Preface

By the campanologist, the playing of tunes is considered to be a childish game; the proper use of bells is to work out mathematical permutations and combinations.
His passion finds its satisfaction in mathematical completeness and mechanical perfection.

DOROTHY L. SAYERS
The Nine Tailors, 1934

This book provides a self-contained introduction to *Axiomatic Set Theory* with main focus on *Infinitary Combinatorics* and the *Forcing Technique*. The book is intended to be used as a textbook in undergraduate and graduate courses of various levels, as well as for self-study. To make the book valuable for experienced researchers also, some historical background and the sources of the main results have been provided in the NOTES, and some topics for further studies are given in the section RELATED RESULTS—where those containing open problems are marked with an asterisk.

The axioms of Set Theory ZFC, consisting of the axioms of *Zermelo–Fraenkel Set Theory* (denoted ZF) and the *Axiom of Choice*, are the foundation of Mathematics in the sense that essentially all Mathematics can be formalised within ZFC. On the other hand, Set Theory can also be considered as a mathematical theory, like Group Theory, rather than the basis for building general mathematical theories. This approach allows us to drop or modify axioms of ZFC in order to get, for example, a Set Theory without the *Axiom of Choice* (see Chapter 4) or in which just a weak form of the *Axiom of Choice* holds (see Chapter 7). In addition, we are also allowed to extend the axiomatic system ZFC in order to get, for example, a Set Theory in which, in addition to the ZFC axioms, we also have *Martin's Axiom* (see Chapter 13), which is a very powerful axiom with many applications for *Infinitary Combinatorics* as well as other fields of Mathematics. However, this approach prevents us from using any kind of Set Theory which goes beyond ZFC, which is used, for example, to prove the existence of a countable model of ZFC (see the *Löwenheim–Skolem Theorem* in Chapter 15).

Most of the results presented in this book are combinatorial results, in particular the results in *Ramsey Theory* (introduced in Chapter 2 and further developed in

Chapter 11), or those results whose proofs have a combinatorial flavour. For example, we get results of the latter type if we work in Set Theory without the *Axiom of Choice*, since in the absence of the *Axiom of Choice*, the proofs must be constructive and therefore typically have a much more combinatorial flavour than proofs in ZFC (examples can be found in Chapters 4 & 7). On the other hand, there are also elegant combinatorial proofs using the *Axiom of Choice*. An example is the proof in Chapter 6, where it is shown that one can divide the solid unit ball into five parts, such that one can build two solid unit balls out of these five parts—another such paradoxical result is given in Chapter 17, where it is shown that it might be possible in ZF to decompose a square into more parts than there are points on the square.

Even though the ZFC axiomatic system is the foundation of Mathematics, by *Gödel's Incompleteness Theorem*—briefly discussed at the end of Chapter 3—no axiomatic system of Mathematics is complete in the sense that every statement can either be proved or disproved; in other words, there are always statements which are independent of the axiomatic system. The main tool to show that a certain statement is independent of the axioms of Set Theory is Cohen's *Forcing Technique*, which he originally developed in the early 1960s in order to show that there are models of ZF in which the *Axiom of Choice* fails (see Chapter 17) and that the *Continuum Hypothesis* is independent of ZFC (see Chapter 14). The *Forcing Technique* is introduced and discussed in great detail in Part II, and in Part III it is used to investigate combinatorial properties of the set of real numbers. This is done by comparing the *Cardinal Characteristics of the Continuum* introduced in Chapter 8.

The following table indicates which of the main topics appear in which chapter, where $***$ means that it is the main topic of that chapter, $**$ means that some new results in that topic are proved or at least that the topic is important for understanding certain proofs, and $*$ means that the topic appears somewhere in that chapter, but not in an essential way:

Chapter	1	2	3	4	5	6	7	8	9	10	11
Forcing Technique	*										
Axiom of Choice & ZF	*	*	***	***	***	**	***				
Ramsey Theory	*	***		*	**		*	**	***	***	***
Cardinal Characteristics	*	*						***	***	*	

Part I

Chapter	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Forcing Technique	***	**	***	*	***	**	**	**	***	**	**	**	**	**	**	**
Axiom of Choice & ZF						***										
Ramsey Theory		**	*				*					*	*	**	***	
Cardinal Characteristics		**					**	**		**	**	**	**	**	**	***

Part II Part III

For example *Ramsey's Theorem*, which is the nucleus of *Ramsey Theory*, is the main topic in Chapter 2, it is used in some proofs in Chapters 4 & 7, it is used as a choice principle in Chapter 5, it is related to two *Cardinal Characteristics* defined in Chapter 8, it is used to define what is called a Ramsey ultrafilter in Chapter 10, it is used in the proof of the *Hales–Jewett Theorem* in Chapter 11, and it is used to formulate a combinatorial feature of Mathias reals in Chapter 24. Furthermore, one can see that *Cardinal Characteristics* are our main tool in Part III in the investigation of combinatorial properties of various forcing notions, even in the cases when—in Chapters 25 & 26—the existence of Ramsey ultrafilters are investigated. Finally, in Chapter 27 we show how *Cardinal Characteristics* can be used to shed new light on a classical problem in Measure Theory. On the other hand, the *Cardinal Characteristics* are used to describe some combinatorial features of different forcing notions. In particular, it will be shown that the cardinal characteristic $\mathfrak{h}$ (introduced in Chapter 8 and investigated in Chapter 9) is closely related to Mathias forcing (introduced in Chapter 24), which is used in Chapter 25 to show that the existence of Ramsey ultrafilters is independent of ZFC.

I tried to write this book like a piece of music, not just writing note by note, but using various themes or voices—like *Ramsey's Theorem* and the cardinal characteristic $\mathfrak{h}$—again and again in different combinations. In this undertaking, I was inspired by the English art of bell ringing and tried to base the order of the themes on Zarlino's introduction to the art of counterpoint.

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