

Chapter 2

An Overview of Fractional Processes and Fractional-Order Signal Processing Techniques

2.1 Fractional Processes

In this monograph, the term fractional processes refers to the following random processes:

- Random processes with long range dependence (LRD);
- Multifractional processes which exhibit local memory/locally self-similar property;
- Random processes with heavy-tailed distributions;
- Random processes which exhibit both LRD and heavy-tailed distribution properties;
- Random processes which exhibit both local memory and heavy-tailed distribution properties.

It is known that a conventional (integer-order) random signal can be considered as the solution of an integer-order differential equation with the white noise as the input excitation. From the perspective of “signals and systems”, a conventional (integer-order) random signal can be regarded as the output of an integer-order differential system or integer-order filter with the white noise as the input signal [114, 204]. Similarly, other studies show in [164, 221, 271] that the fractional signals can be taken as the solutions of constant-order fractional or variable-order fractional differential equations. Therefore, fractional signals can be synthesized by constant-order fractional systems, or variable-order fractional systems with a wGn or a white stable noise as the input signal, where the white stable noise is a cylindrical Wiener processes on Hilbert spaces subordinated by a stable process [38, 121]. In this chapter, fractional processes and FOSP techniques are introduced from the perspective of fractional signals and fractional-order systems.

2.1.1 Fractional Processes and Fractional-Order Systems

Review of Conventional Random Processes and Integer-Order Systems

A continuous-time LTI (linear time invariant) system can be represented by an integer-order ordinary differential equation in the general form [114, 204]

$$\sum_{j=0}^N a_j y^{(j)}(t) = \sum_{i=0}^M b_i f^{(i)}(t), \quad (2.1)$$

where $f(t)$ is the input signal, and $y(t)$ is the output signal of the LTI system with proper initial conditions and $N \geq M$. The transfer function of the continuous LTI system under zero initial conditions is

$$H(s) = \frac{\sum_{i=0}^M b_i s^i}{\sum_{j=0}^N a_j s^j}. \quad (2.2)$$

The output signal $y(t)$ of the LTI system (2.1) can be written as

$$y(t) = \int_0^t h(t - \tau) f(\tau) d\tau, \quad (2.3)$$

under a zero state condition, where $h(t)$ is the impulse response of the LTI system. (2.3) is also called “zero-state response” of (2.1) under input or driving signal $f(t)$. In this monograph, all responses are in the sense of “zero-state response” unless otherwise indicated. A traditional stationary continuous random signal can be expressed as the output of an LTI system with wGn (white Gaussian noise) as the driving input signal,

$$y(t) = \int_0^t h(t - \tau) \omega(\tau) d\tau, \quad (2.4)$$

where $\omega(t)$ is wGn, $h(t)$ is the inverse Laplace transform of transfer function $H(s)$, that is $h(t) = \mathcal{L}^{-1}[H(s)]$. In the same way, a stationary stable continuous random signal with heavy-tailed distribution can be considered as the output of an LTI system with white stable noise as the input

$$y(t) = \int_0^t h(t - \tau) \omega_\alpha(\tau) d\tau, \quad (2.5)$$

where $\omega_\alpha(t)$ is a white stable noise, which will be introduced in Chap. 3.

A linear discrete time-invariant (LDTI) system can be represented by a difference equation of the following general form [114, 204]

$$\sum_{j=0}^N a_j y(n - j) = \sum_{i=0}^M b_i f(n - i), \quad (2.6)$$

where $f(n)$ is the input sequence, and the $y(n)$ is the output sequence of the LDTI system with $m \leq n$. The Z -transfer function of the LDTI system is

$$H(z) = \frac{\sum_{i=0}^M b_i z^{-i}}{\sum_{j=0}^N a_j z^{-j}}. \quad (2.7)$$

A traditional stationary discrete random signal can be expressed as the output of an LDTI system with the discrete wGn as the input,

$$y(n) = \omega(n) * h(n), \quad (2.8)$$

where $\omega(n)$ is a discrete wGn, '*' is the convolution, and $h(n)$ is the inverse Z -transform of $H(z)$, that is $h(n) = \mathcal{Z}^{-1}[H(z)]$.

Similarly, a stationary stable discrete random signal with heavy-tailed distribution can be considered as the output of a discrete LTI system with discrete white stable noise as the input,

$$y(n) = \omega_\alpha(n) * h(n), \quad (2.9)$$

where $\omega_\alpha(n)$ is the discrete white stable noise [215, 253].

Constant-Order Fractional Processes and Constant-Order Fractional Systems

Similar to the integer-order continuous-time LTI system, a constant-order fractional linear continuous time-invariant (FLTI) system can be described by a fractional-order differential equation of the general form [164, 221]

$$\sum_{j=0}^N a_j D^{\nu_j} y(t) = \sum_{i=0}^M b_i D^{\mu_i} f(t), \quad (2.10)$$

where $f(t)$ is the input, $y(t)$ is the output of the FLTI system, and D^α denotes the fractional derivative of order α . The transfer function of the continuous FLTI system under zero initial conditions is [164, 221]

$$H(s) = \frac{\sum_{i=0}^M b_i s^{\nu_i}}{\sum_{j=0}^N a_j s^{\mu_j}}, \quad \text{Re}(s) > 0. \quad (2.11)$$

The output $y(t)$ of an FLTI system can also be described as

$$y(t) = \int_0^t h(t - \tau) f(\tau) d\tau, \quad (2.12)$$

where $h(t)$ is the impulse response of the FLTI system (2.11), and $f(t)$ is the input. A constant-order fractional stationary continuous random signal can be regarded as

the output of an FLTI system with wGn as the input,

$$y(t) = \int_0^t h(t - \tau) \omega(\tau) d\tau, \quad (2.13)$$

where $\omega(t)$ is the wGn, $h(t)$ is the inverse Laplace transform of $H(s)$ in (2.11). In the same way, a constant-order fractional stable continuous random signal can be considered as the output of an FLTI system with the white stable noise as the input,

$$y(t) = \int_0^t h(t - \tau) \omega_\alpha(\tau) d\tau, \quad (2.14)$$

where $\omega_\alpha(t)$ is the white stable noise.

Similar to the LDTI system, a constant-order fractional linear discrete time-invariant (FLDTI) system can be represented by a constant-order fractional difference equation with the general form [164, 220]

$$\sum_{j=0}^N a_j D^{v_j} y(n) = \sum_{i=0}^M b_i D^{\mu_i} f(n), \quad (2.15)$$

where $f(n)$ is the input, $y(n)$ is the output of the FLDTI system, and D^α denotes the fractional difference operator (delay) of order α , that is $D^\alpha y(n) = y(n - \alpha)$. The transfer function of the FLDTI system is [164, 220]

$$H(z) = \frac{\sum_{i=0}^M b_i z^{-v_i}}{\sum_{j=0}^N a_j z^{-\mu_j}}, \quad |z| = 1. \quad (2.16)$$

A constant-order fractional discrete random signal can be considered as the output of an FLDTI system with discrete wGn as the input,

$$y(n) = \omega(n) * h(n), \quad (2.17)$$

where $\omega(n)$ is the discrete wGn, ‘*’ is the convolution, and $h(n)$ is the inverse Z-transform of $H(z)$. A constant-order fractional stable discrete random signal can be considered as the output of a discrete FLDTI system with discrete white stable noise as the input,

$$y(n) = \omega_\alpha(n) * h(n), \quad (2.18)$$

where $\omega_\alpha(n)$ is the discrete white stable noise.

Compared with the constant-order fractional processes, the distributed-order fractional processes and multifractional processes are more complex. Distributed-order fractional processes can be considered as the output of the combination of the constant-order fractional-order systems [180]. Multifractional processes can be considered as the output of a variable-order fractional system which can be represented by a variable-order fractional differential equation. Different from the constant-order fractional systems which can be simply described by transfer functions, the

variable-order fractional systems cannot be simply expressed using the Laplace transform, because it is difficult to calculate the Laplace transformation of variable-order fractional differential equations. Variable-order fractional processes will be discussed in Chap. 4.

2.1.2 Stable Processes

Definition 2.1 A random variable X is stable or stable in the broad sense, if for X_1 and X_2 independent copies of X and any positive constants a and b ,

$$aX_1 + bX_2 \stackrel{d}{=} cX + d, \quad (2.19)$$

for some positive c and $d \in \mathbb{R}$. The random variable is strictly stable or stable in the narrow sense if (2.19) holds with $d = 0$, for all choices of a and b .

A random variable is symmetric stable if it is stable and symmetrically distributed around 0, e.g. $X \stackrel{d}{=} -X$. Here $\stackrel{d}{=}$ means the equivalence in distribution.

Definition 2.2 A real random variable X is $S\alpha S$, if its characteristic function is of the form

$$\varphi(t) = \exp\{jat - \gamma|t|^\alpha\}, \quad (2.20)$$

where $0 < \alpha \leq 2$ is the characteristic exponent, $\gamma > 0$ the dispersion, and $-\infty < a < \infty$ the location parameter.

When $\alpha = 2$, X is Gaussian.

The problem of estimating the parameters of an α -stable distribution is difficult, because majority of the stable family lacks any known closed-form density functions. Since most of the conventional methods in mathematical statistics depend on an explicit form for the density function, these methods cannot be used in estimating the parameters of the α -stable distributions. Fortunately, some numerical methods can be used in the literature for the parameter estimation of symmetric α -stable distributions [215]. The most frequently used method for estimating the parameters of the $S\alpha S$ law with $1 \leq \alpha \leq 2$ is suggested in [92]. Let $F(\cdot)$ be a distribution function. Then, its f fractile x_f is defined by

$$F(x_f) = f, \quad (2.21)$$

where f is restricted to be $0 < f < 1$. The order statistics $X_{(1)}, \dots, X_{(N)}$ of a random sequence X_1, \dots, X_N satisfy $X_{(1)} \leq \dots \leq X_{(N)}$.

Let X_1, \dots, X_N be a random sample sequence from an unknown distribution $F(x)$, whose order statistics are $X_{(1)}, \dots, X_{(N)}$. Specifically, assuming that $0 \leq i \leq$

N and $\frac{2i-1}{2N} \leq f < \frac{2i+1}{2N}$, then

$$\hat{x}_f = X_{(i)} + [X_{(i+1)} - X_{(i)}] \frac{f - q(i)}{q(i+1) - q(i)}, \quad (2.22)$$

where

$$q(i) = \frac{2i-1}{2N}. \quad (2.23)$$

If $i = 0$ or $i = N$, then $\hat{x}_f = X(1)$ and $\hat{x}_f = X(N)$, respectively.

McCulloch generalized the above method to provide consistent estimates for α and c [199]. He also eliminated the asymptotic bias in the Fama-Roll estimators of α and c . Specifically, for the symmetric stable law, the fractile estimate \hat{v}_α is that

$$\hat{v}_\alpha = \frac{\hat{x}_{0.95} - \hat{x}_{0.05}}{\hat{x}_{0.75} - \hat{x}_{0.25}}. \quad (2.24)$$

Thus, a consistent estimate $\hat{\alpha}$ can be found by searching tables in [199], with a matched value of \hat{v}_α . For fixed α , the following quantity

$$v_c = \frac{\hat{x}_{0.75} - \hat{x}_{0.25}}{c}, \quad (2.25)$$

is independent of α . $\hat{x}_{0.75}$ and $\hat{x}_{0.25}$ are all consistent estimators, with the following parameter a consistent estimator of c

$$\hat{c} = \frac{\hat{x}_{0.75} - \hat{x}_{0.25}}{v_c(\hat{\alpha})}. \quad (2.26)$$

McCulloch's method provides consistent estimators for all four parameters, with $-1 \leq \beta \leq 1$ and $\alpha \geq 0.6$ [199].

2.1.3 Fractional Brownian Motion

The definition of 'one-sided' fBm based on the Riemann-Liouville fractional integral, was introduced in [20].

Definition 2.3 The 'one-sided' fBm is defined as

$$B_H(t) = \frac{1}{\Gamma(H + 1/2)} \int_0^t (t - \tau)^{H-1/2} \omega(\tau) d\tau, \quad 1/2 < H < 1, \quad (2.27)$$

where $\omega(t)$ is wGn.

According to the definition of Riemann-Liouville fractional integral, the fBm can be considered as the $(\alpha + 1)$ th integration of wGn.

$$B_H(t) = {}_0D_t^{-1-\alpha} \omega(t). \quad (2.28)$$

So, from the perspective of fractional signals and fractional-order systems, fBm can be generated by $(\alpha + 1)$ th integrator with wGn as the input. Besides the above ‘one-sided’ fBm definition, another frequently used stochastic integral form definition of fBm with index H ($0 < H < 1$) [144, 193] will be introduced in Chap. 3.

The index H is the Hurst parameter which determines the type of fBm. When $H = 0.5$, fBm is the conventional Brownian motion; when $H > 0.5$ the increments of the fBm process are positively correlated [22].

2.1.4 Fractional Gaussian Noise

fGn is the derivative of fBm [193]. So, the fGn can be expressed as the α th order integration of wGn

$$Y_H(t) = {}_0D_t^{-\alpha}\omega(t), \quad (2.29)$$

where $\omega(t)$ is the wGn. The Hurst parameter of fGn is related to α by $H = 1/2 + \alpha$. Therefore, from the perspective of fractional signals and fractional-order systems the fGn can be simulated by the α th integrator with wGn as the input.

fGn has some distinctive properties. The power spectrum of fGn has an inverse power-law form, and the autocorrelation function of fGn has the power-law decay. Different from the i.i.d. random signals characterized by mean, variance or other high-order statistic properties, fGn is mainly characterized by the Hurst parameter (Hurst exponent) $H \in (0, 1)$, which was named after the hydrologist Hurst who pioneered the field of research in the fifties [123]. There are a number of practical methods which can be used to estimate the Hurst parameter. The best known Hurst exponent estimator is the Rescaled Range method (R/S), which was first proposed by Hurst in the hydrological context. A variety of other estimation techniques exist, such as the Aggregated Variance method [22], the Absolute Value method [297], the Periodogram method [97], the fractional Fourier transform (FrFT) based method [60], Koutsoyiannis’ method [153], and so on. A comprehensive evaluation of these Hurst estimators is provided in Chap. 3.

2.1.5 Fractional Stable Motion

The fractional stable motion, which exhibits both the LRD and heavy-tailed distribution properties, is a generalization of fBm. The linear fractional stable motion (LFSM) was studied in [253]. From the perspective of fractional signals and fractional-order systems, the fractional stable motion can be expressed as the output of an $(\alpha + 1)$ th fractional integrator with white stable noise as the input,

$$Y_{\alpha,H}(t) = {}_0D_t^{-1-\lambda}\omega_{\alpha}(t), \quad 0 < \lambda < 1/2, \quad (2.30)$$

where $H = 1/\alpha + \lambda$, and $\omega_{\alpha}(t)$ is the α -stable noise [253].

2.1.6 Fractional Stable Noise

The fractional stable noise provides the increments of fractional stable motion. So, the fractional stable noise can be constructed as the output of an α th fractional integrator with white stable noise as the input

$$Y_{\alpha,H}(t) = {}_0D_t^{-\lambda} \omega_\alpha(t), \quad 0 < \lambda < 1/2, \quad (2.31)$$

where $H = 1/\alpha + \lambda$, and $\omega_\alpha(t)$ is the α -stable noise [253]. The fractional stable noise will be introduced in detail in Chap. 3.

2.1.7 Multifractional Brownian Motion

Based on the definition of ‘one side’ fBm, Lim provided the definition of the Riemann-Liouville fractional integral based mBm in [172].

Definition 2.4 The Riemann-Liouville fractional integral based mBm can be described as

$$B_{H(t)}(t) = \frac{1}{\Gamma(H(t) + 1/2)} \int_0^t (t - \tau)^{H(t)-1/2} \omega(\tau) d\tau, \quad 1/2 < H(t) < 1, \quad (2.32)$$

where $\omega(t)$ is wGn.

Therefore, we can consider mGn as the output of $[\alpha(t) + 1]$ th variable-order fractional integrator with wGn as the input.

$$B_{H(t)}(t) = {}_0D_t^{-1-\alpha(t)} \omega(t). \quad (2.33)$$

Time-dependent local Hölder exponent $H(t)$ is the generalization of the constant Hurst parameter H [232]. Obviously, fBm is a special case of the mBm with a constant Hölder exponent $H(t) = H$. The properties of the mBm will be introduced in Chap. 4.

2.1.8 Multifractional Gaussian Noise

mGn is defined as the derivative of mBm. Therefore, we can consider mGn as the output of $\alpha(t)$ th variable-order fractional integrator with wGn as the input. The mGn $Y_{H(t)}(t)$ can be described as

$$Y_{H(t)}(t) = {}_0D_t^{-\alpha(t)} \omega(t), \quad (2.34)$$

where $\omega(t)$ is wGn. The local Hölder exponent $H(t)$ of mBm is related to $\alpha(t)$ by $H(t) = 1/2 + \alpha(t)$. Similar to the mBm which is the generalization of fBm, mGn is the generalization of fGn, and fGn is a special case of the mGn with a constant local Hölder exponent $H(t) = H$.

2.1.9 Multifractional Stable Motion

The multifractional stable motion, which exhibits both the local self-similarity and heavy-tailed distribution properties, is a generalization of mBm. The multifractional stable motion $Y_{\alpha, H(t)}(t)$ is presented as

$$Y_{\alpha, H(t)}(t) = {}_0D_t^{-1-\lambda(t)} \omega_\alpha(t), \quad 0 < \lambda(t) < 1/2, \quad (2.35)$$

where $\omega_\alpha(t)$ is α -stable noise [253]. The local Hölder exponent $H(t)$ of multifractional stable motion is related to α and $\lambda(t)$ by $H(t) = 1/\alpha + \lambda(t)$. mBm is the special case of the multifractional stable motion with stable distribution parameter $\alpha = 2$.

2.1.10 Multifractional Stable Noise

In the same way, a multifractional stable noise can be considered as the $\lambda(t)$ th integration of an α -stable process. The multifractional stable noise is presented as

$$Y_{\alpha, H(t)}(t) = {}_0D_t^{-\lambda(t)} \omega_\alpha(t), \quad 0 < \lambda(t) < 1/2, \quad (2.36)$$

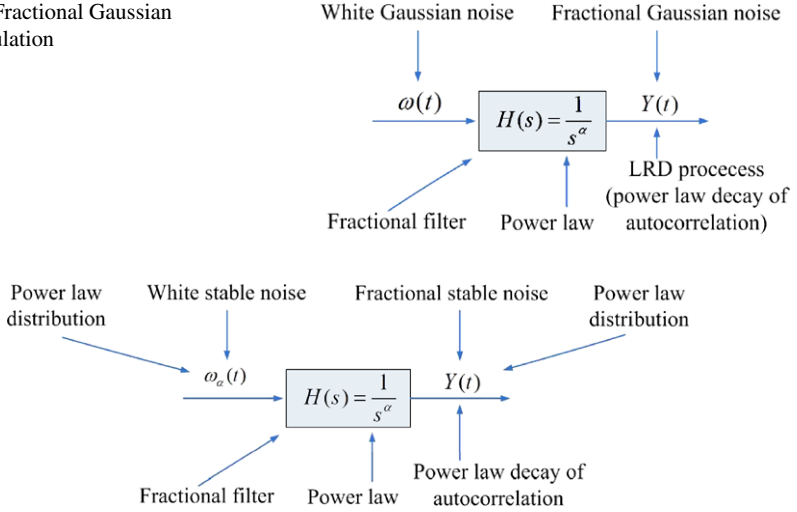
where $\omega_\alpha(t)$ is α -stable noise [253]. Multifractional stable noise exhibits local self-similarity and heavy-tailed distribution. mGn is the special case of the multifractional stable noise with stable distribution parameter $\alpha = 2$.

2.2 Fractional-Order Signal Processing Techniques

In this monograph, like the conventional signal processing methods, FOSP techniques include fractional random signals simulation, fractional filter, fractional systems modeling, and so on. The FOSP techniques are briefly summarized in this section.

2.2.1 Simulation of Fractional Random Processes

As stated above, random processes can be generated by performing time domain integer-order filtering on a white Gaussian process [107, 204]. Similarly, the frac-

Fig. 2.1 Fractional Gaussian noise simulation**Fig. 2.2** Fractional stable noise simulation

tional random processes can be simulated by performing the time domain fractional-order filtering on a white Gaussian process or a white α -stable process. Different types of fractional filters generate different fractional random signals. For example, fractional Gaussian noise and fractional stable noise can be simulated by a constant-order fractional filter. Figures 2.1 and 2.2 illustrate the simulations of fractional Gaussian noise and fractional stable noise, respectively. The constant-order fractional integrated or filtered signals exhibit the LRD property, that is, the power-law decay of the autocorrelation. Similarly, multifractional Gaussian signals and multifractional stable signals can be simulated by variable-order fractional filters. The output signals of the variable-order fractional filters exhibit the local memory property.

2.2.2 Fractional Filter

It has been introduced in the above subsection that the fractional filters can be used to generate the fractional random signals. Similar to the classification of the fractional signals in this monograph, the fractional filters can also be classified into three types: constant-order fractional filters, distributed-order fractional filters, and variable-order fractional filters. Fractional-order filters are different from the integer-order filters. Integer-order filters generate the short-range dependence on the input signal; constant-order fractional filters generate the LRD property; variable-order fractional filters generate the local memory property. The distributed-order filters can be considered as the summation of the constant-order fractional filters. In this monograph, the constant-order and distributed-order fractional filters are

studied. The constant-order fractional filters will be introduced in Chap. 5, and the distributed-order fractional filters will be studied in Chap. 7.

2.2.3 Fractional-Order Systems Modeling

It has been introduced in Sect. 2.1 that a traditional stationary integer-order random signal can be considered as the output of an LTI system with wGn as the input. The continuous-time LTI system can be characterized by a linear difference equation known as an ARMA model in the discrete case. An ARMA(p, q) process X_t is defined as

$$\Phi(B)X_t = \Theta(B)\epsilon_t, \quad (2.37)$$

where ϵ_t is a wGn, and B is the backshift operator. However, the ARMA model can only capture the short-range dependence property of the system. In order to capture the LRD property of the fractional system, the FARIMA(p, d, q) model was proposed [37]. An FARIMA(p, d, q) process X_t is defined as [37]

$$\Phi(B)(1 - B)^d X_t = \Theta(B)\epsilon_t, \quad (2.38)$$

where $d \in (-0.5, 0.5)$, and $(1 - B)^d$ is the fractional differencing operator.

Furthermore, the locally stationary long memory FARIMA(p, d_t, q) model

$$\Phi(B)(1 - B)^{d_t} X_t = \Theta(B)\epsilon_t, \quad (2.39)$$

was suggested in [30], where $\{\epsilon_t\}$ is a wGn and d_t is a time-varying parameter. The locally stationary long memory FARIMA(p, d_t, q) model can capture the local self-similarity of the systems. Besides the above mentioned fractional system models, other fractional models will be introduced in Chaps. 5 and 6.

2.2.4 Realization of Fractional Systems

Realization of fractional systems includes the realization of analogue fractional systems and the realization of digital fractional systems.

Analogue Realization of Fractional Systems

Analogue fractional systems, such as the fractional controllers and fractional filters, can be used widely in engineering. All fractional systems rely on the fractional-order integrator and the fractional-order differentiator as basic elements. Many efforts have been made to design analogue fractional-order integrators and differentiators. Most of these analogue realization methods are based on networks of resistors,

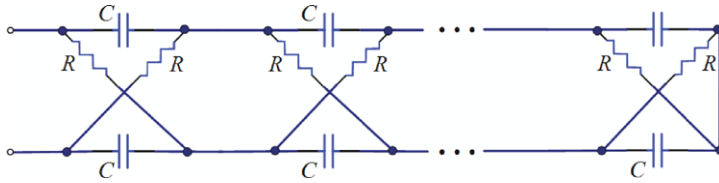


Fig. 2.3 RC transmission line circuit

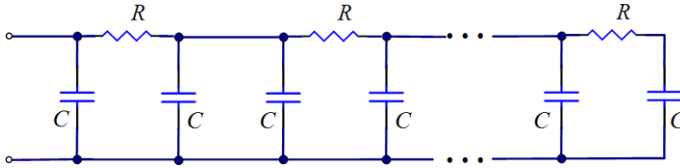


Fig. 2.4 RC domino ladder circuit

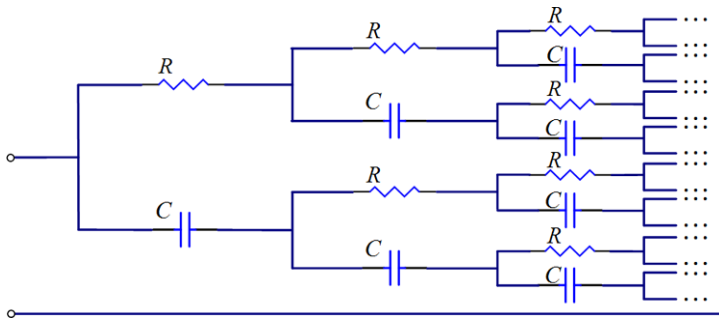
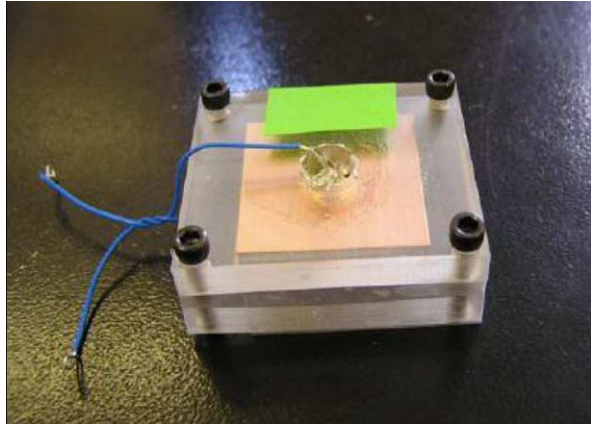


Fig. 2.5 RC binary tree circuit

capacitors or inductors. Figures 2.3, 2.4 and 2.5 illustrate the analogue realization of fractional-order operators using resistor and capacitor networks.

In order to make the analogue fractional device simple and accurate, some researchers have concentrated on smart materials which exhibit realistic fractional behavior. In this monograph, the analogue realization of constant-order fractional-order differentiator/integrator and variable-order fractional differentiator/integrator was based on an electrical element named ‘Fractor’ (Fig. 2.6), manufactured by Bohannan [27, 28]. The Fractor was originally made from Lithium Hydrazinium Sulfate ($\text{LiN}_2\text{H}_5\text{SO}_4$) which exhibits realistic fractional behavior $1/(j\omega C)^\lambda$ over a large range of frequency, where $\alpha \approx 1/2$ [261]. Now, the Fractor is being made from Lithium salts. The analogue realization of fractional systems will be introduced in Chaps. 5 and 6.

Fig. 2.6 Fractor

Digital Realization of Fractional Systems

Based on the definition of fractional calculus, the calculation of the output of a fractional system depends on the long-range history of the input. Because of the limitation of calculation speed and storage space, the digital realization of fractional systems is difficult. The commonly used methods of approximate digital realization of fractional systems are frequency domain methods and time domain methods. Currently both methods offer limited success in fitting the fractional system.

Frequency domain methods include Oustaloup method [227], Carlson method [237], Matsuda method [237], and so on. Frequency-domain fitting techniques can fit the magnitude of the frequency response very well, but cannot guarantee the stable minimum-phase fitting. Time domain methods are mainly based on fitting the impulse response or the step response of the system. An effective time domain impulse response invariant discretization method was discussed in [59, 62, 63, 182]. There, a technique for designing discrete-time infinite impulse response (IIR) filters to approximate the continuous-time fractional-order filters is proposed, keeping the impulse response of the continuous-time fractional-order filter and the impulse response of the approximate discrete-time filter almost the same.

2.2.5 Other Fractional Tools

Besides the above FOSP techniques, there are other FOSP techniques too, such as fractional Hilbert transform, fractional spectrum analysis, fractional B-spline, and so on. These FOSP techniques provide new options for analyzing complex signals.

Fractional Hilbert Transform

The fractional Hilbert transform (FHT), the generalization of the conventional Hilbert transform, was proposed in [176]. FHT has been successfully used in digital

Fig. 2.7 Block diagrams for the different implementations of the fractional Hilbert transform. **(a)** Spatial filter. **(b)** FrFT method. **(c)** Generalized definition

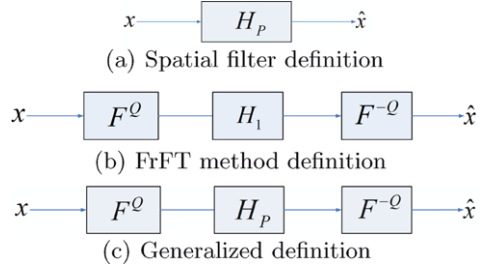


image processing. There are three commonly used definitions of FHT. The first definition is based on modifying the spatial filter with a fractional parameter, and the second one is based upon the fractional Fourier plane for filtering. The third definition is the combination of these two definitions. The transfer function of the first definition is [176]

$$\tilde{H}_P(v) = \exp(+i\phi)u(v) + \exp(-i\phi)u(-v), \quad (2.40)$$

where P is the fractional order, $u(v)$ is a step function, and $\phi = P\pi/2$. The second type FHT is defined as [176]

$$V_Q = F^{-Q} H_1 F^Q, \quad (2.41)$$

where F^α is the FrFT operation of order α , Q is a fractional parameter, and

$$\tilde{H}_1(v) = \exp\left(+i\frac{\pi}{2}\right)u(v) + \exp\left(-i\frac{\pi}{2}\right)u(-v). \quad (2.42)$$

The third definition of FHT is [176]

$$V_Q = F^{-Q} H_P F^Q. \quad (2.43)$$

Figure 2.7 illustrates the three definitions of FHT.

Fractional Power Spectrum Density

Definitions of fractional spectrum density (FPSD) fall into two types: FrFT based and FLOM based. FrFT based FPSD was developed from combining the conventional PSD and the FrFT method. FPSD exhibits distinctive superiority to non-stationary signals. FrFT based fractional power spectrum is defined as

$$P_{\varepsilon\varepsilon}^\alpha(\mu) = \lim_{T \rightarrow \infty} \frac{E|\xi_{\alpha,T}(\mu)|^2}{2T}, \quad (2.44)$$

where $\xi_{\alpha,T}(\mu)$ is the α th FrFT of $\varepsilon_T(t)$, and $\varepsilon_T(t)$ is the truncation function in $[-T, T]$ of the sample function of the random process $\varepsilon(t)$.

FLOM based fractional power spectra include the covariation spectrum and the fractional low-order covariance spectrum [184].

Definition 2.5 The covariations spectrum is defined as [215]

$$\begin{aligned}\tilde{\phi}_{xx}^c(\omega) &= \mathcal{F}[R_{xx}^c(\tau)] = \int_{-\infty}^{\infty} R_{xx}^c(\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} [x(t), x(t-\tau)]_{\alpha} e^{-j\omega\tau} d\tau,\end{aligned}\quad (2.45)$$

where $[x(t), x(t-\tau)]_{\alpha}$ is the covariation defined as

$$[x(t), x(t-\tau)]_{\alpha} = \frac{E[x(t)(x(t-\tau))^{\langle p-1 \rangle}]}{E(|x(t-\tau)|^p)} \gamma_{x(t-\tau)}, \quad 1 \leq p < \alpha, \quad (2.46)$$

where γ_y is the scale parameter of Y , and $z^{\langle \alpha \rangle} = |z|^{\alpha} \text{sgn}(z)$.

Definition 2.6 Fractional low-order covariance spectrum is defined as

$$\tilde{\phi}_{xx}^d(\omega) = \mathcal{F}(R_{xx}^d(\tau)) = \int_{-\infty}^{\infty} R_{xx}^d(\tau) e^{-j\omega\tau} d\tau, \quad (2.47)$$

where

$$R_{xx}^d(\tau) = E[x(t)^{\langle A \rangle} x(t-\tau)^{\langle B \rangle}], \quad 0 \leq A < \frac{\alpha}{2}, \quad 0 \leq B < \frac{\alpha}{2}. \quad (2.48)$$

FLOM based fractional power spectrum techniques have been successfully used in time delay estimation [184].

Fractional Splines

Fractional B-splines can be considered as the generalization of the usual integer-order B-splines. There are three commonly used definitions of fractional B-splines, they are causal fractional B-splines, anti-causal fractional B-splines, and non-causal symmetric fractional B-splines [25, 222, 308]. Fractional causal B-splines are defined by taking the $(a+1)$ th fractional difference of the one-sided power function.

Definition 2.7 The fractional causal B-splines are specified in the Fourier domain

$$\hat{\beta}_+^{\alpha}(\omega) = \left(\frac{1 - e^{j\omega}}{j\omega} \right)^{\alpha+1}. \quad (2.49)$$

Definition 2.8 The anti-causal B-splines of degree α is defined in Fourier domain as

$$\hat{\beta}_-^{\alpha}(\omega) = \left(\frac{e^{j\omega} - 1}{j\omega} \right)^{\alpha+1}. \quad (2.50)$$

Definition 2.9 The non-causal symmetric fractional B-splines of degree α is defined in Fourier domain as

$$\hat{\beta}_*^\alpha(\omega) = \left| \frac{\sin \omega/2}{\omega/2} \right|^{\alpha+1}. \quad (2.51)$$

2.3 Chapter Summary

This chapter provides an overview of basic concepts of fractional processes and FOSP techniques from the perspective of fractional signals and fractional-order systems. Section 2.1 deals with the constant-order fractional-order processes and variable-order fractional processes. All these fractional processes can be generated by fractional-order systems driven by white Gaussian noise. Section 2.2 briefly introduced some FOSP techniques including fractional processes simulation, fractional filter, fractional systems modeling, analogue/digital realization of fractional systems, and other fractional tools. All discussions on FOSP techniques are centered around fractional calculus, FrFT and α -stable distribution. A detailed introduction of constant-order fractional processes and multifractional-processes will be provided in the following two chapters, respectively. The constant-order fractional signal processing techniques, variable-order fractional signal processing techniques and distributed-order filters will be introduced in Chaps. 5, 6 and 7, respectively.

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Techniques and Applications

Sheng, H.; Chen, Y.; Qiu, T.

2012, XXVI, 295 p., Hardcover

ISBN: 978-1-4471-2232-6