + \frac{T}{N}\right)\right] + f_2 \cos\left[k\omega_0\left(t_0 + \frac{2T}{N}\right)\right] + \dots \right.$$
$$\left. + f_{N-2} \cos\left[k\omega_0\left(t_0 + \frac{(N-2)T}{N}\right)\right] + f_{N-1} \cos\left[k\omega_0\left(t_0 + \frac{(N-1)T}{N}\right)\right]\right\}$$

However,

$$\cos\left[k\omega_0\left(t_0 + \frac{jT}{N}\right)\right] = \cos(k\omega_0 t_0)\cos\left(k\omega_0 \frac{jT}{N}\right) - \sin(k\omega_0 t_0)\sin\left(k\omega_0 \frac{jT}{N}\right)$$

the above expression thus becoming

$$a_k \approx \frac{2}{N}\cos(k\omega_0 t_0)\left\{ f_0 + f_1 \cos\left(k\omega_0 \frac{T}{N}\right) + f_2 \cos\left(k\omega_0 \frac{2T}{N}\right) + \dots \right.$$
$$\left. + f_{N-2} \cos\left(k\omega_0 \frac{(N-2)T}{N}\right) + f_{N-1} \cos\left(k\omega_0 \frac{(N-1)T}{N}\right)\right\}$$
$$-\frac{2}{N}\sin(k\omega_0 t_0)\left\{ f_1 \sin\left(k\omega_0 \frac{T}{N}\right) + f_2 \sin\left(k\omega_0 \frac{2T}{N}\right) + \dots \right.$$
$$\left. + f_{N-2} \sin\left(k\omega_0 \frac{(N-2)T}{N}\right) + f_{N-1} \sin\left(k\omega_0 \frac{(N-1)T}{N}\right)\right\}$$

and, in compact form,

$$a_k \approx \frac{2}{N}[\cos(k\omega_0 t_0)]\sum_{j=0}^{N-1} f_j \cos\left(k\omega_0 j\frac{T}{N}\right) - \frac{2}{N}[\sin(k\omega_0 t_0)]\sum_{j=1}^{N-1} f_j \sin\left(k\omega_0 j\frac{T}{N}\right) \tag{2.188}$$

Likewise, for the b_k coefficients, we have

$$b_k \approx \frac{2}{N}[\sin(k\omega_0 t_0)]\sum_{j=0}^{N-1} f_j \cos\left(k\omega_0 j\frac{T}{N}\right) + \frac{2}{N}[\cos(k\omega_0 t_0)]\sum_{j=1}^{N-1} f_j \sin\left(k\omega_0 j\frac{T}{N}\right) \tag{2.189}$$

From Eqs. 2.188 and 2.189 it is apparent that the Fourier coefficients have been approximated by a linear combination of two sums of terms, a sum C_k and a sum S_k, which are defined as

$$C_k \equiv \frac{2}{N}\sum_{j=0}^{N-1} f_j \cos\left(k\omega_0 j\frac{T}{N}\right), \quad S_k \equiv \frac{2}{N}\sum_{j=1}^{N-1} f_j \sin\left(k\omega_0 j\frac{T}{N}\right) \tag{2.190}$$

In particular, for $k = N/2$, when N is *even*,

$$C_{N/2} = \frac{2}{N}\sum_{j=0}^{N-1} f_j \cos\left(\frac{N}{2}\omega_0 j\frac{T}{N}\right) \equiv \frac{2}{N}\sum_{j=0}^{N-1} f_j \cos\left(\omega_0 j\frac{T}{2}\right)$$
$$= \frac{2}{N}\sum_{j=0}^{N-1} f_j \cos(j\pi) = \frac{2}{N}\sum_{j=0}^{N-1} f_j(-1)^j \tag{2.191}$$

$$S_{N/2} = \frac{2}{N}\sum_{j=1}^{N-1} f_j \sin\left(\frac{N}{2}\omega_0 j\frac{T}{N}\right) \equiv \frac{2}{N}\sum_{j=1}^{N-1} f_j \sin\left(\omega_0 j\frac{T}{2}\right)$$
$$= \frac{2}{N}\sum_{j=1}^{N-1} f_j \sin(j\pi) = 0 \tag{2.192}$$

In summary, the Fourier coefficients can be approximated in the form

$$a_k \approx [\cos(k\omega_0 t_0)]C_k - [\sin(k\omega_0 t_0)]S_k \tag{2.193}$$
$$b_k \approx [\sin(k\omega_0 t_0)]C_k + [\cos(k\omega_0 t_0)]S_k \tag{2.194}$$

Interestingly, the foregoing expressions can be cast in a rather revealing vector form, namely,

$$\begin{bmatrix} a_k \\ b_k \end{bmatrix} \approx \begin{bmatrix} \cos(k\omega_0 t_0) & -\sin(k\omega_0 t_0) \\ \sin(k\omega_0 t_0) & \cos(k\omega_0 t_0) \end{bmatrix} \begin{bmatrix} C_k \\ S_k \end{bmatrix}, \quad \text{for} \quad k = 1, 2, \ldots, \text{etc.} \tag{2.195}$$

the 2×2 matrix being orthogonal; this matrix rotates vectors in the plane by an angle $k\omega_0 t_0$. Thus, the *magnitudes* of the two vectors involved in the foregoing equation are *approximately* equal, i.e.,

$$a_k^2 + b_k^2 \approx C_k^2 + S_k^2 \tag{2.196}$$

Note that a_0 is not included in the foregoing vector expression; it appears in Eq. 2.187.

Moreover, the two foregoing sums, C_k and S_k, bear a particular significance in the context of Fourier analysis. They turn out to be the coefficients of the *discrete Fourier transform* of the sequence of discrete values $\mathscr{S} \equiv \{f_j\}_1^{N-1}$. Now, the said transform is the discrete counterpart of the *Fourier transform* of a continuous function $f(t)$, for $-\infty < t < +\infty$, and hence, finds extensive applications in engineering and economics. As a matter of fact, the discrete sequence $\mathscr{S}$ defined above, is known in economics as a *time-series*. The frequency analysis of time-series is important in many areas. For example, in management, the frequency analysis of the variations of market indicators given as time-series provides clues on trends, which helps in the decision-making process. Included in Kahaner et al. [4] is an application of time-series frequency analysis to the meteorological phenomenon known as 'El Niño'.

Because of the frequent occurrence of the C_k and S_k coefficients in many applications, software for their efficient computation is available in many scientific packages, e.g., Matlab, IMSL, Maple, Mathematica, and the like. Kahaner et al.'s book is nowadays complemented with a companion book [5] that offers downloadable software.[12] This software package includes tools for the calculation of the foregoing coefficients in the context of a more general set of routines meant for what is called the *Fast Fourier Transform* (FFT). FFT, or FFT analysis, is used extensively in engineering, science, and economics to determine the frequency contents of either continuous or discrete functions of time, i.e., signals. Astronomer Carl Sagan, in his science fiction bestseller *Contact*, describes a group of information scientists trying to find a clue on the existence of God with the aid of the number π. What these scientists do is a frequency analysis of the occurrence of the digits in this number and, moreover, they try this frequency analysis in various numerical bases. The frequency map that the scientists obtain, with number 11 as a basis, turns out to be a circle!

Coming back to our problem of calculating the Fourier coefficients of a periodic signal, the natural question to ask is up to how many coefficients can we compute from a subdivision of the integration interval of length equal to one period into N subintervals? As discussed by Kahamer et al., if the interval is divided into N subintervals, then we can, at most, compute $\lfloor N/2 \rfloor$ coefficients, where $\lfloor \cdot \rfloor$ stands for the *floor* function of $(\cdot)$, defined as the greatest integer contained in the real interval $[0, N/2]$.

An interesting result, known as *Parseval's Theorem* or *Parseval's identity* [6] is now recalled, that finds extensive applications in the realm of Fourier analysis. We will use it for error estimation in our calculations. This result can be readily derived by squaring both sides of Eq. 2.184a, thus obtaining

$$\begin{aligned} f^2(t) = a_0^2 &+ \left[\sum_1^\infty a_k \cos\left(\frac{2\pi k}{T}t\right)\right]^2 + \left[\sum_1^\infty b_k \sin\left(\frac{2\pi k}{T}t\right)\right]^2 \\ &+ 2a_0 \sum_1^\infty a_k \cos\left(\frac{2\pi k}{T}t\right) + 2a_0 \sum_1^\infty b_k \sin\left(\frac{2\pi k}{T}t\right) \\ &+ 2\left[\sum_1^\infty a_k \cos\left(\frac{2\pi k}{T}t\right)\right]\left[\sum_1^\infty b_k \sin\left(\frac{2\pi k}{T}t\right)\right] \end{aligned} \tag{2.197}$$

In the next step, we integrate both sides of the foregoing equation within one whole period. Without going into the details, it is apparent from the above equation that there will be an infinite sum of terms of five kinds, namely,

1. $a_k^2 \cos^2\left(\frac{2\pi k}{T}t\right)$. The integral over one whole period of these terms is a_k^2 times $T/2$, for every k;

[12] `http://www.mathworks.com/moler`

2. $b_k^2 \sin^2(\frac{2\pi k}{T}t)$. The integral over one whole period of these terms is b_k^2 times $T/2$ as well, for every k;
3. $a_j b_k \cos(\frac{2\pi j}{T}t)\sin\left(\frac{2\pi k}{T}t\right)$, for $j, k = 1, 2, \ldots$, including terms for which $j = k$. The integral of every term of this kind vanishes;
4. $a_j a_k \cos(\frac{2\pi j}{T}t)\cos\left(\frac{2\pi k}{T}t\right)$, for $j, k = 1, 2, \ldots$ and $j \neq k$. The integral of every term of this kind vanishes;
5. $b_j b_k \sin(\frac{2\pi j}{T}t)\sin\left(\frac{2\pi k}{T}t\right)$, for $j, k = 1, 2, \ldots$ and $j \neq k$. The integral of every term of this kind vanishes.

As a result of the foregoing observations, the right-hand side of Eq. 2.197 reduces to the infinite sums of terms of the first two kinds described above, and hence,

$$\int_{-T/2}^{T/2} f^2(t)dt = a_0^2 T + \frac{T}{2}\sum_1^\infty a_k^2 + \frac{T}{2}\sum_1^\infty b_k^2 \tag{2.198a}$$

or, if we divide both sides of the above equation by T,

$$\frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt = a_0^2 + \frac{1}{2}\sum_1^\infty a_k^2 + \frac{1}{2}\sum_1^\infty b_k^2 \tag{2.198b}$$

Now, by taking the square root of both sides of the above equation, we obtain

$$[f(t)]_{\rm rms} = \sqrt{\frac{1}{2}\left(2a_0^2 + \sum_1^\infty a_k^2 + \sum_1^\infty b_k^2\right)} \tag{2.199a}$$

with $[f(t)]_{\rm rms}$ defined as the *root-mean square* of $f(t)$, i.e.,

$$[f(t)]_{\rm rms} = \sqrt{\frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt} \tag{2.199b}$$

Therefore, the *root-mean-square value* of the periodic function $f(t)$ equals the square root of one-half the sum of the squares of its Fourier coefficients, with a_0^2 taken twice. Since root-mean square values are associated with energy—all energy functions are quadratic either in the generalized coordinates or in the generalized speeds—what Parseval's Theorem states is simply one more form of the First Law of Thermodynamics, i.e., the Principle of Conservation of Energy.

Note that the right-hand side of Eq. 2.198b is an infinite sum of squares. As a consequence, if the infinite sums are truncated after N_h harmonics, the N_h-*approximation* of the given function thus resulting leads to an *underestimation* of $[f(t)]_{\rm rms}$, i.e.,

$$a_0^2 + \frac{1}{2}\sum_1^{N_h} a_k^2 + \frac{1}{2}\sum_1^{N_h} b_k^2 < [f(t)]_{\rm rms}^2 \tag{2.200a}$$

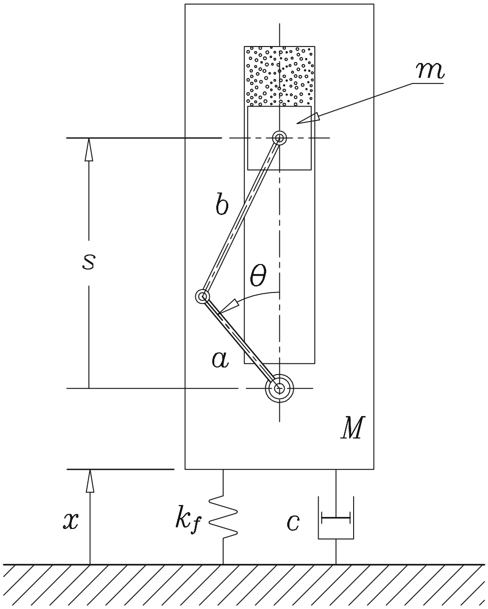


Fig. 2.56 Iconic model of an air compressor mounted on a viscoelastic foundation

Let e_{N_h} be the error in the N_h-approximation of a given periodic function $f(t)$. The *square* of this error is defined as

$$e_{N_h}^2 \equiv [f(t)]_{\text{rms}}^2 - \left(a_0^2 + \frac{1}{2}\sum_1^{N_h} a_k^2 + \frac{1}{2}\sum_1^{N_h} b_k^2 \right) > 0 \tag{2.200b}$$

Obviously, as more and more harmonics are included in the N_h-approximation of a given periodic function, the error decreases monotonically.

In summary, to calculate the Fourier coefficients, we have

Algorithm (Fourier):

1. If N is even, then the coefficients are calculated for $k = 1, \ldots, N/2$, with a_k being calculated for a_0 as well; otherwise, the coefficients are calculated for $k = 1, \ldots, (N-1)/2$.
2. Calculate a_0 as the mean value of the numbers $\{f_j\}_0^{N-1}$.
3. Calculate the C_k and S_k coefficients, for $k = 1, \ldots, \lfloor N/2 \rfloor$ using expressions (2.190).
4. Calculate the a_k and b_k coefficients, for $k = 1, \ldots, \lfloor N/2 \rfloor$ using expressions (2.195).

Example 2.8.5 (Spectral Analysis of the Displacement of an Air Compressor). Shown in Fig. 2.56 is the iconic model of an air compressor mounted on a viscoelastic foundation that is modeled as a spring of stiffness k_f in parallel with a dashpot of coefficient c. The *relative* displacement $s(t)$ of the piston with respect

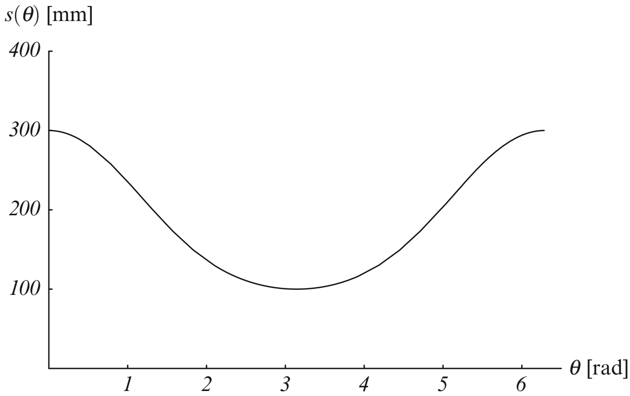


Fig. 2.57 Slider displacement vs. crank angular displacement

to the housing of the compressor is measured as indicated in the same figure and is given below as a function of the angular displacement of the crank, θ:

$$s(\theta) = a\cos\theta + b\sqrt{1-\rho^2\sin^2\theta}$$

where $\rho \equiv a/b$ and the crank turns at a constant angular velocity ω_0.

For the foregoing displacement, which is plotted in Fig. 2.57 for $a = 100\,mm$ and $\rho = 0.5$, compute the Fourier coefficients of the function $s(\theta)$ numerically and give an estimate of the error incurred in the calculations. (As one can readily realize, a longhand Fourier analysis of the foregoing function would be intractable.)

Solution: It is apparent that the signal at hand is even, and hence, its sine coefficients are all zero. Therefore, we only need worry about the a_k coefficients, for $k = 0, 1, \ldots,$ N. Let us now calculate these coefficients for 1, 2, and 4 harmonics. For purposes of illustration, we obtain these coefficients below by longhand calculations for 1 and 2 harmonics, i.e., for $N = 2$ and 4; for $N_h = 4$ or $N = 8$, we resort to numerical calcualtions, but do not include the details here.

1. For $N_h = 1, N = 2$, which yields the sampled values f_0, f_1 and $f_2 = f_0$, as shown in Fig. 2.58. We thus have

$$f(t) \approx a_0 + a_1\cos\omega_0 t$$

the coefficients a_0 and a_1 being calculated below. First, note that

$$f_0 = f(0) = 300, \qquad f_1 = f(t_1) = 100$$

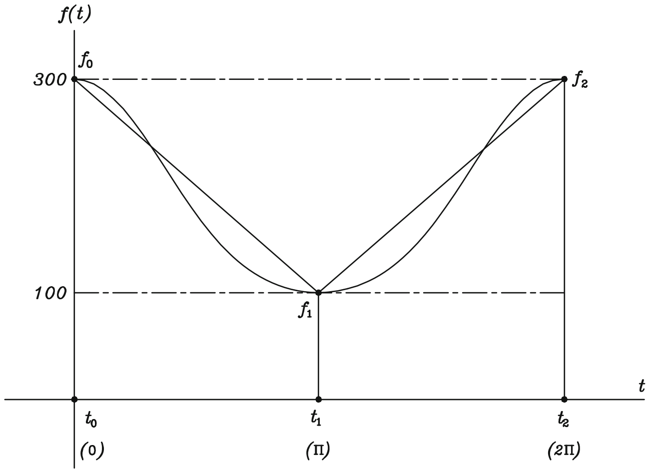


Fig. 2.58 Polygonal approximation of function $s(\theta)$ with $N = 2$ intervals

Therefore,

$$a_0 = \frac{1}{2}(f_0 + f_1) = 200$$

$$C_1 = \frac{2}{2}\sum_{j=0}^{1} f_j \cos\left(\underbrace{k}_{1}\omega_0 j\frac{T}{2}\right) = f_0\underbrace{\cos(0)}_{1} + \underbrace{f_1\cos\left(\underbrace{\omega_0\frac{T}{2}}_{\pi}\right)}_{-1} = f_0 - f_1 = 200$$

$$S_1 = \frac{2}{2}\sum_{j=0}^{1} f_j \sin\left(\omega_0 j\frac{T}{2}\right) = f_0\underbrace{\sin(0)}_{0} + \underbrace{f_1\sin\left(\underbrace{\omega_0\frac{T}{2}}_{\pi}\right)}_{0} = 0$$

Hence,

$$a_1 = C_1\cos\left(\underbrace{\omega_0 t_0}_{0}\right) - S_1\sin\left(\underbrace{\omega_0 t_0}_{0}\right) = C_1 = 200$$

Thus, with one harmonic,

$$f(t) \approx 200(1 + \cos\omega_0 t)$$

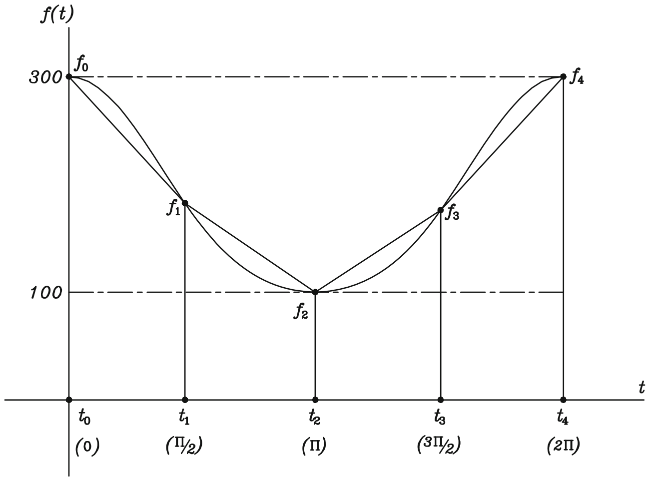


Fig. 2.59 Polygonal approximation of function $s(\theta)$ with $N = 4$ intervals

2. For $N_h = 2$, $N = 4$, which yields the sampled values f_0, f_1, f_2, f_3 and $f_4 = f_0$ of Fig. 2.59. Now we have

$$f(t) \approx a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t$$

In order to calculate the above coefficients, we start by calculating the function values at the sampled instants:

$$f_0 = 300,\ f_1 = a \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} + b \sqrt{1 - \rho^2 \underbrace{\sin^2\left(\frac{\pi}{2}\right)}_{1}} = 200\sqrt{1 - 0.25} = 173.2051 = f_3$$

where we have made use of symmetry to find f_3. Moreover, by inspection,

$$f_2 = 100$$

and hence,

$$a_0 = \frac{1}{4}(300 + 173.2051 + 100 + 173.2051) = 186.6025$$

Furthermore,

$$C_1 = \frac{2}{4}\sum_{j=0}^{3} f_j \cos\left(\omega_0 j \frac{T}{4}\right) = \frac{1}{2}\left[f_0 \cos(0) + f_1 \cos\left(\frac{\pi}{2}\right) + f_2 \cos(\pi) + f_3 \cos\left(\frac{3\pi}{2}\right)\right] \frac{1}{2}(f_0 - f_2) = 100$$

Table 2.1 Fourier coefficients of function $s(\theta)$ of the air compressor

	$N_h = 1$	$N_h = 2$	$N_h = 4$
a_0 (mm)	200.00000000000	186.60254037844	186.84270485857
a_1 (mm)	200.00000000000	100.000000000000	100.000000000000
a_2 (mm)		26.794919243112	13.397459621556
a_3 (mm)			$2.1316282072803\times 10^{-14}$
a_4 (mm)			-0.48032896025319

$$S_1 = \frac{1}{2}\sum_{j=0}^{3} f_j \sin\left(\omega_0 j \frac{T}{4}\right) = \frac{1}{2}\left[f_0 \sin(0) + f_1 \sin\left(\frac{\pi}{2}\right) + f_2 \sin(\pi)\right.$$
$$\left. + f_3 \sin\left(\frac{3\pi}{2}\right)\right] \frac{1}{2}(f_1 - f_3) = 0$$

Likewise,

$$C_2 = \frac{1}{2}\sum_{j=0}^{3} f_j \cos\left(2\omega_0 j \frac{T}{4}\right) = \frac{1}{2}\left[f_0 \cos(0) + f_1 \cos(\pi) + f_2 \cos(2\pi) + f_3 \cos(3\pi)\right]$$
$$= \frac{1}{2}(300 - 173.2051 + 100 - 173.2051) = 26.7949$$
$$S_2 = \frac{1}{2}\sum_{j=0}^{3} f_j \sin\left(2\omega_0 j \frac{T}{4}\right) = \frac{1}{2}\left[f_0 \sin(0) + f_1 \sin(\pi) + f_2 \sin(2\pi)\right.$$
$$\left. + f_3 \sin(3\pi)\right] = 0$$

Therefore,

$$a_1 = C_1 \cos(\omega_0 t_0) - S_1 \sin(\omega_0 t_0) = 100$$
$$a_2 = C_2 \cos(2\omega_0 t_0) - S_2 \sin(2\omega_0 t_0) = 26.7949$$

and hence,

$$f(t) \approx 186.6025 + 100.0000 \cos\omega_0 t + 26.7949 \cos 2\omega_0 t$$

Note that the coefficient of the first harmonic turns out to be nothing but the crank length, which is plausible in light of the expression for $s(t)$ given above, in which this length is the coefficient of $\cos\theta$.

3. The foregoing values, as well as those for $N_h = 4$ ($N = 8$), are given with 14 digits, as obtained with the aid of Matlab, in Table 2.1. We thus have, with four harmonics and four decimals,

$$s(\theta) \approx 186.8427 + 100.0000\cos\theta + 13.3975\cos 2\theta - 0.4803\cos 4\theta$$

where the third harmonic does not appear because a_3 turns out to vanish. Again, notice that the coefficient of the first harmonic is the crank length.

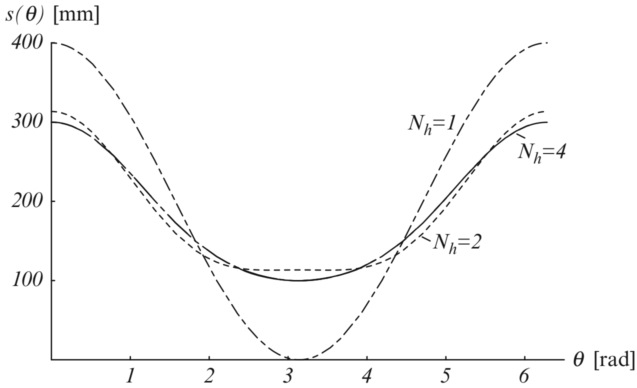


Fig. 2.60 $s(\theta)$-approximation with 1, 2, and 4 Fourier terms

The plots of $s(\theta)$ approximated with $N_h = 1$, 2 and 4 are shown in Fig. 2.60. Note, from this figure, that the value of $s(\theta)$ approximated with one single harmonic overestimates $s(\theta)$ by 33% at its peak values; when $s(\theta)$ is approximated with two harmonics, an overestimation of 4.5% is obtained. The approximation with four harmonics, on the other hand, underestimates $s(\theta)$ by less than 0.01%, but this difference, obviously, is not visible in that figure.

Let us now try to estimate the error incurred in the foregoing calculations, which we can do with the aid of Parseval's identity. To this end, we need the integral of $s^2(\theta)$ over one whole period, which gives us the rms value of $s(\theta)$, i.e.,

$$s_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}\left(a\cos\theta + b\sqrt{1-\rho^2\sin^2\theta}\right)^2 d\theta}$$

Upon expansion of the above integrand, $s^2(\theta)$, we obtain

$$s^2(\theta) = a^2\cos^2\theta + 2ab\sqrt{1-\rho^2\sin^2\theta} + b^2(1-\rho^2\sin^2\theta)$$

It is now apparent that the desired integral cannot be calculated by simple quadrature because of the presence of the square-root term. However, a close look of Table 2.1 will reveal that the crank length a is the coefficient of the first harmonic of the Fourier expansion of $s(\theta)$ with more than one harmonic, and hence, the remaining harmonics of that expansion are bound to give the Fourier expansion of the second term of $s(\theta)$, $b\sqrt{1-\rho^2\sin^2\theta}$. Therefore, the error incurred in approximating $s(\theta)$ by its Fourier expansion is identical to the error incurred in the approximation of

Table 2.2 Fourier coefficients of function $\sqrt{1-\rho^2\sin^2\theta}$ and the estimate of its rms value, $\sqrt{\Sigma}$

	$N_h = 1$	$N_h = 2$	$N_h = 4$
a_0/b	1.0000	0.9330	0.9342
a_2/b		0.1340	0.06699
a_3/b			0.0000
a_4/b			−0.002402
$\sqrt{\Sigma}$	1.0000	0.9378	0.9354

$\sqrt{1-\rho^2\sin^2\theta}$, which will prove to be much easier to compute. Indeed, the rms value of the foregoing square root is

$$\left[\sqrt{1-\rho^2\sin^2\theta}\right]_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}[1-\rho^2\sin^2\theta]d\theta}$$

Thus,

$$\left[\sqrt{1-\rho^2\sin^2\theta}\right]_{\text{rms}} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}\left[1-\frac{\rho^2}{2}(1-\cos 2\theta)\right]d\theta}$$

or

$$\left[\sqrt{1-\rho^2\sin^2\theta}\right]_{\text{rms}} = \sqrt{\frac{1}{2\pi}\left[\left(1-\frac{\rho^2}{2}\right)\theta+\frac{\rho^2}{4}\sin(2\theta)\right]_0^{2\pi}}$$

Finally,

$$\left[\sqrt{1-\rho^2\sin^2\theta}\right]_{\text{rms}} = \sqrt{\frac{1}{2\pi}\left[1-\frac{\rho^2}{2}\right]2\pi} = \sqrt{1-\frac{\rho^2}{2}}$$

For the case at hand, $\rho = 0.5$, and hence,

$$\left[\sqrt{1-\rho^2\sin^2\theta}\right]_{\text{rms}} = \sqrt{0.8750} = 0.9354$$

which is exact to four digits. Table 2.2 includes the values of the Fourier coefficients of the expansion of the foregoing square root, which are identical to those of $[s(\theta)-a]/b$.

An approximation to the Fourier expansion of $s(\theta)$, frequently invoked, is obtained by approximating the square root, for "small" values of ρ, in the form

$$\sqrt{1-\rho^2\sin^2\theta} \approx 1-\frac{\rho^2}{2}\sin^2\theta \equiv 1-\frac{\rho^2}{4}[1-\cos(2\theta)]$$

which thus yields the *approximate Fourier expansion*

$$\sqrt{1-\rho^2\sin^2\theta} \approx 1-\frac{\rho^2}{4}+\frac{\rho^2}{4}\cos(2\theta)$$

whose *approximate Fourier coefficients* are

$$\overline{a}_0 = 1 - \frac{\rho^2}{4} = 0.9375, \quad \overline{a}_1 = 0, \quad \overline{a}_2 = \frac{\rho^2}{4} = 0.0625$$

Therefore, according to Parseval's identity,

$$\begin{aligned}\left[\left(1 - \frac{\rho^2}{4}\right) + \rho\cos\theta + \frac{\rho^2}{4}\cos(2\theta)\right]_{\mathrm{rms}}^2 &= \overline{a}_0^2 + \frac{1}{2}\overline{a}_1^2 + \frac{1}{2}\overline{a}_2^2 \\ &= 0.9375^2 + 0.5 \times 0.0625^2 = 0.8809\end{aligned}$$

and hence,

$$\left[\left(1 - \frac{\rho^2}{4}\right) + \rho\cos\theta + \frac{\rho^2}{4}\cos(2\theta)\right]_{\mathrm{rms}} = 0.9385$$

Displayed in Table 2.2 are the values of the coefficients of the exact Fourier expansion of $\sqrt{1-\rho^2\sin^2\theta}$, item $\sqrt{\Sigma}$ appearing in the same table being defined as

$$\sqrt{\Sigma} \equiv \sqrt{\frac{1}{2}\left[\left(\frac{a_0}{b}\right)^2 + \sum_1^{N_h}\left(\frac{a_k}{b}\right)^2\right]}$$

As compared with the exact value obtained above for the rms value of the square root, 0.9354, the approximation formula gives a fairly acceptable error of about 0.3%. Note that the *exact* Fourier expansion of the same function, as given by Table 2.2, gives an error of about 0.2% when the first two harmonics are considered. Thus, the proposed approximation, which is limited to "small" values of ρ, gives acceptable errors even for a relatively large value of ρ, namely, 0.5.

Finally, note that the exact Fourier expansion of the above square-root function matches the function itself, up to at least the first four digits, when the expansion includes the first four harmonics. The moral of the story is, then, that while simple approximations can give acceptable results, an even higher accuracy can be achieved at a rather low computational overhead.

2.8.3 The Periodic Response of First- and Second-order LTIS

In this section we shall determine the steady-state response of first- and second-order systems to periodic inputs. Such a response will be termed the *periodic response* of the system at hand. As we have seen, such an input can be represented as an infinite series of sine and cosine terms using the Fourier expansion of Eqs. 2.184a, b. Now, Eq. 2.184a can be rewritten as

$$f(t) = a_0 + \sum_1^\infty a_k \cos \omega_k t + \sum_1^\infty b_k \sin \omega_k t \tag{2.201}$$

where

$$\omega_k = \frac{2\pi k}{T} \tag{2.202}$$

Focusing on the terms under the summation signs, we see that they involve harmonic functions of certain frequencies ω_k. The steady-state response of the system to any one of these terms alone has the form

$$M_k^C \cos(\omega_k t + \phi_k) \quad \text{or} \quad M_k^S \sin(\omega_k t + \phi_k) \tag{2.203}$$

where M_k^C, M_k^S and ϕ_k can be determined using the results of Sect. 2.7 with $A = a_k$, $B = b_k$, and $\omega = \omega_k$. Since a_0 can be regarded as a harmonic function of magnitude a_0 and frequency $\omega = 0$, the steady-state response M_0 of the system to the input a_0, can be determined using the results of Sect. 2.7 with $A = a_0$ and $\omega = 0$. The steady-state response of the system to the periodic input $f(t)$ can then be obtained via *superposition*, so that

$$x_P(t) = M_0 + \sum_1^\infty M_k^C \cos(\omega_k t + \phi_k) + \sum_1^\infty M_k^S \sin(\omega_k t + \phi_k) \tag{2.204}$$

or

$$x_P(t) = M_0 + \sum_1^\infty M_k^C \cos\left(\frac{2\pi k}{T} t + \phi_k\right) + \sum_1^\infty M_k^S \sin\left(\frac{2\pi k}{T} t + \phi_k\right) \tag{2.205}$$

In the balance of this section we will determine expressions for M_k^C, M_k^S and ϕ_k for various types of systems.

2.8.3.1 First-order LTI Dynamical Systems

If a first-order LTI dynamical system is driven by a periodic function $f(t)$, then we can model the system dynamics in the form

$$\dot{x} + ax = a_0 + \sum_1^\infty a_k \cos \frac{2\pi k}{T} t + \sum_1^\infty b_k \sin \frac{2\pi k}{T} t \tag{2.206}$$

The term M_0 is determined by setting $\omega = 0$ and multiplying M by a_0 in Eqs. 2.118a, b, to obtain

$$M_0 = \frac{a_0}{a}, \quad \phi_0 = 0 \tag{2.207}$$

Now, M_k^C and ϕ_k are obtained by setting $\omega = 2\pi k/T$ and multiplying M by a_k in Eqs. 2.118a, b, so that

$$M_k^C = \frac{a_k}{\sqrt{a^2 + (2\pi k/T)^2}} \qquad \phi_k = -\arctan\left(\frac{2\pi k}{Ta}\right) \tag{2.208}$$

M_k^S is obtained likewise, by setting $\omega = 2\pi k/T$ and multiplying M by b_k in Eqs. 2.124a, b, which yields

$$M_k^S = \frac{b_k}{\sqrt{a^2 + (2\pi k/T)^2}}, \tag{2.209}$$

2.8.3.2 Second-order Undamped LTI Dynamical Systems

Below we show the model of an undamped second-order system driven by a periodic function expressed in Fourier-expansion form, namely,

$$\ddot{x} + \omega_n^2 x = a_0 + \sum_1^\infty a_k \cos\frac{2\pi k}{T}t + \sum_1^\infty b_k \sin\frac{2\pi k}{T}t \tag{2.210}$$

We derived the frequency response of these systems in Sect. 2.7.3 for harmonic inputs, the associated magnitude and phase expressions being displayed in Eqs. 2.135a, b and 2.144a, b. From these responses, we obtain the frequency response expressions for the system at hand, i.e.,

$$M_0 = \frac{a_0}{\omega_n^2}, \quad \phi_0 = 0 \tag{2.211a}$$

$$M_k^C = \frac{a_k/(\omega_n^2)}{|1 - k^2 r_0^2|}, \quad \phi_k^C = \begin{cases} 0^\circ & \text{if } kr_0 < 1; \\ -180^\circ & \text{if } kr_0 > 1 \end{cases}. \tag{2.211b}$$

$$M_k^S = \frac{b_k/(\omega_n^2)}{|1 - k^2 r_0^2|} \tag{2.211c}$$

where r_0 is defined as the ratio

$$r_0 \equiv \frac{\omega_0}{\omega_n} \equiv \frac{2\pi}{T\omega_n} \tag{2.212}$$

2.8.3.3 Second-order LTI Dynamical Systems

A damped system acted upon by a periodic input leads to the model below:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = a_0 + \sum_1^\infty a_k \cos\frac{2\pi k}{T}t + \sum_1^\infty b_k \sin\frac{2\pi k}{T}t \tag{2.213}$$

The frequency response expressions for the underdamped case were obtained in Sect. 2.7.3. We also pointed out in that subsection that the same amplitude and phase relations hold for critically damped and overdamped systems. From the two sets of magnitude and phase expressions, Eqs. 2.132a, b and 2.142a, b, we obtain the desired frequency-response expressions for the system at hand, as shown below:

$$M_0 = \frac{a_0}{\omega_n^2}, \quad \phi_0 = 0 \tag{2.214a}$$

$$M_k^C = \frac{a_k/\omega_n^2}{\sqrt{(1-k^2r_0^2)^2 + (2\zeta k r_0)^2}} \tag{2.214b}$$

$$\phi_k = -\arctan\left[\frac{2\zeta k r_0}{1-(kr_0)^2}\right] \tag{2.214c}$$

$$M_k^S = \frac{b_k/(\omega_n^2)}{\sqrt{(1-k^2r_0^2)^2 + (2\zeta k r_0)^2}} \tag{2.214d}$$

$$r_0 = \frac{\omega_0}{\omega_n} = \frac{2\pi}{T\omega_n} \tag{2.214e}$$

Example 2.8.6 (The Periodic Response of an Air Compressor). For the numerical values given below, find the steady-state dynamic force transmitted to the ground by the compressor-foundation system of Sect. 2.8.2, with the crank turning at a constant ω_0.

$$a = 100\text{ mm}; \quad \rho = 0.5; \quad m = 10\text{ kg}; \quad M = 200\text{ kg};$$
$$\omega_n = 26.1799\text{ s}^{-1}; \quad \omega_0 = 100\text{ rpm}; \quad \zeta = 0.2$$

Solution: As the reader can readily verify, the mathematical model of the compressor-foundation system takes the form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = -\frac{m}{M+m}\ddot{s}, \quad \omega_n \equiv \sqrt{\frac{k_f}{M+m}}$$

where $s(\theta)$ is given in Example 2.8.5. Now, we need the Fourier expansion of the right-hand side of the foregoing equation. To this end, we can proceed in two ways, namely, (i) by finding $\ddot{s}(t)$ explicitly from the second derivative of $s(\theta)$ given above, and then computing the Fourier coefficients of this function; alternatively, (ii) by differentiation of the terms of the Fourier expansion of $s(\theta)$ found in Example 2.8.5. Since we already have the latter, it is apparent that the second approach should be more expeditious.

Note that $s(\theta)$ was found to be accurately approximated with its first four harmonics, namely,

$$s(\theta) = a_0 + \sum_1^4 a_k \cos(k\theta)$$

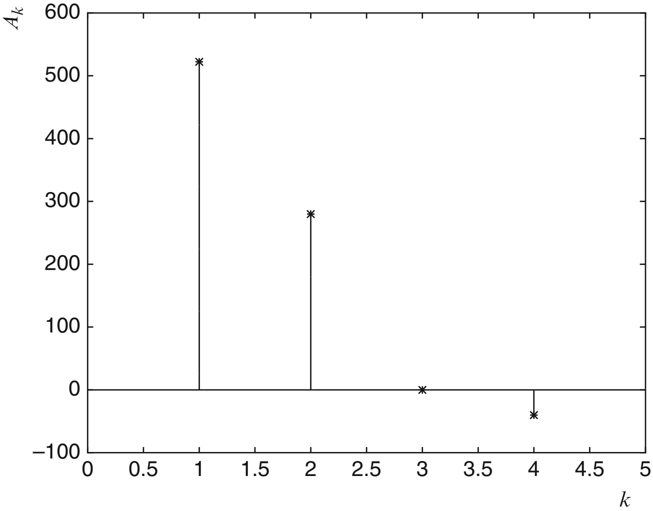


Fig. 2.61 The four Fourier coefficients A_k of the force-transmission analysis

and hence,

$$\dot{s}(\theta) = -\omega_0 \sum_1^4 k a_k \sin(k\theta), \quad \ddot{s}(\theta) = -\omega_0^2 \sum_1^4 k^2 a_k \cos(k\theta)$$

the governing equation thus taking the form

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \sum_1^4 A_k \cos(k\theta)$$

with A_k defined as

$$A_k = \omega_0^2 \frac{m}{M+m} k^2 a_k, \quad k = 1,2,3,4$$

Shown in Fig. 2.61 are the four coefficients A_k, for the given numerical values.

Now, the force transmitted to the foundation, $F(t)$, can be expressed as a sum of harmonics having the form of the expression appearing in Eq. 2.163, i.e.,

$$F(t) = \sum_1^4 F_k \cos(k\omega_0 t + \psi_k)$$

with F_k and ψ_k derived from the expressions given in Eqs. 2.166 and 2.167, with $F_0 = A_k(M+m)$ or $F_0 = \omega_0^2 m k^2 a_k$, for $k = 1,2,3,4$. Therefore,

$$F_k \equiv \frac{A_k(M+m)\sqrt{1+(2\zeta r_k)^2}}{\sqrt{(1-r_k^2)^2 + (2\zeta r_k)^2}}, \quad \psi_k \equiv \tan^{-1}(2\zeta r_k) - \tan^{-1}\left(\frac{2\zeta r_k}{1-r_k^2}\right)$$

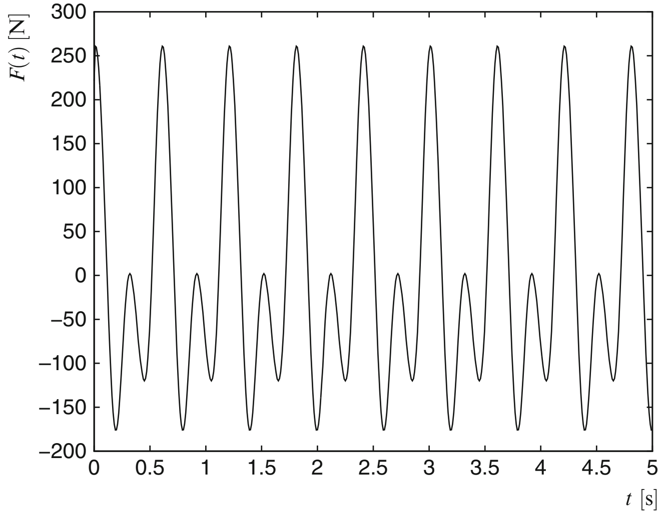


Fig. 2.62 Steady-state force transmitted to the foundation

where the counter k in the foregoing expansion is not to be confused with the stiffness of the foundation, k_f. Moreover, the counter is not needed, for the value of ω_n is given instead. Additionally,

$$r_k \equiv \frac{k\omega_0}{\omega_n}, \quad k = 1, 2, 3, 4$$

The steady-state transmitted force, computed with four harmonics, is plotted in Fig. 2.62.

2.9 The Time Response of Systems with Coulomb Friction

Systems with Coulomb friction are more difficult to handle than those with viscous friction that we studied so far. One reason is that these systems lead to nonlinear models, as we will show presently.

We consider here a very simple model, namely, a mass coupled to an inertial frame through a linear spring and Coulomb friction, as depicted in Fig. 2.63. Here, we model friction as in Chap. 1, namely, we do not distinguish between static and dynamic friction coefficients and take the friction force as proportional to the normal force, the proportionality factor μ, i.e., the friction coefficient, being constant.

Under the foregoing conditions, it is now a simple matter to derive the governing equation of interest, namely,

$$m\ddot{x} + kx = \begin{cases} -\mu N, & \text{if} \quad \dot{x} > 0; \\ \mu N, & \text{if} \quad \dot{x} < 0; \end{cases}$$

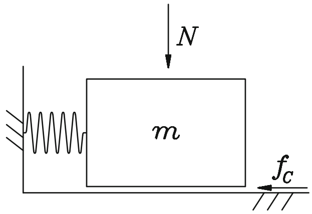


Fig. 2.63 Mass-spring system subjected to Coulomb friction

which readily leads to

$$m\ddot{x}+kx=-\mu N\operatorname{sgn}(\dot{x}) \tag{2.215}$$

or

$$\ddot{x}+\omega_n^2 x=-\frac{\mu N}{m}\operatorname{sgn}(\dot{x}) \tag{2.216}$$

Such a model is, then, said to be *piecewise linear*. We have thus, piecewise, an undamped second-order system acted upon by a constant force of magnitude $f_0=\mu N/m$. Clearly, as the velocity changes sign, so does the friction force, always opposing the motion. It is then expected that, within a period in which the velocity does not change sign, the friction force will slow down the mass, which will eventually come to a standstill. Now, because of the potential energy stored in the spring upon reaching the standstill position, the mass will resume motion as long as the force supplied by the spring is large enough to overcome the friction force. Motion will eventually stop when the spring force is not large enough to overcome the friction force. Thus, in order to find the desired time response, we can assume, without loss of generality, that we start observing the system while the mass is stationary, and so, we can assume the initial conditions $x(0)=x_0>0$ and $\dot{x}(0)=0$. We then have

$$\ddot{x}+\omega_n^2 x=\mu\frac{N}{m}, \qquad x(0)=x_0>0, \quad \dot{x}(0)=0 \tag{2.217}$$

The response of the above system is readily derived from previous results. Indeed, this response has the form given in Eq. 2.88, with $v_0=0$ and $f(t)=\mu N/m=$ const, i.e.,

$$x(t)=(\cos\omega_n t)x_0+\int_0^t \mu\frac{N}{m}\frac{1}{\omega_n}(\sin\omega_n(t-\tau))d\tau \tag{2.218}$$

and hence,

$$x(t)=\frac{\mu N}{m\omega_n^2}+(\cos\omega_n t)\left(x_0-\frac{\mu N}{m\omega_n^2}\right) \tag{2.219}$$

while the velocity is

$$\dot{x}(t)=-\omega_n(\sin\omega_n t)\left(x_0-\frac{\mu N}{m\omega_n^2}\right) \tag{2.220}$$

Therefore, the mass will reach a standstill at a time $t = t_1 = \pi/\omega_n$. Furthermore, upon substitution of t_1 in the expression for $x(t)$, Eq. 2.219, an expression for $x_1 \equiv x(t_1)$ is readily derived, namely,

$$x_1 = \frac{\mu N}{m\omega_n^2} - \left(x_0 - \mu\frac{N}{m\omega_n^2}\right) = -x_0 + 2\frac{\mu N}{m\omega_n^2}$$

Now we regard $t = t_1$ as the initial time for the ensuing motion. The governing equation from which this motion is derived is identical to the former, Eq. 2.217, with the difference that now the non-homogeneous term changes sign, the response thus becoming

$$x(t) = -\frac{\mu N}{m\omega_n^2} + (\cos\omega_n t)\left(x_0 - 3\mu\frac{N}{m\omega_n^2}\right), \quad t \geq t_1$$

Likewise,

$$\dot{x}(t) = -\omega_n(\sin\omega_n t)\left(x_0 - 3\mu\frac{N}{m\omega_n^2}\right)$$

Therefore, the mass reaches a standstill at a time $t_2 > t_1$ given by $t_2 = 2\pi/\omega_n$, at which $x(t)$ attains the value $x_2 \equiv x(t_2)$, given by

$$x_2 = x_0 - 4\frac{\mu N}{m\omega_n^2}$$

The motion for $t > t_2$ is derived likewise. It is apparent by now that the mass will attain a standstill at $t = t_3 = 3\pi/\omega_n$, at which

$$x_3 \equiv x(t_3) = -x_0 + 6\frac{\mu N}{m\omega_n^2}$$

and hence, the mass attains rest at a sequence of positions $\{x_i\}_1^F$ given by

$$x_i = (-1)^i\left(x_0 - 2i\frac{\mu N}{m\omega_n^2}\right), \quad i = 1,\ldots,F$$

x_F, the final position, being reached when the friction force is greater than the force provided by the spring while the mass is at rest, the mass thereby stopping for good. The value F is the lowest integer F for which

$$k|x_F| < \mu N < k|x_{F-1}|$$

or

$$|x_F| < \frac{\mu N}{k} < |x_{F-1}|$$

A plot of the above time response is shown in Fig. 2.64.

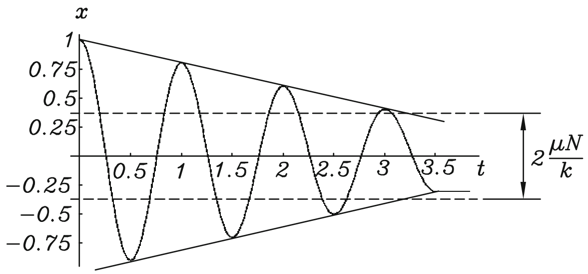


Fig. 2.64 Time response of mass-spring system subjected to Coulomb friction

Note that the time response of the system at hand is piecewise harmonic, with decreasing amplitude. However, contrary to viscous damping, which produces an exponential decrement of the motion amplitude, Coulomb damping produces a linear decrement. Hence, the nature of a damped system can be identified by inspection, i.e., by looking at its decrement.

Finally, it can be readily proven that the sequence of values $\{x_i\}_1^F$ is governed by the recursive relation given below:

$$x_{i+1} = -x_i + 2\frac{\mu N}{m\omega_n^2}, \quad x_0 = x_0, \quad i = 1, \ldots, F \tag{2.221}$$

with x_F observing the constraint derived above.

2.10 Exercises

2.1. We refer to the tugboat-barge system of Example 2.2.1 under the failure conditions described therein, but with the numerical data given below: $\tau_B = 10$ s, $\tau_T = 8$ s, $v_0 = 5$ m/s, the length of the tugging cable being 8 m. Determine the time elapsed until the occurrence of a collision. *Hint: A graphical solution is suggested here.*

2.2. The iconic model of an overhead material-transport system is sketched in Fig. 2.65. The system consists of a massless, rigid wheel driven at a controlled constant speed w along a bent track, in such a way that the load, of mass m, can be safely assumed to undergo pure translation. Note that the load is attached to the wheel by means of an undamped suspension. Find the value of w beyond which the wheel will jump off the track after reaching the horizontal track section.

2.3. Refer to the aircraft modeled upon landing as shown in Fig. 2.13. At touchdown, the fuselage lies a height $h_0 = 1/(2\pi)$ m above its static-equilibrium level (SEL), while the approach velocity has a horizontal component u_0 that can safely

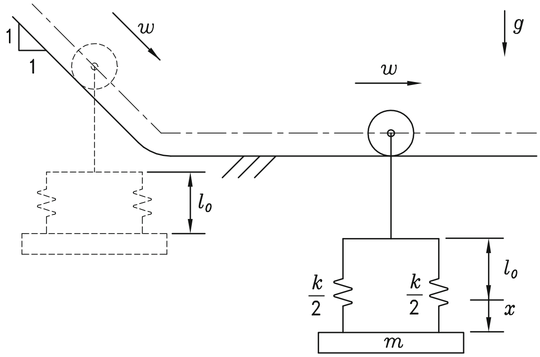


Fig. 2.65 An overhead material-transport system with an undamped suspension

be assumed constant during touchdown and a vertical component v_0 of $4\sqrt{2}$ m/s. If $\omega_n = 1$ Hz and $\zeta = \sqrt{2}/2$,

(a) Find the maximum height h_1 reached by the fuselage from the SEL right after landing
(b) Find the maximum force transmitted to the ground

2.4. Consider the overhead crane of Example 1.6.5. We want to stabilize the unstable rod-up equilibrium configuration with *analog feedback* applied by a torsional spring of stiffness k at the pin of attachment of the rod with the slider. Find the minimum value $k_{\min}$ of k that will make the feedback system stable, under the assumption that the spring is unloaded in the rod-up equilibrium configuration. Now, for a value of $k = 2k_{\min}$ and constant $\ddot{u} = 0.5g$, sketch the time response of the system under the perturbations (1) $\delta\theta_0 = 0.02$ rad, $\delta\dot{\theta}_0 = 0$; (2) $\delta\theta_0 = 0$, $\delta\dot{\theta}_0 = 0.02$ rad/s; and (3) $\delta\theta_0 = 0.01$ rad, $\delta\dot{\theta}_0 = 0.06$ rad/s. Comment on the relationship among the three results. Assume that the pin provides a light damping of 10%, i.e., the damping ratio is 0.1, and $g/l = 4\ \mathrm{s}^{-2}$.

2.5. We refer to the system of Exercise 1.2. An experiment is conducted to determine the stiffness of the shaft. The experiment consists of letting the bar fall freely from its top position, i.e., with the motor exerting a zero torque on the pinion. As the bar reaches its bottom position, the motor is blocked, thereby fixing the left end of the shaft to an inertial frame. It is then noted that the link undergoes 80 small-amplitude oscillations, behaving as a rigid body, about its lowest position in 10 s. Moreover, the light damping always present, in the form of air drag, material hysteresis, and the like, produces a decrement of the amplitude of the oscillations such that, after 80 oscillations, the amplitude is only 5% of its original value. With this information, and knowing that the bar is 1 m long and weighs 117.72 N, find the stiffness of the shaft and the coefficient of damping, under the assumption that damping is linear.

2.6. The system of Example 2.4.2 is revisited here. Under the assumption that $\omega_n = 3.1451$ rad/s and $\zeta = 0.04711$, find the minimum height h from which the mass m_2

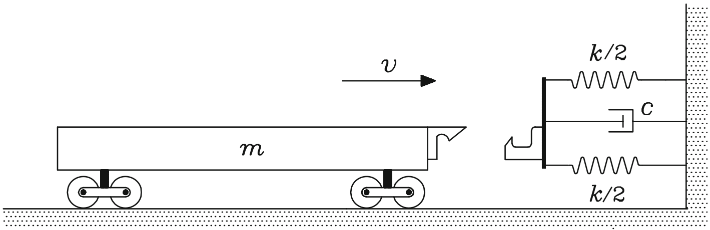


Fig. 2.66 A railroad car approaching a bumper

should be dropped in order to produce a strong-enough rebound of the pad that will exert a pull of the springs on the ground. Assume that the coefficient of restitution of the ball with the pad is $e = 0.5$. Find the instant t_{min} at which the pull is expected to occur. *Note: Because of the change in conditions, the plot of Fig. 2.25 has nothing to do with this problem.*

2.7. Boxes are transported on a horizontal track of rollers. Upon placement of a box of mass m on the rollers, the box is given a push that moves it at a speed v_0. Find the value of v_0 that will allow the box to travel a given distance d before its speed drops below 0.5 v_0, under the assumption that the lubricant of the rollers produces linearly viscous damping of a known coefficient c(Ns/m).

If a batch of different boxes, ordered by weight, is to be placed on the rollers at equal time-intervals T, with equal initial speed v_0, how would you order the placement of the boxes, in order to prevent collisions between boxes, i.e., from heavier to lighter, alternating them, etc.? Explain your rationale.

2.8. The railroad car shown in Fig. 2.66 is released with a speed of 20 km/h a distance of 50 m from the bumper to the right. The car weighs 100 kN, and its axles turn on bearings providing a linearly viscous damping that decreases its speed by 80% just before hitting the bumper. Once the bumper has been hit, assume that the car engages the bumper without backlash and without energy losses. Then, the compression of the springs is recorded. It is found that the spring compression reaches a first peak of 250 mm and a second one of only 35 mm. Find:

(a) The damping coefficient of the car axles
(b) The damping ratio of the car-bumper system
(c) The natural frequency of the same system
(d) The stiffness of the bumper
(e) The damping coefficient of the bumper

2.9. Assume that the railroad car of Fig. 2.66 weighs 98.10 kN, and travels at a velocity $v(t)$ to the right. Upon hitting the bumper, coupling is assumed to occur instantly and without backlash. Under these ideal conditions, assume further that the wheels roll without slipping on the railway, the only sink of energy being the lubricant in the axles, which can be assumed to produce linearly viscous damping

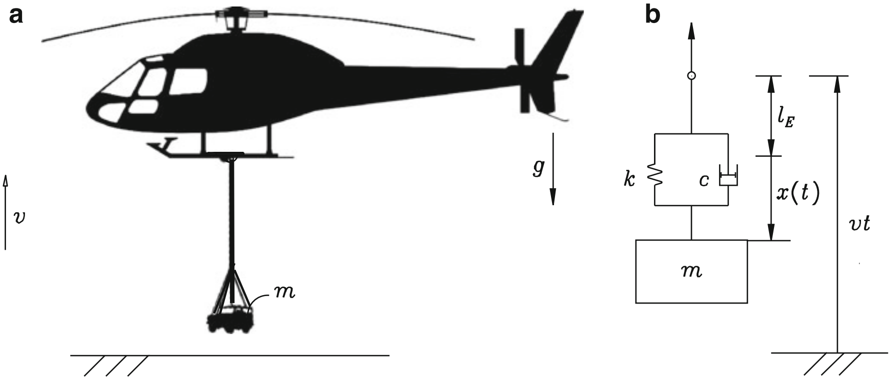


Fig. 2.67 A helicopter lifting a load: (**a**) the physical system (silhouette of the Aérospatiale—Eurocopter—AS 355 Écureil Helicopter); (**b**) the iconic model

of coefficient $c_W = 0.7$ kNs/m. The car is released with a velocity v_R and, 10 s afterwards, it is observed to hit the bumper with a velocity of 10 m/s.

(a) Find the value of v_R.
(b) The bumper is to be designed so that it will provide linearly viscous damping with a damping ratio of $\zeta = 0.5$ and undamped oscillations of $1/(2\pi)$ Hz. Specify the spring stiffness and the dashpot coefficient.
(c) Find the maximum force transmitted to the bumper, when the car approaches it as described above, i.e., with a velocity of 10 m/s.
(d) Sketch the displacement of the car as a function of time after coupling has taken place, and determine the maximum deflection experienced by the bumper.
(e) What design changes (i.e., new values of k and c) would you recommend in order to bring the maximum deflection of item (d) down to 10 % of the value found above, without changing the damping ratio? Under these changes, what is the maximum force transmitted for the given approach velocity?

2.10. Shown in Fig. 2.67a[13] is a helicopter lifting a load of mass m at a constant lifting speed v, l_E being the length of the spring when the load is airborne and under static equilibrium. The lifting mechanism is modeled in Fig. 2.67b as a linear mass-spring-dashpot system. Moreover, the force f exerted by the cable on the mass can be expressed in the form

$$f(t) = kx + c\dot{x}$$

We define the *dynamic factor* D_f as the dimensionless quantity by which the weight of the load has to be multiplied, when designing the cable, to account for the dynamic effects of lifting. We want to determine this factor by finding the maximum force exerted on the cable once the load is airborne. To this end,

[13]Reproduced with authorization from http://commons.wikimedia.org/wiki/ File:Helicopter_silhouette_AS-355.svg

(a) Show that the condition under which the force in the cable attains a stationary value (maximum, minimum, or saddle point) is given by

$$(4\zeta^2 - 1)\dot{x} + 2\zeta\omega_n x = 0$$

(b) Find an expression for D_f, if we know that $\zeta = 0.5$

2.11. For the motor-clutch system described in Example 1.6.10, assume that the rotor can be safely modeled as a homogeneous cylinder of mass $m = 5$ kg and radius $r = 100$ mm. In designing this system, we want to meet the specification below: when the rotor is turning at a constant angular velocity ω, it is required that, upon turning the motor off, the rotor angular velocity be reduced to at least 1% of its original value in 100 ms. What is the minimum value of the dashpot coefficient that will produce this behavior?

2.12. (To be assigned only if Problem 1.8 was previously assigned.) In designing the suspension of Problem 1.8, a test is conducted on the suspension such that the wheel is displaced from its equilibrium state by a small amount y_0 and is then released at rest. It is required that the wheel return to within 1% of its equilibrium configuration after the second oscillation. If the natural frequency of the suspension is 1 Hz, what is the required value of the damping ratio? Under these conditions, for $y_0 = 50$ mm and $\dot{y}_0 = 0$, plot the time response of the system for the first three oscillations.

2.13. A fluid clutch connects a load, e.g., the whole inertia of an automobile, to an engine, as shown in Fig. 1.23. Assume that the torque transmitted through the clutch, when it is engaged, is proportional to the difference in speed of the input and output shafts (proportionality constant c). Assume also that the speed ω of the engine shaft is constant and unaffected by the load. If the load is a pure inertia J and the system is underdamped, find the time response of the load angular velocity $\omega_R = \dot{\theta}$, after the clutch is suddenly engaged.[14]

2.14. A wheelchair crossing a ditch is modeled as the underdamped mass-spring-dashpot system shown in Fig. 2.68.

(a) Determine the time response $x(t)$ of its vertical motion, upon traversing the ditch at a constant speed v, in terms of the physical parameters of the system and the geometry of the ditch.
(b) Plot $x(t)$ vs. time at a suitable scale, for values of

$$\omega_n = 1 \text{ Hz}, \quad \zeta = \frac{\sqrt{2}}{2}, \quad \frac{v}{\lambda\omega_n} = 0.01, \frac{1}{2\pi}, 10.0$$

Comment on the results.

[14]This exercise is drawn from a similar one in [3].

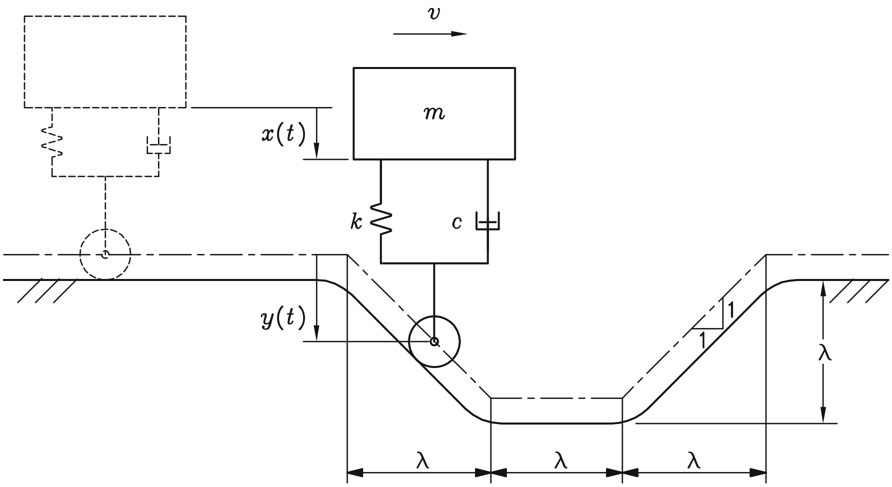


Fig. 2.68 The iconic model of a wheelchair crossing a ditch

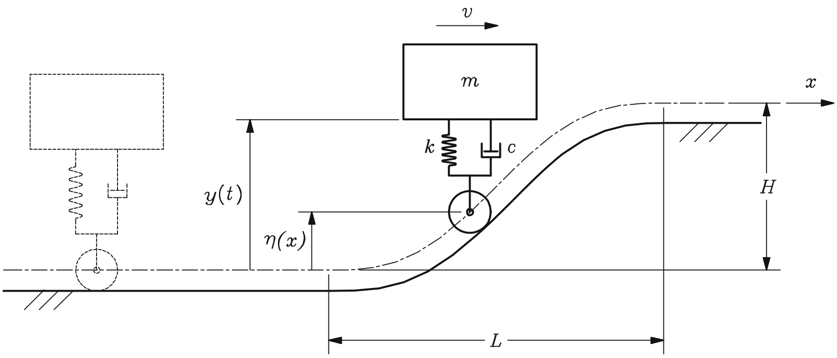


Fig. 2.69 A terrestrial vehicle climbing a cycloidal ramp

2.15. A terrestrial vehicle is modeled as the underdamped mass-spring-dashpot system shown in Fig. 2.69. Determine the time response $y(t)$ of its vertical motion, upon overcoming a *cycloidal* slope at a constant speed v, in terms of the physical parameters of the system and the geometry of the slope. The cycloidal slope is modeled via the function $\eta(x)$, defined as

$$\eta(x) \equiv \begin{cases} 0, & \text{for} \quad x \leq 0 \\ H[x/L - 1/(2\pi)\sin(2\pi x/L)], & \text{for} \quad 0 \leq x \leq L \\ H, & \text{for} \quad x \geq L \end{cases}$$

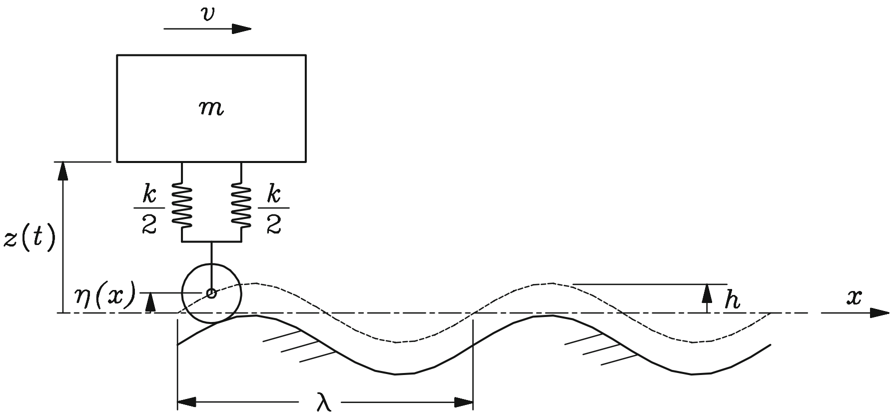


Fig. 2.70 Vehicle with undamped suspension traveling on a wavy road

Furthermore, plot the response of the system under the conditions: $c = 0$ and $\omega_n = 1$ Hz, for three values of v, $v = 0.01\omega_n L$, $v = \omega_n L/(2\pi)$, and $v = 10\omega_n L$, $L = 1$m. Comment on your results.

2.16. Shown in Fig. 2.70 is the iconic model of a vehicle with an undamped suspension—intended to model worn-out "shocks"—traveling on a wavy road whose profile is modeled as a sinusoidal wave $\eta(x) = h\sin(2\pi x/\lambda)$.

(a) In order to prevent the mass from oscillating with an amplitude greater than $5h$, a range of the values of the constant velocity v should be avoided; find this range in terms of the given parameters.
(b) Now, we want to add shock absorbers to the suspension, while knowing that the weight of the vehicle body is 10 kN and that the damped frequency of the system should be 1 Hz. Find the stiffness and the dashpot coefficient of the suspension that will prevent the mass from moving with amplitudes greater than $2.5h$ under *any* constant velocity v.

2.17. An aircraft turbine mounted on a testbed is modeled as shown in Fig. 2.71a. The turbine provides a compressive force $f(t)$ of the saturation type,[15] as plotted in Fig. 2.71b. Derive the time response of the system, $x(t)$, under zero initial conditions.

2.18. A Baja vehicle weighing 5,000 N is being designed with a suspension that has to provide a natural frequency of 1 Hz. Moreover, it is required that, upon letting the vehicle fall from a certain height h_0, the vehicle body bounce back to successive height peaks h_1 and h_2, so that h_2 be 20% of h_1. Find the dashpot coefficient and the spring stiffness of the suspension.

[15] The saturation function $\mathrm{sat}(x)$ was introduced in Eq. 1.37 and plotted in Fig. 1.26b.

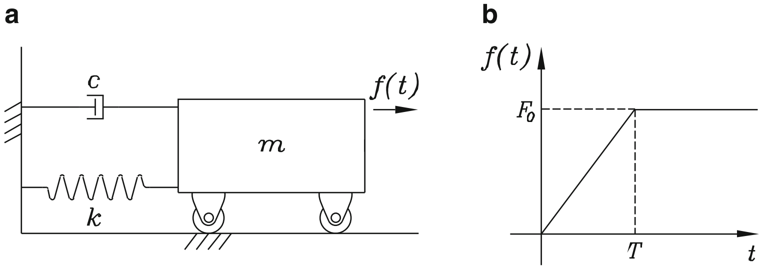


Fig. 2.71 Model for an aircraft turbine

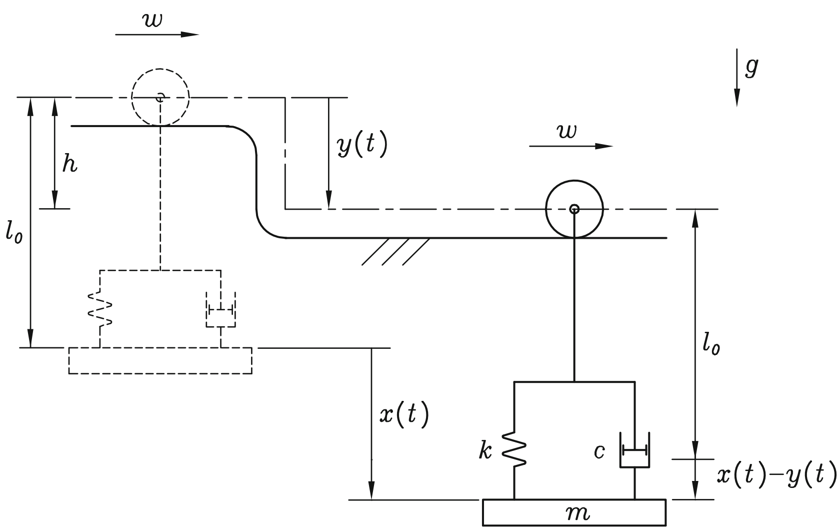


Fig. 2.72 An overhead material-transport system with an underdamped suspension

2.19. We study here the behavior of the *underdamped* material-handling system of Fig. 2.72 as the wheel goes down a step of height h along its track at $t = 0$.

(a) Derive the mathematical model of the system.
(b) Find the time response of the system.
(c) Find the values of $x(0^+)$ and $\dot{x}(0^+)$.

2.20. Shown in Fig. 2.73 is the model of a terrestrial vehicle with an undamped suspension, traveling at a constant speed v_0 on a bumpy road that is known to have a periodic profile. For purposes of our analysis, we model the road profile in the form

$$\eta(x) = h\left|\sin\left(\frac{\pi}{\lambda}x\right)\right|$$

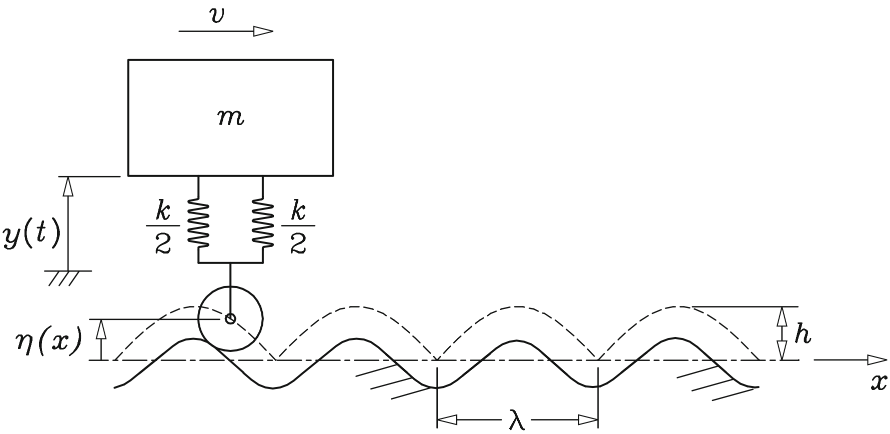


Fig. 2.73 Terrestrial vehicle traveling on an bumpy road

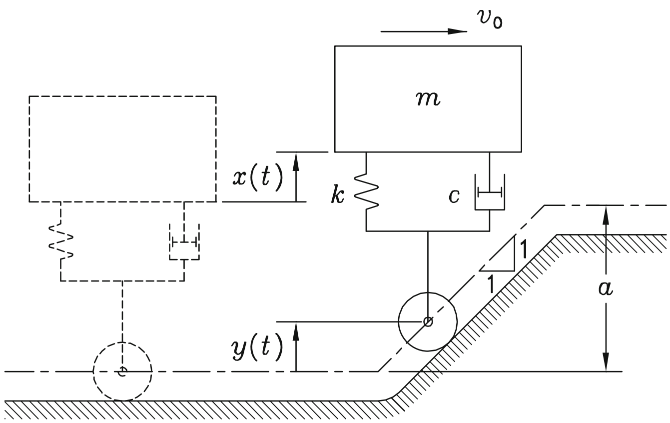


Fig. 2.74 Terrestrial vehicle overcoming a 45° ramp

for a bump height h and a wavelength λ. Upon introducing the coordinate y measuring the vertical position of the mass with respect to its equilibrium configuration, the governing equation takes the form

$$\ddot{y} + \omega_n^2 y = \omega_n^2 \eta$$

Find the steady-state response $y(t)$ of the system in terms of the parameters of the system and of the road profile. *Hint:* $2\sin a \cos b = \sin(a+b) + \sin(a-b)$.

2.21. An *all-terrain terrestrial vehicle* with an underdamped suspension is modeled as in Fig. 2.74. It is meant to overcome a 45° slope as shown in the same figure, while traveling at a constant speed v_0. The variable x measures the vertical displacement

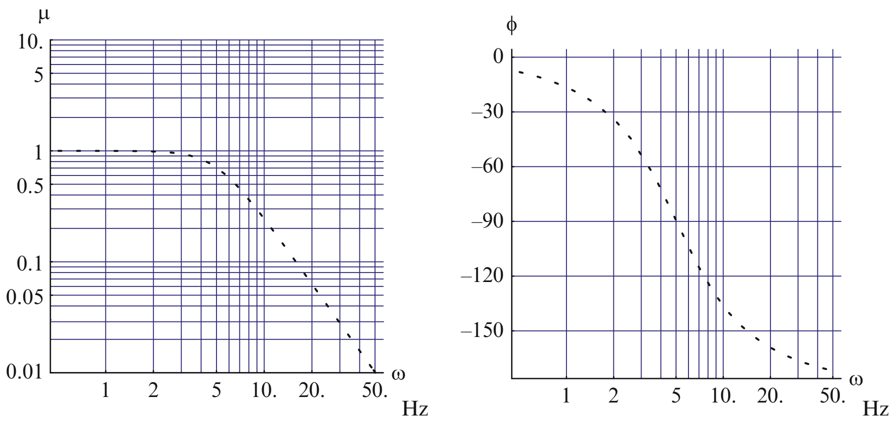


Fig. 2.75 Magnitude (μ) and phase (ϕ) of the tests conducted on a vehicle suspension

of the vehicle with respect to an inertial frame. This frame is located so that the origin corresponds to the position of static equilibrium of the vehicle when it is at rest on the ground before hitting the ramp. The variable y represents the height of the ramp from the ground. Derive the time response of the vehicle.

2.22. We refer here to the suspension system of Problem 1.8. The linearized model about the equilibrium configuration $y = 0$ takes the form

$$m_E\ddot{y} + c_E\dot{y} + k_E y = f(t)$$

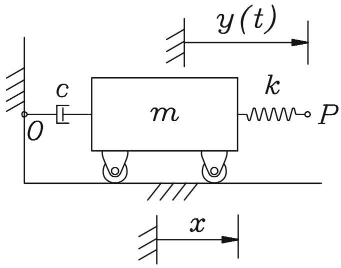
where y is a "small-amplitude" displacement from equilibrium, while m_E, c_E, and k_E are the equivalent mass, equivalent dashpot coefficient, and equivalent stiffness of the linearized system, and $f(t)$ is an external vertical force. Expressions for m_E, c_E, and k_E are given as

$$m_E = 19m, \quad c_E = \frac{3}{4}c, \quad k_E = \frac{3}{4}k$$

The suspension system is tested on a machine that applies a vertical force $f(t) = F_0 \cos \omega t$ on the wheel with ω changing in a broad spectrum of frequencies, thus obtaining the plots shown in Fig. 2.75. For the displacement $y(t)$ determine the values of c and k, if $m = 5$ kg.

2.23. The tool of a NC lathe is modeled as a linearly elastic, massless spring, while the positioning mechanism is modeled as a mass-dashpot system, as shown in Fig. 2.76. When the workpiece is turning at a constant angular velocity ω_f, the workpiece exerts a displacement $y(t) = A\cos\omega_f t$ on the tip of the tool due to the irregularities in the workpiece profile, where A is assumed to be "small." Find the frequency ω_f for which the force transmitted to the machine-tool frame

Fig. 2.76 Model of a NC-lathe-tool system



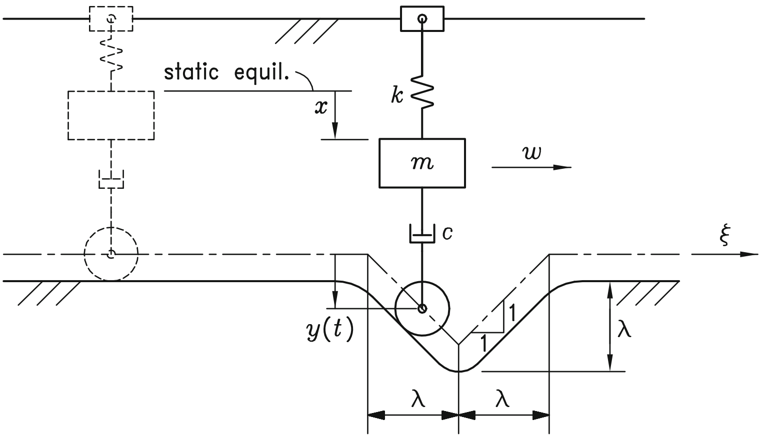


Fig. 2.77 Iconic model of a coordinate-measuring machine

is a maximum. In this model, note that the workpiece cannot, properly speaking, exert a pull on the tool. However, the spring is assumed to be under compression all the time, the displacement $y(t)$ becoming positive when the contact point of the workpiece is to the right of its nominal position; the latter is defined, in turn, as the one that the contact point would have if the workpiece were perfectly circular.[16]

2.24. Shown in Fig. 2.77 is a rough iconic model of a coordinate-measuring machine, as it is probing a nominally flat surface at a constant velocity w, while approaching a groove that can be safely modeled as a *triangular ditch*. Note that the slider is massless and can move freely and without friction on its horizontal guideway. Also note that the probe is always in contact with the surface. Under the assumption that the system at hand is underdamped, and for a set of numerical values that yield the model

$$\ddot{x} + 2\dot{x} + 2x = 200[u(t) - 2u(t-T) + u(t-2T)]$$

[16]This exercise is drawn from a similar one in [3].

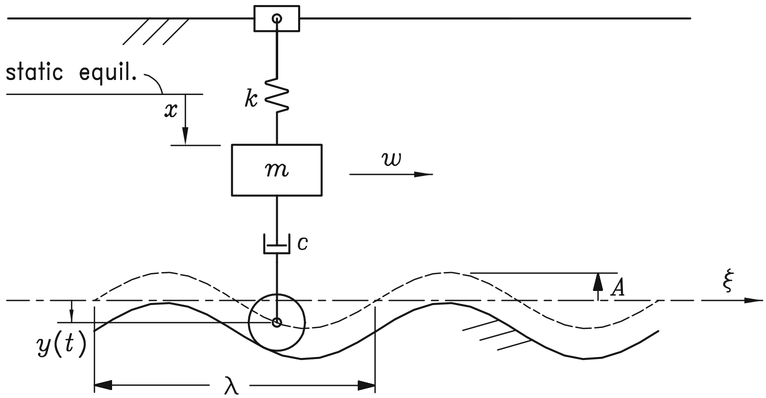


Fig. 2.78 Iconic model of a coordinate-measuring machine

find the time response of the system in terms of the corresponding responses to abrupt and impulsive excitations.

2.25. Shown in Fig. 2.78 is the same model of the coordinate-measuring machine of Exercise 2.24 as it probes a nominally flat surface at a constant velocity w. The surface shows small irregularities that can be approximated as a harmonic wave of amplitude A. Note that: (1) the slider is massless; (2) the slider can move freely and without friction on its horizontal guideway; and (3) the probe is always in contact with the surface. Under the foregoing assumptions,

(a) show that the mathematical model of the system can be cast in the form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t)$$

and find an expression for $f(t)$.

(b) Under the assumption that the system at hand is underdamped, and for a set of numerical values that yield the model

$$\ddot{x} + 2\dot{x} + 2x = 200\cos\omega t \quad [\mu m/s^2]$$

find an expression for ω in terms of the parameters of the problem.

(c) Find the value of the steady-state peak force exerted by the probe on the surface, assuming $m = 1$ kg.

2.26. Find an expression for the time response of the underdamped system of Fig. 2.79 traveling at a constant velocity w upon encountering a ramp of angle α, in terms of the response(s) of the same system to abrupt and impulsive excitations.

2.27. Shown in Fig. 2.76 is the iconic model of the tool-carrying mechanism of a lathe, with point P indicating the contact point between tool and workpiece.

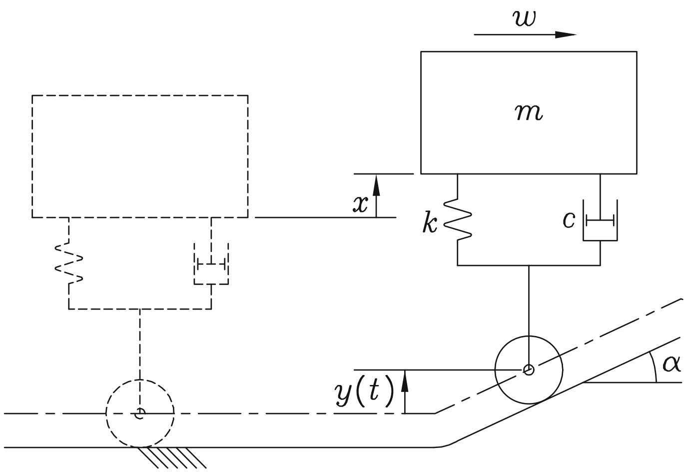


Fig. 2.79 An underdamped system encountering a ramp

The latter subjects the tip of the tool to a harmonic displacement $y(t) = A\cos\omega t$, with $A = 500$ µm, that is due to the irregularities on the machined surface. The mathematical model of this system takes the form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 y(t)$$

If we know that $m = 1$ kg, $\zeta = 0.5$, and $\omega_n = 200/\pi$ Hz for the model at hand, find the range of cutting speeds ω that will produce a steady-state force exerted at point O higher than 48 N, with O denoting the mounting of the tool carrier on the frame of the machine.
Hint: The use of Bode plots is highly recommended here.

2.28. Shown in Fig. 2.76 is the iconic model of the tool-carrying mechanism of a lathe, with point P indicating the contact point between tool and workpiece. The presence of an irregularity on the latter subjects the tip of the tool to a *triangular bump*, as shown in Fig. 2.80. The mathematical model of this system takes the form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t)$$

(a) Find an expression for $f(t)$ in terms of the parameters of the system.
(b) Now, for a given set of numerical values of the natural frequency and the damping ratio, the foregoing system is acted upon by a function $f(t)$—not necessarily the same as that of item (a)—of the form

$$f(t) = r(at) - 2r(at - aT)$$

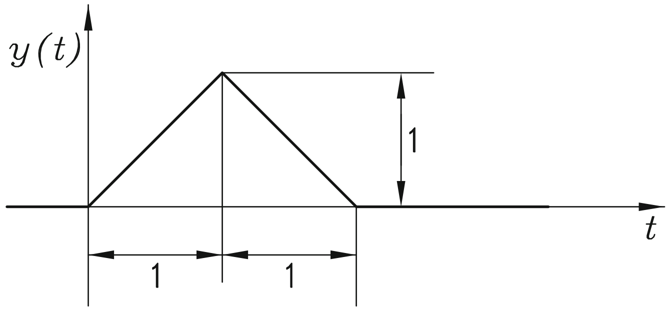


Fig. 2.80 Triangular bump profile

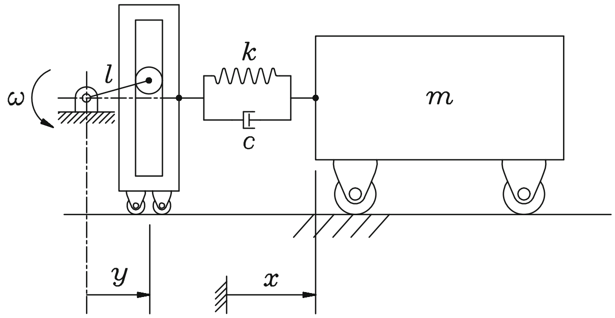


Fig. 2.81 A simplified model of a press mechanism

where $r(t)$ is the unit ramp, and T is a constant. Sketch the foregoing excitation and find the time response of the system at hand to this excitation.

2.29. A simple model of the mechanism of a press is shown in Fig. 2.81. In this model, the Scotch yoke of the left is driven by a crank that turns at a constant angular speed ω, while the spring is undeformed when $x = 0$ and $y = 0$.

(a) Under the current design it has been found that, when $\omega = \sqrt{k/m}$, the mass oscillates with an amplitude $M = 5l$. From this observation, can you estimate the damping ratio ζ?
(b) A modification is being proposed, that consists of changing the spring to one with a new stiffness k_{new}. Choose k_{new}, without modifying c, so that the amplitudes of the oscillations of the mass and the Scotch yoke coincide.

2.30. Derive the ramp response of (1) first-order systems and (2) *undamped* second-order systems. What is the steady-state component of these responses? *Skip (1) if first-order systems were not covered by your instructor.*

2.31. Derive the ramp response of (1) underdamped, (2) *critically damped* and (3) *overdamped* second-order systems. What is the steady-state component of these responses? *Skip (1) if first-order systems were not covered by your instructor.*

2.32. A wheelchair designed for climbers of pre-Columbian pyramids is to be analyzed to verify whether it meets its design specifications. To this end, we use the Fourier expansion of the pyramid profile derived in Example 2.8.4 and base our analysis on the first two harmonics. The forward velocity of the wheelchair is to be kept at a constant $3\,\mathrm{m/s}$; the length of the pyramid steps is assumed to be $\lambda/2 = 0.5\,\mathrm{m}$; the natural frequency of the chair is 1 Hz; and the damping ratio is 0.7071.

(a) Find the magnitude and the phase of the harmonics of the steady-state response.
(b) What is the error incurred in the approximation of the pyramid profile when taking only the first two harmonics?
(c) What about the error in the approximation of the steady-state response with only two harmonics?

2.33. A production machine carries a pneumatic hammer which is to be designed so that, upon hitting a nail with a force $F_0 \cos \omega t$, with $F_0 = 100\,\mathrm{N}$ and $\omega = 10\,\mathrm{Hz}$, the machine support experiences a harmonic force with an amplitude of only $10\,\mathrm{N}$.

(a) If the hammer weighs 200 N, and the hammer-suspension system has a natural frequency of 1 Hz, what damping ratio do you recommend to meet the design specifications?
(b) If the nail is assumed to provide a periodic force in the form of a square wave with intermittent values of 0 N and 100 N over equal periods of 0.1 s, find the steady-state response of the hammer by taking only the first two harmonics of the Fourier expansion of the periodic force.
(c) What is the error incurred in the approximation of the force with only two harmonics?
(d) What about the error in the approximation of the force transmitted to the user?

2.34. The air compressor of Example 2.8.6 is revisited here. The velocity $\dot{s}$ of the piston with respect to the housing is given below as a function of the angular displacement θ of the crank, in dimensionless form:

$$\frac{\dot{s}}{\omega a} = \left[1 + \rho \frac{\cos\theta}{\sqrt{1 - \rho^2 \cos^2\theta}} \right] \sin\theta$$

where ω denotes the constant angular velocity of the crank, and $\rho \equiv a/b$. Moreover, the foregoing dimensionless velocity is plotted for $\rho = 0.5$ in Fig. 2.82a. The acceleration $\ddot{s}$ appearing in the right-hand side of the mathematical model of that example can be found from the above expression by straightforward differentiation. However, as one can readily realize, a Fourier analysis of the foregoing velocity, not to speak of its time-derivative, would be untractable with longhand calculations. Here we have a typical example of an algebraically difficult problem that can be rendered tractable with a reasonable approximation. Indeed, we can approximate the velocity plot of Fig. 2.82a by a triangular wave, as depicted in Fig. 2.82b.

Express the right-hand side of the corresponding mathematical model as a series of harmonic functions, under the assumption that the crank turns at a constant rate ω_0 rad/s, for a piston velocity approximated as in Fig. 2.82b.

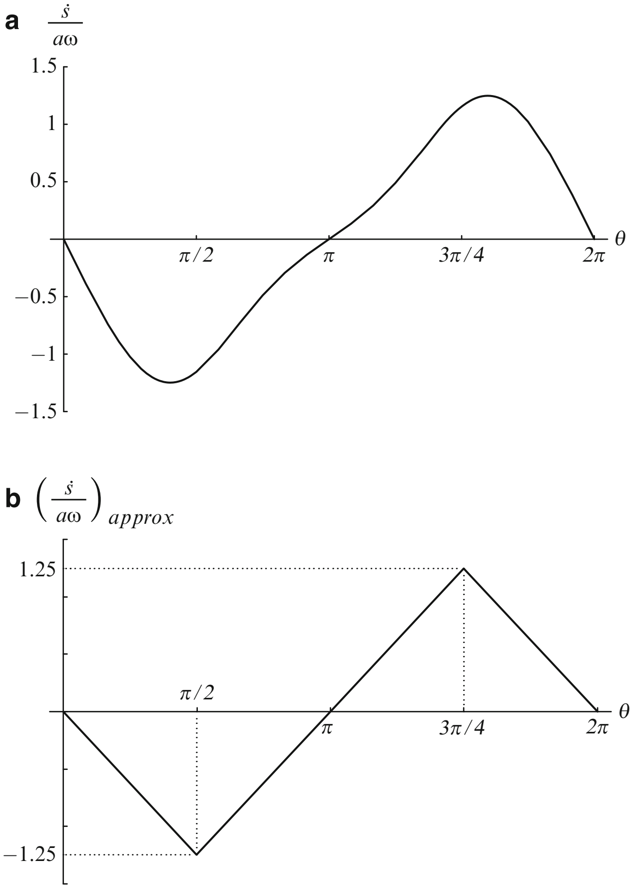


Fig. 2.82 Plot of (**a**) nondimensional piston velocity and (**b**) its piecewise linear approximation

2.35. A vehicle with worn-out shock absorbers is represented in Fig. 2.83 as it rolls through a bumpy road whose bumps follow a circular pattern of radius r. The vehicle travels with an unperturbed constant speed $v = 20\omega_n r$, where $\omega_n = \sqrt{k/m}$. We want to estimate how much the body of the vehicle will "jump" as it traverses the road. To this end, we conduct an *approximate* Fourier analysis of the road profile using a trapezoidal integration of the profile, which yields the expansion below:

$$\eta(x) \approx \tilde{a}_0 + \tilde{a}_1 \cos\left(\frac{\pi x}{r}\right) + \tilde{a}_2 \cos\left(\frac{2\pi x}{r}\right)$$

where $\tilde{a}_k$, for $k = 0, 1, 2$ are the *approximate* Fourier coefficients of the road profile, computed using the trapezoidal rule. Our computer-algebra calculations yield

$$\tilde{a}_0 = \frac{1+\sqrt{3}}{4}r, \quad \tilde{a}_1 = -\frac{1}{2}r, \quad \tilde{a}_2 = \frac{1-\sqrt{3}}{2}r$$

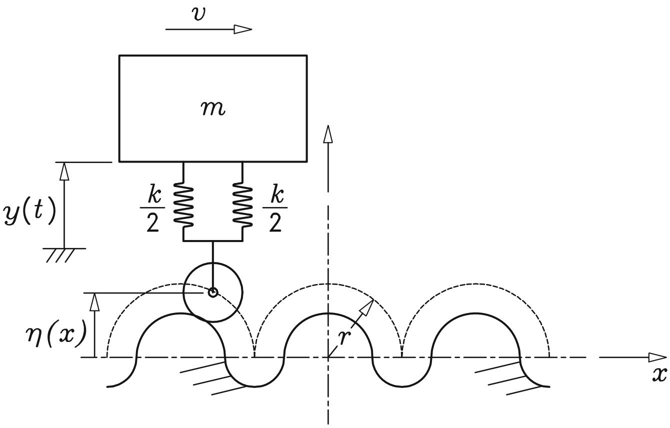


Fig. 2.83 An undamped vehicle traversing a bumpy road

(a) For starters, we want to estimate the error of the above approximation of the road profile. Find an expression for that error.

(b) Under the above approximation, the (steady-state) periodic response of the vehicle can be approximated as

$$y(t) \approx M_0 + M_1 \cos\left(\frac{\pi v t}{r} + \phi_1\right) + M_2 \cos\left(\frac{2\pi v t}{r} + \phi_2\right)$$

Find expressions for M_0, M_1, M_2, ϕ_1, and ϕ_2.

(c) Estimate the amount of "jumping" by means of the root-mean square value of $y(t)$, i.e., find an expression for $[y(t)]_{\text{rms}}$, where, in order to avoid algebraic mistakes and spending precious time with the required algebra, you need not substitute the expression for $\{M_i\}_0^2$ and $\{\phi_i\}_1^2$ found in item (b). *Hint: Express $y(t)$ as a* truncated *Fourier series, i.e., as a sum of (1) a constant term, (2) two cosine terms, and (3) two sine terms, none of these with a phase angle.*

2.36. A fluid clutch, similar to that of Fig. 1.23, is considered here, with a Geneva wheel—see Fig. 4.13—mounted between the motor M and the driving disk rotating at an angular velocity ω. Moreover, the load is driven by the above-mentioned disk through a viscous fluid that transmits a moment proportional to the difference in angular velocities of the two disks, the constant of proportionality being the viscous friction coefficient c. The system can be modeled as shown below:

$$\dot{\omega}_R + \frac{1}{\tau}\omega_R = \frac{1}{\tau}\omega$$

where $\omega \equiv \dot{\phi}$, $\omega_R \equiv \dot{\theta}$, and τ is the time constant of the system, defined as $\tau \equiv J/c$. Moreover, the profile of the angular displacement ϕ is approximated as indicated in Fig. 2.84.

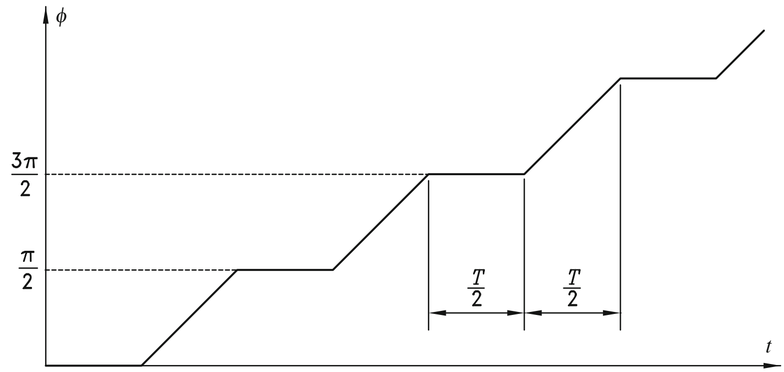


Fig. 2.84 Displacement produced by a Geneva wheel

Find an expression for the time response $\omega_R(t)$ after a long-enough time has elapsed so as to allow for the decay of all transients, i.e., at steady-state.

2.37. A fluid clutch connects a load, e.g., the whole inertia of an automobile, to an engine, as shown in Fig. 1.23. Assume that the torque transmitted through the clutch, when engaged, is proportional to the difference in speed of the input and output shafts (proportionality constant c). Assume also that the speed $\omega(t)$ of the input shaft is unaffected by the load. If the load is a pure inertia J, find the steady-state angular velocity $\omega_R = \dot{\theta}$ of the load, if the input angular velocity $\omega(t)$ is given by[17]

$$\omega(t) = \frac{c}{J}\left|\sin\frac{2\pi bt}{J}\right|$$

2.38. Shown in Fig. 2.85 is a rotor of moment of inertia J that turns at a rate $\dot{\theta}$, as it is driven by a shaft via a universal joint. If the input axis of the universal joint is driven, in turn, at a constant angular velocity ω_0, the output axis delivers a periodic angular velocity σ, which is transmitted to the left end of an elastic shaft of stiffness k. That is, if the constant angle between input and output axes is labelled α, and the angular displacement of the input shaft is denoted by ψ, then,

$$\dot{\phi} = \sigma = \frac{\cos\alpha}{1-\sin^2\alpha\sin^2\psi}\omega_0, \quad \omega_0 \equiv \dot{\psi}$$

Moreover, the relation between the output angle ϕ and the input angle ψ is given by

$$\tan\phi = -\frac{\cot\psi}{\cos\alpha}$$

[17] This exercise is drawn from a similar one in [3].

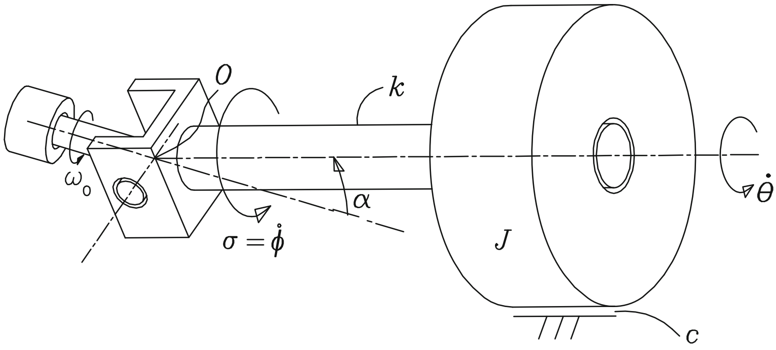


Fig. 2.85 Rotor driven by a universal joint

Using *Parseval's Formula*, we can estimate the error in the approximation of the function $\sigma(t)$ by comparing the rms value of $\sigma(t)$, $\tilde{\sigma}$, with the sum of the squares of the coefficients of the series, and hence, the square of the error in the approximation takes the form

$$e^2 = a_0^2 + \frac{1}{2}\sum_1^{N_h} a_k^2 + \frac{1}{2}\sum_1^{N_h} b_k^2 - \tilde{\sigma}^2$$

where $\tilde{\sigma}$ is calculated as

$$\tilde{\sigma} = \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)dt}$$

It is highly recommended that you calculate this integral using numerical quadrature, as implemented in a Matlab routine (`quad` or `quad8`), or that you use another reliable commercial routine *with truncation-error control*, which is indicated via a user-prescribed tolerance. Assign your tolerance judiciously: if you choose a very loose tolerance, your computed integral will contain an inadmissible high error; if too tight, the procedure may take too long to finish. As well, it is recommended that you calculate the Fourier coefficients using the computational scheme given in Sect. 2.8.2.

Produce a table of error e vs. N_h, for $N_h = 1$, 2, 3, 4, 5, 10, and 20. Comment on your tabulated results.

Now, find the steady-state response $\dot{\theta}(t)$ for N_h harmonics of the Fourier expansion of $\sigma(t)$, using a suitable value of N_h, that you should choose based on the table that you produced above. Plot time response vs. time in an interval $[0, 4T]$. For this part of the problem, use the model

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \omega_n^2\phi$$

For the calculation, use the numerical values

$$\alpha = 30^\circ, \quad \zeta = 0.1, \quad \omega_n = 10p, \quad p = 0.3\ \mathrm{s}^{-1}$$

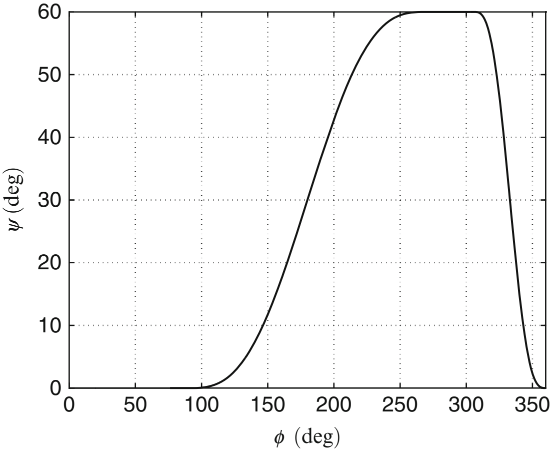


Fig. 2.86 Follower motion program of quick-return cam mechanism

2.39. The oscillating follower of a quick-return cam mechanism goes through a *lower dwell* in the first 25% of its cycle; then, it performs a *working stroke* of 60° of amplitude for 50% of its cycle; it goes further through an *upper dwell* for 10% of its cycle, and returns to its lower dwell in the remaining 15% of its cycle. One period of the periodic angular displacement of the follower be represented by $\phi = \phi(\psi)$, where ψ is the angular displacement of the cam, which rotates at a constant rate $\dot{\psi} = \omega = \text{const}$, the *motion program* $\phi(\psi)$ being given below. Moreover, the follower drives an elastic shaft connected to a load of moment of inertia J mounted on bearings that provide a linearly viscous dissipative torque of coefficient c, the displacement of the load being denoted by θ. The follower motion program, displayed in Fig. 2.86, is given by

$$\phi(\psi) = \begin{cases} 0, & \text{for} \quad 0 \le \psi \le \dfrac{\pi}{2}; \\ \dfrac{\pi}{3}\left[\dfrac{\psi}{\pi} - \dfrac{1}{2} - \dfrac{1}{2\pi}\sin(2\psi - \pi)\right], & \text{for} \quad \dfrac{\pi}{2} \le \psi \le \dfrac{3\pi}{2}; \\ \dfrac{\pi}{3}, & \text{for} \quad \dfrac{3\pi}{2} \le \psi \le \dfrac{17\pi}{10}; \\ \dfrac{\pi}{3}\left[\dfrac{20}{3} - \dfrac{10\psi}{3\pi} - \dfrac{1}{2\pi}\sin\left(\dfrac{40\pi - 20\psi}{3}\right)\right], & \text{for} \quad \dfrac{17\pi}{10} \le \psi \le 2\pi \end{cases}$$

while the mathematical model of the follower-load system takes the form

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \omega_n^2\phi$$

and the torque experienced by the shaft, $\tau(t)$, is

$$\tau(t) = k[\theta(t) - \phi(t)]$$

The numerical values of the variables involved are given as

$$\zeta = 0.1, \quad \omega_n = 10\omega, \quad \omega = 300 \text{ rpm}$$

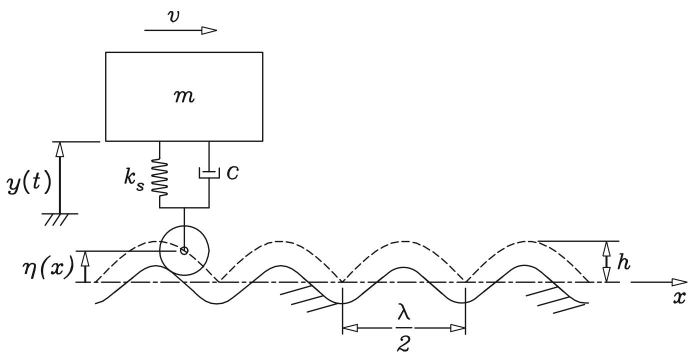


Fig. 2.87 A terrestrial vehicle traveling on a wavy road

(a) Using *Parseval's Formula*, estimate the error in the Fourier approximation of the function $\phi(t)$ by comparing the rms value $\tilde{\phi}$ of $\phi(t)$ with the sum of the squares of the coefficients of the truncated Fourier series. The difference of the two quantities produces the square of the error, e^2. Now, produce a table of error e vs. N_h, for $N_h = 1, 2, 3, 4, 5, 10$, and 20. Comment on your tabulated results.

(b) Then, find the steady-state response $\theta(t)$ for N_h harmonics of the Fourier expansion of $\phi(t)$, using a suitable value of N_h, which you should choose based on the table that you produced above. Plot the steady-state response vs. time in the interval $[0, 4T]$, where T is the duration of one cycle.

(c) Now plot the steady-state value of the torque experienced by the elastic shaft in dimensionless form, i.e., plot $\tau(t)/(J\omega_n^2)$ vs. time, in the time-interval $[0, 4T]$.

2.40. A terrestrial vehicle traveling at a constant velocity v on a wavy road with a profile that can be approximated fairly well by $h|\sin(2\pi x/\lambda)|$ is shown in Fig. 2.87, its mathematical model taking the form,

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 2\zeta\omega_n\dot{\eta}(t) + \omega_n^2\eta(t)$$

where $\omega_n \equiv \sqrt{k_s/m}$, $\zeta \equiv c/(2m\omega_n)$, and $0 < \zeta < 1$, while

$$\eta = h|\sin(2\pi x/\lambda)|.$$

(a) With $\omega_0 \equiv 2\pi v/\lambda$, express the right-hand side of the mathematical model in the form

$$2\zeta\omega_n\dot{\eta}(t) + \omega_n^2\eta(t) = \sum_{k=0}^{\infty} C_k \cos(k\omega_0 t + \psi_k)$$

and give expressions for C_k and ψ_k.

(b) Using only the first two harmonics of the Fourier expansion of the right-hand side of the mathematical model, find the **steady-state** response of the system.

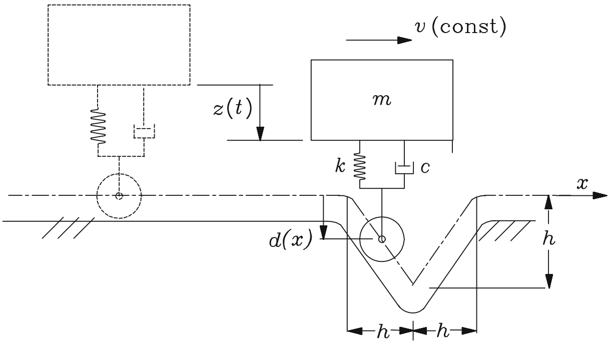


Fig. 2.88 The model of an automobile suspension encountering a triangular ditch

2.41. The model of an automobile suspension is depicted in Fig. 2.88. Find the time response of the model as the vehicle approaches a ditch that is modeled by a triangular pulse. Assume that the traveling speed v of the vehicle is constant, and that this speed is not affected by the ditch. Use the relations

$$\zeta = \frac{\sqrt{2}}{2}, \quad \omega_n \equiv \sqrt{\frac{k}{m}} = \frac{v}{h} .$$

References

1. Moler CB, Van Loan C (1978) Nineteen dubious ways to compute the exponential of a matrix. SIAM Rev 20(4):801–836
2. Chicurel-Uziel E (2007) Dirac delta representation by parametric equations. Applications to impulsive vibration systems. J Sound Vibr 305:134–150
3. Cannon RH (1967) Dynamics of physical systems. McGraw-Hill Book Co., New York
4. Kahaner D, Moler C, Nash S (1989) Numerical methods and software. Prentice Hall, Inc., Englewood Cliffs, NJ
5. Moler C (2004) Numerical computing with MATLAB, Electronic edition. The MathWorks, Inc., Nantick
6. Strang G (1986) Introduction to applied mathematics. Wessley-Cambridge Press, Wessley, pp 274–276

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