Chapter 2
Using D-Spectra in Network Monte Carlo: Estimation of System Reliability and Component Importance

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Abstract We present a combinatorial definition of network-type system destruction spectrum (signature) and component importance D-spectra, and demonstrate how Birnbaum importance measures (BIMs) can be expressed via these spectra. We demonstrate an efficient algorithm which calculates simultaneously system reliability spectrum and BIM spectra for all its components. Use of BIMs in optimal system design is discussed.

Keywords D-spectrum · Signature · System reliability · Birnbaum importance measure · Importance D-spectrum · Network design

2.1 Introduction

It is a well-known fact that there exists a wide gap between theoretical analysis of reliability problems and the ability to compute reliability parameters for large or even moderate networks. Our purpose is to describe a Monte Carlo approach which allows efficient estimation of various network reliability parameters and may also be used in the optimal network design.

This approach is based on using the so-called *D-spectrum* for network reliability and the *importance D-spectrum* for computing network component

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importance measures. These notions were introduced and investigated by Elperin et al. (1991), Gertsbakh and Shpungin (2004, 2008, 2009). Examples of simulated D-spectra for a family of complete graphs and the dodecahedron networks were presented in Elperin et al. (1991) and named network *internal distributions*. Numerically, they are identical to so-called *signatures* introduced six years earlier by Samaniego (1985).

The paper is organized as follows. In Sect. 2.2, we give the basic notions and definitions of D-spectra. In Sect. 2.3, we present an efficient algorithm which calculates simultaneously system reliability D-spectrum and BIM D-spectra for all system components. In Sect. 2.4, we describe the use of BIMs D-spectra in network design.

2.2 Basic Notions and Definitions

2.2.1 Network and Its Reliability

By network $\mathbf{N} = (V, E, T)$ we denote an undirected graph with a node-set V, $|V| = n$, an edge-set E, $|E| = m$, and a set $T \subseteq V$ of special nodes called *terminals*. Each element a (node or edge) is associated with a probability p_a of being *up* and probability $q_a = 1 - p_a$ of being *down*. We postulate that element failures are mutually independent events. In a given network **N** the state of **N** is induced by all its elements which are in the *up* state.

In this paper we deal with *terminal connectivity* operational criterion. By this criterion the network state is *UP* if any pair of terminals is connected by the elements in the *up* state. Otherwise, the state is *DOWN*. The terminal connectivity has the property of being monotone: each subset of the *DOWN* state is a *DOWN* state, and each superset of the *UP* state is an *UP* state.

We define the network reliability $R(N)$ as the probability that the network is in *UP* state.

2.2.2 Birnbaum Importance Measure

Let us introduce the Birnbaum Importance Measure (BIM) (Barlow and Proschan 1975) of system component $j, j = 1, 2, \ldots, k$. If system reliability is a function $R = \Psi(p_1, p_2, \ldots, p_k)$ of component reliability p_j, then BIM of component j is defined as

$$BIM_j = \frac{\partial \Psi(p_1, \ldots, p_k)}{\partial p_j} = \Psi(p_1, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_k) - \Psi(p_1, \ldots, p_{j-1}, 0, p_{j+1}, \ldots, p_k). \tag{2.1}$$

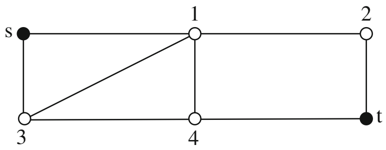


Fig. 2.1 Network with two terminals

BIM has a transparent probabilistic meaning: it is the gain in system reliability received from replacing a *down* component j by an absolutely reliable one. BIM_j gives the approximation to the system reliability increment δR resulted from component j reliability increment by δp_j.

The first expression on the right-hand side of (2.1) is the reliability of a system in which component j is permanently *up* and the second term is the reliability of the system in which component j is permanently *down*.

The use of BIM in practice was limited since usually the reliability function $\Psi(\cdot)$ is not available in explicit form. It turns out however that for the case of equal component reliability $p_j \equiv p$, there is a surprising connection between the network spectrum and the reliability function which allows to estimate the component BIMs without knowing the analytic form of the reliability function (Gertsbakh and Shpungin 2008).

2.3 Network Reliability and BIM D-Spectrum

2.3.1 Reliability Destruction Spectrum (D-Spectrum)

Definition 1 Let π be a permutation of all unreliable elements (edges or nodes) $e_1, \ldots, e_k : \pi = (e_{i_1}, e_{i_2}, \ldots, e_{i_k})$. Start with a network with all elements being *up* and “erase” the elements in the order they appear in π, from left to right. Stop at the first element e_{i_r} when the network becomes *DOWN*. The ordinal number r of this element is called the *anchor* of permutation π and denoted $r(\pi)$.#

Example 1 Consider the network shown in Fig. 2.1. In this network with two terminals, the edges are reliable and nodes 1,2,3,4 fail with probability $q = 1 - p$. Consider an arbitrary permutation π of node numbers, e.g. $\pi = (3, 2, 4, 1)$. Let us begin the following destruction process. We start with a network with all elements in the *up* state, and erase one node after another in the order prescribed by π, from left to right. Erasing a node means erasing all edges incident to this node. The network becomes *DOWN* after erasing the third node, i.e. node 4. So we have $r(\pi) = 4$.

Note that the anchor value for given π depends only on the network structure and its *DOWN* definition. It is completely separated from the stochastic mechanism which governs the node or edge failures in a real network destruction process.

Definition 2 Let x_i be the total number of permutations such that their anchor equals i. The set

$$D = \left\{ d_1 = \frac{x_1}{k!}, d_2 = \frac{x_2}{k!}, \ldots, d_k = \frac{x_k}{k!} \right\} \tag{2.2}$$

is called the *D-spectrum of the network.*#

For example, it is easy to check that for the network in Fig. 2.1, the D-spectrum is $\left\{0, {}^1\!/_2, {}^1\!/_2, 0\right\}$.

Remark "D" in Definition 2 refers to the "destruction" process of erasing network elements from left to right in the permutation π. D-spectrum is a distribution of the anchor value, and obviously $\sum_{i=1}^{k} d_i = 1$. Numerically, the D-spectrum coincides with the so-called *signature* introduced first in (Samaniego 1985). It was proved there (see also Samaniego 2007) that if system components fail independently and their lifetimes X_i have identical continuous distribution function $F(t)$, then the system lifetime distribution $F_s(t) = \sum_{i=1}^{k} d_i \cdot F_{i:k}(t)$ where $F_{i:k}$ is the cumulative distribution function of the ith order statistics in random sample $X_1, X_2, \ldots, X_k$.#

Definition 3 Let $y_b = \sum_{i=1}^{b} d_i, b = 1, 2, \ldots, k$. Then the set $(y_1, y_2, \ldots, y_k)$ is called the *cumulative D-spectrum.*#

Theorem 1 *If all* $p_i \equiv p$, *then network static reliability* $R(\mathbf{N})$ *can be expressed in the following form:*

$$R(\mathbf{N}) = 1 - \sum_{i=1}^{k} y_i \frac{k! q^i p^{k-i}}{i!(k-i)!}.\# \tag{2.3}$$

We omit the proof of this statement which can be found in Gertsbakh and Shpungin (2009), Chap. 8. This theorem is equivalent to the following interesting combinatorial fact: $y_i \cdot k!/(i!(k-i)!) = C(i)$, where $C(i)$ is the total number of cut sets of size i in the system. A statement equivalent to Theorem 1 is presented in Samaniego (2007), Sect. 6.1.

2.3.2 BIM Spectrum

We use the abbreviation BIM for the Birnbaum Importance Measure.

Definition 4 Denote by $Z_{i,j}$ the number of permutations satisfying the following two conditions:

- If the first i elements in the permutation are *down*, then the network is *DOWN*;
- Element j (node or edge) is among the first i elements of the permutation.

Table 2.1 $BIM\Diamond S$ for network

i	$z_{i,1}$	$z_{i,2}$	$z_{i,3}$	$z_{i,4}$
1	0	0	0	0
2	1/3	1/6	1/6	1/3
3	3/4	3/4	3/4	3/4
4	1	1	1	1

The collection of $z_{i,j} = Z_{i,j}/k!$ values, $i = 1, 2, \ldots, k; j = 1, 2, \ldots, k$, is called BIM-spectrum of the network and denoted $BIM\Diamond S$. The set of $z_{i,j}$ values for fixed j and $i = 1, \ldots, k$ is called the BIM_j D-spectrum, or the importance spectrum of component j.

Example 1 (continued) Let us turn to the network in Fig. 2.1 and calculate one of the $Z_{i,j}$ values, say $Z_{2,2}$. The permutations which satisfy the conditions of the above definition, are the following: $(2, 4, 1, 3)$, $(2, 4, 3, 1)$, $(4, 2, 1, 3)$, $(4, 2, 3, 1)$. So we have $Z_{2,2} = 4$. The Table 2.1 presents the $BIM\Diamond S$ for our network.

The columns in this table are the BIM_j spectra.

The proof of the following theorem is presented in Gertsbakh and Shpungin (2009), Chap. 10.

Theorem 2 *BIM_j, $j = 1, 2, \ldots, k$, equals*

$$BIM_j = \sum_{i=1}^{k} \frac{k!(z_{i,j} q^{i-1} p^{k-i} - (y_i - z_{i,j}) q^i p^{k-i-1})}{i!(k-i)!}. \# \tag{2.4}$$

Note that $y_k - z_{k,j} = 0$, which means that in the second term of the numerator of (2.4) *one can assume that index i changes from 1 to $k-1$.*

The exact calculation of network D-spectrum and BIM-spectra is a formidable task. We suggest estimating the spectra using Monte Carlo approach. Below we present an algorithm which simultaneously estimates the D-spectrum and the BIM-spectra for all network components.

The core of Algorithm 1 is Operator 3 which finds out the anchor of the random permutation of network elements simulated by Operator 2. A single permutation and its anchor allow obtaining one observation replica for estimating the D-spectrum and all BIM-spectra. A substantial feature of this operator is that only a *single* spanning tree is constructed to find out the anchor for given π. More details about this algorithm can be found in Gertsbakh and Shpungin (2004, 2009).

The algorithm is adjusted to the case of n unreliable edges in the network.

Algorithm 1 Computing BIM Spectra and D-Spectrum

1. **`Initialize`** `all` a_i *and* $b_{i,j}$ *to be zero*, $i = 1, \ldots, n; j = 1, \ldots, n$.
2. **`Simulate`** `permutation` $\pi \in \Pi_E$. (Π_E `is the set of all edge permutations.`)

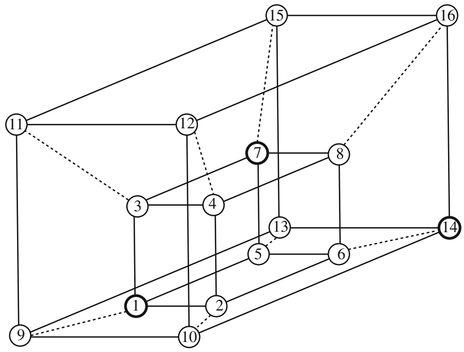


Fig. 2.2 Hypercube H_4 with 3 terminals

Table 2.2 Network H_4: cumulative reliability spectrum

i	y_i	i	y_i	i	y_i	i	y_i
1	0	9	0.011709	17	0.404952	25	0.994868
2	0	10	0.020265	18	0.547724	26	0.998277
3	0	11	0.035539	19	0.687263	27	0.999590
4	0.000079	12	0.053547	20	0.802654	28	0.999947
5	0.000426	13	0.083682	21	0.885786	29	1.0
6	0.001297	14	0.128297	22	0.939116	30	1.0
7	0.003053	15	0.192645	23	0.970280	31	1.0
8	0.006217	16	0.284164	24	0.986978	32	1.0

3. **Find out** the minimal index of the edge $r = r(\pi)$ such that the first r edges in π create network *DOWN* state (r is the anchor of π.)
4. **Put** $a_r := a_r + 1$.
5. **Find** all j such that e_j occupies one of the first r positions in π, and for each such j put $b_{r,j} := b_{r,j} + 1$.
6. **Put** $r := r + 1$. **If** $r \le k$, **GOTO** 4.
7. **Repeat** 2–6 M times.
8. **Estimate** $y_i, z_{i,j}$ via $\hat{y}_i = \frac{a_i}{M}$, $\hat{z}_{i,j} = \frac{b_{i,j}}{M}$.

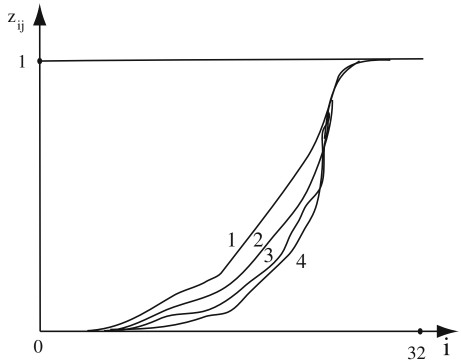
Example 2 Consider a three-terminal hypercube H_4 with 16 reliable nodes and 32 unreliable edges. This hypercube is shown on Fig. 2.2. Let its terminals be nodes 1, 7, and 14. Edges will be denoted by (k,s) where k and s are node numbers. Table 2.2 presents the values of spectrum for this network. Table 2.3 presents hypercube reliability R for different values of p. The values of R may be computed via the spectrum.

It follows from the simulated BIMs spectra (not presented here) that in our network there are several groups of edges having approximately equal estimated BIMs within each group, while the groups can be partially ranked according to their importance measures. The first group consists of edges (1, 5) and (5, 7)

Table 2.3 Network H_4: reliability for different values of p

p	R	p	R	p	R
0.1	0.000529	0.4	0.344084	0.7	0.966237
0.15	0.003855	0.45	0.498603	0.75	0.985145
0.2	0.015373	0.5	0.647305	0.8	0.994341
0.25	0.046276	0.55	0.775587	0.85	0.998327
0.3	0.107647	0.6	0.869351	0.9	0.999694
0.35	0.207249	0.65	0.930455	0.95	0.999977

Fig. 2.3 BIMs spectra for various edges



connecting terminals 1 and 7. This group dominates all other edges by its BIMs for all values of p. The second group consists of two edges (6, 14) and (13, 14). Their BIMs dominate the remaining edge BIMs also for all p values. The third group of approximately identical BIMs constitute four edges {(1, 3), (3, 7), (5, 6), (5, 13)}. The third group, however, does not dominate another group of edges uniformly for all p values. A similar behavior exhibits the fourth group of the rest edges.

Figure 2.3 presents the graphs of accumulated BIM-spectra for edges representing each of the four groups of edges.

2.4 BIM-Based Heuristic for Network Optimal Design

In this section we consider network optimal design problem in the following form. Suppose that s, $1 \leq s \leq k$, elements (nodes or edges) can be simultaneously replaced by more reliable ones having the *up* probability $p^* > p$. The problem is to choose the s candidates for the reinforcement in order to maximize the network reliability.

The solution of this problem is not trivial and involves the notion of *joint importance measure*. We will not investigate here this issue but instead provide a well-working heuristic method. This method uses the estimated network element importance measures.

Table 2.4 Network reliability after four edges replacement

Edges	$p = 0.7$	$p = 0.8$	$p = 0.9$	$p = 0.95$
(1,5), (5,7), (6,14), (13,14)	0.904522	0.93793	0.968138	0.982422
(1,3), (3,7), (5,6), (5,13)	0.893605	0.917477	0.940232	0.952468
(1,2), (1,9), (7,8), (7,15)	0.899953	0.923532	0.942001	0.948896
(2,6), (6,8), (9,13), (10,14)	0.884582	0.898797	0.911858	0.917419

The background for the proposed method is the following theorem (Gertbakh and Shpungin 2008, 2009).

Theorem 3 *Suppose we are given the BIM◇S for our network. Let us fix two indices* α *and* $\beta \neq \alpha$*, and the corresponding* $z_{i,\alpha}$ *and* $z_{i,\beta}$ *values from the BIM◇S.*

- If for all $i, i = 1, 2, \ldots, k$, $z_{i,\alpha} \geq z_{i,\beta}$, then $BIM_\alpha \geq BIM_\beta$ for all p values.
- Suppose that the first sentence does not take place, and let s be the maximal index such that $z_{s,\alpha} \neq z_{s,\beta}$. Without loss of generality, assume that $z_{s,\alpha} > z_{s,\beta}$.

Then there exists p_0 such that for $p \geq p_0$ $BIM_\alpha > BIM_\beta$.

The following heuristic algorithm gives rather good, although not necessary optimal solution for the problem of choosing s elements for replacement. The advantage of this algorithm is its simplicity and computational efficiency.

Algorithm 2 Heuristic Replacement

1. **Estimate** the BIM's values for all elements.
2. **Range** the elements by their BIM's values, from the ''best'' to the ''worst''.
3. **Take** the first ''best'' s elements and **replace** them by more reliable ones.

Example 3 Consider the network H_4 in Fig. 2.2, with three terminals: 1, 7, and 14. Suppose that all edges *up* probabilities equal 0.6. The appropriate network computed reliability is $R = 0.86917$. Let us replace four edges by more reliable ones in accordance with the above heuristic algorithm. For the first, we try the edges with the highest BIMs, from the first group (such are two edges) and from the second (also two edges). After this we take edges from the third, fourth, and fifth groups. We see from Table 2.4 that the replacing by the edges with the highest BIMs is better than all other replacings. Replacing by the edges from the second group is slightly worse than from the third group, for the probabilities $p = 0.7$, 0.8, 0.9, and slightly better for $p = 0.95$.

2.5 Conclusions

For the case of networks with equally reliable components (nodes, or edges or both), we have developed a new efficient computation scheme for evaluating the BIMs. Its main idea is based on calculating, via a Monte Carlo scheme, network combinatorial invariant called spectrum and on the connection between the spectrum and the component BIMs.

We have proposed a new efficient approach to one of the problems in optimal network design.

The techniques developed in this approach can be viewed as a first step toward developing other efficient algorithms for optimal network reliability design.

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