Chapter 2
Overture: Ramsey's Theorem

Musicians in the past, as well as the best of the moderns, believed that a counterpoint or other musical composition should begin on a perfect consonance, that is, a unison, fifth, octave, or compound of one of these.

GIOSEFFO ZARLINO
Le Istitutioni Harmoniche, 1558

The Nucleus of Ramsey Theory

Most of this text is concerned with sets of subsets of the natural numbers, so, let us start there: The set $\{0, 1, 2, \dots\}$ of **natural numbers** (or of non-negative integers) is denoted by ω. It is convenient to consider a natural number n as an n-element subset of ω, namely as the set of all numbers smaller than n, so, $n = \{k \in \omega : k < n\}$. In particular, $0 = \emptyset$, where $\emptyset$ is the **empty set**. For any $n \in \omega$ and any set S, let $[S]^n$ denote the set of all n-element subsets of S (*e.g.*, $[S]^0 = \{\emptyset\}$). Further, the set of all finite subsets of a set S is denoted by $[S]^{<\omega}$.

For a finite set S let $|S|$ denote the number of elements in S, also called the **cardinality** of S.

A set S is called **countable** if there is an enumeration of S, *i.e.*, if $S = \emptyset$ or $S = \{x_i : i \in \omega\}$. In particular, every finite set is countable. However, when we say that a set is countable we usually mean that it is a countably infinite set. For any set S, $[S]^\omega$ denotes the set of all countably infinite subsets of S, in particular, since every infinite subset of ω is countable, $[\omega]^\omega$ is the set of *all* infinite subsets of ω.

Let S be an arbitrary non-empty set. A binary relation "$\sim$" on S is an **equivalence relation** if it is

- *reflexive* (*i.e.*, for all $x \in S$: $x \sim x$),
- *symmetric* (*i.e.*, for all $x, y \in S$: $x \sim y \leftrightarrow y \sim x$), and
- *transitive* (*i.e.*, for all $x, y, z \in S$: $x \sim y \wedge y \sim z \ \rightarrow \ x \sim z$).

The **equivalence class** of an element $x \in S$, denoted $[x]\tilde{}$, is the set $\{y \in S : x \sim y\}$. We would like to recall the fact that, since "$\sim$" is an equivalence relation, for any

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$x, y \in S$ we have *either* $[x]\tilde{} = [y]\tilde{}$ *or* $[x]\tilde{} \cap [y]\tilde{} = \emptyset$. A set $A \subseteq S$ is a set of **representatives** if for each equivalence class $[x]\tilde{}$ we have $|A \cap [x]\tilde{}| = 1$; in other words, A has exactly one element in common with each equivalence class. It is worth mentioning that in general, the existence of a set of representatives relies on the Axiom of Choice (see Chapter 5).

For sets A and B, let ${}^{A}B$ denote the set of all functions $f : A \to B$. For $f \in {}^{A}B$ and $S \subseteq A$ let $f[S] := \{f(x) : x \in S\}$ and let $f|_S \in {}^{S}B$ (the restriction of f to S) be such that for all $x \in S$, $f(x) = f|_S(x)$.

Further, for sets A and B, let the set-theoretic difference of A and B be the set $A \setminus B := \{a \in A : a \notin B\}$.

For some positive $n \in \omega$, let us colour all n-element subsets of ω with three colours, say red, blue, and yellow. In other words, each n-element set of natural numbers $\{k_1, \ldots, k_n\}$ is coloured either red, or blue, or yellow. Now one can ask whether there is an infinite subset H of ω such that all its n-element subsets have the same colour (*i.e.*, $[H]^n$ is **monochromatic**). Such a set we would call **homogeneous** (for the given colouring). In the terminology above, this question reads as follows: Given any colouring (*i.e.*, function) $\pi : [\omega]^n \to 3$, where $3 = \{0, 1, 2\}$, does there exist a set $H \in [\omega]^\omega$ such that $\pi|_{[H]^n}$ is constant? Alternatively, one can define an equivalence relation "$\sim$" on $[\omega]^n$ by stipulating $x \sim y$ *iff* $\pi(x) = \pi(y)$ and ask whether there exists a set $H \in [\omega]^\omega$ such that $[H]^n$ is included in one equivalence class. The answer to this question is given by RAMSEY'S THEOREM 2.1 below, but before we state and prove this theorem, let us say a few words about its background.

Ramsey proved his theorem in order to investigate a problem in formal logic, namely the problem of finding a regular procedure to determine the truth or falsity of a given logical formula in the language of *First-Order Logic*, which is also the language of Set Theory (*cf.* Chapter 3). However, RAMSEY'S THEOREM is a purely combinatorial statement and was the nucleus—but not the earliest result—of a whole combinatorial theory, the so-called *Ramsey Theory*. We would also like to mention that Ramsey's original theorem, which will be discussed later, is somewhat stronger than the theorem stated below but is, like König's Lemma, not provable without assuming some form of the Axiom of Choice (see PROPOSITION 7.8).

THEOREM 2.1 (RAMSEY'S THEOREM). *For any number $n \in \omega$, for any positive number $r \in \omega$, for any $S \in [\omega]^\omega$, and for any colouring $\pi : [S]^n \to r$, there is always an $H \in [S]^\omega$ such that H is homogeneous for π, i.e., the set $[H]^n$ is monochromatic.*

Before we prove RAMSEY'S THEOREM, let us consider a few examples: In the first example we colour the set of prime numbers $\mathbb{P}$ with two colours. A **Wieferich prime** is a prime number p such that p^2 divides $2^{p-1} - 1$, denoted $p^2 \mid 2^{p-1} - 1$. Recall that by FERMAT'S LITTLE THEOREM we have $p \mid 2^{p-1} - 1$ for any prime p. Now, define the 2-colouring π_1 of $\mathbb{P}$ by stipulating

$$\pi_1(p) = \begin{cases} 0 & \text{if } p \text{ is a Wieferich prime,} \\ 1 & \text{otherwise.} \end{cases}$$

Let $H_0 = \{p \in \mathbb{P} : p^2 \mid 2^{p-1} - 1\}$ and $H_1 = \mathbb{P} \setminus H_0$. The only numbers which are known to belong to H_0 are 1093 and 3511. On the other hand, it is not known whether H_1 is infinite. However, by the Infinite Pigeon-Hole Principle we know that at least one of the two sets H_0 and H_1 is infinite, which gives us a homogeneous set for π_1.

As a second example, define the 2-colouring π_2 of the set of 2-element subsets of $\{7l : l \in \omega\}$ by stipulating

$$\pi_2\big(\{n, m\}\big) = \begin{cases} 0 & \text{if } n^m + m^n + 1 \text{ is prime,} \\ 1 & \text{otherwise.} \end{cases}$$

An easy calculation modulo 3 shows that the set $H = \{42k + 14 : k \in \omega\} \subseteq \{7l : l \in \omega\}$ is homogeneous for π_2; in fact, for all $\{n, m\} \in [H]^2$ we have $3 \mid (n^m + m^n + 1)$.

Before we give a third example, we prove the following special case of RAMSEY'S THEOREM.

PROPOSITION 2.2. *For any positive number $r \in \omega$, for any $S \in [\omega]^\omega$, and for any colouring $\pi : [S]^2 \to r$, there is always an $H \in [S]^\omega$ such that $[H]^2$ is monochromatic.*

Proof. The proof is in fact just a consequence of the Infinite Pigeon-Hole Principle; firstly, the Infinite Pigeon-Hole Principle is used to construct homogeneous sets for certain 2-colourings τ and then it is used to show the existence of a homogeneous set for π.

Let $S_0 = S$ and let $a_0 = \min(S_0)$. Define the r-colouring $\tau_0 : S_0 \setminus \{a_0\} \to r$ by stipulating $\tau_0(b) := \pi(\{a_0, b\})$. By the Infinite Pigeon-Hole Principle there is an infinite set $S_1 \subseteq S_0 \setminus \{a_0\}$ such that $\tau_0|_{S_1}$ is constant (*i.e.*, $\tau_0|_{S_1}$ is a constant function) and let $\rho_0 := \tau_0(b)$, where b is any member of S_1. Now, let $a_1 = \min(S_1)$ and define the r-colouring $\tau_1 : S_1 \setminus \{a_1\} \to r$ by stipulating $\tau_1(b) := \pi(\{a_1, b\})$. Again we find an infinite set $S_2 \subseteq S_1 \setminus \{a_1\}$ such that $\tau_1|_{S_2}$ is constant and let $\rho_1 := \tau_1(b)$, where b is any member of S_2. Proceeding this way we finally get infinite sequences $a_0 < a_1 < \ldots < a_n < \ldots$ and $\rho_0, \rho_1, \ldots$. Notice that by construction, for all $n \in \omega$ and all $k > n$ we have $\pi(\{a_n, a_k\}) = \tau_n(a_k) = \rho_n$. Define the r-colouring $\tau : \{a_n : n \in \omega\} \to r$ by stipulating $\tau(a_n) := \rho_n$. Again by the Infinite Pigeon-Hole Principle there is an infinite set $H \subseteq \{a_n : n \in \omega\}$ such that $\tau|_H$ is constant, which implies that H is homogeneous for π, *i.e.*, $[H]^2$ is monochromatic. $\dashv$

As a third example, consider the 17-colouring π_3 of the set of 9-element subsets of $\mathbb{P}$ defined by stipulating

$$\pi_3\big(\{p_1, \ldots, p_9\}\big) = c \quad \Longleftrightarrow \quad p_1 \cdot p_2 \cdot \ldots \cdot p_9 \equiv c \mod 17.$$

For $0 \leq k \leq 16$ let $P_k = \{p \in \mathbb{P} : p \equiv k \bmod 17\}$. Then, by Dirichlet's theorem on primes in arithmetic progression, P_k is infinite whenever $\gcd(k, 17) = 1$, *i.e.*, for all positive numbers $k \leq 16$. Thus, by an easy calculation modulo 17 we find for $1 \leq k \leq 16$, that P_k is homogeneous for π_3.

Now we give a complete proof of RAMSEY'S THEOREM 2.1:

Proof of Ramsey's Theorem. The proof is by induction on n. For $n = 2$ we get PROPOSITION 2.2. So, we assume that the statement is true for $n \geq 2$ and prove it for $n+1$. Let $\pi : [\omega]^{n+1} \to r$ be any r-colouring of $[\omega]^{n+1}$. For each integer $a \in \omega$ let π_a be the r-colouring of $[\omega \setminus \{a\}]^n$ defined as follows:

$$\pi_a(x) = \pi\big(x \cup \{a\}\big).$$

By induction hypothesis, for each $S' \in [\omega]^\omega$ and for each $a \in S'$ there is an $H_a^{S'} \in [S' \setminus \{a\}]^\omega$ such that $H_a^{S'}$ is homogeneous for π_a. Construct now an infinite sequence $a_0 < a_1 < \ldots < a_i < \ldots$ of natural numbers and an infinite sequence $S_0 \supseteq S_1 \supseteq \ldots \supseteq S_i \supseteq \ldots$ of infinite subsets of ω as follows: Let $S_0 = S$ and $a_0 = \min(S)$, and in general let

$$S_{i+1} = H_{a_i}^{S_i}, \quad \text{and} \quad a_{i+1} = \min\{a \in S_{i+1} : a > a_i\}.$$

It is clear that for each $i \in \omega$, the set $[\{a_m : m > i\}]^n$ is monochromatic for π_{a_i}; let $\tau(a_i)$ be its colour (*i.e.*, τ is a colouring of $\{a_i : i \in \omega\}$ with at most r colours). By the Infinite Pigeon-Hole Principle there is an $H \subseteq \{a_i : i \in \omega\}$ such that τ is constant on H, which implies that $\pi|_{[H]^{n+1}}$ is constant, too. Indeed, for any $x_0 < \ldots < x_n$ in H we have $\pi(\{x_0, \ldots, x_n\}) = \pi_{x_0}(\{x_1, \ldots, x_n\}) = \tau(x_0)$, which completes the proof. $\dashv$

Corollaries of Ramsey's Theorem

In finite Combinatorics, the most important consequence of RAMSEY'S THEOREM 2.1 is its finite version:

COROLLARY 2.3 (FINITE RAMSEY THEOREM). *For all $m, n, r \in \omega$, where $r \geq 1$ and $n \leq m$, there exists an $N \in \omega$, where $N \geq m$, such that for every colouring of $[N]^n$ with r colours, there exists a set $H \in [N]^m$, all of whose n-element subsets have the same colour.*

Proof. Assume towards a contradiction that the FINITE RAMSEY THEOREM fails. So, there are $m, n, r \in \omega$, where $r \geq 1$ and $n \leq m$, such that for all $N \in \omega$ with $N \geq m$ there is a colouring $\pi_N : [N]^n \to r$ such that no $H \in [N]^m$ is homogeneous, *i.e.*, $[H]^n$ is not monochromatic. We shall construct an r-colouring π of $[\omega]^n$ such that no infinite subset of ω is homogeneous for π, contradicting RAMSEY'S THEOREM. The r-colouring π will be induced by an infinite branch through a finitely branching tree, where the infinite branch is obtained by König's Lemma. Thus, we first need an infinite, finitely branching tree. For this, consider the following graph G: The vertex set of G consists of $\emptyset$ and all colourings $\pi_N : [N]^n \to r$, where $N \geq m$, such that no $H \in [N]^m$ is homogeneous for π_N. There is an edge between $\emptyset$ and each r-colouring π_m of $[m]^n$, and there is an edge between the colourings π_N and π_{N+1}

iff $\pi_N \equiv \pi_{N+1}|_N$ (*i.e.*, for all $x \in [N]^n$, $\pi_{N+1}(x) = \pi_N(x)$). In particular, there is no edge between two different r-colouring of $[N]^n$. By our assumption, the graph G is infinite. Further, by construction, it is cycle-free, connected, finitely branching, and has a root, namely $\emptyset$. In other words, G is an infinite, finitely branching tree and therefore, by König's Lemma, contains an infinite branch of r-colourings, say $(\emptyset, \pi_m, \pi_{m+1}, \ldots, \pi_{m+i}, \ldots)$, where for all $i, j \in \omega$, the colouring π_{m+i+j} is an extension of the colouring π_{m+i}.

At this point we would like to mention that since for any $N \in \omega$ the set of all r-colouring of $[N]^n$ can be ordered, for example lexicographically, we do not need any non-trivial form of the Axiom of Choice to construct an infinite branch.

Now, the infinite branch $(\emptyset, \pi_m, \pi_{m+1}, \ldots)$ induces an r-colouring π of $[\omega]^n$ such that no m-element subset of ω is homogeneous. In particular, there is no infinite set $H \in [\omega]^\omega$ such that $\pi|_{[H]^n}$ is constant, which is a contradiction to RAMSEY'S THEOREM 2.1 and completes the proof. ⊣

The following corollary is a geometrical consequence of the FINITE RAMSEY THEOREM 2.3:

COROLLARY 2.4. *For every positive integer n there exists an $N \in \omega$ with the following property: If P is a set of N points in the Euclidean plane without three collinear points, then P contains n points which form the vertices of a convex n-gon.*

Proof. By the FINITE RAMSEY THEOREM 2.3, let N be such that for every 2-colouring of $[N]^3$ there is a set $H \in [N]^n$ such that $[H]^3$ is monochromatic. Now let N points in the plane be given, and number them from 1 to N in an arbitrary but fixed way. Colour a triple (i, j, k), where $i < j < k$, red, if travelling from i to j to k is in clockwise direction; otherwise, colour it blue. By the choice of N, there are n ordered points so that every triple has the same colour (*i.e.*, orientation) from which one verifies easily (*e.g.*, by considering the convex hull of the n points) that these points form the vertices of a convex n-gon. ⊣

The following theorem—discovered more than a decade before RAMSEY'S THEOREM—is perhaps the earliest result in Ramsey Theory:

COROLLARY 2.5 (SCHUR'S THEOREM). *If the positive integers are finitely coloured (i.e., coloured with finitely many colours), then there are three distinct positive integers x, y, z of the same colour, with $x + y = z$.*

Proof. Let r be a positive integer and let π be any r-colouring of $\omega \setminus \{0\}$. Let $N \in \omega$ be such that for every r-colouring of $[N]^2$ there is a homogeneous 3-element subset of N. Define the colouring $\pi^* : [N]^2 \to r$ by stipulating $\pi^*(i, j) = \pi(|i - j|)$, where $|i - j|$ is the modulus or absolute value of the difference $i - j$. Since N contains a homogeneous 3-element subset (for π^*), there is a triple $0 \le i < j < k < N$ such that $\pi^*(i, j) = \pi^*(j, k) = \pi^*(i, k)$, which implies that the numbers $x = j - i$, $y = k - j$, and $z = k - i$, have the same colour, and in addition we have $x + y = z$. ⊣

The next result is a purely number-theoretical result and follows quite easily from RAMSEY'S THEOREM. However, somewhat surprisingly, it is unprovable in Number Theory, or more precisely, in *Peano Arithmetic* (which will be discussed in Chapter 3). Before we can state the corollary, we have to introduce the following notion: A non-empty set $S \subseteq \omega$ is called *large* if S has more than $\min(S)$ elements. Further, for $n, m \in \omega$ let $[n, m] := \{i \in \omega : n \leq i \leq m\}$.

COROLLARY 2.6. *For all $n, k, r \in \omega$ with $r \geq 1$, there is an $m \in \omega$ such that for any r-colouring of $[[n, m]]^k$, there exists a large homogeneous set.*

Proof. Let $n, k, r \in \omega$, where $r \geq 1$, be some arbitrary but fixed numbers. Let $\pi : [\omega \setminus n]^k \to r$ be any r-colouring of the k-element subsets of $\{i \in \omega : i \geq n\}$. By RAMSEY'S THEOREM 2.1 there exists an infinite homogeneous set $H \in [\omega \setminus n]^\omega$. Let $a = \min(H)$ and let S denote the least $a + 1$ elements of H. Then S is large and $[S]^k$ is monochromatic.

The existence of a finite number m with the required properties now follows—using König's Lemma—in the very same way as the FINITE RAMSEY THEOREM followed from RAMSEY'S THEOREM (see the proof of the FINITE RAMSEY THEOREM 2.3). ⊣

Generalisations of Ramsey's Theorem

Even though Ramsey's theorems are very powerful combinatorial results, they can still be generalised. The following result will be used later in Chapter 7 in order to prove that the Prime Ideal Theorem—introduced in Chapter 5—holds in the ordered Mostowski permutation model (but it will not be used anywhere else in this book).

In order to illustrate the next theorem, as well as to show that it is optimal to some extent, we consider the following two examples: Firstly, define the 2-colouring π_1 of $[\omega]^2 \times [\omega]^3 \times [\omega]^1$ by stipulating

$$\pi_1\big(\{x_1, x_2\}, \{y_1, y_2, y_3\}, \{z_1\}\big) = \begin{cases} 1 & \text{if } 2^{x_1 \cdot x_2} + 13^{y_1 \cdot y_2 \cdot y_3} + 17^{z_1} - 3 \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $H_1 = \{3 \cdot k : k \in \omega\}$, $H_2 = \{2 \cdot k : k \in \omega\}$, and $H_3 = \{6 \cdot k : k \in \omega\}$. Then an easy calculation modulo 7 shows that $[H_1]^2 \times [H_2]^3 \times [H_3]^1$ is an infinite monochromatic set.

Secondly, define the 2-colouring π_2 of $[\omega]^1 \times [\omega]^1$ by stipulating

$$\pi_2\big(\{x\}, \{y\}\big) = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that whenever H_1 and H_2 are *infinite* subsets of ω, then $[H_1]^1 \times [H_2]^1$ is not monochromatic; on the other hand, we easily find arbitrarily large *finite* sets $M_1, M_2 \subseteq \omega$ such that $[M_1]^1 \times [M_2]^1$ is monochromatic.

Thus, if $[\omega]^{n_1} \times \ldots \times [\omega]^{n_l}$ is coloured with r colours, then, in general, we cannot expect to find infinite subsets of ω, say $H_1, \ldots, H_l$, such that $[H_1]^{n_1} \times \ldots \times [H_l]^{n_l}$ is monochromatic; but we always find arbitrarily large finite subsets of ω:

THEOREM 2.7. *Let $r, l, n_1, \ldots, n_l \in \omega$ with $r \geq 1$ be given. For every $m \in \omega$ with $m \geq \max\{n_1, \ldots, n_l\}$ there is some $N \in \omega$ such that whenever $[N]^{n_1} \times \ldots \times [N]^{n_l}$ is coloured with r colours, then there are $M_1, \ldots, M_l \in [N]^m$ such that $[M_1]^{n_1} \times \ldots \times [M_l]^{n_l}$ is monochromatic.*

Proof. The proof is by induction on l and the induction step uses a so-called *product-argument*. For $l = 1$ the statement is equivalent to the FINITE RAMSEY THEOREM 2.3. So, assume that the statement is true for $l \geq 1$ and let us prove it for $l + 1$. By induction hypothesis, for every $r \geq 1$ there is an N_l (depending on r) such that for *every* r-colouring of $[N_l]^{n_1} \times \ldots \times [N_l]^{n_l}$ there are $M_1, \ldots, M_l \in [N_l]^m$ such that $[M_1]^{n_1} \times \ldots \times [M_l]^{n_l}$ is monochromatic. Now, the crucial idea in order to apply the FINITE RAMSEY THEOREM is to consider the coloured l-tuples in $([N_l]^m)^l$ as new colours. More precisely, let u_l be the number of different l-tuples in $([N_l]^m)^l$ and let $r_l := u_l \cdot r$. Notice that each colour in r_l corresponds to a pair $\langle t, c\rangle$, where t is an l-tuple in $([N_l]^m)^l$ and c is one of r colours. Notice also that r_l is very large compared to r. Now, by the FINITE RAMSEY THEOREM 2.3, there is a number $N_{l+1} \in \omega$ such that whenever $[N_{l+1}]^{n_{l+1}}$ is coloured with r_l colours, then there exists an $M_{l+1} \in [N_{l+1}]^m$ such that $[M_{l+1}]^{n_{l+1}}$ is monochromatic. Let $N = \max\{N_l, N_{l+1}\}$ and let π be any r-colouring of $[N_l]^{n_1} \times \ldots \times [N_l]^{n_l} \times [N]^{n_{l+1}}$. For every $F \in [N]^{n_{l+1}}$ let π^F be the r-colouring of $[N_l]^{n_1} \times \ldots \times [N_l]^{n_l}$ defined by stipulating

$$\pi^F(X) = \pi\big(\langle X, F\rangle\big).$$

By the definition of N, for every $F \in [N]^{n_{l+1}}$ there is a lexicographically first l-tuple $(M_1^F, \ldots, M_l^F) \in ([N_l]^m)^l$ such that $[M_1^F]^{n_1} \times \ldots \times [M_l^F]^{n_l}$ is monochromatic for π^F. By definition of r_l we can define an r_l-colouring π_{l+1} on $[N]^{n_{l+1}}$ as follows: Every set $F \in [N]^{n_{l+1}}$ is coloured according to the l-tuple $t = (M_1^F, \ldots, M_l^F)$ (which can be encoded as one of u_l numbers) and the colour $c = \pi^F(X)$, where X is any element of the set $[M_1^F]^{n_1} \times \ldots \times [M_l^F]^{n_l}$; because $[M_1^F]^{n_1} \times \ldots \times [M_l^F]^{n_l}$ is monochromatic for π^F, c is well-defined and one of r colours. In other words, for every $F \in [N]^{n_{l+1}}$, $\pi_{l+1}(F)$ correspond to a pair $\langle t, c\rangle$, where $t \in ([N_l]^m)^l$ and c is one of r colours. Finally, by definition of N, there is a set $M_{l+1} \in [N]^m$ such that $[M_{l+1}]^{n_{l+1}}$ is monochromatic for π_{l+1}, which implies that for all $F, F_1, F_2 \in [M_{l+1}]^{n_{l+1}}$ we get that

- $[M_1^F]^{n_1} \times \ldots \times [M_l^F]^{n_l}$ is monochromatic for π^F,
- $(M_1^{F_1}, \ldots, M_l^{F_1}) = (M_1^{F_2}, \ldots, M_l^{F_2})$,
- and restricted to the set $[M_1^F]^{n_1} \times \ldots \times [M_l^F]^{n_l}$, the colourings $\pi_l^{F_1}$ and $\pi_l^{F_2}$ are identical.

Hence, there are $M_1, \ldots, M_{l+1} \in [N]^m$ such that $\pi|_{[M_1]^{n_1} \times \ldots \times [M_{l+1}]^{n_{l+1}}}$ is constant, which completes the proof. $\dashv$

A very strong generalisation of RAMSEY'S THEOREM in terms of partitions is the PARTITION RAMSEY THEOREM 11.4. However, since the proof of this generalisation is quite involved, we postpone the discussion of that result until Chapter 11 and consider now some other possible generalisations of RAMSEY'S THEOREM: Firstly one could finitely colour all finite subsets of ω, secondly one could colour $[\omega]^n$ with infinitely many colours, and finally, one could finitely colour all the infinite subsets of ω. However, below we shall see that none of these generalisations works, but first, let us consider Ramsey's original theorem, which is—at least in the absence of the Axiom of Choice—also a generalisation of RAMSEY'S THEOREM.

Ramsey's Original Theorem. The theorem which Ramsey proved originally is somewhat stronger than what we proved above. In our terminology, it states as follows:

RAMSEY'S ORIGINAL THEOREM. *For any infinite set A, for any number $n \in \omega$, for any positive number $r \in \omega$, and for any colouring $\pi : [A]^n \to r$, there is an infinite set $H \subseteq A$ such that $[H]^n$ is monochromatic.*

Notice that the difference is just that the infinite set A is not necessarily a subset of ω, and therefore, it does not necessarily contain a countable infinite subset. However, this difference is crucial, since one can show that, like König's Lemma, this statement is not provable without assuming some form of the Axiom of Choice (AC). On the other hand, if one has AC, then every infinite set has a countably infinite subset, and so RAMSEY'S THEOREM implies the original version. Ramsey was aware of this fact and stated explicitly that he is assuming the *axiom of selections* (*i.e.*, AC). Even though we do not need full AC in order to prove RAMSEY'S ORIGINAL THEOREM, there is no way to avoid some non-trivial kind of choice, since there are models of Set Theory in which RAMSEY'S ORIGINAL THEOREM fails (*cf.* PROPOSITION 7.8). Consequently, RAMSEY'S ORIGINAL THEOREM can be used as a choice principle, which will be discussed in Chapter 5.

***Finite Colourings of* $[\omega]^{<\omega}$.** Assume we have coloured all the finite subsets of ω with two colours, say red and blue. Can we be sure that there is an infinite subset of ω such that all its finite subsets have the same colour? The answer to this question is negative and it is not hard to find a counterexample (*e.g.*, colour a set $x \in [\omega]^{<\omega}$ blue, if $|x|$ is even; otherwise, colour it red).

Thus, let us ask for slightly less. Is there at least an infinite subset of ω such that for each $n \in \omega$, all its n-element subsets have the same colour? The answer to this question is also negative: Colour a non-empty set $x \in [\omega]^{<\omega}$ red, if x has more than $\min(x)$ elements (*i.e.*, x is large); otherwise, colour it blue. Now, let I be an infinite subset of ω and let $n = \min(I)$. We leave it as an exercise to the reader to verify that $[I]^{n+1}$ is dichromatic.

The picture changes if we are asking just for an almost homogeneous sets: An infinite set $H \subseteq \omega$ is called **almost homogeneous** for a colouring $\pi : [\omega]^n \to r$ (where $n \in \omega$ and r is a positive integer), if there is a finite set $K \subseteq \omega$ such that $H \setminus K$ is homogeneous for π. Now, for a positive integer r consider any colouring

$\bar{\pi} : [\omega]^{<\omega} \to r$. Then, for each $n \in \omega$, $\bar{\pi}|_{[\omega]^n}$ is a colouring $\pi_n : [\omega]^n \to r$. Is there an infinite set $H \subseteq \omega$ which is almost homogeneous for all π_n's simultaneously? The answer to this question is affirmative and is given by the following result.

PROPOSITION 2.8. *Let $\{r_k : k \in \omega\}$ and $\{n_k : k \in \omega\}$ be two (possibly finite) sets of positive integers, and for each $k \in \omega$ let $\pi_k : [\omega]^{n_k} \to r_k$ be a colouring. Then there exists an infinite set $H \subseteq \omega$ which is almost homogeneous for each π_k ($k \in \omega$).*

Proof. A first attempt to construct the required almost homogeneous set would be to start with an $I_0 \in [\omega]^\omega$ which is homogeneous for π_0, then take an $I_1 \in [I_0]^\omega$ which is homogeneous for π_1, *et cetera*, and finally take the intersection of all the I_k's. Even though this attempt fails—since it is very likely that we end up with the empty set—it is the right direction. In fact, if the intersection of the I_k's would be non-empty, it would be homogeneous for all π_k's, which is more than what is required. In order to end up with an infinite set we just have to modify the above approach—the trick, which is used almost always when the word "almost" is involved, is called *diagonalisation*.

The proof is by induction on k: By RAMSEY'S THEOREM 2.1 there exists an $H_0 \in [\omega]^\omega$ which is homogeneous for π_0. Assume we have already constructed $H_k \in [\omega]^\omega$ (for some $k \geq 0$) such that H_k is homogeneous for π_k. Let $a_k = \min(H_k)$ and let $S_k = H_k \setminus \{a_k\}$. Then, again by RAMSEY'S THEOREM 2.1, there exists an $H_{k+1} \in [S_k]^\omega$ such that H_{k+1} is homogeneous for π_{k+1}. Let $H = \{a_k : k \in \omega\}$. Then, by construction, for every $k \in \omega$ we see that $H \setminus \{a_0, \dots, a_{k-1}\}$ is homogeneous for π_k, which implies that H is almost homogeneous for all π_k's simultaneously. ⊣

Now we could ask what is the least number of 2-colourings of 2-element subsets of ω we need in order to make sure that no single infinite subset of ω is almost homogeneous for all colourings simultaneously? By PROPOSITION 2.8 we know that countably many colourings are not sufficient, but as we will see later, the axioms of Set Theory do not decide how large this number is (*cf.* Chapter 18).

The dual question would be as follows: How large must a family of infinite subsets of ω be, in order to make sure that for each 2-colouring of the 2-element subsets of ω we find a set in the family which is homogeneous for this colouring? Again, the axioms of Set Theory do not decide how large this number is (*cf.* Chapter 18).

Going to the Infinite. There are two parameters involved in a colouring $\pi : [\omega]^n \to r$, namely n and r. Let first consider the case when $n = 2$ and $r = \omega$. In this case, we obviously cannot hope for any infinite homogeneous or almost homogeneous set. However, there are still infinite subsets of ω which are homogeneous in a broader sense which leads to the CANONICAL RAMSEY THEOREM. Even though the CANONICAL RAMSEY THEOREM is a proper generalisation of RAMSEY'S THEOREM, we will not discuss it here (but see RELATED RESULT 0).

In the case when $n = \omega$ and $r = 2$ we cannot hope for an infinite homogeneous set, as the following example illustrates (compare this result with Chapter 5 | RELATED RESULT 38):

In the presence of the Axiom of Choice *there is a* 2*-colouring of* $[\omega]^\omega$ *such that there is no infinite set, all whose infinite subsets have the same colour.*

The idea is to construct (or more precisely, to prove the existence of) a colouring of $[\omega]^\omega$ with say red and blue in such a way that whenever an infinite set $x \in [\omega]^\omega$ is coloured blue, then for each $a \in x$, $x \setminus \{a\}$ is coloured red, and vice versa.

For this, define an equivalence relation on $[\omega]^\omega$ as follows: for $x, y \in [\omega]^\omega$ let

$$x \sim y \quad \iff \quad x \triangle y \text{ is finite}$$

where $x \triangle y = (x \setminus y) \cup (y \setminus x)$ is the **symmetric difference** of x and y. It is easily checked that the relation "$\sim$" is indeed an equivalence relation on $[\omega]^\omega$. Further, let $\mathscr{A} \subseteq [\omega]^\omega$ be any set of representatives, *i.e.*, $\mathscr{A}$ has exactly one element in common with each equivalence class. Since the existence of the set $\mathscr{A}$ relies on the Axiom of Choice, the given proof is not entirely constructive.

Colour now an infinite set $x \in [\omega]^\omega$ blue, if $|x \triangle r_x|$ is even, where $r_x \in (\mathscr{A} \cap [x]^{\tilde{}})$; otherwise, colour it red. Since two sets $x, y \in [\omega]^\omega$ with finite symmetric difference are always equivalent, every infinite subset of ω must contain blue as well as red coloured infinite subsets.

So, there is a colouring $\pi : [\omega]^\omega \to \{0, 1\}$ such that for no $x \in [\omega]^\omega$, $\pi|_{[x]^\omega}$ is constant. On the other hand, if the colouring is not too sophisticated we may find a homogeneous set: For $\mathscr{A} \subseteq [\omega]^\omega$ define $\pi_{\mathscr{A}} : [\omega]^\omega \to \{0, 1\}$ by stipulating $\pi_{\mathscr{A}}(x) = 1$ *iff* $x \in \mathscr{A}$. Now we say that the set $\mathscr{A} \subseteq [\omega]^\omega$ has the **Ramsey property** if there exists an $x_h \in [\omega]^\omega$ such that $\pi_{\mathscr{A}}|_{[x]^\omega}$ is constant. In other words, $\mathscr{A} \subseteq [\omega]^\omega$ has the Ramsey property if and only if there exists an $x_h \in [\omega]^\omega$ such that either $[x_h]^\omega \subseteq \mathscr{A}$ or $x_h]^\omega \cap \mathscr{A} = \emptyset$. The Ramsey property is related to the cardinal $\mathfrak{h}$ (*cf.* Chapter 8) and will be discussed in Chapter 9.

A slightly weaker property than the Ramsey property is the so-called *doughnut property*: If a and b are subsets of ω such that $b \setminus a$ is infinite, then we call the set $[a, b]^\omega := \{x \in [\omega]^\omega : a \subseteq x \subseteq b\}$ a **doughnut**. (Why such sets are called "doughnuts" is left to the reader's imagination.) Now, a set $\mathscr{A} \subseteq [\omega]^\omega$ is said to have the **doughnut property** if there exists an doughnut $[a, b]^\omega$ (for some a and b) such that either $[a, b]^\omega \subseteq \mathscr{A}$ or $[a, b]^\omega \cap \mathscr{A} = \emptyset$. Obviously, every set with the Ramsey property has also the doughnut property (consider doughnuts of the form $[\emptyset, b]^\omega$). On the other hand, it is not difficult to show that, in the presence of the Axiom of Choice, there are sets with the doughnut property which fail to have the Ramsey property (just modify the example given above).

Notes

Ramsey's Theorem. Frank Plumpton Ramsey (1903–1930), the elder brother of Arthur Michael Ramsey (who was Archbishop of Canterbury from 1961 to 1974), proved his famous theorem in [34] and the part of the volume in which his article appeared was issued on the 16th of December in 1929, but the volume itself belongs to the years 1929 and 1930 (which caused some confusion about the year Ramsey's article was actually published). However, Ramsey submitted his paper already in November 1928. For Ramsey's paper and its relation to First-Order Logic, as well as for an introduction to Ramsey Theory in general, we refer the reader to the classical textbook by Graham, Rothschild, and Spencer [16] (for Ramsey's other papers on Logic see [35]). In [34], RAMSEY'S THEOREM 2.1 appears as THEOREM A and the FINITE RAMSEY THEOREM 2.3 is proved as a corollary and appears as THEOREM B. Although RAMSEY'S THEOREM is accurately attributed to Ramsey, its popularisation stems from the classical paper of Erdős and Szekerés [9], where they proved (independently of Ramsey) COROLLARY 2.4—which can be seen as a variant of the FINITE RAMSEY THEOREM 2.3 in a geometrical context (see also Morris and Soltan [27]). The elegant proof we gave for COROLLARY 2.4 is due to Tarsy (*cf.* Lewin [25] or Graham, Rothschild, and Spencer [16, p. 26]).

Schur's Theorem. Schur's original paper [36] was motivated by FERMAT'S LAST THEOREM, and he actually proved the following result: *For all natural numbers m, if p is prime and sufficiently large, then the equation $x^m + y^m = z^m$ has a non-zero solution in the integers modulo p.* A proof of this theorem can also be found in Graham, Rothschild, and Spencer [16, Section 3.1]. For some historical background and for the early development of Ramsey Theory (before Ramsey) see Soifer [38].

The Paris–Harrington Result. As mentioned above, COROLLARY 2.6 is true but unprovable in *Peano Arithmetic* (also called *First-Order Arithmetic*). This result was the first natural example of such a statement and is due to Paris and Harrington [31] (see also Graham, Rothschild, and Spencer [16, Section 6.3]). For other statements of that type see Paris [30].

It is worth mentioning that Peano Arithmetic is, in a suitable sense, equivalent to Zermelo–Fraenkel Set Theory with the Axiom of Infinity replaced by its negation, which is a reasonable formalisation of standard combinatorial reasoning about finite sets.

Rado's Generalisation of the Finite Ramsey Theorem. THEOREM 2.7, which is the only proper generalisation of the FINITE RAMSEY THEOREM shown in this book so far, is due to Rado [32] (see also page 113, Problems 4 & 5 of Jech [23]).

Ramsey Sets and Doughnuts. Even though the Ramsey property and the doughnut property look very similar, there are sets which have the Ramsey property, but which fail to have the doughnut property. For the relation between the doughnut property and other regularity properties see for example Halbeisen [18] or Brendle, Halbeisen, and Löwe [4] (see also Chapter 9 | RELATED RESULT 60).

RELATED RESULTS

0. *Canonical Ramsey Theorem.* The following result, known as the CANONICAL RAMSEY THEOREM, is due to Erdős and Rado (*cf.* [8, Theorem I]): *Whenever*

we have a colouring π *of* $[\omega]^n$*, for some* $n \in \omega$*, with an arbitrary (e.g., infinite) set of colours, there exist an infinite set* $H \subseteq \omega$ *and a set* $I \subseteq \{1, 2, \ldots, n\}$ *such that for any ordered* n*-element subsets* $\{k_1 < \ldots < k_n\}, \{l_1 < \ldots < l_n\} \in [H]^n$ *we have* $\pi(\{k_1, \ldots, k_n\}) = \pi(\{l_1, \ldots, l_n\}) \iff k_i = l_i$ *for all* $i \in I$. The 2^n possible choices for I correspond to the so-called *canonical colourings* of $[\omega]^n$. As an example let us consider the case when $n = 2$: Let π be an arbitrary colouring of $[\omega]^2$ and let $H \in [\omega]^\omega$ and $I \subseteq \{1, 2\}$ be as above. Then we are in exactly one of the following four cases for *all* $\{k_1 < k_2\}, \{l_1 < l_2\} \in [H]^2$ (*cf.* [8, Theorem II]):
(1) If $I = \emptyset$, then $\pi(\{k_1, k_2\}) = \pi(\{l_1, l_2\})$.
(2) If $I = \{1, 2\}$, then $\pi(\{k_1, k_2\}) = \pi(\{l_1, l_2\})$ *iff* $\{k_1, k_2\} = \{l_1, l_2\}$.
(3) If $I = \{1\}$, then $\pi(\{k_1, k_2\}) = \pi(\{l_1, l_2\})$ *iff* $k_1 = l_1$.
(4) If $I = \{2\}$, then $\pi(\{k_1, k_2\}) = \pi(\{l_1, l_2\})$ *iff* $k_2 = l_2$.
Obviously, if π is a finite colouring of $[\omega]^n$, then we are always in case (1), which gives us just RAMSEY'S THEOREM 2.1.

1. *Ramsey numbers*. The least number of people that must be invited to a party, in order to make sure that n of them mutually shook hands before or m of them mutually did not shake hands before, is denoted by $R(n, m)$, and the numbers $R(n, m)$ are called **Ramsey numbers**. Notice that by the FINITE RAMSEY THEOREM, Ramsey numbers $R(n, m)$ exist for all integers $n, m \in \omega$. Very few Ramsey numbers are actually known. It is easy to show that $R(2, 3) = 3$ (in general, $R(2, n) = n$), and we leave it as an exercise to show that $R(3, 3) = 6$. A comprehensive list of what is known about small Ramsey numbers is maintained by Radziszowski [33].

2. *Monochromatic triangles in* K_6*-free graphs*. Erdős and Hajnal [10] asked for a graph which contains no K_6 (*i.e.*, no complete graph on 6 vertices) but has the property that whenever its edges are 2-coloured there must be a monochromatic triangle. A minimal example for such a graph was provided by Graham [14]: On the one hand he showed that if a 5-cycle is deleted from a K_8, then the resulting graph contains no K_6 and has the property that whenever its edges are 2-coloured there is a monochromatic triangle. On the other hand, if a graph on 7 vertices contains no K_6, then there is a 2-colouring of the edges with no monochromatic triangle.

3. *Hindman's Theorem*. If $F \in [\omega]^{<\omega}$, then we write $\sum F$ for $\sum_{a \in F} a$, where as usual we define $\sum \emptyset := 0$. HINDMAN'S THEOREM states that if ω is finitely coloured, then there is an $x \in [\omega]^\omega$ such that $\{\sum F : F \in [x]^{<\omega} \wedge F \neq \emptyset\}$ is monochromatic (*cf.* Hindman [21, Theorem 3.1] or Hindman and Strauss [22, Corollary 5.10] where references to alternative proofs are given on page 102). Using HINDMAN'S THEOREM as a strong Pigeon-Hole Principle, Milliken proved in [26] a strengthened version of RAMSEY'S THEOREM 2.1 which includes HINDMAN'S THEOREM as well as RAMSEY'S THEOREM 2.1. Since Milliken's result was proved independently by Taylor (*cf.* [39]), it is usually called MILLIKEN–TAYLOR THEOREM. In order to state this result we have to

introduce some notation. Two finite sets $K_1, K_2 \subseteq \omega$ are said to be *unmeshed* if $\max(K_1) < \min(K_2)$ or $\max(K_2) < \min(K_1)$. If I and H are two sets of pairwise unmeshed finite subsets of ω and every member of I is the union of (finitely many) members of H, then we write $I \sqsubseteq H$. Further, let $\langle\omega\rangle^\omega$ denote the set of all infinite sets of pairwise unmeshed finite subsets of ω, and for $H \in \langle\omega\rangle^\omega$ let $\langle H\rangle^n := \{I : |I| = n \textit{ and } I \sqsubseteq H\}$. Now, the MILLIKEN–TAYLOR THEOREM states as follows: *If all the n-element sets of pairwise unmeshed finite subsets of ω are finitely coloured, then there exists an $H \in \langle\omega\rangle^\omega$ such that $\langle H\rangle^n$ is monochromatic.*

4. *Colourings of the plane*. Erdős [7] proved that there is a colouring of the Euclidean plane with countably many colours, such that any two points at a rational distance have different colours. This result was strengthened by Komjáth [24] in the following way: *Let $\mathbb{Q}$ be the set of rational numbers and let $Q := \{(q, 0) : q \in \mathbb{Q}\}$ be a copy of the rationals in the Euclidean plane. Then there exists a colouring of the Euclidean plane with countably many colours, such that for any rigid motion σ of the plane, every colour occurs in $\sigma[Q] = \{\sigma(p) : p \in Q\}$ exactly once.*

5. *Finite colourings of* $\mathbb{Q}$. If we colour the rational numbers $\mathbb{Q}$ with finitely many colours, is there always an infinite homogeneous set which is order-isomorphic to $\mathbb{Q}$? In general, this is not the case: Let $\{q_n : n \in \omega\}$ be an enumeration of $\mathbb{Q}$ (see Chapter 4, in particular RELATED RESULT 14) and colour a pair $\{q_i, q_j\}$ blue if $q_i < q_j \leftrightarrow i < j$, otherwise, colour it red. Then it is easy to see that an infinite homogeneous set which is order-isomorphic to $\mathbb{Q}$ would yield an infinite decreasing sequence of natural numbers, which is obviously not possible. On the other hand, for every positive integer $n \in \omega$ there is a smallest number $t_n \in \omega$ such that if $[\mathbb{Q}]^n$ is finitely coloured then there is an infinite set $X \subseteq \mathbb{Q}$ which is order-isomorphic to $\mathbb{Q}$ such that $[X]^n$ is coloured with at most t_n colours. For this see Devlin [6] or Vuksanović [41], where it is shown that such numbers exist and that the sequence of numbers t_n coincides with the so-called *tangent numbers* (*cf.* Sloane [37, A000182]). In particular, $t_1 = 1$ and for $n \geq 2$ we have $t_n = \sum_{i=1}^{n-1} \binom{2n-2}{2i-1} t_i t_{n-i}$.

6. *Symmetry and colourings*. Banakh and Protasov investigated in [2] the following problem: Is it true that for every n-colouring of the group $\mathbb{Z}^n$ there exists an infinite monochromatic subset of $\mathbb{Z}^n$ which is symmetric with respect to a central reflection. It turns out that the answer is always positive (for all n). However, there exists a 4-colouring of $\mathbb{Z}^3$ without infinite, symmetric, monochromatic set. For more general results we refer the reader to Banakh, Verbitski, and Vorobets [3].

7. *Wieferich primes**. The so-called Wieferich primes were first introduced by Wieferich in [42] in relation to FERMAT'S LAST THEOREM. As mentioned above, the only known Wieferich primes (less than $1.25 \cdot 10^{15}$) are 1093 and 3511 (found in 1913 and 1922, respectively). It is not known if there are infinitely many primes of this type, even though it is conjectured that this is the

case (see for example Halbeisen and Hungerbühler [19]). Moreover, it is not even known whether there are infinitely many non-Wieferich primes—although it is very likely to be the case.

8. *Sums and products*. As a consequence of RAMSEY'S THEOREM we see that if ω is finitely coloured, then there are infinite sequences of positive integers $(x_0, x_1, \ldots, x_k, \ldots)$ and $(y_0, y_1, \ldots, y_k, \ldots)$ such that $\{x_i + x_j : i, j \in \omega \wedge i < j\}$ as well as $\{y_i \cdot y_j : i, j \in \omega \wedge i < j\}$ is monochromatic (but not necessarily of the same colour). On the other hand, it is known (*cf.* Hindman and Strauss [22, Chapter 17.2]) that *one can colour the positive integers with finitely many colours in such a way that there is* no *infinite sequence* $(x_0, x_1, \ldots, x_k, \ldots)$ *such that* $\{x_i + x_j : i, j \in \omega \wedge i < j\} \cup \{x_i \cdot x_j : i, j \in \omega \wedge i < j\}$ *is monochromatic*.

9. *The graph of pairwise sums and products**. One can show that if ω is 2-coloured, then there are infinitely many pairs of distinct positive integers x and y such that $x + y$ has the same colour as $x \cdot y$. For this consider the graph on ω with n joined to m if for some distinct $x, y \in \omega$ we have $x + y = n$ and $x \cdot y = m$. Now, notice that it is enough to show that this so-called *graph of pairwise sums and products* contains infinitely many triangles (*cf.* Halbeisen [17]).

   Suppose now that ω is finitely coloured. Are there two distinct positive integers x and y such that $x + y$ has the same colour as $x \cdot y$? This problem—which is equivalent to asking whether the chromatic number of the graph of pairwise sums and products is finite or infinite—is still open (*cf.* Hindman and Strauss [22, Question 17.18]). A partial result is given in Halbeisen [17], where it is shown that such numbers x and y exist if ω is 3-coloured.

10. *Problems in Ramsey Theory**. For a variety of open problems from Ramsey Theory we refer the reader to Graham [15] (it might be worth mentioning that Graham is offering modest rewards for most of the presented problems).

11. *Applications of Ramsey Theory to Banach Space Theory*. There are many—and sometimes quite unexpected—applications of Ramsey Theory to Banach Space Theory (see for example Odell [28], Gowers [13], or Argyros and Todorčević [1]). Let us mention just the following two applications:

    An unexpected application of Ramsey Theory to Banach Space Theory is due to Brunel and Sucheston [5]: *If* $x_1, x_2, \ldots$ *is an infinite normalised basic sequence in a Banach space* X *and* $\varepsilon_n \searrow 0$ (*a sequence of positive real numbers which tends to* 0), *then one can find an infinite subsequence* $y_1, y_2, \ldots$ *of* $x_1, x_2, \ldots$ *which has the following property*: *For any positive* $n \in \omega$, *any sequence of scalars* $(a_1, \ldots, a_n) \in [-1, 1]^n$ *and any natural numbers* $n \leq i_0 < \ldots < i_{n-1}$ *and* $n \leq j_0 < \ldots < j_{n-1}$ *we have*

$$\left| \left\| \sum_{k=1}^{n} a_k y_{i_k} \right\| - \left\| \sum_{k=1}^{n} a_k y_{j_k} \right\| \right| < \varepsilon_n .$$

The limit $\| \sum_{k=1}^{n} a_k \tilde{e}_k \|$ we obtain for each finite sequence $(a_1, \ldots, a_n) \in [-1, 1]^n$ leads to the sequence $\tilde{e}_1, \tilde{e}_2, \ldots$, and the Banach space generated by $\tilde{e}_1, \tilde{e}_2, \ldots$ is called a *spreading model* of X. The notion of spreading models was generalised (*e.g.*, using the MILLIKEN–TAYLOR THEOREM) and investigated by Halbeisen and Odell in [20].

Another example is due to Gowers [11, 12] (see also Todorčević [40, Section 2.3]), who discovered the long sought *Block Ramsey Theorem*—a genuinely new Ramsey-type result—for Banach spaces, which he used to prove his famous DICHOTOMY THEOREM (see also Gowers [13, Section 5] or Odell [29]): *Every Banach space X contains a subspace Y which either has an unconditional basis or is hereditarily indecomposable* (*i.e., Y contains no subspaces having a non-trivial complemented subspace*).

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