Chapter 2
Definition of Reset Control System and Basic Results

2.1 Preliminaries and Problem Setup

The main focus of the book will be on the use of single-input single-output reset compensators having as reset condition the classical condition of zero input. Therefore, the main goal will be to analyze and design reset control systems with linear and time invariant base systems. This chapter will be devoted to the definition of a reset control system, developing basic conditions for a system to be well-posed. In addition, an analysis of the dependence of zero crossing instants with respect to initial conditions will be given, showing the complex patterns that may result depending on the dimension of the after-reset surface.

Consider the feedback control system of Fig. 2.1, where the system P, with state $\mathbf{x}_p$, is described by

$$(P) \quad \begin{cases} \dot{\mathbf{x}}_p(t) = A_p\mathbf{x}_p(t) + B_p u(t), & \mathbf{x}_p(0) = \mathbf{x}_{p0}, \\ y(t) = C_p\mathbf{x}_p(t) \end{cases} \tag{2.1}$$

and the *reset* compensator R, with state $\mathbf{x}_r$, is modeled in principle by the impulsive differential equation

$$(R) \quad \begin{cases} \dot{\mathbf{x}}_r(t) = A_r\mathbf{x}_r(t) + B_r e(t) & \text{if } e(t) \neq 0, \\ \mathbf{x}_r(t^+) = A_\rho\mathbf{x}_r(t) & \text{if } e(t) = 0, \end{cases} \tag{2.2}$$

where $\mathbf{x}_r(0) = \mathbf{x}_{r0}$ and $v(t) = C_r\mathbf{x}_r(t)$. Here n_p is the dimension of the state $\mathbf{x}_p$, and n_r is the dimension of the state $\mathbf{x}_r$. In addition, $\mathbf{x}_r(t^+)$, or simply $\mathbf{x}_r^+$, is the value $\mathbf{x}_r(t+\varepsilon)$ with $\varepsilon \to 0^+$. A_ρ is a diagonal matrix with diagonal elements $(A_\rho)_{ii} = 0$ if the compensator state component $(\mathbf{x}_r)_i$ is to be reset, and $(A_\rho)_{ii} = 1$ otherwise, $i = 1, \ldots, n_r$. In general, it is assumed that the first $n_{\bar{\rho}}$ compensator state components are not reset, and the last n_ρ compensator states are reset or set to zero at the reset instants t in which the compensator input $e(t)$ is zero. Thus, $n_r = n_{\bar{\rho}} + n_\rho$ and A_ρ is given by $A_\rho = \text{diag}(I_{n_{\bar{\rho}}}, O_{n_\rho})$. For the particular case corresponding to $n_{\bar{\rho}} = 0$

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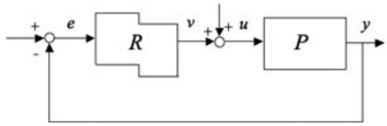

Fig. 2.1 Reset controller R applied to an LTI plant

and $n_r = n_\rho$, that is, if A_ρ is the zero matrix, the reset compensator will be referred to as *full reset*. Otherwise, it will be referred to as *partial reset*.

The closed-loop autonomous unforced system is given by $e(t) = -y(t)$, $u(t) = v(t)$, where by definition the closed-loop state of dimension $n = n_p + n_r$ is

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_p \\ \mathbf{x}_r \end{pmatrix}. \tag{2.3}$$

The result is the closed-loop system (without exogenous inputs)

$$\begin{cases} \dot{\mathbf{x}}(t) = A\mathbf{x}(t) & \text{if } \mathbf{x}(t) \notin \mathcal{M}, \\ \mathbf{x}(t^+) = A_R\mathbf{x}(t) & \text{if } \mathbf{x}(t) \in \mathcal{M} \end{cases} \tag{2.4}$$

with $\mathbf{x}(0) = \mathbf{x}_0$ and $y(t) = C\mathbf{x}(t)$, and where $\mathbf{x}_0 = \left(\begin{smallmatrix} \mathbf{x}_{p0} \\ \mathbf{x}_{r0} \end{smallmatrix}\right)$, $A = \left(\begin{smallmatrix} A_p & B_pC_r \\ -B_rC_p & A_r \end{smallmatrix}\right)$, $A_R = \operatorname{diag}(I_{n_p}, A_\rho) = \operatorname{diag}(I_{n_p}, (I_{n_{\bar\rho}}, O_{n_\rho}))$, and $C = (C_p\ 0)$.

Therefore, to complete the closed-loop system equations, the set $\mathcal{M}$, which will be referred to later as the reset surface, needs to be defined. Another set, the after-reset surface $\mathcal{M}_{\mathcal{R}}$, also plays an important role in the definition of closed-loop system solutions. Note that reset actions occur when the state $\mathbf{x}(t)$ contacts the reset surface $\mathcal{M}$ at some instant t, that is, $\mathbf{x}(t) \in \mathcal{M}$, and then the state jumps to $A_R\mathbf{x}(t) \in \mathcal{M}_{\mathcal{R}}$. In general, the set $\mathcal{M}_{\mathcal{R}}$ will be defined as

$$\mathcal{M}_{\mathcal{R}} = \mathcal{R}(A_R) \cap \mathcal{N}(C), \tag{2.5}$$

where $\mathcal{R}(X)$ and $\mathcal{N}(X)$ stand for the image and the null subspace of the linear operator given by the matrix X, respectively. Thus, $\mathcal{M}_{\mathcal{R}}$ is the set of states $\mathbf{x}$ that belong both to the null space of C (and then the output is $y = C\mathbf{x} = 0$) and to the image space of A_R (they are the after-reset states). In addition, the set $\mathcal{M}$ is defined as

$$\mathcal{M} = \mathcal{N}(C) \setminus \mathcal{M}_{\mathcal{R}} \tag{2.6}$$

where the after-reset states are removed from the reset surface. Otherwise, an infinite number of resets would be produced after a reset action; in general, the reset system (2.4) can exhibit rather complex behaviors that have been referred to as beating or pulse phenomena in the literature on impulsive systems [1, 10]. They are related to the fact that reset instants may not be well defined or may not be distinct; in addition, even if the reset instants are well defined and are distinct, they may converge to a finite number and then the reset system exhibits Zenoness. This topic will be analyzed in detail in Sect. 2.2.2; however, a common solution in practice to avoid these behaviors is to use time regularization.

A time regularization solution can be simply constructed adopting the scheme proposed in [12]: the system (2.4) is modified including a temporal restriction over the reset instants by simply avoiding resets if some minimum time between resets Δ_m has not passed. Thus, the closed-loop system will be described by

$$\begin{cases} \dot{\Delta}(t) = 1, & \dot{\mathbf{x}}(t) = A\mathbf{x}(t), & (\mathbf{x}(t) \notin \mathscr{M}) \vee (\Delta \leq \Delta_m), \\ \Delta(t^+) = 0, & \mathbf{x}(t^+) = A_R\mathbf{x}(t), & (\mathbf{x}(t) \in \mathscr{M}) \wedge (\Delta > \Delta_m) \end{cases} \tag{2.7}$$

with $\Delta(0) = 0$, $\mathbf{x}(0) = \mathbf{x}_0$, and $y(t) = C\mathbf{x}(t)$, and where the reset action is only performed if the state $\mathbf{x}(t)$ contacts the reset surface $\mathscr{M}$ at some reset instant t_j, $j = 1, 2, \ldots$, and in addition every reset interval $\Delta_k = t_k - t_{k-1}$ satisfies $\Delta_k > \Delta_m$, $k = 1, 2, \ldots$, where $\Delta_m > 0$ is some given constant.

2.1.1 Reset Control System Solutions

The reset control system given by (2.4) or (2.7) is a special class of a system with impulse effects, or an impulsive system. There is a large literature on impulsive systems [1, 5, 8, 10, 11, 14]. Most of the work has been done in the area of systems with impulses at fixed instants, or systems with impulses dependent on the state. The case in which reset and after-reset surfaces are time independent, which is usually referred to as autonomous impulsive systems, has attracted considerably less attention in spite of being relevant for engineering applications including control systems, this being the case of reset control systems. The framework developed in [8] will be (partly) adopted here. The LTI system described by the first equation in (2.4) will be referred to as the *linear base system*, or simply the *base system*, while the second equation in (2.4) will be referred to as the *resetting law*.

Let $\mathscr{D} \subset \mathbb{R}^n$ be an initial conditions set, and $\mathscr{I}_{\mathbf{x}_0} \subseteq [0, \infty)$, with $\mathbf{x}_0 \in \mathbb{R}^n$, a dense subset of $[0, \infty)$ such that $\mathscr{J}_{\mathbf{x}_0} = [0, \infty) \setminus \mathscr{I}_{\mathbf{x}_0}$ is a countable set with a finite or infinite number of elements; in fact, this set will be the set of reset times corresponding to the initial condition $\mathbf{x}_0$. In general, an initial condition $\mathbf{x}_0$ may produce a finite or infinite number of resets, depending on whether the set $\mathscr{J}_{\mathbf{x}_0}$ is finite or infinite. A function $\mathbf{x} : [0, \infty) \times \mathscr{D} \to \mathbb{R}^n$ is a *solution* to the reset control system (2.4) if the following conditions are satisfied:

1. $\mathbf{x}(\cdot, \mathbf{x}_0)$ is left-continuous in t, that is, $\lim_{\tau \to t^-} \mathbf{x}(\tau, \mathbf{x}_0) = \mathbf{x}(t, \mathbf{x}_0)$ for all $\mathbf{x}_0 \in \mathscr{D}$ and $t \in (0, \infty)$.
2. $\mathbf{x}(\cdot, \mathbf{x}_0)$ is differentiable in t, and $\frac{d\mathbf{x}(t,\mathbf{x}_0)}{dt} = A\mathbf{x}(t, \mathbf{x}_0)$, for all $t \in \mathscr{I}_{\mathbf{x}_0}$.
3. $\mathbf{x}(t^+, \mathbf{x}_0) = A_R\mathbf{x}(t, \mathbf{x}_0)$, for all $t \in \mathscr{J}_{\mathbf{x}_0}$.

In addition, functions $\tau_k : \mathscr{D} \to [0, \infty)$, $k = 1, 2, \ldots$, are defined such that $\tau_k(\mathbf{x}_0)$ is the kth reset instant of the solution $\mathbf{x}(\cdot, \mathbf{x}_0)$. Therefore, functions $\Delta_k : \mathbb{R}^n \to [0, \infty)$, $k = 1, 2, \ldots$, are defined such that $\Delta_k(x_0) = \tau_k(\mathbf{x}_0) - \tau_{k-1}(\mathbf{x}_0)$ is the kth reset interval, with $\tau_0(\mathbf{x}_0) = 0$ for any $\mathbf{x}_0 \in \mathbb{R}^n$. The following result is directly adapted from [1] with minimal effort.

Proposition 2.1 *For any initial condition* $\mathbf{x}_0 \in \mathscr{D}$, *assume that* $\tau_1(\mathbf{x}_0) < \tau_2(\mathbf{x}_0) < \cdots < \tau_k(\mathbf{x}_0) < \cdots$, *and* $\tau_k(\mathbf{x}_0) \to \infty$ *as* $k \to \infty$, *then there exists a unique solution* $\mathbf{x}(\cdot, \mathbf{x}_0)$ ($\mathbf{x}(\cdot)$ *in short*) *to the reset control system* (2.4) *that can be written in the form* $\mathbf{x}(t, \mathbf{x}_0) = W(t, \mathbf{x}_0)\mathbf{x}_0$ *for* $t > 0$, *where the transition matrix* $W(t, \mathbf{x}_0)$ *is given by*

$$W(t, \mathbf{x}_0) = e^{A(t-\tau_k(\mathbf{x}_0))} A_R e^{A(\tau_k(\mathbf{x}_0)-\tau_{k-1}(\mathbf{x}_0))} A_R \cdots A_R e^{A\tau_1(\mathbf{x}_0)} \tag{2.8}$$

for $t \in (\tau_k(\mathbf{x}_0), \tau_{k+1}(\mathbf{x}_0)]$.

Proof Since reset instants are directly determined by the initial condition, for a given initial condition $x_0 \in \mathscr{D}$ and its reset instants $t_k = \tau_k(\mathbf{x}_0), k = 1, 2, \ldots$, the system (2.4) is an impulsive system with impulses at fixed instants $t_1, t_2, \ldots$, where by assumption $0 = t_0 < t_1 < t_2 < \cdots$. A direct application of Theorem 3.6 and Corollary 3.2 in [1] gives the result. □

Note that in general if the formulation (2.7) for reset system is used then the reset intervals are lower-bounded by the time regularization constant Δ_m, that is, $\Delta_k(\mathbf{x}_0) > \Delta_m$, $k = 1, 2, \ldots$ for any $\mathbf{x}_0 \in \mathbb{R}$, and thus the assumptions of Proposition 2.1 are clearly satisfied. If time regularization is not performed, deadlock and beating could be present if $\tau_{k+1}(\mathbf{x}_0) = \tau_k(\mathbf{x}_0)$ for some initial condition $\mathbf{x}_0$ and some k; or even a Zeno solution may be obtained if $\tau_k(\mathbf{x}_0) < \infty$ for $k \to \infty$.

Alternatively, if the initial instant is $t_0 \neq 0$ and the initial condition is $\mathbf{x}(t_0) = \mathbf{x}_0$, then a solution $\mathbf{x}(t, t_0, x_0)$ is defined in a similar way.

In Sect. 2.2, a detailed analysis of beating, deadlock, and Zeno solutions in the reset system (2.4) will be given. It will be shown that under mild conditions it does not have these behaviors and that time regularization is not necessary to have well-defined solutions for forward time.

2.1.2 Characterization of Reset Intervals

An important question that has not been previously approached in the reset control literature is the analysis of reset instants for a given reset control system. For example, given the base system of (2.4), it is not evident if every initial condition will cross the reset surface $\mathscr{M}$ at some finite instant. Usually, since at least a reset action is wanted for initial conditions in some set $\mathscr{D}$, a common practice is to design the base closed-loop system to have a pair of dominant complex poles, and thus to force crossings after some arbitrarily large time.

In the following, it will be shown that this practice is, in fact, theoretically supported, and in addition an upper bound over resets intervals will be computed. In general, the problem of computing crossings of the solution of a linear system governed by $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, with an initial condition $\mathbf{x}_0 \in \mathbb{R}$, with a given hyperplane, is a particular instance of the reachability problem for linear systems. In general, for arbitrary values of the state matrix A, it has been shown to be an open problem, referred to as the continuous Skolem–Pisot problem [4, 9]. As discussed above, this is a central problem in reset control where a base system has to be designed for the reset system to perform crossings with a reset surface.

For the analysis of the crossings with the hyperplane defined by the row vector C, that is, the instants $t > 0$ at which $Ce^{At}\mathbf{x}_0 = 0$, it is convenient to use the equation $Ce^{(A+\lambda I)t}\mathbf{x}_0 = 0$ for some $\lambda \in \mathbb{R}$, obtaining the same results [4]. This is a simple, but key simplifying result because all the eigenvalues of A can be assumed to have non-positive real parts without loss of generality, for some λ properly chosen. In addition, it has also been shown [4] that if for some initial condition $\mathbf{x}_0$ the matrix A does not have dominant real eigenvalues and $(A, C, \mathbf{x}_0)$ is reduced, the continuous Skolem–Pisot problem always has a solution for that initial condition.

By definition, an eigenvalue of A is dominant if it is the rightmost placed eigenvalue of A in the complex plane. In addition, the term $Ce^{At}\mathbf{x}_0$ can always be split as $Ce^{At}\mathbf{x}_0 = y_1(t) + y_2(t)$, where by definition the dominant term $y_1(t)$ is not identically zero if $(C, A, \mathbf{x}_0)$ is reduced, and in addition $y_2(t)$ tends to zero exponentially fast as t increases. This means that a reset control system with such a matrix A will always produce crossings for the initial condition $\mathbf{x}_0$. Note that the set of states $\mathbf{x}_0$ for which (A, C, x_0) is not reduced has a linear subspace structure, and will be referred to as $\bar{\mathscr{R}}$. Note that if the modes corresponding to dominant eigenvalues are unobservable then $(A, C, \mathbf{x}_0)$ is not reduced for any $\mathbf{x}_0 \in \mathbb{R}^n$, and thus $\bar{\mathscr{R}} = \mathbb{R}^n$, but in the case they are observable, $(A, C, \mathbf{x}_0)$ may not be reduced for some initial condition $\mathbf{x}_0 \in \mathbb{R}^n$, and in general $\bar{\mathscr{R}}$ is not the empty set.

In general, if the matrix $A \in \mathbb{R}^{n\times n}$ has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$ such that $k_i = \text{index}(\lambda_i)$, then it can be expressed as

$$A = (P_1 \ \dots \ P_s\,)J \begin{pmatrix} Q_1 \\ \vdots \\ Q_s \end{pmatrix} = \sum_{i=0}^{s} P_i J(\lambda_i) Q_i, \tag{2.9}$$

where J is the Jordan form, and $J(\lambda_i)$ the Jordan segment associated to the eigenvalue λ_i. In addition, for a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\lambda_i)$, $f'(\lambda_i)$, $\dots$, $f^{(k_i-1)}(\lambda_i)$ exist for each $i = 1, \dots, s$, the value of $f(A)$ can be determined by using a generalization of the spectral theorem for non-diagonalizable matrices

$$f(A) = \sum_{i=1}^{s}\sum_{j=0}^{k_i-1} \frac{f^{(j)}(\lambda_i)}{j!}(A - \lambda_i I)^j G_i, \tag{2.10}$$

where $G_i = P_i Q_i$ is the spectral projector corresponding to the eigenvalue λ_i.

In the case in which the matrix A is diagonalizable, the expression of $f(A)$ is much simpler since $\text{index}(\lambda_i) = 1$, for $i = 1, \dots, s$. In this case,

$$f(A) = \sum_{i=1}^{s} f(\lambda_i) G_i, \tag{2.11}$$

where in addition the spectral projectors are simply given by $G_i = \mathbf{v}_i \mathbf{w}_i^T$, with $\mathbf{v}_i$ and $\mathbf{v}_j$ being the right-hand eigenvector and the left-hand eigenvector corresponding to the eigenvalue λ_i, respectively.

In the following, it will be shown how a reset control system produces an infinite number of resets, and that the reset intervals are upper-bounded, if the base system does not have real dominant eigenvalues.

Proposition 2.2 *Consider the reset control system* (2.4), *where the matrix A does not have real dominant eigenvalues, and a nonempty closed set of initial conditions $\mathscr{D} \subset \mathbb{R}^n \setminus \bar{\mathscr{R}}$, then reset intervals are uniformly upper-bounded, that is, $\Delta_k(x_0) = \tau_{k+1}(x_0) - \tau_k(x_0) < \Delta_M$, $k = 1, 2, \ldots,$ for any $x_0 \in \mathscr{D}$ and some finite constant $\Delta_M > 0$.*

Proof It is assumed, without loss of generality, that A has dominant imaginary eigenvalues. The case in which A has a pair of dominant eigenvalues will be considered in the following (with multiplicity not necessarily equal to one), the more general case of multiple dominant eigenvalues is a bit more involved but follows a similar reasoning. The eigenvalues of A are ordered in such a way that its spectrum is $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_s\} = \{i\beta_1, -i\beta_1, \alpha_3 + i\beta_3, \ldots, \alpha_s + i\beta_s\}$, and $\alpha_l < 0, l = 3, \ldots, s$. In addition, let $k_l, l = 1, \ldots, s$, be the index of each eigenvalue. Then, the output $y(t)$ of the reset control system can be expressed (using (2.9) and (2.10) for computing e^{At}) as

$$y(t) = \sum_{l=1}^{s} \sum_{j=0}^{k_l-1} \frac{t^j e^{\lambda_l t}}{j!} C(A - \lambda_l I)^j G_l \mathbf{x}_0, \tag{2.12}$$

where G_l is the spectral projector associated to the eigenvalue λ_l. It can be split in two parts as $y(t) = y_1(t) + y_2(t)$, where

$$y_1(t) = \sum_{l=1}^{2} \sum_{j=0}^{k_l-1} \frac{t^j e^{i\beta_l t}}{j!} C(A - i\beta_l I)^j G_l \mathbf{x}_0 \tag{2.13}$$

and

$$y_2(t) = \sum_{l=3}^{s} \sum_{j=0}^{k_l-1} \frac{t^j e^{\alpha_l t} e^{i\beta_l t}}{j!} C(A - (\alpha_l + i\beta_l)I)^j G_l \mathbf{x}_0. \tag{2.14}$$

Here $y_2(t)$ tends to zero exponentially fast as t increases since $\alpha_l < 0, l = n + 1, \ldots, s$, and thus it is clear that for any $\varepsilon_2 > 0$ it is always possible to choose an instant t_2 large enough such that $|y_2(t)| < \varepsilon_2 \|x_0\|$, for $t \geq t_2$.

In addition, defining constant matrices $Z_{lj} = \sum_{j=0}^{k_l-1} \frac{(A - i\beta_l I)^j G_l}{j!} \in \mathbb{C}$ and matrix polynomials $P_l(t) = (\sum_{j=0}^{k_l-1} Z_{lj} t^j)$, the dominant term of the output $y_1(t)$ can be expressed as

$$\begin{aligned} y_1(t) &= C\big(P_1(t)e^{i\beta_l t} + P_2(t)e^{-i\beta_l t}\big)x_0 \\ &= C(P_{1,R}(t)\cos(\beta_1 t) + P_{1,I}(t)\sin(\beta_1 t))\mathbf{x}_0, \end{aligned} \tag{2.15}$$

where $P_2(t) = \bar{P}_1(t)$, $P_{1,R} = 2\,\mathrm{Real}\{P_1\}$, and $P_{1,I} = 2\,\mathrm{Imag}\{P_1\}$. Now, for instants $t_m = m\frac{2\pi}{\beta_1}$, with $m \in \mathbb{N}$, the value of the dominant term is given simply by $y_1(t_m) = CP_{1,R}(t_m)\mathbf{x}_0$, and since $(C, A, \mathbf{x}_0)$ is reduced for any $x_0 \in \mathscr{D}$, and $\mathscr{D}$ is closed, there must exist some lower bound $\varepsilon_1 > 0$ such that $|y_1(t_m)| > \varepsilon_1 \|x_0\|$, for any m and $x_0 \in \mathscr{D}$. This can be shown by contradiction, if it were false then it would exist a sequence of states $\{x_{01}, x_{02}, \dots\} \to x_0$ such as for any $t > 0$, $\|CP_{1,R}(t)x_{0n}\| \to 0$ as $n \to \infty$. Thus, since $\mathscr{D}$ is closed it must contain a state x_0 such as (C, A, x_0) is not reduced, which is a contradiction. Also note that for an m_1 large enough, the sign of $y_1(t_m)$ is constant for $m > m_1$ (for example, choosing t_{m_1} as the Cauchy's bound of the roots of the polynomial $CP_{1,R}(t)\mathbf{x}_0$). It will be assumed that $y_1(t_{m_1}) > 0$, otherwise $t_m = m\frac{\pi}{\beta}$, $m \in \mathbb{N}$, may be chosen. Therefore, it is possible to find a large enough instant t_2 such as $\varepsilon_2 < \varepsilon_1$ and thus $|\frac{y_2(t_m)}{y_1(t_m)}| < \frac{\varepsilon_2\|\mathbf{x}_0\|}{\varepsilon_1\|\mathbf{x}_0\|} = \frac{\varepsilon_2}{\varepsilon_1} < 1$, and then it is true that $y(t_m) = y_1(t_m)(1 + \frac{y_2(t_m)}{y_1(t_m)}) > 0$, provided that $t_m > \max\{t_{m_1}, t_2\}$. A similar reasoning may be applied to show that $y(t_n) < 0$, with $t_n = n\frac{\pi}{\beta_1}$, for some $t_n > \max\{t_{n_1}, t_2\}$, where n_1 is computed in a similar way. As a result, the output of the reset system $y(t) = Ce^{At}\mathbf{x}_0$, for some initial condition $\mathbf{x}_0 \in \mathbb{R}^n$, is equal to 0 for $t < t_m$, with the upper bound t_m being independent of the initial condition $\|\mathbf{x}_0\|$. Then, since the reset action is only active for $t > \Delta_m$, $\Delta_1(\mathbf{x}_0) = \tau_1(\mathbf{x}_0) < \Delta_m + t_m =: \Delta_M$. Obviously, the rest of the reset intervals are also upper-bounded by Δ_M, and thus the proof is complete. □

2.2 Zenoness, Beating, and Deadlock

In control practice, it is required that the reset control system solutions $\mathbf{x}(t, \mathbf{x}_0)$ be well defined in the sense that they exist and are unique for any $t > 0$, as given in Proposition 2.1. As it has been previously discussed, this may be complicated by the fact that these solutions may exhibit complex phenomena such as non-continuability of solutions or *deadlock*, *beating* or *livelock*, and *Zenoness*. These concepts will be used as defined in [8]. Deadlock occurs when the state $\mathbf{x}(t)$ cannot evolve in time because no continuation, continuous or discrete, is possible. To avoid deadlock, a typical assumption (adapted from [8]) is that

(A1) $\forall \mathbf{x}(t) \in \mathscr{M}_{\mathscr{R}}$, $\exists \varepsilon > 0$ such that $\forall \delta \in (0, \varepsilon)$ $\mathbf{x}(t + \delta) \notin \mathscr{M}$,

that is, the after-reset states evolve with the continuous base dynamics for some finite time interval. Beating appears when the system solution encounters the reset surface after resetting, which in our case is simply avoided by assuming that

(A2) $\mathscr{M}_{\mathscr{R}} \cap \mathscr{M} = \emptyset$,

that is, assuming that the after-reset states are not elements of the reset surface. Finally, a Zeno solution exists if a system solution has infinitely many reset actions in a finite time.

By definition, the reset control system (2.4), with sets $\mathscr{M}_{\mathscr{R}}$ and $\mathscr{M}$, is well-posed if it satisfies conditions (A1) and (A2) (and thus it does not exhibit beating or deadlock, but Zeno solutions may exist in principle). Note that in the reset control system (2.4), the reset action occurs at the instants t at which the output $y(t)$ is zero. The set $\mathscr{M}_{\mathscr{R}}$ is defined as in (2.5), that is, as the set of states that belong both to the null space of C (and then the output is $y = Cx = 0$) and to the image space of A_R (they are the after reset states). In addition, the set $\mathscr{M}$ will be defined as in (2.6), where the points that are the after-reset states are removed from the reset surface to satisfy condition (A2), otherwise an infinite number of resets may be produced after a reset action, and beating would be present.

2.2.1 Well-posedness: Beating and Deadlock

To have well-defined solutions to the reset systems as given in last section, reset instants have to be well defined and distinct. In general, two phenomena that have to be avoided are deadlock (non-continuability of solutions), and beating or live-lock. Another important type of solutions like Zeno solutions will be treated in next section. In general, a reset control system as defined by (2.4)–(2.6) does not exhibit beating, once the surfaces $\mathscr{M}_{\mathscr{R}}$ and $\mathscr{M}$ are defined according to (2.5) and (2.6), respectively. On the other hand, additional assumptions have to be made to avoid deadlock.

Proposition 2.3 *The reset control system* (2.4)–(2.6) *is well-posed if the after-reset surface $\mathscr{M}_{\mathscr{R}}$ is a subset of the observable subspace of the linear base system, that is,*

$$\mathscr{M}_{\mathscr{R}} \cap \mathscr{N}(\mathscr{O}_{base}) = \{\mathbf{0}\} \tag{2.16}$$

where

$$\mathscr{O}_{base} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

Proof Beating is avoided by defining the after-reset surface $\mathscr{M}_R$ by (2.6). Thus, the proof is centered around deadlock. In general, given any initial condition $\mathbf{x}_0 \in \mathscr{D}$, the reset surface $\mathscr{M}$ is first contacted at the instant $t_1 = \tau_1(\mathbf{x}_0)$. Then the reset instant t_1 is simply given by $t_1 = \inf\{t > 0 | \mathbf{x}(t, \mathbf{x}_0) \in \mathscr{M}\}$, and thus is a solution of the equation $Ce^{At_1}\mathbf{x}_0 = 0$. In addition, it must be true that $e^{At_1}\mathbf{x}_0 \notin \mathscr{R}(A_R)$ according to the definitions of the after-reset and reset surfaces as given by (2.5) and (2.6), respectively. For simplicity, consider first that the closed-loop state matrix A has distinct eigenvalues, then the matrix exponential may be computed by using the Caley–Hamilton method, that is,

$$e^{At_1} = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}, \tag{2.17}$$

where α_i, $i = 0, \ldots, n-1$, are given by

$$\begin{cases} e^{\lambda_1 t_1} = \alpha_0 + \alpha_1 \lambda_1 + \cdots + \alpha_{n-1} \lambda_1^{n-1}, \\ e^{\lambda_2 t_1} = \alpha_0 + \alpha_1 \lambda_2 + \cdots + \alpha_{n-1} \lambda_2^{n-1}, \\ \quad \ldots \\ e^{\lambda_n t_1} = \alpha_0 + \alpha_1 \lambda_n + \cdots + \alpha_{n-1} \lambda_n^{n-1}. \end{cases} \tag{2.18}$$

Using the vector notation $\boldsymbol{\lambda}^T = (\lambda_1\ \lambda_2\ \ldots\ \lambda_n)$, $\boldsymbol{\alpha}^T = (\alpha_0\ \alpha_1\ \ldots\ \alpha_{n-1})$ and $e^{\boldsymbol{\lambda} t_1} = \sum_{i=1}^n e^{\lambda_i t_i} \mathbf{e}_i$, where $\mathbf{e}_i$ stands for the unit vector $(0\ \ldots\ 0\ 1\ 0\ \ldots\ 0)^T$ in which the ith component is 1, (2.18) can be compactly written as

$$e^{\boldsymbol{\lambda} t_1} = V(\boldsymbol{\lambda})^T \boldsymbol{\alpha}, \tag{2.19}$$

where $V(\boldsymbol{\lambda})$ is a (nonsingular) Vandermonde matrix. Now, by eliminating $\boldsymbol{\alpha}$ from (2.17) and (2.19) the equation $0 = Ce^{At_1}\mathbf{x}_0$ is transformed into

$$0 = \boldsymbol{\alpha}^T \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \mathbf{x}_0 = e^{\boldsymbol{\lambda} t_1 T} U(\boldsymbol{\lambda}) \mathscr{O}_{base} \mathbf{x}_0, \tag{2.20}$$

where $U(\boldsymbol{\lambda}) = V(\boldsymbol{\lambda})^{-1}$. Now, if condition (2.16) is satisfied, then the right-hand side of (2.20) is an analytical function (in fact, a sum of exponentials) that is not zero for all $t \geq 0$. As a result, it has no isolated zeros, and then the reset instant t_1 is lower-bounded. In other words, solutions of (2.20) may not be obtained for arbitrarily small values of the reset instant t_1, and thus deadlock does not occur if condition (2.16) is satisfied.

In the case in which the eigenvalues of A may be repeated, a similar argument may be applied. Note that e^{At_1} may be written as the infinite series $D(A) = \sum_{i=0}^{\infty} \frac{A^i t_1^i}{i!}$. Thus the polynomial $D(\lambda) = \sum_{i=0}^{\infty} \frac{\lambda^i t_1^i}{i!}$ can be factorized as $D(\lambda) = Q(\lambda)P(\lambda) + R(\lambda)$, with $R(\lambda) = 0$, or $\deg(R) < \deg(P) = n$. In addition, R has degree no greater than $n-1$, and thus $R(\lambda) = \sum_{j=0}^{n-1} \alpha_j \lambda^j$. Since the characteristic polynomial is zero for the eigenvalues of A, $D(\lambda_k) = R(\lambda_k)$ for $k = 0, 1, \ldots, n-1$. And then $D(\lambda_k) = \sum_{i=0}^{\infty} \frac{\lambda_k^i t_1^i}{i!} = e^{\lambda_k t_1} = R(\lambda_k) = \sum_{j=0}^{n-1} \alpha_j \lambda_k^j$ for $k = 0, 1, \ldots, n-1$. This can be compactly expressed as $V^T(\boldsymbol{\lambda})\boldsymbol{\alpha} = e^{\boldsymbol{\lambda} t_1}$, and the expression (2.19) is obtained.

Now, if A has r different eigenvalues with respective multiplicity order n_i, and as a consequence the characteristic polynomial is $p(\lambda) = \prod_{i=1}^r (\lambda - \lambda_i)^{n_i}$, then again there exist unique polynomials Q and R such as $D(\lambda) = Q(\lambda)P(\lambda) + R(\lambda)$ where $D(\lambda) = e^{\lambda t_1}$ and $R = 0$ or $\deg(R) < \deg(P)$. Here R can be expressed as $R(\lambda) = \sum_{i=0}^{n-1} \alpha_i \lambda^i$, where the coefficients are unique. Since p and its derivatives up to order

n_r are zero at λ_i,

$$\frac{d^j D(\lambda_i)}{d\lambda^j} = \frac{d^j R(\lambda_i)}{d\lambda^j} \quad \forall i = 1, 2, \ldots, r, \ \forall j = 0, 1, \ldots, n_i - 1. \tag{2.21}$$

This can be expressed by $\boldsymbol{\mu} = W\boldsymbol{\alpha}$, where

$$\boldsymbol{\mu} = \sum_{i=1}^{r} \sum_{j=0}^{n_i-1} \mathbf{e}_i e^{\lambda_i t_1} \otimes \mathbf{e}_j \lambda_i^j, \tag{2.22}$$

$$W = \left(\sum_{i=1}^{r} \sum_{j=0}^{n_i-1} \mathbf{e}_i \otimes \mathbf{e}_j \mathbf{e}_i^T \frac{\partial^j V(\boldsymbol{\lambda})}{\partial \lambda_i^j} \right). \tag{2.23}$$

By using arguments based on the Lagrange–Hermite interpolation problem, it can be shown that, in fact, the matrix W is invertible. And then an expression similar to (2.20) may be obtained. Using similar arguments as those after (2.20), the proposition is proved for the general case of repeated eigenvalues. □

Obviously, a reset control system will be well-posed if the base linear system is observable. But some unobservable base linear systems can also define well-posed reset control systems. Therefore, note that in the proof of Proposition 2.3 no particular structure of the matrices A, C, and A_R has been used. Thus, the result is in general valid for any reset system given by (2.4) with arbitrary values of those matrices. In the following, two examples corresponding to an ill-posed (not well-posed) reset system and a well-posed reset control system are given.

2.2.1.1 Example (Ill-posed Reset System)

This example is used in [12] for analyzing some weak points in the definition of reset systems given in [3]. Consider a reset system (2.4) with the following system matrices

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 \ 0 \ 0 \end{pmatrix}, \tag{2.24}$$

where in addition the sets $\mathcal{M}_{\mathcal{R}}$ and $\mathcal{M}$ are defined according to (2.5) and (2.6) as

$$\mathcal{M}_{\mathcal{R}} = \mathcal{R}(A_R) \cap \mathcal{N}(C) = \text{span}\{(0, 1, 0)^T\},$$

and

$$\mathcal{M} = \mathcal{N}(C) \setminus \mathcal{M}_{\mathcal{R}} = \text{span}\{(0, 1, 0)^T, (0, 0, 1)^T\} \setminus \text{span}\{(0, 1, 0)^T\}.$$

Note that this reset system cannot be realized as a control reset system with the structure of Fig. 2.1.

In [12], it is correctly argued that for any initial condition $\mathbf{x}_0 = (0, a, 0)^T \in \mathscr{M}_{\mathscr{R}}$, the solution is ill-defined since the continuous dynamics makes the system instantly evolve to the set $\mathscr{M}_{\mathscr{R}}$ and then the system instantly resets once the reset surface is reached, making infinitely many resets without leaving the reset surface; in fact, there is deadlock. Note that this is due to the fact that the after-reset surface $\mathscr{M}_{\mathscr{R}}$ is a subset of the unobservable subspace of the linear base system, which is given in this case by

$$\mathscr{N}\left(\begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}\right) = \mathscr{N}\left(\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right) = \mathscr{N}(C) \supset \mathscr{M}_{\mathscr{R}}. \tag{2.25}$$

2.2.1.2 Example (Well-posed Reset Control System)

This example, adapted from [3], shows how an unobservable linear base system may define a well-posed reset system, as long as the unobservable subspace does not contain after-reset states. Consider a reset control system (2.4)–(2.6) with

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -0.2 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \qquad A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 \; 1 \; 0 \end{pmatrix} \tag{2.26}$$

that has an unobservable mode corresponding to a stable pole–zero cancellation in the linear base system, where the plant has a transfer function $P(s) = \frac{s+1}{s(s+0.2)}$, and the base compensator is $C(s) = \frac{1}{s+1}$ (corresponding to a first order reset element—FORE). In addition, the after-reset and reset surfaces are given by $\mathscr{M}_{\mathscr{R}} = \mathscr{R}(A_R) \cap \mathscr{N}(C) = \text{span}\{(1, 0, 0)^T\}$ and $\mathscr{M} = \mathscr{N}(C) \setminus \mathscr{M}_{\mathscr{R}} = \text{span}\{(1, 0, 0)^T, (0, 0, 1)^T\} \setminus \text{span}\{(1, 0, 0)^T\}$, respectively. In this case, the set $\mathscr{M}_{\mathscr{R}}$ is not a subset of the linear base system unobservable subspace given by

$$\mathscr{N}\left(\begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}\right) = \mathscr{N}\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & -0.2 & 1 \\ -0.2 & -0.96 & -0.2 \end{pmatrix}\right) = \text{span}\left\{(1, 0, -1)^T\right\}. \tag{2.27}$$

As a result, Proposition 2.3 may be used to ensure that the system is well-posed. Figure 2.2 shows the system solutions corresponding to two initial conditions.

2.2.2 Zeno Solutions

In principle, the reset control system (2.4) may exhibit Zeno solutions even in the case where it is well-posed (assuming that condition (2.16) is satisfied). Zeno solutions are solutions to (2.4) that have an infinite number of jumps in a compact time interval. However, as it will be shown in the following, condition (2.16) is sufficient to avoid the existence of Zeno solutions.

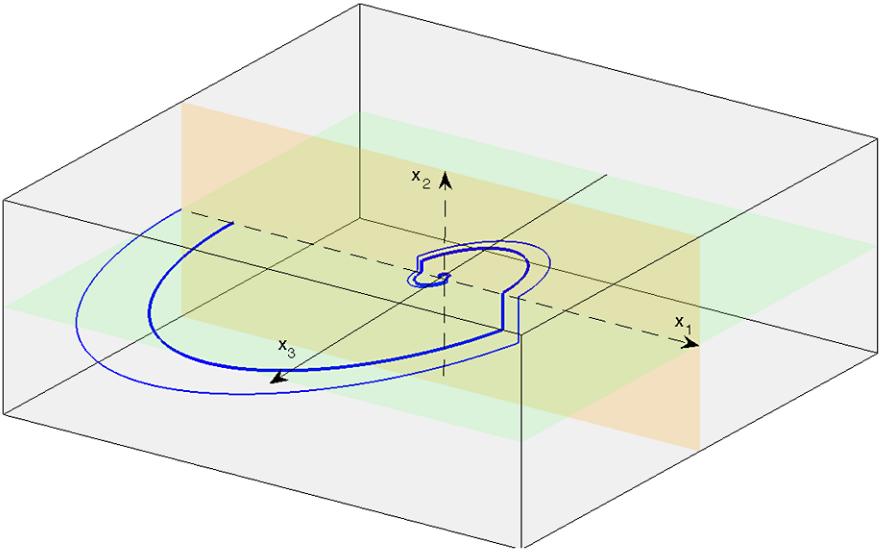

Fig. 2.2 System solution for the well-posed reset system example

Proposition 2.4 *The reset system* (2.4)–(2.6) *does not have Zeno solutions if* $\mathscr{M}_{\mathscr{R}} \cap \mathscr{N}(\mathscr{O}_{base}) = \{\mathbf{0}\}$.

Proof The basic idea of the proof consists of showing that the reset system (2.4)–(2.6), with an initial condition in $\mathscr{M}_{\mathscr{R}}$, can only have finite sequences of reset intervals $\Delta_k, k = 1, 2, \ldots, m-1$ such as $\Delta_{m-1} < \Delta_{m-2} < \cdots < \Delta_1 = \varepsilon$, for some $\varepsilon > 0$ arbitrarily small but fixed, and some finite positive integer m. In fact, in the following it will be shown that at most there will be sequences of length $m-1$, with m being the dimension of the after-reset surface $\mathscr{M}_{\mathscr{R}}$.

Without loss of generality, it is considered that the plant state equations (2.1) are given in an observer canonical form, that is,

$$A_p = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n_p-1} \end{pmatrix}, \qquad B_p = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n_p-1} \end{pmatrix}, \qquad C_p = \begin{pmatrix} 0\,0 \ldots 1 \end{pmatrix}, \tag{2.28}$$

then $C = (0, 0, \ldots, 1, 0, \ldots, 0)$ and thus

$$\mathscr{O}_{base} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n_p-1} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \cdots & 0 & 1 & -a_{n_p-1} & X & \ldots & X \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -a_{n_p-1} & \cdots & X & X & X & X & \cdots & X \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}, \tag{2.29}$$

where X stands for a non (necessarily) zero term.

For simplicity, the case of full reset is approached at first. Thus, an after-reset state $\mathbf{x} \in \mathscr{M}_{\mathscr{R}}$ is given by

$$\mathbf{x} = (x_1, x_2, \ldots, x_{n_p-2}, x_{n_p-1}, 0, 0, \ldots, 0)^T \tag{2.30}$$

for some values $x_1, \ldots, x_{n_p-1} \in \mathbf{R}$, with n_p being the number of plant states. Thus, $m = n_p - 1$ in the case of full reset.

Let us start with the case $m = 1$, which corresponds to second order plants and full reset compensators of arbitrary order. In this case, if the reset control system is well-posed, it is well known that the reset is periodic after reaching the set $\mathscr{M}_{\mathscr{R}}$ from any initial condition; thus starting from $\mathscr{M}_{\mathscr{R}}$, the reset will be periodic and Zeno solutions are not possible. In fact, there exists no initial condition in the set $\mathscr{M}_{\mathscr{R}}$ that contacts $\mathscr{M}$ in an arbitrarily small time since reset intervals are constant.

The case $m = 2$ is analyzed in the following. Consider an initial condition $\mathbf{x}_0 = \mathbf{x}^1 \in \mathscr{M}_{\mathscr{R}}$, that is, $\mathbf{x}^1 = (x_1, x_2, 0, 0, \ldots, 0)^T$. If the solution $\mathbf{x}(t, 0, \mathbf{x}^1)$ contacts the reset surface $\mathscr{M}_{\mathscr{R}}$ at time $t_1 = \varepsilon_1$, thus $\Delta_1 = \varepsilon_1$, for some $\varepsilon_1 > 0$ arbitrarily small, then

$$0 = Ce^{A\varepsilon_1}\mathbf{x}^1 = C\mathbf{x}^1 + \varepsilon_1 CA\mathbf{x}^1 + \frac{\varepsilon_1^2}{2}CA^2\mathbf{x}^1 + \cdots . \tag{2.31}$$

Since the control system (2.4)–(2.6) is well-posed, the right-hand side of (2.31) is not identically zero for any $\mathbf{x}^1 \in \mathscr{M}_R$. Now using the special structure given in (2.29), one obtains

$$0 = x_2 + \frac{\varepsilon_1}{2}x_1 + O\big(\varepsilon_1^2\big), \tag{2.32}$$

where the terms of order ε_1^2 and higher maybe be neglected, in principle. Note that for (2.32) to be satisfied for an arbitrarily small $\varepsilon_1 > 0$ it must be true that $x_1 \neq 0$ and $x_2 \neq 0$.

In addition, the following after-reset state $\mathbf{x}^2$ is given by $\mathbf{x}^2 = A_R\mathbf{x}(t_1, 0, \mathbf{x}^1) = (x_1 + O(\varepsilon_1^2), x_2 + \varepsilon_1 x_1 + O(\varepsilon_1^2), 0, 0, \ldots, 0)^T$. Repeating the argument, the solution $\mathbf{x}(t, t_1, \mathbf{x}^2)$ will again contact $\mathscr{M}$ at the instant $t_2 = t_1 + \Delta_2$. If $\Delta_2 = \varepsilon_2 \leq \varepsilon_1$ for some $\varepsilon_2 > 0$, then it is verified that

$$0 = x_2 + \varepsilon_1 x_1 + \frac{\varepsilon_2}{2}x_1 + O\big(\varepsilon_1^2\big), \tag{2.33}$$

where the properties $O(\varepsilon_2^2) = O(\varepsilon_1^2)$ for $\varepsilon_2 \leq \varepsilon_1$ and $O(k\varepsilon) = O(\varepsilon)$, for a real constant k, have been used. Now, using (2.32) and (2.33), the result is that given some $\varepsilon_1 > 0$ arbitrarily small, $\varepsilon_2 = -\varepsilon_1 + O(\varepsilon_1^2) < 0$, which is absurd. Thus, by contradiction it is true that $\varepsilon_2 > \varepsilon_1$, and thus any initial condition in the set $\mathscr{M}_{\mathscr{R}}$ that produces a first reset interval $\varepsilon_1 > 0$ arbitrarily small, gives a larger second reset interval $\varepsilon_2 > 0$. Thus Zeno solutions do not exist in this case either.

In the rest of the proof, the terms $O(\varepsilon_1^m)$ are directly neglected for simplicity. For the case $m = 3$, consider an initial condition $\mathbf{x}_0 = \mathbf{x}^1 = (x_1, x_2, x_3, 0, 0, \ldots, 0)^T \in \mathscr{M}_{\mathscr{R}}$. Applying a similar argument, the result is now that if a sequence of decreasing reset intervals $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ exists, with $\varepsilon_1 \geq \varepsilon_2 \geq \varepsilon_3 > 0$ and ε_1 being arbitrarily small, then

$$\begin{aligned} x_3 + \frac{\varepsilon_1}{2}x_2 + \frac{\varepsilon_1^2}{6}x_1 &= 0, \\ x_3 + \left(\varepsilon_1 + \frac{\varepsilon_2}{2}\right)x_2 + \left(\frac{\varepsilon_1^2}{2} + \frac{\varepsilon_1\varepsilon_2}{2} + \frac{\varepsilon_2^2}{6}\right)x_1 &= 0, \\ x_3 + \left(\varepsilon_1 + \varepsilon_2 + \frac{\varepsilon_3}{2}\right)x_2 + \left(\frac{(\varepsilon_1+\varepsilon_2)^2}{2} + \frac{(\varepsilon_1+\varepsilon_2)\varepsilon_3}{2} + \frac{\varepsilon_3^2}{6}\right)x_1 &= 0, \end{aligned} \tag{2.34}$$

and now, eliminating x_1, x_2, and x_3, after some computation, ε_3 is given as a function of ε_1 and ε_2 by the second order equation

$$(\varepsilon_1 + \varepsilon_2)\varepsilon_2 + (\varepsilon_1 + 2\varepsilon_2)\varepsilon_3 + \varepsilon_3^2 = 0 \tag{2.35}$$

having the solutions $\varepsilon_3 = -\varepsilon_2 < 0$ and $\varepsilon_3 = -(\varepsilon_1 + \varepsilon_2) < 0$, which is a contradiction. Thus, no initial condition in $\mathscr{M}_{\mathscr{R}}$ can produce a sequence of resets intervals that converge to zero, and again Zeno solutions do not exist for the case $m = 3$.

For the general case in which the dimension of $\mathscr{M}_{\mathscr{R}}$ is m, with initial state $\mathbf{x}_0 = \mathbf{x}^1 = (x_1, x_2, \ldots, x_m, 0, 0, \ldots, 0)^T$, a similar reasoning results in the set of equations

$$\begin{aligned} \sum_{k=1}^{m} \frac{\varepsilon_1^{m-1}}{(m+1-k)!}x_k &= 0, \\ \sum_{i=1}^{m}\sum_{k=1}^{i} \frac{\varepsilon_2^{m-i}\varepsilon_1^{i-k}}{(m+1-i)!(i-k)!}x_k &= 0, \\ &\cdots \\ \sum_{i=1}^{m}\sum_{k=1}^{i} \frac{\varepsilon_m^{m-i}(\varepsilon_1 + \cdots + \varepsilon_{m-1})^{i-k}}{(m+1-i)!(i-k)!}x_k &= 0, \end{aligned} \tag{2.36}$$

which results in an algebraic equation of order m in ε_m, with the solutions $\varepsilon_m = -\varepsilon_{m-1}$, $\varepsilon_m = -(\varepsilon_{m-1} + \varepsilon_{m-2}), \ldots, \varepsilon_m = -(\varepsilon_{m-1} + \varepsilon_{m-2} + \cdots + \varepsilon_1)$. And again, a sequence of reset intervals $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$ with $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_m > 0$ and ε_1 arbitrarily small cannot exist, showing that a Zeno solution is not possible in the full-reset case.

The case of partial-reset can be conveniently transformed into the full-reset form by a change of coordinates, by a simple resorting of coordinates so that the bijectivity is guaranteed. We will consider the system structure decomposition by writing the states as $\mathbf{x}^T = (\mathbf{x}_p^T, \mathbf{x}_{\bar{\rho}}{}^T, \mathbf{x}_\rho^T)$ where $\mathbf{x}_p \in \mathbb{R}^{n_p}$ stands for the states of the plant, $\mathbf{x}_{\bar{\rho}} \in \mathbb{R}^{n_{\bar{\rho}}}$ for the non-resetting compensator states, and $\mathbf{x}_\rho \in \mathbb{R}^{n_\rho}$ for the resetting compensator states. Define the linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$T\mathbf{x} = T\begin{pmatrix}\mathbf{x}_p\\ \mathbf{x}_{\bar\rho}\\ \mathbf{x}_\rho\end{pmatrix} = \begin{pmatrix}\mathbf{x}_{\bar\rho}\\ \mathbf{x}_p\\ \mathbf{x}_\rho\end{pmatrix} = \mathbf{z}, \tag{2.37}$$

that is,

$$T = \begin{pmatrix} 0_{n_{\bar\rho}\times n_p} & I_{n_{\bar\rho}\times n_{\bar\rho}} & 0_{n_{\bar\rho}\times n_\rho}\\ I_{n_p\times n_p} & 0_{n_p\times n_{\bar\rho}} & 0_{n_p\times n_\rho}\\ 0_{n_\rho\times n_p} & 0_{n_\rho\times n_{\bar\rho}} & I_{n_\rho\times n_\rho}\end{pmatrix}. \tag{2.38}$$

Note that T is a square matrix, all of whose entries are 0 or 1, and in each row and column of T there is precisely one 1. This means that T is a permutation matrix. Clearly, such a matrix is unitary, hence orthogonal, so $T^T = T^{-1}$. The nonsingular matrix T allows us to rewrite the dynamical system via a similarity transformation (congruence transformation):

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \bar{A}\mathbf{z}(t) && \text{if } \mathbf{z}(t) \notin \tilde{\mathcal{M}},\\ \mathbf{z}(t^+) &= \bar{A}_R\mathbf{z}(t) && \text{if } \mathbf{z}(t) \in \tilde{\mathcal{M}},\\ y(t) &= CT^T\mathbf{z}(t) = \bar{C}\mathbf{z}(t), \end{aligned} \tag{2.39}$$

where $\bar{A} = TAT^T$, $\bar{A}_R = TA_RT^T$, and $\bar{C} = CT^T$, and in addition the reset surface is transformed into $\tilde{\mathcal{M}} = \{\mathbf{z} \in \mathbb{R}^n : T^T z \in \mathcal{M}\}$. Note that $\bar{C} = CT^T = \mathbf{e}_{n_p+n_{\bar\rho}}$ so that the output is not changed by the transformation, i.e., $y(t) = z_{n_p+n_{\bar\rho}}(t)$ as expected. Henceforth, (2.39) is in full-reset form. Since observability is invariant under similarity transformations, it is clear that (2.4)–(2.6) is well-posed if and only if (2.39) is well-posed. Finally, to complete the proof it is necessary to show that the observability matrix has the structure given in (2.29) (using state transformations if needed). This is simply done by considering the substate $\mathbf{z}_1 = \binom{\mathbf{x}_{\bar\rho}}{\mathbf{x}_p}$. In general, there always exists a state transformation of $\mathbf{z} = \binom{\mathbf{z}_1}{\mathbf{x}_\rho}$ to $\mathbf{w} = \binom{\mathbf{w}_1}{\mathbf{x}_\rho}$ such that the state submatrix corresponding to $\mathbf{z}_1$ is in the observability staircase form, and thus the observability matrix has the structure given in (2.29) once unobservable states are eliminated. This concludes the proof. $\square$

2.2.2.1 Example: Well-posed Reset Control System with Partial Reset

Consider a reset control system (2.4)–(2.6) where the plant, with state $\mathbf{x}_p = \binom{x_1}{x_2}$, is given by the state space model

$$A_p = \begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}, \qquad B_p = \begin{pmatrix}1\\ 0\end{pmatrix}, \qquad C_p = \begin{pmatrix}0 & 1\end{pmatrix}, \tag{2.40}$$

and the reset compensator, with state $\mathbf{x}_r = \binom{x_3}{x_4}$, is given by

$$A_r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad C_r = \begin{pmatrix} 1 & 1 \end{pmatrix}, \qquad A_\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.41}$$

that is, the reset control system has a partial reset compensator: it is a parallel connection of an integrator and a Clegg integrator, where only the state x_4 is set to zero at the reset instants.

The closed-loop system is defined by the matrices

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad A_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0\,1\,0\,0 \end{pmatrix}, \tag{2.42}$$

and the closed-loop state $\mathbf{x} = (x_1\ x_2\ x_3\ x_4)^T$.

This reset control system is well-posed, since

$$\mathscr{M}_{\mathscr{R}} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \qquad \mathscr{N}(\mathscr{O}_{base}) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}, \tag{2.43}$$

and then $\mathscr{M}_{\mathscr{R}} \cap \mathscr{N}(\mathscr{O}_{base}) = \{\mathbf{0}\}$. Following the reasoning given in the proof of Proposition 2.4, the closed-loop state $\mathbf{x}$ can be transformed into a state $\mathbf{z}$ in which the observability matrix has the form (2.29). In this case, this is obtained with $\mathbf{z} = (x_3\ x_1\ x_2\ x_4)^T$. Thus, the initial conditions that produce a crossing in an arbitrarily small time $\varepsilon > 0$ are of the form $\mathbf{z}^1 = (1 - \frac{\varepsilon}{2}\ 0\ 0)^T$, or equivalently,

$$\mathbf{x}^1 = \left(-\frac{\varepsilon}{2}\ 0\ 1\ 0 \right)^T. \tag{2.44}$$

Now, the second after-reset state is given by

$$\mathbf{x}^2 = A_R e^{A\varepsilon} \left(-\frac{\varepsilon}{2}\ 0\ 1\ 0 \right)^T = \left(\frac{\varepsilon}{2}\ 0\ 1\ 0 \right)^T, \tag{2.45}$$

and, according to Proposition 2.4, $\mathbf{x}^2$ cannot produce a new crossing in a time less than or equal to ε. This fact can be verified by computing solutions to the implicit equation $0 = Ce^{At}(\alpha\ 0\ 1\ 0)^T$ for t, given $\alpha \in \mathbb{R}$. The solution is shown in Fig. 2.3, where $t = \tau_1((\alpha\ 0\ 1\ 0)^T)$ is given.

Note that for t to have an arbitrarily small value, an initial condition $\mathbf{x}^1$ in the after-reset surface must have the form (2.44), that is, $\alpha = -\varepsilon/2$ in Fig. 2.3. Then, as a result the state after the first reset $\mathbf{x}^2$ has the form (2.45), that is, $\alpha = +\varepsilon/2$ (see also Fig. 2.3). And then the value of the second reset instant can be obtained from Fig. 2.3. The result is that if the first reset instant is arbitrarily small, then the second reset instant is arbitrarily close to 3.15.

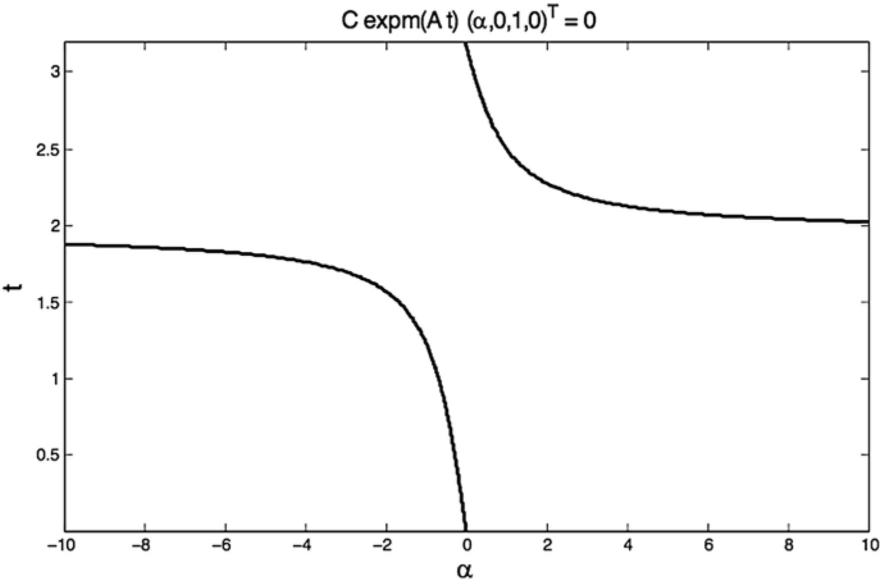

Fig. 2.3 First reset instant as a function of α

2.3 Reset Instants and the After-Reset Surface Dimension

In general, reset instants $t_k = \tau_k(x_0)$, $k = 1, 2, \ldots$, can take different and complex patterns for different initial conditions $x_0 \in \mathscr{D}$, and in fact this is a key property of reset control systems since it determines the way in which the reset instants evolve and also some important properties. For example, the fact that $\tau_1(\mathbf{x}_0)$ has a discontinuity in the example of the last section (Fig. 2.3) is directly related with the non-existence of Zeno solutions.

A useful property is that functions $\tau_k(\mathbf{x}_0)$ are homogeneous (of degree 0) since $\tau_k(\alpha\mathbf{x}_0) = \tau_k(\mathbf{x}_0)$, for any $\alpha > 0$ and $k = 1, 2, \ldots$. If the set of initial conditions is $\mathscr{D}$, this means that the computation of reset intervals can be simplified, for example, to those states that are elements of the unit ball (centered at the origin). On the other hand, the set of not reduced states $\bar{\mathscr{R}}$, as defined before Proposition 2.2, can be obtained by using the spectral projectors of the corresponding eigenvalues.

In the rest of this section, it is assumed that the reset control systems are well-posed and that their base systems have a state matrix A with a simple pair of complex dominant eigenvalues $\lambda_1 = \alpha_1 + i\beta_1$ and $\lambda_2 = \alpha_1 - i\beta_1$ (with index 1). As far as the computation of the set of not reduced states $\bar{\mathscr{R}}$ is concerned, it may be assumed that $\alpha_1 = 0$ without loss of generality. Thus, for an initial condition $\mathbf{x}_0$ the output dominant term before the first reset instant is given by (using (2.15))

$$y_1(t) = C\big(\mathrm{Re}\{G_1\}\cos(\beta_1 t) + \mathrm{Im}\{G_1\}\sin(\beta_1 t)\big)\mathbf{x}_0, \tag{2.46}$$

where G_1 is the spectral projector given by $G_1 = \mathbf{v}_1 \mathbf{w}_1^T$, and $\mathbf{v}_1$ and $\mathbf{w}_1$ are the right and the left eigenvectors, respectively. Thus, the set $\bar{\mathscr{R}}$ can be simply computed as

$$\bar{\mathscr{R}} = \mathscr{N}\left\{\begin{pmatrix} C\,\mathrm{Re}\{G_1\} \\ C\,\mathrm{Im}\{G_1\} \end{pmatrix}\right\}, \tag{2.47}$$

where $\mathrm{Re}\{G_1\}$ and $\mathrm{Im}\{G_1\}$ stands for the real and imaginary parts of G_1, respectively. And thus the set of reduced states is given simply by $\mathbb{R}^n \setminus \bar{\mathscr{R}}$.

The order of the reset control system, and in particular the dimension of the after-reset surface, is key in the analysis of the reset instants, thus in the following different cases corresponding to dimensions of the after-reset surface 1, 2, and ≥ 3 are treated separately.

2.3.1 $\dim(\mathscr{M}_{\mathscr{R}}) = 1$

In this case, since the after-reset surface has dimension 1, any initial condition $\mathbf{x}_0 \in \mathscr{M}_R$ can be generated by one vector $\mathbf{u}$, that is, $\mathbf{x}_0 = \alpha\mathbf{u}$ for some $\alpha \in \mathbb{R}$. Thus, simply by using the homogeneity property of the functions $t_k = \tau_k(\mathbf{x}_0)$, $k = 1, 2, \ldots,$ it is clear that $\tau_k(\mathbf{x}_0) = \tau_k(\alpha\mathbf{u}) = \tau_k(\mathbf{u})$, that is, they are all constant functions over the set $\mathscr{M}_R$, as long as $(C, A, \mathbf{u})$ is reduced (otherwise there are no crossings). In other words, starting from an initial condition in the after-reset surface, the reset instants are periodic.

For initial conditions that are not elements of the after-reset surface, the first reset instant is in general different. Using Proposition 2.2, if the reset system has a pair of dominant complex eigenvalues then the first reset instant is uniformly bounded, that is, $\|\tau_1(\mathbf{x}_0)\| < \Delta$, for some upper bound $\Delta > 0$ and for any initial condition x_0 such that $(C, A, \mathbf{x}_0)$ is reduced. In the case in which $(C, A, \mathbf{x}_0)$ is not reduced, $\tau_1(\mathbf{x}_0)$ is not necessarily bounded, that is, the initial condition may not cross the reset surface.

2.3.1.1 Example

Consider the reset control system (2.4)–(2.6) where the plant, with state $\mathbf{x}_p = \binom{x_1}{x_2}$, is given by the state space model

$$A_p = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \qquad B_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad C_p = \begin{pmatrix} 0 & 1 \end{pmatrix}, \tag{2.48}$$

and the reset compensator, with state $\mathbf{x}_r = x_3$, is given by

$$A_r = -1, \qquad B_r = 1, \qquad C_r = 1, \qquad A_\rho = 0, \tag{2.49}$$

that is, the reset compensator is FORE, and the state x_3 is set to zero at the reset instants.

The closed-loop system is given by the matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \qquad A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 \; 1 \; 0 \end{pmatrix}, \tag{2.50}$$

and the closed-loop state is $\mathbf{x} = (x_1 \; x_2 \; x_3)^T$.

It can be easily checked that the system is well-posed since the base linear system is observable. By definition, the after-reset surface is given by

$$\mathscr{M}_R = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}. \tag{2.51}$$

Now, the subspace of not reduced states $\bar{\mathscr{R}}$ is computed. The closed-loop state matrix A has the eigenvalues $\lambda_1 = -0.12 + j0.74$, $\lambda_2 = -0.12 - j0.74$, and $\lambda_3 = -1.75$. Thus, the reset control system has two complex dominant eigenvalues, with their spectral projectors being

$$G_1 = \begin{pmatrix} 0.41 - j0.28 & 0.15 + j0.34 & 0.12 - j0.41 \\ 0.12 - 0.j41 & 0.29 + j0.14 & -0.16 - j0.34 \\ 0.16 + j0.34 & -0.27 + j0.07 & 0.29 + j0.14 \end{pmatrix}, \qquad G_2 = G_1^{\star}. \tag{2.52}$$

The subspace of not reduced states is $\bar{\mathscr{R}} = \mathscr{N}\left(\left(\begin{smallmatrix} C\,\mathrm{Re}\{G_1\} \\ C\,\mathrm{Im}\{G_1\} \end{smallmatrix}\right)\right)$, which in this example has dimension 1 and is given by

$$\bar{\mathscr{R}} = \text{span}\left\{ \begin{pmatrix} -0.41 \\ 0.55 \\ 0.72 \end{pmatrix} \right\}. \tag{2.53}$$

Since, according to (2.51) and (2.53), every nonzero after-reset state is reduced, it results in that the reset instants are periodic as discussed above. However, note that if the initial condition is not an after-reset state, two types of solutions may occur (see Figs. 2.4–2.5):

- solutions with no crossings if

$$\mathbf{x}_0 = \alpha \begin{pmatrix} -0.41 \\ 0.55 \\ 0.72 \end{pmatrix},$$

  for some real number α, and
- solutions with a first crossing at a finite time and then an infinite number of periodic crossings.

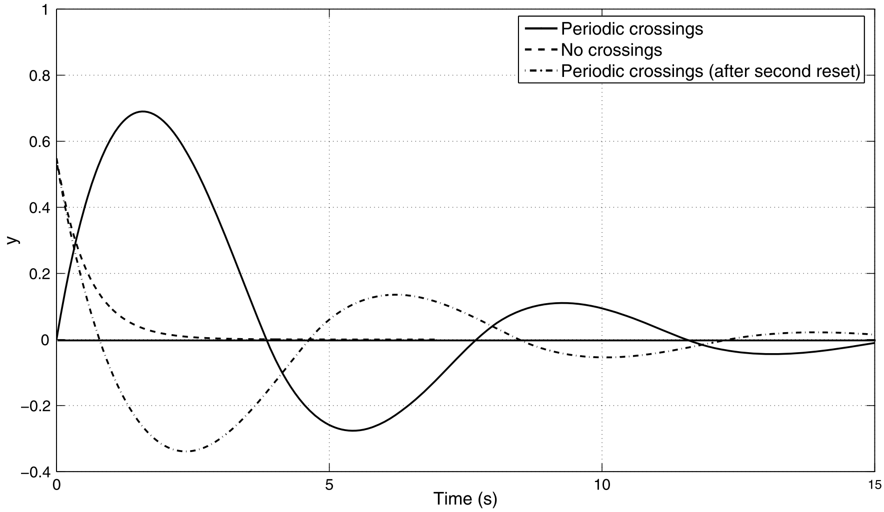

Fig. 2.4 Closed-loop output for three different initial conditions

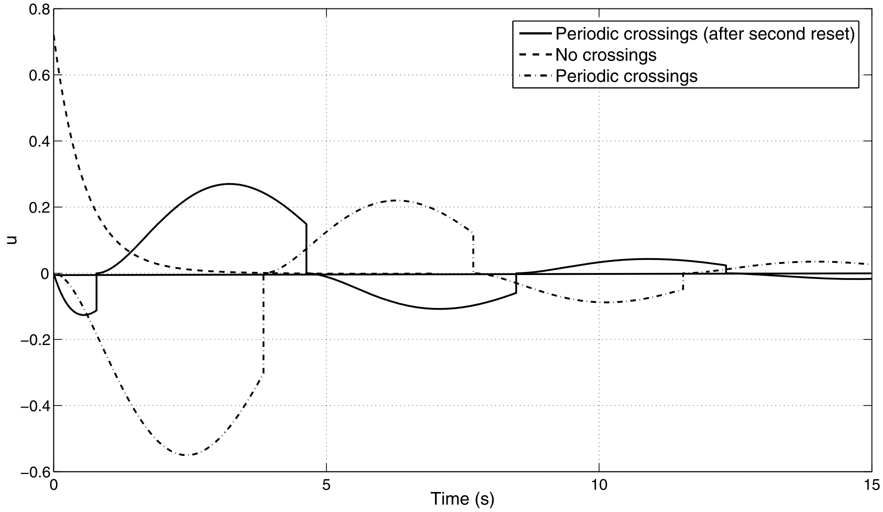

Fig. 2.5 Control input for three different initial conditions

2.3.2 $\dim(\mathscr{M}_{\mathscr{R}}) = 2$

The case of $\dim(\mathscr{M}_{\mathscr{R}}) = 2$ is slightly more involved since, as it will be described below, the resets instants are not periodic in general, and a large variety of solutions may appear regarding the number and structure of crossings associated to a given initial condition. By simplicity, the particular case of a third order plant and a full reset compensator is considered. Without loss of generality, consider an observer canonical form realization of the plant, that is,

$$A_p = \begin{pmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{pmatrix}, \qquad B_p = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}, \qquad C_p = \begin{pmatrix} 0 \; 0 \; 1 \end{pmatrix}. \tag{2.54}$$

Thus an after-reset state $\mathbf{x} \in \mathscr{M}_{\mathscr{R}}$ is given by

$$\mathbf{x} = (x_1, x_2, 0, 0, \dots, 0)^T \tag{2.55}$$

for some values $\binom{x_1}{x_2} \in \mathbb{R}^2$. Using again the fact that $t_k = \tau_k(x_0)$, $k = 1, 2, \dots$, are homogeneous, for the systems solutions with an initial condition in the after-reset surface it is enough to check the reset instants in some subset of $\mathbb{R}^2$, for example, the unit circle. Using polar coordinates $\binom{x_1}{x_2} = \binom{\rho\cos\theta}{\rho\sin\theta}$, for $\rho \in [0, \infty)$, $\theta \in [0, 2\pi)$, it is true that

$$\tau_k\left(\begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) = \tau_k\left(\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) \tag{2.56}$$

for $k = 1, 2, \dots$. As a result the reset instants are functions of the single parameter θ.

In general, the system's solutions may exhibit no crossings, a finite number of crossings, or infinitely many crossings. If the initial condition is an after-reset state, that is, $x_0 \in \mathscr{M}_{\mathscr{R}}$, no crossings or a finite number of crossings may occur if some of the after-reset states are not reduced. Otherwise, an infinite number of crossings is produced.

2.3.2.1 Example

Consider again the reset control system with a FORE compensator, with parameters $A_r = -1$, $B_r = 1$, and $C_r = 1$, and a plant given by the state space model

$$A_p = \begin{pmatrix} 0 & 0 & -0.35 \\ 1 & 0 & -2.40 \\ 0 & 1 & -4.35 \end{pmatrix}, \qquad B_p = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \qquad C_p = \begin{pmatrix} 0 \; 0 \; 1 \end{pmatrix}. \tag{2.57}$$

The reset control system is well-posed since the base linear system is observable. Using the parameter θ as in (2.56), the function τ_1 can be computed by solving the implicit equation

$$Ce^{At_1}\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \\ 0 \end{pmatrix} = 0 \tag{2.58}$$

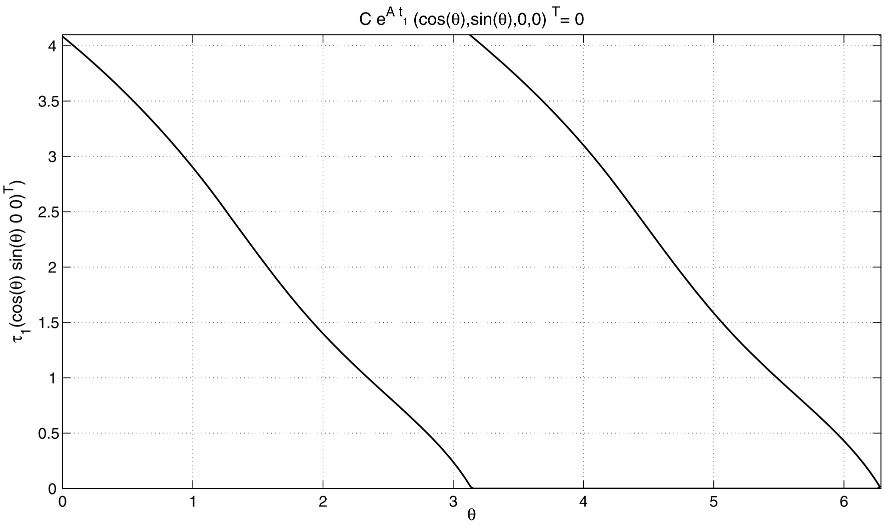

Fig. 2.6 Reset instants corresponding to after-reset states as a function of the parameter θ

for $t_1 = \tau_1(\theta)$. The result is given in Fig. 2.6, where it can be seen that the mapping τ_1 has a discontinuity at $\theta = \pi$, and in addition it results in the reset instants being uniformly bounded for initial conditions in the after-reset surface. This is due to the fact that all the after-reset states are reduced, as it will be seen below.

The after-reset surface is given by

$$\mathscr{M}_R = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \tag{2.59}$$

and the subspace of not reduced states by

$$\bar{\mathscr{R}} = \mathscr{N}\left(\begin{pmatrix} C\,\mathrm{Re}\{G_1\} \\ C\,\mathrm{Im}\{G_1\} \end{pmatrix} \right) = \text{span}\left\{ \begin{pmatrix} 0.20 \\ 0.31 \\ 0.92 \\ 0.16 \end{pmatrix}, \begin{pmatrix} -0.81 \\ 0.47 \\ -0.05 \\ 0.35 \end{pmatrix} \right\}, \tag{2.60}$$

thus it can be easily checked that $\mathscr{M}_R \cap \bar{\mathscr{R}} = \{\emptyset\}$, which means that all the after-reset states are reduced. In Figs. 2.7 and 2.8, several simulations corresponding to different initial conditions in the after-reset surface $\mathscr{M}_R$ are shown, including closed-loop outputs and control inputs. Note that since the after-reset states are reduced, there are an infinite number of resets corresponding to each initial condition, and as indicated above, a bound over the reset intervals may be found.

In the case in which the initial condition is not an after-reset surface state, it may occur that no crossings are produced. This is, in fact, the case for initial conditions in the set $\bar{\mathscr{R}}$ given by (2.60). Otherwise, an infinite number of crossings are produced

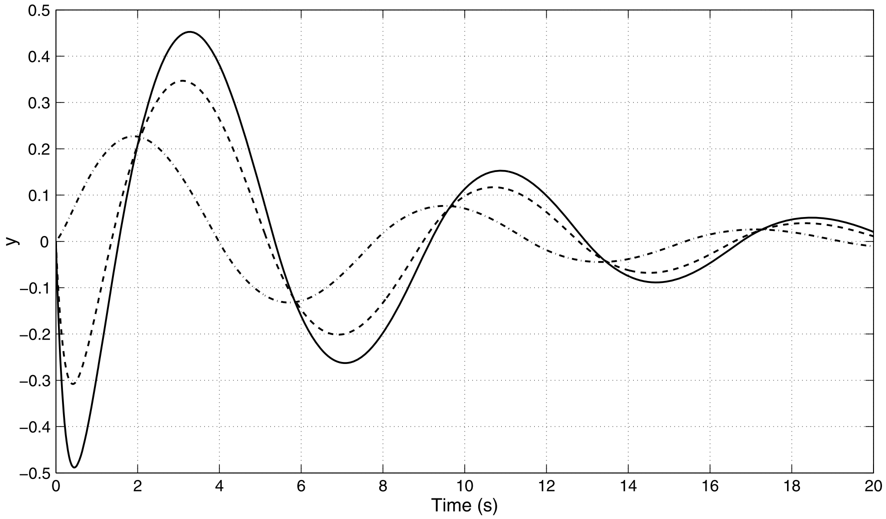

Fig. 2.7 Closed-loop output for three initial conditions in the after-reset surface

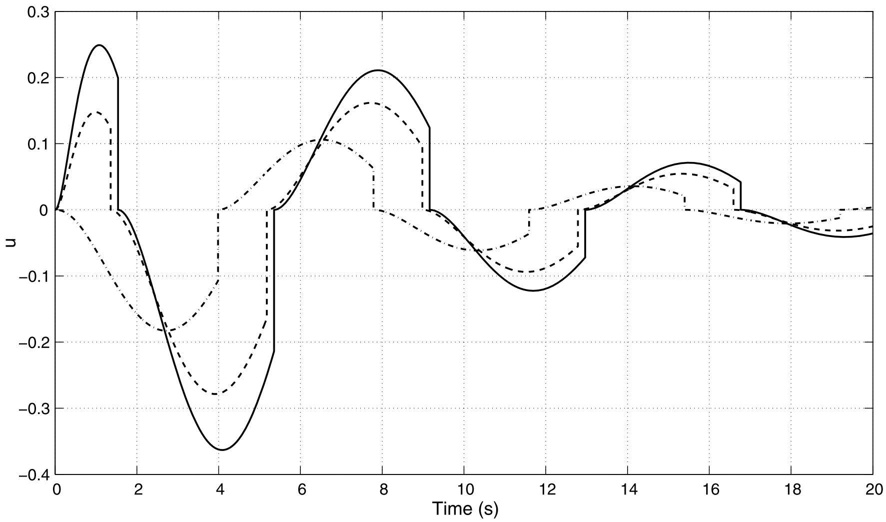

Fig. 2.8 Control input for three initial conditions in the after-reset surface

in this example. Different simulations for not reduced initial conditions are shown in Figs. 2.9–2.10

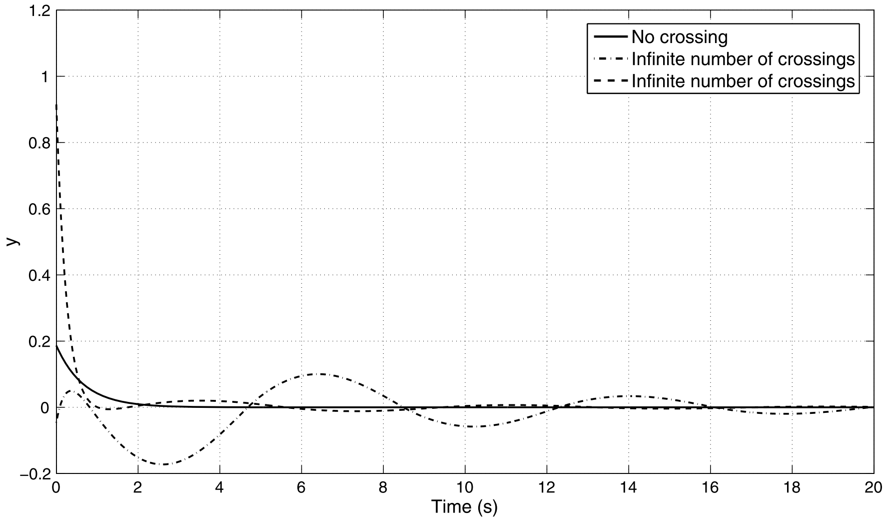

Fig. 2.9 Closed-loop output for three initial conditions (not reduced)

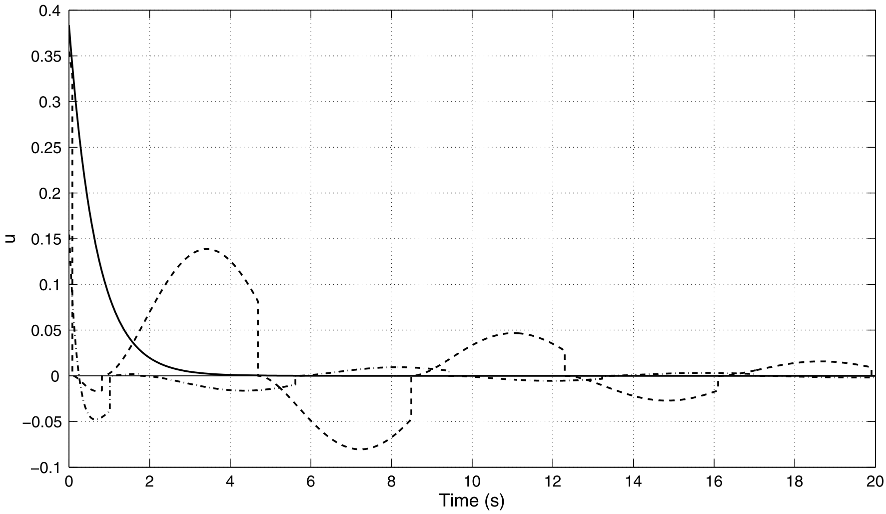

Fig. 2.10 Control input for three initial conditions (not reduced)

2.3.3 $\dim(\mathscr{M}_{\mathscr{R}}) \geq 3$

For higher order reset control systems, the functions τ_k, $k = 1, 2, \ldots$, depend on more than one parameter, and in general they can be described by the values of τ_1 over the unit ball in $\mathbb{R}^m$, where m is the number of the after-reset states. In general, the function τ_1 can exhibit an infinite number of discontinuities over that domain even in the case $m=2$, resulting in very complex patterns of the resets instants as a function of the initial condition. A detailed analysis of these patterns is still an open issue. In the following, an example corresponding to a second order partial reset compensator and a third order plant is analyzed.

2.3.3.1 Example

In this example, a PI + CI compensator is used. It consists of a parallel connection of a PI compensator and a Clegg integrator, and it is a partial reset compensator. Consider the PI + CI compensator with a state space realization given by

$$A_r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_r = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$C_r = \begin{pmatrix} 0.2 \; 0.2 \end{pmatrix}, \qquad D_r = 0.2, \qquad A_\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.61}$$

and a third order plant given by

$$A_p = \begin{pmatrix} -2.20 & -1.32 & -0.72 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad B_p = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad C_p = \begin{pmatrix} 0 \; 0 \; 1.33 \end{pmatrix}. \tag{2.62}$$

In addition, closed-loop state matrices are

$$A = \begin{pmatrix} -2.20 & -1.32 & -0.99 & 0.20 & 0.2 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1.33 & 0 & 0 \\ 0 & 0 & -1.33 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 \; 0 \; 1.33 \; 0 \; 0 \end{pmatrix}, \tag{2.63}$$

and

$$A_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.64}$$

In this case, the base system is unobservable, with the unobservable subspace in this case being given by

$$\mathscr{N}(\mathscr{O}_{base}) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}; \tag{2.65}$$

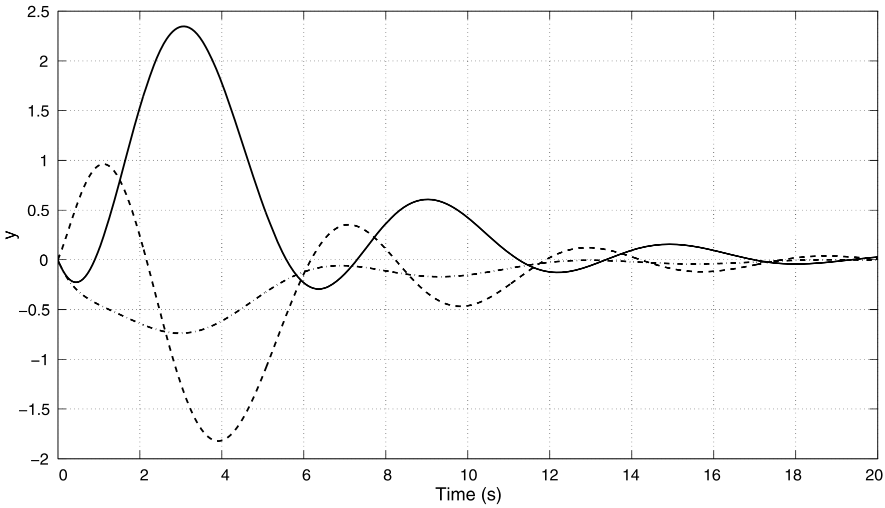

Fig. 2.11 Closed-loop output for three initial conditions in the after reset surface

however, the reset control system is well-posed since the after-reset surface is

$$\mathscr{M}_R = \text{span}\left\{\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}\right\} \tag{2.66}$$

and $\mathscr{M}_{\mathscr{R}} \cap \mathscr{N}(\mathscr{O}_{base}) = \{\mathbf{0}\}$. In addition, it can be checked that the after reset states are not reduced, in other words, assuming initial conditions in the after-reset surface there are always an infinite number of crossings.

In Fig. 2.11, closed-loop outputs corresponding to different initial conditions are obtained. Also in Fig. 2.12 the different control inputs are shown. Note that in general, as the dimension of the after-reset surface increases, the structure of the reset instants becomes not that simple, for example, in the interval $[0, 15]s$ one initial condition does not produce a crossing while the others produce five reset actions. As a conclusion, as the dimension of the after reset surface increases, the reset instants occur with more complex patterns.

2.4 Reset Control Systems with Exogenous Inputs

As it is usual in control practice, reset control systems are driven by external or exogenous inputs such as reference or disturbance signals. In this case, well-posedness of the reset control system can be analyzed using the arguments given in previous sections, if the exogenous inputs are generated by an exosystem.

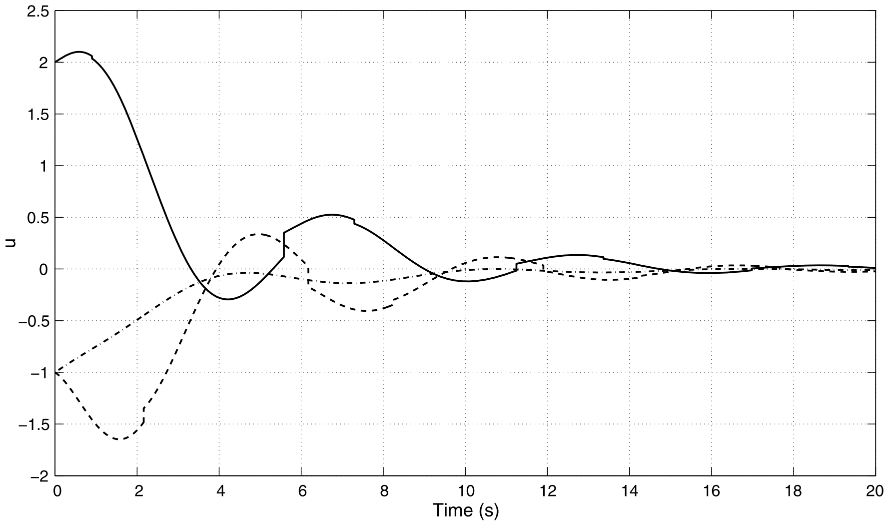

Fig. 2.12 Control input for three initial conditions in the after-reset surface

Consider the reset control system of Fig. 2.1, where the plant and the reset compensator are given by (2.1) and (2.2), respectively, and with a reference input r and a disturbance input d generated by exosystems, with the state space models

$$\begin{cases} \dot{\mathbf{w}}_1(t) = A_1\mathbf{w}_1(t), & \mathbf{w}_1(0) = \mathbf{w}_{10}, \\ r(t) = C_1\mathbf{w}_1(t), \end{cases} \tag{2.67}$$

with $\mathbf{w}_1 \in \mathbb{R}^{m_1}$, and

$$\begin{cases} \dot{\mathbf{w}}_2(t) = A_2\mathbf{w}_2(t), & \mathbf{w}_2(0) = \mathbf{w}_{20}, \\ d(t) = C_2\mathbf{w}_2(t), \end{cases} \tag{2.68}$$

with $\mathbf{w}_2 \in \mathbb{R}^{m_2}$. These exosystems allow the generation of signals like steps, ramps, sinusoids, ... Now, the feedback connection is given by $e = r - y$ and $u = v + d$, and the base linear closed-loop system may be described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \begin{pmatrix} 0 \\ B_r \end{pmatrix} r(t) + \begin{pmatrix} 0 \\ B_p \end{pmatrix} d(t) \tag{2.69}$$

where $\mathbf{x} = \binom{\mathbf{x}_p}{\mathbf{x}_r}$, and $\mathbf{x}_p$ and $\mathbf{x}_r$ are the plant and compensator states. In addition, the reset instants are defined as those instants t at which the closed-loop output $y(t) = C\mathbf{x}(t)$ is equal to the reference signal $r(t) = C_1\mathbf{w}_1(t)$. Define the augmented state $\mathbf{z}$ as $\mathbf{z} = (\mathbf{w}_1^T \ \mathbf{w}_2^T \ \mathbf{x}^T)^T$. Then, the reset map can be defined in the augmented state space by

$$\bar{A}_R = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & A_\rho \end{pmatrix} \tag{2.70}$$

and the after-surface $\bar{\mathscr{M}}_R$ and the reset surface $\bar{\mathscr{M}}$ as

$$\begin{aligned} \bar{\mathscr{M}}_R &= \mathscr{R}(\bar{A}_R) \cap \mathscr{N}(\bar{C}), \\ \bar{\mathscr{M}} &= \mathscr{N}(\bar{C}) \setminus \bar{\mathscr{M}}_R \end{aligned} \tag{2.71}$$

with $\bar{C} = (C_1 \; 0 \; -C_p \; 0)$. Finally, the closed-loop system in the augmented state space is given by

$$\begin{cases} \dot{\mathbf{z}}(t) = \bar{A}\mathbf{z}(t) & \text{if } \mathbf{z}(t) \notin \mathscr{M}, \\ \mathbf{z}(t^+) = \bar{A}_R \mathbf{z}(t) & \text{if } \mathbf{z}(t) \in \mathscr{M} \end{cases} \tag{2.72}$$

with the state space matrix

$$\bar{A} = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & B_p C_2 & A_p & B_p C_r \\ B_r C_1 & 0 & -B_r C_p & A_r \end{pmatrix}. \tag{2.73}$$

In the augmented space state representation, Propositions 2.3 and 2.4 can be directly applied, giving the next result.

Proposition 2.5 *Consider the reset control system of Fig. 2.1, with the plant and reset compensator given by (2.1) and (2.2), respectively, and with inputs r and d generated by the exosystems (2.67)–(2.68). If $\bar{\mathscr{M}}_R \cap \mathscr{N}(\bar{\mathscr{O}}_{base}) = \{\mathbf{0}\}$, where $\bar{\mathscr{O}}_{base}$ is the observability matrix*

$$\bar{\mathscr{O}}_{base} = \begin{pmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n+m_1+m_2-1} \end{pmatrix} \tag{2.74}$$

then

1. *The reset system is well-posed.*
2. *The reset system does not have Zeno solutions.*

Proof The augmented state can be partitioned as $\mathbf{z} = \binom{\mathbf{z}_1}{\mathbf{x}_r}$, and thus the state matrix (2.73) is partitioned as

$$\bar{A} = \begin{pmatrix} \bar{A}_p & \bar{B}_p C_r \\ B_r \bar{C}_p & A_r \end{pmatrix}, \tag{2.75}$$

where

$$\bar{A}_p = \begin{pmatrix} A_1 & 0 & u & 0 \\ 0 & A_2 & u & 0 \\ 0 & B_p C_2 u A_p \end{pmatrix}, \qquad \bar{B}_p = \begin{pmatrix} 0 \\ 0 \\ B_p \end{pmatrix}, \qquad \bar{C}_p = \begin{pmatrix} C_1 \; 0 \; -C_p \end{pmatrix}. \tag{2.76}$$

Part 1 of the proposition is a direct application of Proposition 2.3 to the system (2.72) defined by the matrices $\bar{A}$, $\bar{A}_R$, and $\bar{C}$. On the other hand, since the state matrix $\bar{A}$ has the structure given in (2.75), it always possible to make a state transformation of $\mathbf{z}_1$ in such a way that corresponding principal minor be in the observer form. Thus, the reasoning used in the proof of Proposition 2.4 may be used to prove Part 2. □

In general, a simple sufficient condition for the well-posedness of a reset control system with exogenous inputs it that the augmented closed-loop system (2.72) be observable, that is, the matrix $\bar{\mathcal{O}}_{base}$ be full rank.

2.4.1 A Well-posed Reset Control System with Exogenous Input

A simple example is shown here to illustrate Proposition 2.5. Consider the reset control system of Fig. 2.1, consisting of the feedback interconnection of a Clegg integrator and an integrator. The Clegg integrator has the state equations

$$\begin{cases} \dot{x}_r(t) = r(t), & x_r(t) - r(t) \neq 0, \\ x_r(t^+) = 0, & x_r(t) - r(t) = 0, \\ v(t) = x_r(t). \end{cases} \tag{2.77}$$

In addition, consider a sinusoidal reference input $r(t) = a\sin(\omega t + \phi)$, for some given constants a, $\omega > 0$, and ϕ. It is given by the exosystem

$$\begin{cases} \dot{\mathbf{w}}_1(t) = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \mathbf{w}_1(t), & \mathbf{w}_1(0) = \begin{pmatrix} a\sin\phi \\ a\cos\phi \end{pmatrix}, \\ r(t) = (1\; 0)\mathbf{w}_1. \end{cases} \tag{2.78}$$

Since a disturbance signal is not considered in this example, Proposition 2.5 can be used by eliminating the row and column blocks corresponding to the disturbance exosystem in the matrices $\bar{A}$, $\bar{A}_R$, and $\bar{C}$. The result is

$$\bar{A} = \begin{pmatrix} 0 & \omega & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \qquad \bar{C} = \begin{pmatrix} 1 \; 0 \; -1 \; 0 \end{pmatrix}. \tag{2.79}$$

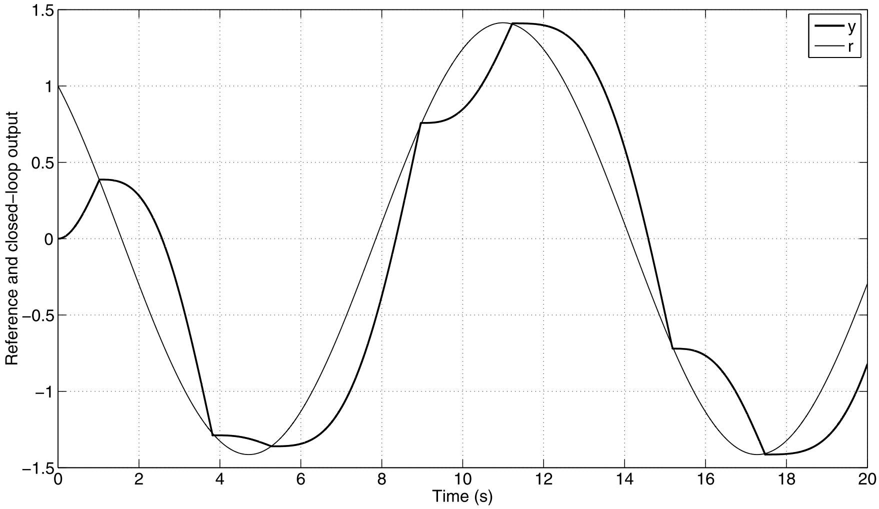

Fig. 2.13 Reference signal and closed-loop output corresponding to a well-posed reset control system

Now, the observability matrix of the (augmented) base system is

$$\bar{\mathcal{O}}_{base} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & \omega & 0 & -1 \\ -(1+\omega^2) & 0 & 1 & 0 \\ 0 & -\omega(1+\omega^2) & 0 & 1 \end{pmatrix} \tag{2.80}$$

which is full rank for any $\omega > 0$ (and does not depend on a), and thus the system is well-posed for any sinusoidal reference input. Figures 2.13 and 2.14 show a simulation of the reset control system for $\omega = 0.5$ rad/s, with an initial condition $\mathbf{w}_1(0) = (1 \ \ -1)^T$ for the exosystem, and with zero initial condition for the Clegg integrator and the integrator.

2.4.2 A Reset Control System with Zeno Solutions

In general, Proposition 2.5 gives a simple and checkable condition for well-posedness of the reset control system of Fig. 2.1 with exogenous inputs, for the plant and compensator given by (2.1) and (2.2), respectively. For well-posedness, the result may be also used for any reset system that can be expressed as (2.72) with arbitrary values of $\bar{A}$, $\bar{A}_R$, and $\bar{C}$, since no particular structure of these matrices is used to prove the result. However, for avoiding Zeno solutions the structure of these matrices, related with (2.1) and (2.2), is a central part of the result. Thus, a reset control system with a different structure may have Zeno solutions, this is the case, for example, corresponding to a non-strictly proper plant in Fig. 2.1.

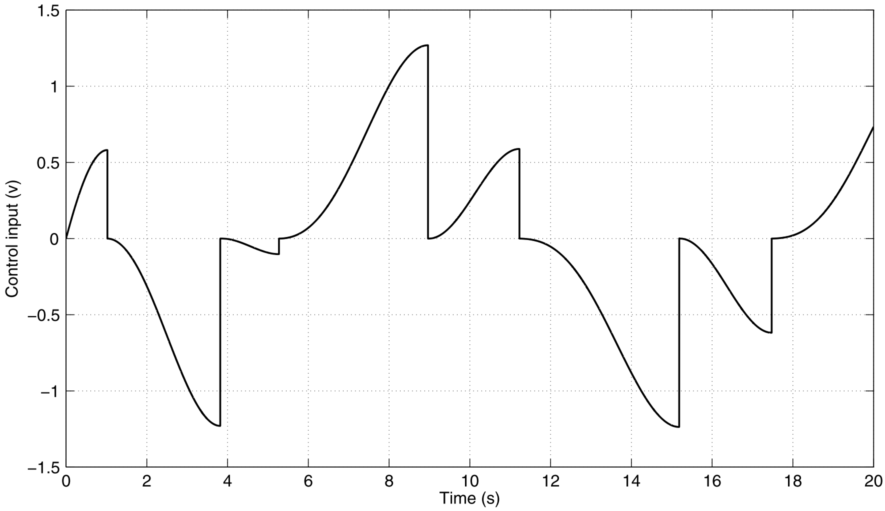

Fig. 2.14 Control input corresponding to a well-posed reset control system

In the following, an example developed in [6] is shown here to illustrate the existence of Zeno solutions in reset control systems with external inputs, where the plant is simply $P(s) = 1$. The resulting feedback system is a Clegg integrator with unity feedback. In addition, a sinusoidal reference input is also considered. The state space models (2.77) and (2.78) are again used, as a result the augmented state matrices are in this case (the row and column block corresponding to the plant are simply removed)

$$\bar{A} = \begin{pmatrix} 0 & \omega & 0 \\ -\omega & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \qquad \bar{A}_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \bar{C} = \begin{pmatrix} 1 \; 0 \; -1 \end{pmatrix}, \tag{2.81}$$

and the observability matrix of the augmented base system is given by

$$\bar{\mathscr{O}}_{base} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & \omega & 1 \\ 1-\omega^2 & -\omega & -1 \end{pmatrix} \tag{2.82}$$

which is full rank for any $\omega > 0$. Although Proposition 2.5 cannot be used in this case because the plant does not fit the model (2.2), the well-posedness can be assured since $\bar{\mathscr{O}}_{base}$ is full rank. However, the well-posedness argument cannot be used for assessing the existence of Zeno solutions. In fact, this system has a Zeno solution with sequences of reset instants $\{t_k\}$, $k = 1, 2, \ldots$, converging to $t^* = n\pi$ for every integer $n \geq 1$.

Figure 2.15 (top) shows the output response $y(t)$ of the closed loop (and output of the Clegg integrator) to the sinusoidal reference $w(t) = \sin \omega t$. It can be shown that a Zeno behavior appears to the left of every $t = k\pi$, $k = 1, 2, 3, \ldots$. The solution

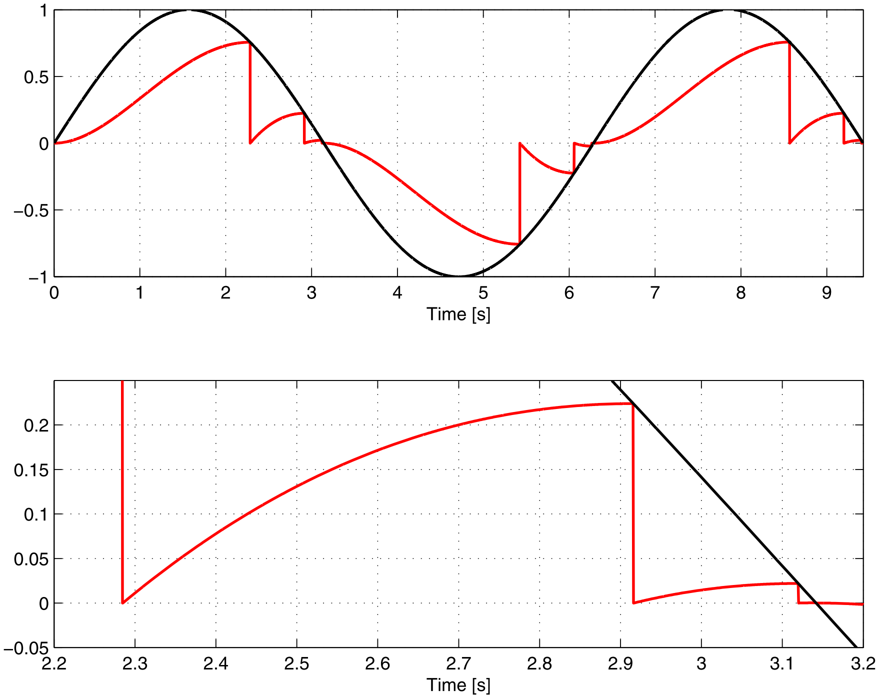

Fig. 2.15 (*Top*) Response $y(t)$ to a sinusoidal reference of a feedback loop with a Clegg integrator; (*Bottom*) detailed zoom of the three first visible resets close to $t = \pi$

starting from the right of every $y(k\pi) = 0$ escapes the Zeno behavior until it again reaches another Zeno point at $t = (k+1)\pi$.

Figure 2.15 (bottom) plots a detailed zoom of the first three visible resets close to $t = \pi$. These reset times are $t_1 \approx 2.28$, $t_2 \approx 2.92$, and $t_3 \approx 3.12$. In fact, there exists an infinite sequence of resets $\{t_k\}$ converging to $t_\infty = \pi$.

It is illustrative to see in Fig. 2.16 the time response of this system in a three-dimensional plot that shows the time evolution of the state $(w_{11}, w_{12}, x_p)^\top = (r, \dot{r}, y)^\top$. The first two coordinates are from the exosystem, and the first one is the reference $r(t)$ to the closed loop, in this case $r(t) = \sin \omega t$ and $\dot{r}(t) = \cos \omega t$, with $\omega = 1$. Thus the trajectory lies entirely within the cylinder $r^2 + \dot{r}^2 = 1$.

The three-dimensional plot reveals clearly that, in the state space $(r, \dot{r}, y)$, there are actually two Zeno accumulation points, namely $Z_1 = (0, 1, 0)$ and $Z_2 = (0, -1, 0)$. The first one corresponds in the time domain (Fig. 2.15) to the zero crossings of the reference for $t = 2k\pi$ and the second one for $t = (2k+1)\pi$, with k integer.

Figure 2.16 also shows two relevant planes: the null space $\mathscr{N}(C)$ of $C = (1, 0, -1)$, that is, the plane $y = r$ that triggers the reset action, and the plane $\mathscr{M}_R$ given by $y = 0$ where the state is projected immediately after a reset $y(t_k^+) = 0$.

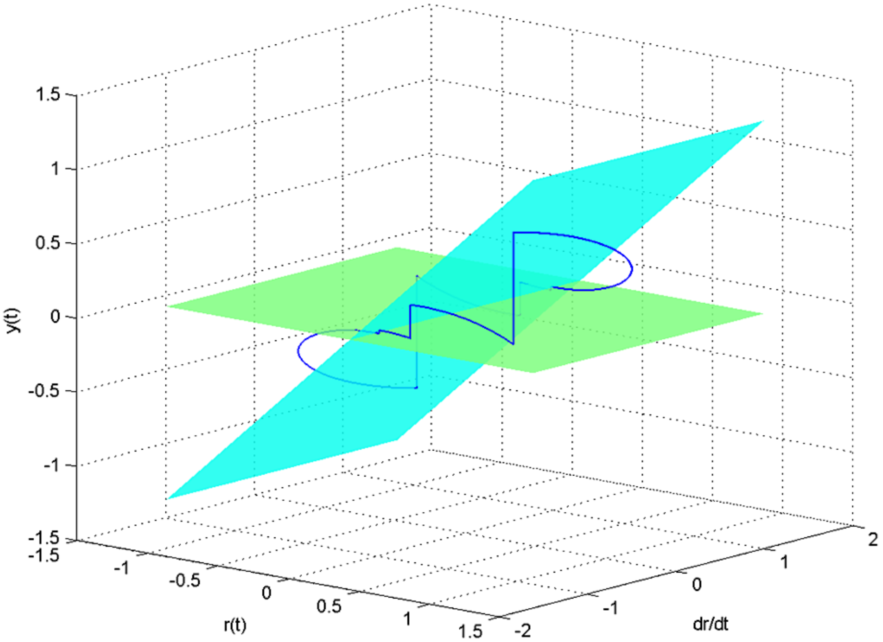

Fig. 2.16 Trajectory $(r(t), \dot{r}(t), y(t))$ with Zeno behavior

The analysis of Zeno behavior in dynamical systems is an involved topic, and we will not address it here rigorously. Notice from the previous proposition in this section that the reset control systems that we are interested in (well-posed) do not have Zeno behavior. However, in order to give an idea of the asymptotic behavior close to the Zeno limit, let us complete the example with a simplified study based on the Poincaré map. We refer the interested reader to the literature, for example, [8], [7], and [13] for a full exploration of these features.

From Fig. 2.16 it is clear that from any initial condition the state evolves until it reaches the reset condition at $(r_k, \dot{r}_k, y_k)$ with $y_k = r_k$ and then resets to the point $\mathbf{x}_k = (r_k, \dot{r}_k, 0)$. Then, it flows again until a new reset condition holds and a new reset moves the state to $\mathbf{x}_{k+1} = (r_{k+1}, \dot{r}_{k+1}, 0)$. The sequence of after-reset points $\{\mathbf{x}_k\}$ defines a discrete-time map, or iteration $\mathbf{x}_k \to \mathbf{x}_{k+1}$, called the *Poincaré map* of the reset system. Note that, since $r_k^2 + \dot{r}_k^2 = 1$ for all k, the Poincaré sequence $\{\mathbf{x}_k\} \in \mathscr{P}$ is here defined on a one-dimensional manifold $\mathscr{P}$ given by $r^2 + \dot{r}^2 = 1$ and $y = 0$.

Since $\mathscr{P}$ is one-dimensional, the Poincaré map can be determined also from the time domain plots of $r(t)$ and $y(t)$ in Fig. 2.15, for if we determine the sequence of reset times $\{t_k\}$, then the Poincaré sequence is $\{\mathbf{x}_k\} = \{(\sin t_k, \cos t_k, 0)\}$. In Fig. 2.15 (bottom), we see the first three reset times $t_1 \approx 2.28$, $t_2 \approx 2.92$, and $t_3 \approx 3.12$. These t_k tend from the left to $t_\infty = \pi$. Let us suppose that we start at $t = t_k$ with $y(t_k) = 0$.

The evolution of the Clegg integrator state is governed by:

$$\dot{y} = r(t) - y(t) = \sin t - y(t), \qquad y(t_k) = 0.$$

Since t_k tends to π, a key simplifying assumption is that, in the limit, we can approximate $\sin t$ by $-(t-\pi)$. This simplification is also helpful because the Poincaré maps give rise to implicit equations not solvable analytically. In this way, close to $t = \pi$, we have:

$$\dot{y} = -(t-\pi) - y(t), \qquad y(t_k) = 0,$$

having the solution

$$y(t) = (t_k - \pi - 1)e^{-(t-t_k)} - t + \pi + 1, \quad t \geq t_k.$$

To determine the next reset time t_{k+1}, we have to impose the condition $y(t_{k+1}) = r(t_{k+1})$. Again we replace $r(t_{k+1}) = \sin t_{k+1}$ by $-(t_{k+1} - \pi)$. Thus the Poincaré map $t_k \to t_{k+1}$ is given implicitly by the condition

$$-(t_{k+1} - \pi) = (t_k - \pi - 1)e^{-(t_{k+1}-t_k)} - t_{k+1} + \pi + 1.$$

Introduce the change of variables $t_k - \pi = d_k$, with $d_k \to 0^-$. Using the asymptotic approximation $e^{-(t_{k+1}-t_k)} = e^{-(d_{k+1}-d_k)} \approx 1 - (d_{k+1} - d_k)$ gives rise to

$$-d_{k+1} = (d_k - 1)(1 - d_{k+1} + d_k) - d_{k+1} + 1,$$

from which we can solve explicitly $d_{k+1} = d_k^2/(d_k - 1)$. This law for $d_k \to 0^-$ approaches $d_{k+1} = -d_k^2$, or equivalently, approaches the limit relation

$$|t_{k+1} - \pi| = |t_k - \pi|^2$$

which proves that there exists an infinite sequence of reset times t_k tending to π and governed, in the limit, by a quadratic recurrence law. This explains why in Fig. 2.15 there are only a few visible resets: to detect the reset at $t_k = \pi + d_k$, we should implement a simulation step size smaller than $|d_k|$, that, from the quadratic law $|d_{k+1}| = |d_k|^2$, decreases very fast with k, implying a strong computational cost.

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