Preface

Manifolds crop up everywhere in mathematics. These generalizations of curves and surfaces to arbitrarily many dimensions provide the mathematical context for understanding "space" in all of its manifestations. Today, the tools of manifold theory are indispensable in most major subfields of pure mathematics, and are becoming increasingly important in such diverse fields as genetics, robotics, econometrics, statistics, computer graphics, biomedical imaging, and, of course, the undisputed leader among consumers (and inspirers) of mathematics—theoretical physics. No longer the province of differential geometers alone, smooth manifold technology is now a basic skill that all mathematics students should acquire as early as possible.

Over the past century or two, mathematicians have developed a wondrous collection of conceptual machines that enable us to peer ever more deeply into the invisible world of geometry in higher dimensions. Once their operation is mastered, these powerful machines enable us to think geometrically about the 6-dimensional solution set of a polynomial equation in four complex variables, or the 10-dimensional manifold of 5×5 orthogonal matrices, as easily as we think about the familiar 2-dimensional sphere in $\mathbb{R}^3$. The price we pay for this power, however, is that the machines are assembled from layer upon layer of abstract structure. Starting with the familiar raw materials of Euclidean spaces, linear algebra, multivariable calculus, and differential equations, one must progress through topological spaces, smooth atlases, tangent bundles, immersed and embedded submanifolds, vector fields, flows, cotangent bundles, tensors, Riemannian metrics, differential forms, foliations, Lie derivatives, Lie groups, Lie algebras, and more—just to get to the point where one can even think about studying specialized applications of manifold theory such as comparison theory, gauge theory, symplectic topology, or Ricci flow.

This book is designed as a first-year graduate text on manifold theory, for students who already have a solid acquaintance with undergraduate linear algebra, real analysis, and topology. I have tried to focus on the portions of manifold theory that will be needed by most people who go on to use manifolds in mathematical or scientific research. I introduce and use all of the standard tools of the subject, and prove most of its fundamental theorems, while avoiding unnecessary generalization or specialization. I try to keep the approach as concrete as possible, with pictures and intuitive discussions of how one should think geometrically about the abstract concepts, but without shying away from the powerful tools that modern mathematics has to offer. To fit in all of the basics and still maintain a reasonably sane pace, I have had to omit or barely touch on a number of important topics, such as complex manifolds, infinite-dimensional manifolds, connections, geodesics, curvature, fiber bundles, sheaves, characteristic classes, and Hodge theory. Think of them as dessert, to be savored after completing this book as the main course.

To convey the book's compass, it is easiest to describe where it starts and where it ends. The starting line is drawn just after topology: I assume that the reader has had a rigorous introduction to general topology, including the fundamental group and covering spaces. One convenient source for this material is my *Introduction to Topological Manifolds* [LeeTM], which I wrote partly with the aim of providing the topological background needed for this book. There are other books that cover similar material well; I am especially fond of the second edition of Munkres's *Topology* [Mun00]. The finish line is drawn just after a broad and solid background has been established, but before getting into the more specialized aspects of any particular subject. In particular, I introduce Riemannian metrics, but I do not go into connections, geodesics, or curvature. There are many Riemannian geometry books for the interested student to take up next, including one that I wrote [LeeRM] with the goal of moving expediently in a one-quarter course from basic smooth manifold theory to nontrivial geometric theorems about curvature and topology. Similar material is covered in the last two chapters of the recent book by Jeffrey Lee (no relation) [LeeJeff09], and do Carmo [dC92] covers a bit more. For more ambitious readers, I recommend the beautiful books by Petersen [Pet06], Sharpe [Sha97], and Chavel [Cha06].

This subject is often called "differential geometry." I have deliberately avoided using that term to describe what this book is about, however, because the term applies more properly to the study of smooth manifolds endowed with some extra structure—such as Lie groups, Riemannian manifolds, symplectic manifolds, vector bundles, foliations—and of their properties that are invariant under structure-preserving maps. Although I do give all of these geometric structures their due (after all, smooth manifold theory is pretty sterile without some geometric applications), I felt that it was more honest not to suggest that the book is primarily about one or all of these geometries. Instead, it is about developing the general tools for working with smooth manifolds, so that the reader can go on to work in whatever field of differential geometry or its cousins he or she feels drawn to.

There is no canonical linear path through this material. I have chosen an ordering of topics designed to establish a good technical foundation in the first half of the book, so that I can discuss interesting applications in the second half. Once the first twelve chapters have been completed, there is some flexibility in ordering the remaining chapters. For example, Chapter 13 (Riemannian Metrics) can be postponed if desired, although some sections of Chapters 15 and 16 would have to be postponed as well. On the other hand, Chapters 19–21 (Distributions and Foliations, The Exponential Map, and Quotient Manifolds, respectively) could in principle be inserted any time after Chapter 14, and much of the material can be covered even earlier if you are willing to skip over the references to differential forms. And the final chapter (Symplectic Manifolds) would make sense any time after Chapter 17, or even after Chapter 14 if you skip the references to de Rham cohomology.

As you might have guessed from the size of the book, and will quickly confirm when you start reading it, my style tends toward more detailed explanations and proofs than one typically finds in graduate textbooks. I realize this is not to every instructor's taste, but in my experience most students appreciate having the details spelled out when they are first learning the subject. The detailed proofs in the book provide students with useful models of rigor, and can free up class time for discussion of the meanings and motivations behind the definitions as well as the "big ideas" underlying some of the more difficult proofs. There are plenty of opportunities in the exercises and problems for students to provide arguments of their own.

I should say something about my choices of conventions and notations. The old joke that "differential geometry is the study of properties that are invariant under change of notation" is funny primarily because it is alarmingly close to the truth. Every geometer has his or her favorite system of notation, and while the systems are all in some sense formally isomorphic, the transformations required to get from one to another are often not at all obvious to students. Because one of my central goals is to prepare students to read advanced texts and research articles in differential geometry, I have tried to choose notations and conventions that are as close to the mainstream as I can make them without sacrificing too much internal consistency. (One difference between this edition and the previous one is that I have changed a number of my notational conventions to make them more consistent with mainstream mathematical usage.) When there are multiple conventions in common use (such as for the wedge product or the Laplace operator), I explain what the alternatives are and alert the student to be aware of which convention is in use by any given writer. Striving for too much consistency in this subject can be a mistake, however, and I have eschewed absolute consistency whenever I felt it would get in the way of ease of understanding. I have also introduced some common shortcuts at an early stage, such as the Einstein summation convention and the systematic confounding of maps with their coordinate representations, both of which tend to drive students crazy at first, but pay off enormously in efficiency later.

Prerequisites

This subject draws on most of the topics that are covered in a typical undergraduate mathematics education. The appendices (which most readers should read, or at least skim, first) contain a cursory summary of prerequisite material on topology, linear algebra, calculus, and differential equations. Although students who have not seen this material before will not learn it from reading the appendices, I hope readers will appreciate having all of the background material collected in one place. Besides giving me a convenient way to refer to results that I want to assume as known, it also gives the reader a splendid opportunity to brush up on topics that were once (hopefully) understood but may have faded.

Exercises and Problems

This book has a rather large number of exercises and problems for the student to work out. Embedded in the text of each chapter are questions labeled as “Exercises.” These are (mostly) short opportunities to fill in gaps in the text. Some of them are routine verifications that would be tedious to write out in full, but are not quite trivial enough to warrant tossing off as obvious. I recommend that serious readers take the time at least to stop and convince themselves that they fully understand what is involved in doing each exercise, if not to write out a complete solution, because it will make their reading of the text far more fruitful.

At the end of each chapter is a collection of (mostly) longer and harder questions labeled as “Problems.” These are the ones from which I select written homework assignments when I teach this material. Many of them will take hours for students to work through. Only by doing a substantial number of these problems can one hope to absorb this material deeply. I have tried insofar as possible to choose problems that are enlightening in some way and have interesting consequences in their own right. When the result of a problem is used in an essential way in the text, the page where it is used is noted at the end of the problem statement.

I have deliberately not provided written solutions to any of the problems, either in the back of the book or on the Internet. In my experience, if written solutions to problems are available, even the most conscientious students find it very hard to resist the temptation to look at the solutions as soon as they get stuck. But it is exactly at that stage of being stuck that students learn most effectively, by struggling to get unstuck and eventually finding a path through the thicket. Reading someone else’s solution too early can give one a comforting, but ultimately misleading, sense of understanding. If you really feel you have run out of ideas, talk with an instructor, a fellow student, or one of the online mathematical discussion communities such as *math.stackexchange.com*. Even if someone else gives you a suggestion that turns out to be the key to getting unstuck, you will still learn much more from absorbing the suggestion and working out the details on your own than you would from reading someone else’s polished proof.

About the Second Edition

Those who are familiar with the first edition of this book will notice first that the topics have been substantially rearranged. This is primarily because I decided it was worthwhile to introduce the two most important analytic tools (the rank theorem and the fundamental theorem on flows) much earlier, so that they can be used throughout the book rather than being relegated to later chapters.

A few new topics have been added, notably Sard’s theorem, some transversality theorems, a proof that infinitesimal Lie group actions generate global group actions, a more thorough study of first-order partial differential equations, a brief treatment of degree theory for smooth maps between compact manifolds, and an introduction to contact structures. I have consolidated the introductory treatments of Lie groups, Riemannian metrics, and symplectic manifolds in chapters of their own, to make it easier to concentrate on the special features of those subjects when they are first introduced (although Lie groups and Riemannian metrics still appear repeatedly in later chapters). In addition, manifolds with boundary are now treated much more systematically throughout the book.

Apart from additions and rearrangement, there are thousands of small changes and also some large ones. Parts of every chapter have been substantially rewritten to improve clarity. Some proofs that seemed too labored in the original have been streamlined, while others that seemed unclear have been expanded. I have modified some of my notations, usually moving toward more consistency with common notations in the literature. There is a new notation index just before the subject index.

There are also some typographical improvements in this edition. Most importantly, mathematical terms are now typeset in ***bold italics*** when they are officially defined, to reflect the fact that definitions are just as important as theorems and proofs but fit better into the flow of paragraphs rather than being called out with special headings. The exercises in the text are now indicated more clearly with a special symbol (►), and numbered consecutively with the theorems to make them easier to find. The symbol $\square$, in addition to marking the ends of proofs, now also marks the ends of statements of corollaries that follow so easily that they do not need proofs; and I have introduced the symbol // to mark the ends of numbered examples. The entire book is now set in Times Roman, supplemented by the excellent *MathTime Professional II* mathematics fonts from Personal TeX, Inc.

Acknowledgments

Many people have contributed to the development of this book in indispensable ways. I would like to mention Tom Duchamp, Jim Isenberg, and Steve Mitchell, all of whom generously shared their own notes and ideas about teaching this subject; and Gary Sandine, who made lots of helpful suggestions and created more than a third of the illustrations in the book. In addition, I would like to thank the many others who have read the book and sent their corrections and suggestions to me. (In the Internet age, textbook writing becomes ever more a collaborative venture.) And most of all, I owe a debt of gratitude to Judith Arms, who has improved the book in countless ways with her thoughtful and penetrating suggestions.

For the sake of future readers, I hope each reader will take the time to keep notes of any mistakes or passages that are awkward or unclear, and let me know about them as soon as it is convenient for you. I will keep an up-to-date list of corrections on my website, whose address is listed below. (Sad experience suggests that there will be plenty of corrections despite my best efforts to root them out in advance.) If that site becomes unavailable for any reason, the publisher will know where to find me. Happy reading!

Seattle, Washington, USA John M. Lee
www.math.washington.edu/~lee

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Lee, J.
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