Chapter 2
The Restricted Linear Group

A celebrated result of H.J.S. Smith [47, 86] shows that when Λ is a commutative integral domain which possesses a Euclidean algorithm then an arbitrary $m \times m$ matrix X over Λ can be expressed as a product $X = E_+ D E_-$ where D is diagonal and E_+, E_- are products of elementary unimodular matrices. This chapter is a general study of rings whose matrices possess an analogue of such a Smith Normal Form.

2.1 Some Identities Between Elementary Matrices

Given a ring Λ we denote by $M_n(\Lambda)$ the ring of $(n \times n)$-matrices over Λ and by $GL_n(\Lambda)$ the group of invertible $n \times n$-matrices over Λ. For each $n \geq 2$, $M_n(\Lambda)$ has the canonical Λ-basis $\epsilon(i,j)_{1 \leq i,j \leq n}$ given by $\epsilon(i,j)_{r,s} = \delta_{ir}\delta_{js}$. The elementary invertible matrices $E(i,j;\lambda)$ $(\lambda \in \Lambda)$ and $D(i,\delta)$ $(\delta \in \Lambda^*)$ which perform row and column operations are expressed in terms of the basic matrices as follows;

$$E(i,j;\lambda) = \mathrm{I}_n + \lambda\epsilon(i,j) \quad (i \neq j);$$

$$D(i,\delta) = \mathrm{I}_n + (\delta - 1)\epsilon(i,i).$$

There are a number of familiar identities between these matrices:

$$E(i,j;\lambda)E(i,j;\mu) = E(i,j;\lambda + \mu); \tag{2.1}$$

$$E(i,j;\lambda)^{-1} = E(i,j;-\lambda); \tag{2.2}$$

$$[E(i,j;\lambda), E(j,k;\mu)] = E(i,k;\lambda\mu) \quad (i \neq k); \tag{2.3}$$

$$[E(i,j;\lambda), E(k,l;\mu)] = 1 \quad (\{i,j\} \cap \{k,l\} = \emptyset). \tag{2.4}$$

Here we are taking the commutator $[X,Y]$ to be $[X,Y] = XYX^{-1}Y^{-1}$.

$$D(i,\lambda)D(i,\mu) = D(i,\lambda\mu); \tag{2.5}$$

$$D(i,\lambda)^{-1} = D(i,\lambda^{-1}); \tag{2.6}$$

F.E.A. Johnson, *Syzygies and Homotopy Theory*, Algebra and Applications 17,
DOI 10.1007/978-1-4471-2294-4_2, © Springer-Verlag London Limited 2012

$$D(i,\lambda)E(i,j;\mu) = E(i,j;\lambda\mu)D(i,\lambda); \tag{2.7}$$

$$D(i,\lambda)E(j,k;\mu) = E(j,k;\mu)D(i,\lambda) \quad (i \notin \{j,k\}). \tag{2.8}$$

For 2×2 matrices we have the following identity where $u \in \Lambda^*$;

$$\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & u^{-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -u & 1 \end{bmatrix}\begin{bmatrix} 1 & u^{-1} \\ 0 & 1 \end{bmatrix}.$$

It generalises to the following identity with $i \neq j$:

$$\begin{aligned} \Delta(i,u)\Delta(j,u^{-1}) = E(j,i;1)E(i,j;-1)E(j,i;1)E(i,j;u^{-1}) \\ \times E(j,i;-u)E(i,j;u^{-1}). \end{aligned} \tag{2.9}$$

Let Σ_n denote the group of permutations on $\{1,\ldots,n\}$. For each $\sigma \in \Sigma_n$ there is an $n \times n$ permutation matrix $P(\sigma)$ defined by

$$P(\sigma)_{r,s} = \delta_{r,\sigma(s)}.$$

It is straightforward to see that:

$$P \text{ defines an injective homomorphism } P : \Sigma_n \to GL_n(\Lambda). \tag{2.10}$$

The permutation matrices $P(\sigma)$ can be expressed as products of matrices of the form $D(i,-1)$ and $E(i,j;\pm 1)$. As Σ_n is generated by the transpositions (i,j) it suffices to express each $P(i,j)$ as a product of this type. In fact, we have:

$$P(\sigma) = D(j,-1)E(i,j;1)E(j,i;-1)E(i,j;1). \tag{2.11}$$

It is useful to record how the permutation matrices $P(\sigma)$ interact with the $E(i,j;\lambda)$ and $D(i,\delta)$. First the action on the basic matrices $\epsilon(i,j)$;

$$P(\sigma)\epsilon(i,j) = \epsilon(\sigma(i),\sigma(j))P(\sigma). \tag{2.12}$$

This easily implies

$$P(\sigma)E(i,j;\lambda) = E(\sigma(i),\sigma(j);\lambda)P(\sigma). \tag{2.13}$$

Alternatively expressed:

$$E(i,j;\lambda)P(\sigma) = P(\sigma)E(\sigma^{-1}(i),\sigma^{-1}(j);\lambda). \tag{2.14}$$

Whilst

$$P(\sigma)D(i,\delta) = D(\sigma(i),\delta)P(\sigma). \tag{2.15}$$

In the special case where $\lambda \in \Lambda^*$ we have the matrix equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -\lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda^{-1} & 1 \end{pmatrix}$$

which generalises to

$$P(i,j)E(i,j;\lambda) = E(i,j;\lambda^{-1})D(i,-\lambda^{-1})D(j,\lambda)E(j,i;\lambda^{-1}). \tag{2.16}$$

2.2 The Restricted Linear Group

For $n \geq 2$ we denote by $D_n(\Lambda)$ the subgroup of $GL_n(\Lambda)$ defined by

$$D_n(\Lambda) = \{D(1,\delta) : \delta \in \Lambda^*\};$$

that is

$$D_n(\Lambda) = \left\{ \begin{bmatrix} \delta & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \right\}$$

and by $E_n(\Lambda)$ the subgroup of $GL_n(\Lambda)$ generated by the matrices $E(i,j;\lambda)$ $(\lambda \in \Lambda)$. From (2.7) and (2.8)

$$D(1,\delta)E(i,j;\lambda)D(1,\delta)^{-1} = \begin{cases} E(1,j;\delta\lambda) & i=1, \\ E(i,1;\lambda\delta^{-1}) & j=1, \\ E(i,j;\lambda) & i \notin \{i,j\}. \end{cases}$$

We see that:

$$D_n(\Lambda) \text{ normalises } E_n(\Lambda). \tag{2.17}$$

We define the *restricted linear group* $GE_n(\Lambda)$ to be the subgroup of $GL_n(\Lambda)$ given as the internal product

$$GE_n(\Lambda) = D_n(\Lambda) \cdot E_n(\Lambda).$$

In general $GE_n(\Lambda)$ is a proper subgroup of $GL_n(\Lambda)$ and Λ is said to be *weakly Euclidean* when $GE_n(\Lambda) = GL_n(\Lambda)$ for all $n \geq 2$. We shall examine this notion at greater length in Sects. 2.4 and 2.5. From (2.17) we get:

$$E_n(\Lambda) \text{ is a normal subgroup of } GE_n(\Lambda). \tag{2.18}$$

We put $\widehat{\Sigma}_n = \{P(\sigma) : \sigma \in \Sigma_n\}$.

Proposition 2.19 $\widehat{\Sigma}_n \subset GE_n(\Lambda)$.

Proof It follows from (2.11) that $P(i,j) \in E_n(\Lambda)$ for each transposition (i,j). The conclusion follows as Σ_n is generated by transpositions. □

It is useful to have different descriptions of $GE_n(\Lambda)$. For $\delta_1, \dots, \delta_n \in \Lambda^*$ let $\Delta(\delta_1, \dots, \delta_n)$ denote the diagonal matrix

$$\Delta(\delta_1, \dots, \delta_n) = \begin{bmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_n \end{bmatrix}$$

and put $\Delta_n(\Lambda) = \{\Delta(\delta_1, \dots, \delta_n) : \delta_i \in \Lambda^*\}$.

Proposition 2.20 *$\Delta_n(\Lambda)$ is a subgroup of $GE_n(\Lambda)$.*

Proof It is straightforward to see that $\Delta_n(\Lambda)$ is a subgroup of $GL_n(\Lambda)$. Note that $D(j, \delta_j) \in GE_n(\Lambda)$ as, by (2.15),

$$D(j, \delta) = P(1, j)D(1, \delta)P(1, j)$$

and $D(i, \delta), P(1, j) \in GE_n(\Lambda)$. The conclusion follows as $\Delta(\delta_1, \dots, \delta_n) = \prod_{j=1}^{n} D(j, \delta_j)$. □

By (2.7) we have $\Delta(\delta_1, \dots, \delta_n)E(i, j; \lambda)\Delta(\delta_1, \dots, \delta_n)^{-1} = E(i, j; \delta_i \lambda \delta_j^{-1})$ from which we see that:

$$\Delta_n(\Lambda) \text{ normalises } E_n(\Lambda). \tag{2.21}$$

Now $\Delta_n(\Lambda)$, $E_n(\Lambda)$ are subgroups of $GE_n(\Lambda) = D_n(\Lambda)E_n(\Lambda)$ and $D_n(\Lambda) \subset \Delta_n(\Lambda)$. We obtain another description of $GE_n(\Lambda)$.

$$GE_n(\Lambda) = \Delta_n(\Lambda) \cdot E_n(\Lambda). \tag{2.22}$$

The constructions GE_n, E_n are functorial under ring homomorphisms. In particular, given a surjective ring homomorphism $\pi : A \to B$ the induced map $\pi_* : E_n(A) \to E_n(B)$ is also surjective. However unless the induced map on units $\pi_* : A^* \to B^*$ is also surjective then the induced homomorphism $\pi_* : GE_n(A) \to GE_n(B)$, need not be surjective. We say that ring homomorphism $\pi : A \to B$ has the *lifting property for units* when the induced map on units $\pi_* : A^* \to B^*$ is surjective. Then we have:

Proposition 2.23 *Let $\pi : A \to B$ be a surjective ring homomorphism with the lifting property for units; then $\pi_* : GE_n(A) \to GE_n(B)$ is surjective.*

2.3 Matrices with a Smith Normal Form

If Λ is a ring and $\alpha_1, \dots, \alpha_n \in \Lambda$ write $\Delta(\alpha_1, \dots, \alpha_n)$ for the diagonal matrix

$$\Delta(\alpha_1, \dots, \alpha_n) = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}.$$

We shall say that $X \in M_n(\Lambda)$ *has a Smith Normal Form* when

$$X = E_+ \Delta(\alpha_1, \dots, \alpha_n) E_-$$

for some $E_+, E_- \in E_n(\Lambda)$ and some $\alpha_1, \dots, \alpha_n \in \Lambda$. More generally, if $X, Y \in M_n(\Lambda)$ we write $X \sim Y$ when $X = E_+ Y E_-$ for some $E_+, E_- \in E_n(\Lambda)$. Evidently we have:

Proposition 2.24 *Let $X \sim Y \in M_n(\Lambda)$; if Λ is commutative then* $\det(X) = \det(Y)$.

The identities of Sect. 2.1 allow us to move units around; let $\alpha_1, \dots, \alpha_n \in \Lambda$ and $u_1, \dots, u_n \in \Lambda^*$. Writing $D(r) = \Delta(1, u_r)\Delta(r, u_r^{-1})$, $D = D(2)D(3)\cdots D(n)$ and $\mathbf{u} = u_1 \cdots u_n$ we see that $\Delta(\alpha_1 u_1, \dots, \alpha_n u_n) D = \Delta(\alpha_1 \mathbf{u}, \alpha_2, \dots, \alpha_n)$. However, each $D(r) \in E_n(\Lambda)$ by (2.9); thus for $u_1, \dots, u_n \in \Lambda^*$:

$$\Delta(\alpha_1 u_1, \dots, \alpha_n u_n) \sim \Delta(\alpha_1 \mathbf{u}, \alpha_2, \dots, \alpha_n) \quad \text{where } \mathbf{u} = u_1 \cdots u_n. \tag{2.25}$$

Similarly for $v_1, \dots, v_n \in \Lambda^*$:

$$\Delta(v_1 \alpha_1, \dots, v_n \alpha_n) \sim \Delta(\mathbf{v} \alpha_1, \alpha_2, \dots, \alpha_n) \quad \text{where } \mathbf{v} = v_n \cdots v_1. \tag{2.26}$$

Write $T(i, j) = E(i, j; 1)E(j, i; -1)E(i, j; 1) \in E_n(\Lambda)$; when τ is the transposition which interchanges the indices i and j we have

$$\Delta(\alpha_{\tau(1)}, \dots, \alpha_{\tau(n)}) = T(i, j)\Delta(\alpha_1, \dots, \alpha_n)T(j, i).$$

Writing an arbitrary permutation σ as a product of transpositions we see that

$$\Delta(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}) \sim \Delta(\alpha_1, \dots, \alpha_n). \tag{2.27}$$

The following is useful:

Proposition 2.28 *Let Λ be a commutative ring and suppose that $X \in M_n(\Lambda)$ has a Smith Normal Form where $n \geq 2$; if* $\det(X)$ *is indecomposable in Λ then*

$$X \sim \Delta(\det(X), 1, \dots, 1).$$

Proof As X has a Smith Normal Form then $X \sim \Delta(\alpha_1, \dots, \alpha_n)$ for some $\alpha_1, \dots, \alpha_n \in \Lambda$. By Proposition 2.24 $\det(X) = \prod_{r=1}^n \alpha_r$. As $\det(X)$ is indecomposable it follows that there is an index t such that $\alpha_r \in \Lambda^*$ for $r \neq t$. Denoting by σ the transposition $(1, t)$ we see from (2.27) that $X \sim \Delta(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})$ where $\alpha_{\sigma(r)} \in \Lambda^*$ for $r \geq 2$. By (2.25) $X \sim \Delta(\prod_r^n \alpha_{\sigma(r)}, 1, \dots, 1)$. However $\prod_r^n \alpha_{\sigma(r)} = \prod_{r=1}^n \alpha_r = \det(X)$. $\square$

The Smith Normal Form is compatible with products of rings as we now proceed to show. Suppose $\Lambda = \Lambda_1 \times \Lambda_2$ is a direct product of rings. Then the projections

$\pi_r : \Lambda \to \Lambda_r$ induce ring homomorphisms $\pi_r : M_m(\Lambda) \to M_m(\Lambda_r)$ so that

$$(\pi_1, \pi_2) : M_m(\Lambda) \to M_m(\Lambda_1) \times M_m(\Lambda_2) \text{ is a ring isomorphism.} \tag{2.29}$$

In consequence:

$$(\pi_1, \pi_2) : GL_m(\Lambda) \to GL_m(\Lambda_1) \times GL_m(\Lambda_2) \text{ is an isomorphism of groups.} \tag{2.30}$$

Requiring slightly more care is:

Proposition 2.31 (π_1, π_2) *induces an isomorphism* $E_m(\Lambda) \xrightarrow{\simeq} E_m(\Lambda_1) \times E_m(\Lambda_2)$.

Proof $\pi_r(E(k, l; (\lambda_1, \lambda_2))) = E(k, l; \lambda_r)$ so that (π_1, π_2) induces a group homomorphism $(\pi_1, \pi_2) : E_m(\Lambda) \to E_m(\Lambda_1) \times E_m(\Lambda_2)$. Evidently (π_1, π_2) is injective since it is already the restriction of a ring isomorphism. To show that (π_1, π_2) is onto $E_m(\Lambda_1) \times E_m(\Lambda_2)$ we first show that:

$$\text{If } X \in E_m(\Lambda_1) \text{ then there exists } \widetilde{X} \in E_m(\Lambda) \text{ such that } (\pi_1, \pi_2)(\widetilde{X}) = (X, \mathrm{Id}). \tag{$*$}$$

To prove $(*)$ write $X \in E_m(\Lambda_1)$ as a product $X = E(k_1, l_1; \lambda_1) \cdots E(k_N, l_N; \lambda_N)$ and write $\widetilde{X} = E(k_1, l_1; (\lambda_1, 0)) \cdots E(k_N, l_N; (\lambda_N, 0))$. One sees easily that $(\pi_1, \pi_2)(\widetilde{X}) = (X, \mathrm{Id})$. Similarly:

$$\text{If } Y \in E_m(\Lambda_2) \text{ then there exists } \widetilde{Y} \in E_m(\Lambda) \text{ such that } (\pi_1, \pi_2)(\widetilde{Y}) = (\mathrm{Id}, Y). \tag{$**$}$$

Surjectivity of (π_1, π_2) now follows, for if $(X, Y) \in E_m(\Lambda_1) \times E_m(\Lambda_2)$ then

$$(X, Y) = (X, \mathrm{Id})(\mathrm{Id}, Y) = (\pi_1, \pi_2)(\widetilde{X})(\pi_1, \pi_2)(\widetilde{Y}) = (\pi_1, \pi_2)(\widetilde{X}\widetilde{Y}). \qquad \square$$

The existence of Smith Normal Forms is compatible with products in the strongest sense; let $X \in M_m(\Lambda)$ and put $X_r = \pi_r(X) \in M_m(\Lambda_r)$ for $r = 1, 2$. Suppose given sequences $\alpha_s^r \in \Lambda_r$ ($s = 1, \dots, m$, $r = 1, 2$); with this notation:

Theorem 2.32 *For any permutations* σ, τ *we have*:

$$X \sim \begin{pmatrix} (\alpha^1_{\sigma(1)}, \alpha^2_{\tau(1)}) & & & \\ & (\alpha^1_{\sigma(2)}, \alpha^2_{\tau(2)}) & & \\ & & \ddots & \\ & & & (\alpha^1_{\sigma(n)}, \alpha^2_{\tau(n)}) \end{pmatrix}$$

$$\iff \quad X_r \sim \begin{pmatrix} \alpha_1^r & & & \\ & \alpha_2^r & & \\ & & \ddots & \\ & & & \alpha_n^r \end{pmatrix}.$$

We shall say that the ring Λ is *generalized Euclidean* when for all $m \geq 2$,

$$M_m(\Lambda) = E_m(\Lambda)\mathcal{D}_m(\Lambda)E_m(\Lambda). \tag{2.33}$$

The terminology *generalized Euclidean* is taken from the classical sufficient conditions for this to occur, namely those of the Smith Normal Form, which may be re-expressed thus:

Proposition 2.34 (Smith [86]) *Let Λ be a commutative integral domain with a Euclidean algorithm; then Λ is generalized Euclidean.*

It follows from Theorem 2.32 that:

Proposition 2.35 *Let Λ_1, Λ_2 be generalized Euclidean rings; then $\Lambda_1 \times \Lambda_2$ is also generalized Euclidean.*

To consider an example, we denote by $\mathbf{Z}_n$ the ring of residues mod n; that is, $\mathbf{Z}_n = \mathbf{Z}/n$. If n is square free we may write it as a product of distinct primes $n = p_1 p_2 \cdots p_k$. Then $\mathbf{Z}_n \cong \mathbf{F}_{p_1} \times \cdots \times \mathbf{F}_{p_k}$ where $\mathbf{F}_p$ is the field with p elements and so

$$\mathbf{Z}_n[t, t^{-1}] \cong \mathbf{F}_{p_1}[t, t^{-1}] \times \cdots \times \mathbf{F}_{p_k}[t, t^{-1}].$$

However, each $\mathbf{F}_{p_r}[t, t^{-1}]$ is a Euclidean ring and so is generalized Euclidean by Proposition 2.34; from Proposition 2.35 we see that:

$$\text{If } n \text{ is square free then } \mathbf{Z}_n[t, t^{-1}] \text{ is generalized Euclidean.} \tag{2.36}$$

Another useful result is:

Proposition 2.37 *Let Λ_1 be a generalized Euclidean ring; if $\varphi : \Lambda_1 \to \Lambda_2$ is a surjective ring homomorphism then Λ_2 is also generalized Euclidean ring.*

Proof Let $A \in M_m(\Lambda_2)$. First choose $\widetilde{A} \in M_m(\Lambda_1)$ such that $\varphi_*(\widetilde{A}) = A$. Since Λ_1 is generalized Euclidean, we may write $\widetilde{A} = P_1 D P_2$ where $P_i \in E_m(\Lambda_1)$ and $D \in \mathcal{D}_m(\Lambda_1)$. Then $\varphi_*(P_i) \in E_m(\Lambda_2)$, $\varphi_*(D) \in \mathcal{D}_m(\Lambda_2)$ and the required decomposition is given by

$$A = \varphi_*(P_1)\varphi_*(D)\varphi_*(P_2).$$

□

Next consider diagonal matrices

$$\Delta = \begin{pmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_m \end{pmatrix} \in M_m(\Lambda).$$

We say that Δ is of *restricted type* when $\delta_2, \ldots, \delta_m \in \Lambda^*$ and that $\Delta \in M_m(\Lambda)$ is of *very restricted type* when $\delta_2 = \cdots = \delta_m = 1$. Likewise we say that a matrix $X \in M_m(\Lambda)$ has a Smith Normal Form of *restricted type* (*resp. very restricted type*) when $X \sim \Delta$ where Δ is a diagonal matrix of *restricted type* (*resp. very restricted type*). From (2.25) and (2.27) it follows easily that if $X \in M_m(\Lambda)$ then:

$$\left\{\begin{matrix} X \text{ has a Smith Normal} \\ \text{Form of restricted type} \end{matrix}\right\} \iff \left\{\begin{matrix} X \text{ has a Smith Normal} \\ \text{Form of very restricted type} \end{matrix}\right\}. \tag{2.38}$$

Given a ring Λ a two sided ideal J in Λ is of *radical type* when $u + j \in \Lambda^*$ for any $j \in J$ and $u \in \Lambda^*$. We say that a ring homomorphism $\varphi : A \to B$ has the *strong lifting property for units* when, in addition to the lifting property for units, φ satisfies

$$\alpha \in A^* \iff \pi(\alpha) \in B^*. \tag{2.39}$$

It is straightforward to see that if $\varphi : A \to B$ is a surjective ring homomorphism with the lifting property for units then:

$$\varphi \text{ has the strong lifting property for units} \iff \text{Ker}(\varphi) \text{ is of radical type.} \tag{2.40}$$

If J is a two-sided ideal in Λ we shall say that a matrix $U = (u_{ij}) \in M_m(\Lambda)$ has *restricted form* with respect to J when $u_{22}, \ldots, u_{mm} \in \Lambda^*$ whilst $u_{ij} \in J$ when $i \neq j$ and $i, j \geq 2$; otherwise expressed, the image $\overline{U}$ in Λ/J then takes the form

$$\overline{U} = \begin{pmatrix} * & * & * & * \\ * & \overline{u_{22}} & 0 & 0 \\ * & 0 & \ddots & 0 \\ * & 0 & 0 & \overline{u_{mm}} \end{pmatrix}.$$

Now suppose that J is a two sided ideal of radical type in Λ and that $U \in M_m(\Lambda)$ has restricted form with respect to J. As r descends from m to 2 we may, by means of suitable row and column operations, successively kill the rth row and column leaving units in the (r, r)th places; we obtain:

Proposition 2.41 *Let J be a two sided ideal of radical type in Λ; if $U \in M_m(\Lambda)$ has restricted form with respect to J then U has a Smith Normal Form of restricted type.*

Lemma 2.42 (Lifting Lemma) *Let $\varphi : \Lambda \to \breve{\Lambda}$ be a surjective ring homomorphism with the strong lifting property for units; if $X \in M_m(\Lambda)$ is such that $\varphi(X)$ has a Smith Normal Form of restricted type then X also has a Smith Normal Form of restricted type.*

Proof By hypothesis, $\varphi(X)$ has a Smith Normal Form of restricted type which, by (2.38), we may take to be of the form

$$\begin{pmatrix} \xi & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

Formally, there exist $E_+, E_- \in E_m(\breve{\Lambda})$ such that $E_+\varphi(X)E_- = \Delta(\xi, 1, \dots, 1)$. As φ is surjective we may now choose $E'_+, E'_- \in E_m(\Lambda)$ such that $\varphi(E'_\sigma) = E_\sigma$. One sees easily that $E'_+XE'_-$ has restricted form with respect to the two sided ideal $\mathrm{Ker}(\varphi)$. However, as φ has the strong lifting property for units then $\mathrm{Ker}(\varphi)$ is of radical type. By Proposition 2.41, $E'_+XE'_-$ has a Smith Normal Form of restricted type. Hence X also has a Smith Normal Form of restricted type. □

2.4 Weakly Euclidean Rings

The ring Λ is said to be *weakly Euclidean* when $GL_n(\Lambda) = GE_n(\Lambda)$ for all $n \geq 2$. As $\Delta_m(\Lambda)$ normalises $E_m(\Lambda)$ this is equivalent to requiring that each $X \in GL_n(\Lambda)$ has a Smith Normal Form. We may now establish a Recognition Criterion for this property:

Proposition 2.43 *Let $\varphi : A \to B$ be a surjective ring homomorphism where B is weakly Euclidean; if φ has the strong lifting property for units then A is also weakly Euclidean.*

Proof Let $X \in GL_m(A)$; then $\varphi(X) \in GL_m(B)$. By hypothesis on B, $\varphi(X)$ is a product

$$\varphi(X) = D(\breve{\delta}, 1)E,$$

where $\breve{\delta} \in B^*$ and $E \in E_m(B)$. In particular, $\varphi(X)$ has a Smith Normal Form of very restricted type. By Lemma 2.42 and (2.38), X also has a Smith Normal Form of very restricted type so we may write $X = E_+D(\delta, 1)E_-$ for some $E_+, E_- \in E_m(A)$. As X is invertible then $\delta \in A^*$. It follows from (2.17) that $E'_+ = D(\delta^{-1}, 1)E_+D(\delta, 1) \in E_m(A)$ and so $X = D(\delta, 1)E$ where $E = E'_+E_- \in E_m(A)$. □

There is a useful generalization of the notion of weakly Euclidean ring; for any ring Λ the group $GL_m(\Lambda)$ imbeds via stabilization in $GL_n(\Lambda)$ for $m \leq n$; moreover $GL_m(\Lambda) \cdot E_n(\Lambda)$ is a subgroup of $GL_n(\Lambda)$ and is normalised by $GL_m(\Lambda)$. We say that Λ is *m-weakly Euclidean* when $GL_n(\Lambda) = GL_m(\Lambda) \cdot E_n(\Lambda) =$ for $m \leq n$. Evidently if Λ is m-weakly Euclidean then it is Λ is n-weakly Euclidean when $m \leq n$. Moreover, Λ is 1-weakly Euclidean precisely when it is weakly Euclidean.

2.5 Examples of Weakly Euclidean Rings

The standard theory of reduction by row operations shows that:

$$\text{Any division ring is weakly Euclidean.} \tag{2.44}$$

Moreover, from Smith's Theorem Proposition 2.34, it follows a fortiori that:

$$\text{Any commutative Euclidean domain is weakly Euclidean.} \tag{2.45}$$

Let L be a (possibly noncommutative) local ring; it is straightforward to see that the canonical homomorphism $\pi : L \to L/\mathrm{rad}(L)$ has the strong lifting property for units. Moreover, as $L/\mathrm{rad}(L)$ is a division ring it is weakly Euclidean. Applying Proposition 2.43 we get the following which first seems to have been observed by Klingenberg [64].

Corollary 2.46 *If L is a (possibly noncommutative) local ring then L is weakly Euclidean.*

Note that weakly Euclidean rings are closed under products; that is:

Proposition 2.47 *If $R_1, \ldots, R_m$ are weakly Euclidean rings then $R_1 \times \cdots \times R_m$ is also weakly Euclidean.*

We note also the following adjunct to Morita theory:

Theorem 2.48 *Let R be a weakly Euclidean ring; then for each $n \geq 1$, the ring $M_n(R)$ of $n \times n$ matrices over R is also weakly Euclidean.*

Proof If V is an R-module then for each $m \geq 1$ put $V^{(m)} = \underbrace{V \oplus \cdots \oplus V}_{m}$; for each $m \geq 1$ there is a ring isomorphism $\Psi : \mathrm{End}_R(V^{(m)}) \xrightarrow{\simeq} M_m(\mathrm{End}_R(V))$ given by

$$\Psi(\alpha)_{rs} = \pi_s \alpha i_r,$$

where $i_s : V \to V^{(m)}$ is the inclusion of the sth summand and $\pi_r : V^{(m)} \to V$ is projection onto the rth-factor. When V is a free module of rank n with basis $\{\epsilon_k\}_{1 \leq k \leq n}$ we identify $\mathrm{End}_R(V)$ with $M_n(R)$. Moreover $V^{(m)}$ then has the basis $\{E_t\}_{1 \leq t \leq mn}$ where, on writing $t = n(a-1) + b$ with $1 \leq a \leq m$ and $1 \leq b \leq n$,

$$E_t = i_a(\epsilon_b)$$

enabling us to identify $\mathrm{End}_R(V^{(m)})$ with $M_{mn}(R)$. Ψ then becomes the 'block decomposition' isomorphism

$$\Psi : M_{mn}(R) \xrightarrow{\simeq} M_m(M_n(R))$$

inducing a group isomorphism $\Psi : GL_{mn}(R) \xrightarrow{\simeq} GL_m(M_n(R))$. It will suffice to show that

$$\Psi(GE_{mn}(R)) \subset GE_m(M_n(R)).$$

For then, by the weakly Euclidean hypothesis on R, $GE_{mn}(R) = GL_{mn}(R)$ and so

$$GL_m(M_n(R)) = \Psi(GL_{mn}(R)) \subset GE_m(M_n(R))$$

and hence $GL_m(M_n(R)) = GE_m(M_n(R))$ showing that $M_n(R)$ is weakly Euclidean. To show that $\Psi(GE_{mn}(R)) \subset GE_m(M_n(R))$ first observe that $\Psi(\Delta_{mn}(R)) \subset \Delta_m(M_n(R))$. As $GE_{mn}(R) = \Delta_{mn}(R) \cdot E_{mn}(R)$ it therefore suffices to show that

$$\Psi(E(s,t;\lambda)) \in GE_m(M_n(R))$$

for $1 \le s,t \le mn$ and $s \ne t$, $\lambda \in R$. Write $s = n(a-1)+b$, $t = n(a'-1)+b'$ with $1 \le a, a' \le m$ and $1 \le b, b' \le n$. If $a = a'$ then $\Psi(E(s,t;\lambda)) \in \Delta_m(M_n(R))$ whilst if $a \ne a'$ $\Psi(E(s,t;\lambda)) \in E_m(M_n(R))$. □

For $n \ge 2$ we denote by F_n the free *nonabelian* group of rank n, whilst F_1 will denote the infinite cyclic group C_∞. We have the following theorem of Cohn [17]:

Theorem 2.49 *For any division ring D the group ring $D[F_n]$ is weakly Euclidean.*

Let R be a ring and G a group; we say that the group ring $R[G]$ has *only trivial units* when each $\lambda \in R[G]^*$ has the form $\lambda = ug$ for $u \in R^*$ and $g \in G$.

Proposition 2.50 *Let $\pi : A \to B$ be a surjective ring homomorphism with the strong lifting property for units and suppose that the ideal* $\mathrm{Ker}(\pi)$ *is nilpotent*; *if G is a group for which $B[G]$ has only trivial units then the induced homomorphism $\pi_* : A[G] \to B[G]$ has the strong lifting property for units.*

Proof Putting $J = \mathrm{Ker}(\pi)$ observe that $\mathrm{Ker}(\pi_*) = J[G] = \{\sum_{g\in G} \xi_g g : \xi_g \in J\}$. Now, by hypothesis, $J^M = \{0\}$ for some $M \ge 1$. It follows that if $X \in \mathrm{Ker}(\pi_*)$ then $X^M = 0$. Hence $1 - X \in A[G]^*$ as

$$(1-X)(1+X+X^2+\cdots+X^{M-1}) = 1.$$

Now let $\alpha \in A[G]$ satisfy $\pi_*(\alpha) \in B[G]^*$. We must show that $\alpha \in A[G]^*$. By hypothesis, $B[G]^*$ has only trivial units; that is $\pi_*(\alpha) = ug$ for some $u \in B^*$ and $g \in G$. As π is surjective we may choose $v \in A$ such that $\pi(v) = u$. It then follows that $v \in A^*$ as π has the strong lifting property for units. Put $\gamma = v^{-1}\alpha g^{-1}$; and $X = 1 - \gamma$; then $\gamma = 1 - X$ where $X \in \mathrm{Ker}(\pi_*)$, so that $\gamma \in A[G]^*$ by the above and $\alpha = v\gamma g \in A[G]^*$ as required. □

If G is a group and X, Y are subsets of G we say that $g \in G$ is represented as a product in X, Y when $g = xy$ for some $x \in X$ and $y \in Y$. We say that $g \in G$ is *uniquely represented* as a product in X, Y when, in addition, if $g = x'y'$ with $x' \in X$ and $y' \in Y$ then $x = x'$ and $y = y'$. We say that G satisfies the *two unique products* condition (abbreviated to $\mathcal{TUP}$) when, given finite subsets X, Y of G with $2 \le |X|$ and $2 \le |Y|$, at least two elements of G are uniquely represented as products in X, Y. In Appendix C we give a fuller account of this topic; in particular we will see:

Theorem 2.51 *Let G be a $\mathcal{TUP}$ group; then for any (possibly noncommutative) integral domain A, $A[G]$ has only trivial units.*

It is known (cf. Appendix C) that the free group F_n satisfies the $\mathcal{TUP}$ condition; thus for any division ring D the group ring $D[F_n]$ has only trivial units. Suppose that L is a local ring in which $\mathrm{rad}(L)$ is nilpotent and put $D = L/\mathrm{rad}(L)$; by applying Proposition 2.50 to the induced homomorphism $L[F_n] \to D[F_n]$ we may extend Theorem 2.49 as follows:

Corollary 2.52 *If L is a local ring for which* $\mathrm{rad}(L)$ *is nilpotent then the group ring $L[F_n]$ is weakly Euclidean.*

With very little change the requirement that $\mathrm{rad}(L)$ be nilpotent may be replaced by the hypothesis that L is complete to obtain:

Proposition 2.53 *If $\widehat{L}$ is a complete local ring then $\widehat{L}[F_n]$ is weakly Euclidean for any $n \ge 1$.*

We may generate a useful class of examples by the triangular algebra construction. If R is a ring and n is an integer ≥ 2 we define

$$\mathcal{T}_n(R) = \{X \in M_n(R) : X_{ij} = 0 \text{ for } i < j\};$$

that is $\mathcal{T}_n(R)$ consists of elements of the form

$$\begin{pmatrix} X_{11} & & & & \\ & X_{22} & & 0 & \\ & & \ddots & & \\ & * & & \ddots & \\ & & & & X_{nn} \end{pmatrix}.$$

Observe that $\mathcal{T}_n(R)$ is a subring of $M_n(R)$. Moreover, there is an obvious surjective ring homomorphism

$$\delta : \mathcal{T}_n(R) \to \underbrace{R \times \cdots \times R}_{n}; \quad \delta(X) = (X_{11}, X_{22}, \ldots, X_{nn}).$$

It is straightforward to check that δ has the strong lifting property for units. From this observation together with Propositions 2.43 and 2.47 we see that:

Proposition 2.54 *If R is weakly Euclidean then $\mathcal{T}_n(R)$ is also weakly Euclidean.*

2.6 The Dieudonné Determinant

If $P(\sigma)$ is a permutation matrix then by (2.15)

$$P(\sigma)\Delta(\delta_1,\ldots,\delta_n)P(\sigma)^{-1} = \Delta(\delta_{\sigma(1)},\ldots,\delta_{\sigma(n)}).$$

In particular, in $GE_n(\Lambda)$ the subgroup $\widehat{\Sigma}_n$ of permutation matrices normalises the subgroup $\Delta_n(\Lambda)$ of diagonal matrices. We define the *core subgroup* $\mathcal{C}_n(\Lambda)$ of $GE_n(\Lambda)$ to be the internal product

$$\mathcal{C}_n(\Lambda) = \Delta_n(\Lambda)\cdot\widehat{\Sigma}_n. \tag{2.55}$$

We have seen, in (2.21), that $\Delta_n(\Lambda)$ normalises $E_n(\Lambda)$. It follows from (2.13) that $\widehat{\Sigma}_n$ also normalises $E_n(\Lambda)$. Thus $\mathcal{C}_n(\Lambda)$ normalises $E_n(\Lambda)$. However, $GE_n(\Lambda) = \Delta_n(\Lambda)\cdot E_n(\Lambda)$ so that, a fortiori,

$$GE_n(\Lambda) = \mathcal{C}_n(\Lambda)\cdot E_n(\Lambda). \tag{2.56}$$

It is clear that $\Delta_n(\Lambda)\cap\widehat{\Sigma}_n = \{1\}$ so that the decomposition (2.55) is actually a semi-direct product. Alternatively we can regard $\mathcal{C}_n(\Lambda)$ as a semidirect product

$$\mathcal{C}_n(\Lambda) \cong (\Lambda^*)^n \bullet \Sigma_n, \tag{2.57}$$

where Σ_n acts on $(\Lambda^*)^n$ via $\sigma\cdot(\delta_1,\ldots,\delta_n) = (\delta_{\sigma(1)},\ldots,\delta_{\sigma(n)})$. Let $(\Lambda^*)^{ab}$ denote the abelianization of the unit group $(\Lambda^*)^{ab}$; that is

$$(\Lambda^*)^{ab} = \Lambda^*/[\Lambda^*,\Lambda^*]. \tag{2.58}$$

If $\delta\in\Lambda^*$ we denote by $[\delta]$ its class in $(\Lambda^*)^{ab}$. One sees easily that the multiplication map

$$\mu : (\Lambda^*)^n \to (\Lambda^*)^{ab};\qquad \mu(\delta_1,\ldots,\delta_n) = [\delta_1],\ldots,[\delta_n]$$

is a homomorphism. Moreover, μ extends to a homomorphism from the semidirect product

$$\widehat{\mu} : (\Lambda^*)^n \bullet \Sigma_n \to (\Lambda^*)^{ab}\quad \text{by } \widehat{\mu}(\delta_1,\ldots,\delta_n;\sigma) = [\mathrm{sign}(\sigma)][\delta_1],\ldots,[\delta_n].$$

We define the *proto-determinant* $\mathrm{prot}_n : \mathcal{C}_n(\Lambda)\to(\Lambda^*)^{ab}$ to be the homomorphism which corresponds to $\widehat{\mu}$ under the isomorphism (2.57); that is:

$$\mathrm{prot}_n\,(\Delta(\delta_1,\ldots,\delta_n)\cdot P(\sigma)) = [\mathrm{sign}(\sigma)][\delta_1],\ldots,[\delta_n]. \tag{2.59}$$

The proto-determinant is compatible with stabilization. For any integers k,n with $1\le k$ the stabilization inclusion $s_{n,k} : GL_n(\Lambda)\to GL_{n+k}(\Lambda)$

$$s_{n,k}(A) = \begin{pmatrix} A & 0\\ 0 & I_k\end{pmatrix}$$

induces the subsidiary inclusions $s_{n,k} : E_n(\Lambda) \to E_{n+k}(\Lambda)$, $s_{n,k} : \Delta_n(\Lambda) \to \Delta_{n+k}(\Lambda)$, $s_{n,k} : GE_n(\Lambda) \to GE_{n+k}(\Lambda)$, $s_{n,k} : \widehat{\Sigma}_n \to \widehat{\Sigma}_{n+k}$ and hence also $s_{n,k} : \mathcal{C}_n(\Lambda) \to \mathcal{C}_{n+k}(\Lambda)$ and the following diagram commutes for each k, n with $1 \le k$

$$\begin{array}{ccc} \mathcal{C}_{n+k}(\Lambda) & \stackrel{\mathrm{prot}_{n+k}}{\to} & (\Lambda^*)^{ab} \\ \uparrow s_{n,k} & & \uparrow \mathrm{Id} \\ \mathcal{C}_n(\Lambda) & \stackrel{\mathrm{prot}_n}{\to} & (\Lambda^*)^{ab} \end{array}$$

By a *weak determinant* for Λ we mean a family $\{\det_n\}_{2\le n}$ of group homomorphisms

$$\det_n : GE_n(\Lambda) \to (\Lambda^*)^{ab}$$

such that

(i) $\det_n$ extends prot_n; that is $\det_{n|\mathcal{C}_n(\Lambda)} = \mathrm{prot}_n$;
(ii) the family $\{\det_n\}_{2\le n}$ is compatible with stabilization; that is, the diagram below commutes for each k, n with $1 \le k$:

$$\begin{array}{ccc} GE_{n+k}(\Lambda) & \stackrel{\det_{n+k}}{\to} & (\Lambda^*)^{ab} \\ \uparrow s_{n,k} & & \uparrow \mathrm{Id} \\ GE_n(\Lambda) & \stackrel{\det_n}{\to} & (\Lambda^*)^{ab} \end{array}$$

Observe that:

Proposition 2.60 *If* $\{\det_n\}_{2\le n}$ *is a weak determinant for* Λ *then* $\det_n(E) = 1$ *for each* $E \in E_n(\Lambda)$.

Proof It suffices to show that $\det_n(E(i, j; \lambda)) = 1$ for each generator $E(i, j; \lambda)$. When $n \ge 3$ then, by (2.3), for some $r \le n$, $E(i, j; \lambda) = [E(i, r; \lambda), E(r, j; 1)]$ so that

$$\begin{aligned} \det_n(E(i, j; \lambda)) &= \det_n(E(i, r; \lambda))\det_n(E(r, j; 1)) \\ &\quad \times \det_n(E(i, r; \lambda))^{-1}\det_n(E(r, j; 1))^{-1} \end{aligned}$$

and the conclusion follows as $\det_n$ takes values in the abelian group $(\Lambda^*)^{ab}$. For $n = 2$, the result follows by stabilization as

$$\det_2(E(i, j; \lambda)) = \det_3(s_{3,2}(E(i, j; \lambda))) = 1. \qquad \square$$

It follows that a weak determinant for Λ is unique; that is:

Proposition 2.61 *If* $\{\det_n\}_{2\le n}$ *and* $\{\det'_n\}_{2\le n}$ *are weak determinants for* Λ *then* $\det_n = \det'_n$ *for each* $n \ge 2$.

Proof Write $X \in GE_n(\Lambda)$ in the form $X = C \cdot E$ where $C \in \mathcal{C}_n(\Lambda)$ and $E \in E_n(\Lambda)$. Then $\det_n(C) = \mathrm{prot}_n(C)$ and $\det_n(E) = 1$ so that

$$\det_n(X) = \det_n(C) \cdot \det_n(E) = \mathrm{prot}_n(C).$$

Repeating the calculation gives $\det'_n(X) = \mathrm{prot}_n(C)$ so that $\det'_n = \det_n$. □

2.7 Equivalent Formulations of the Dieudonné Condition

We say that Λ is a *Dieudonné ring* when Λ possesses a weak determinant. In this section we derive a simple necessary and sufficient condition for Λ to be Dieudonné, namely that for each $n \geq 2$

$$D_n(\Lambda) \cap E_n(\Lambda) = D_n([\Lambda^*, \Lambda^*]),$$

where

$$D_n([\Lambda^*, \Lambda^*] = \{D(1, h) : h \in [\Lambda^*, \Lambda^*]\}.$$

We shall also show that when $\{\det_n\}_{2 \leq n}$ is a weak determinant for Λ then the sequence

$$1 \to E_n(\Lambda) \subset GE_n(\Lambda) \xrightarrow{\det_n} (\Lambda^*)^{ab} \to 1$$

is exact. We begin by observing:

Proposition 2.62 *$D_n([\Lambda^*, \Lambda^*]) \subset E_n(\Lambda)$ for each $n \geq 2$.*

Proof First consider the case $n = 2$. Then for $\alpha, \beta \in \Lambda^*$,

$$\begin{bmatrix} \alpha\beta\alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha\beta^{-1} \\ 0 & 1 \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & 0 \\ -\beta\alpha^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha\beta^{-1} \\ 0 & 1 \end{bmatrix}$$

from which we see that:

$$\begin{bmatrix} \alpha\beta\alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{bmatrix} \in E_2(\Lambda). \tag{I}$$

Replacing β by β^{-1} and taking $\alpha = 1$ we obtain the following, which is also a consequence of (2.9)

$$\begin{bmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{bmatrix} \in E_2(\Lambda). \tag{II}$$

Taking the product $\left[\begin{smallmatrix} \alpha\beta\alpha^{-1}\beta^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \alpha\beta\alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{smallmatrix}\right]\left[\begin{smallmatrix} \beta^{-1} & 0 \\ 0 & \beta \end{smallmatrix}\right]$ we conclude that

$$\begin{bmatrix} \alpha\beta\alpha^{-1}\beta^{-1} & 0 \\ 0 & 1 \end{bmatrix} \in E_2(\Lambda). \tag{III}$$

The statement for $n = 2$ now follows as $[\Lambda^*, \Lambda^*]$ is generated by the basic commutators $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ whilst for $n > 2$ the conclusion follows by stabilization as

$$D_n([\Lambda^*, \Lambda^*]) = s_{n,2}(D_2([\Lambda^*, \Lambda^*])) \subset s_{n,2}(E_2(\Lambda) \subset E_n(\Lambda). \qquad \square$$

It should cause no confusion to identify Λ^* with $D_n(\Lambda)$ and $[\Lambda^*, \Lambda^*]$ with $D_n([\Lambda^*, \Lambda^*])$. In particular we shall write

$$GE_n(\Lambda) = \Lambda^* \cdot E_n(\Lambda) \quad \text{and} \quad GE(\Lambda) = \Lambda^* \cdot E(\Lambda),$$

where

$$E(\Lambda) = \varinjlim E_n(\Lambda) \quad \text{and} \quad GE(\Lambda) = \varinjlim GE_n(\Lambda).$$

Whenever $2 \le k < n$ we have, by Proposition 2.62, a sequence of inclusions

$$[\Lambda^*, \Lambda^*] \subset \Lambda^* \cap E_k(\Lambda) \subset \Lambda^* \cap E_n(\Lambda) \subset \Lambda^* \cap E(\Lambda).$$

Let $\natural_n : (\Lambda^*)^{ab} \to GE_n(\Lambda)/E_n(\Lambda)$ be the composition $\natural_n = \nu_n \circ [\,]_n$ where

$$[\,]_n : \Lambda^*/[\Lambda^*, \Lambda^*] \to \Lambda^*/\Lambda^* \cap E_n(\Lambda)$$

is the natural surjection and

$$\nu_n : \Lambda^*/\Lambda^* \cap E_n(\Lambda) \to \Lambda^* E_n(\Lambda)/E_n(\Lambda) = GE_n(\Lambda)/E_n(\Lambda)$$

is the Noether isomorphism $\nu_n([\delta]_n) = \delta \cdot E_n(\Lambda)$. Let $\natural_\infty$ be the composition $\natural_\infty = \nu_\infty \circ [\,]_\infty : (\Lambda^*)^{ab} \to GE(\Lambda)/E(\Lambda)$ where $\nu_\infty = \varinjlim \nu_n$ and $[\,]_\infty = \varinjlim [\,]_n$. We get a commutative diagram with exact rows

$$\left\{\begin{array}{ccccccccc}
1 \to & \Lambda^* \cap E(\Lambda)/[\Lambda^*, \Lambda^*] & \to & (\Lambda^*)^{ab} & \stackrel{\natural_\infty}{\to} & GE(\Lambda)/E(\Lambda) & \to 1 \\
& \uparrow i_{\infty,n} & & \|\mathrm{Id} & & \uparrow s_{\infty,n} & \\
1 \to & \Lambda^* \cap E_n(\Lambda)/[\Lambda^*, \Lambda^*] & \to & (\Lambda^*)^{ab} & \stackrel{\natural_n}{\to} & GE_n(\Lambda)/E_n(\Lambda) & \to 1 \\
& \uparrow i_{n,k} & & \|\mathrm{Id} & & \uparrow s_{n,k} & \\
1 \to & \Lambda^* \cap E_k(\Lambda)/[\Lambda^*, \Lambda^*] & \to & (\Lambda^*)^{ab} & \stackrel{\natural_k}{\to} & GE_k(\Lambda)/E_k(\Lambda) & \to 1
\end{array}\right. \tag{2.66}$$

where the mappings $i_{\infty,n}$, $i_{n,k}$, $s_{\infty,n}$, $s_{n,k}$ are induced by stabilization. Note that $i_{\infty,n}$, $i_{n,k}$ are injective whilst $s_{\infty,n}$, $s_{n,k}$ are surjective.

The homomorphism $\Lambda^* \to K_1(\Lambda) = GL(\Lambda)/E(\Lambda)$; $\delta \mapsto D(1, \delta) \cdot E(\Lambda)$ induces a canonical mapping $i : (\Lambda^*)^{ab} \to K_1(\Lambda)$. As we now see, the Dieudonné condition on a ring Λ can be expressed in a number of ways. The conditions below are comprehensive without being exhaustive.

Theorem 2.64 *For any ring Λ the conditions below are equivalent*:

(i) *Λ admits a weak determinant*;
(ii) $\Lambda^* \cap E_n(\Lambda) = [\Lambda^*, \Lambda^*]$ *for each* $n \ge 2$;

(iii) $\Lambda^* \cap E(\Lambda) = [\Lambda^*, \Lambda^*]$;
(iv) *the canonical mapping* $\natural_n : (\Lambda^*)^{ab} \to GE_n(\Lambda)/E_n(\Lambda)$ *is an isomorphism for each* $n \geq 2$;
(v) *the canonical mapping* $\natural_\infty : (\Lambda^*)^{ab} \to GE(\Lambda)/E(\Lambda)$ *is an isomorphism*;
(vi) *the canonical mapping* $i : (\Lambda^*)^{ab} \to K_1(\Lambda)$ *is injective*.

Proof The equivalence of (ii) and (iii) follows directly from the definition of $E(\Lambda)$ as $\varinjlim E_n(\Lambda)$ whilst the equivalence of (ii) with (iv) and (iii) with (v) follows directly from the exactness of the rows in (2.66). Thus it suffices to show that (i) $\Longleftrightarrow$ (ii) and (iii) $\Longleftrightarrow$ (vi).

(i) $\Longrightarrow$ (ii) Let $\{\det_n\}_{2\leq n}$ be a weak determinant for Λ. As $\det_n(E) = 1$ for any $E \in E_n(\Lambda)$ then $\det_n$ induces a homomorphism $(\det_n)_* : GE_n(\Lambda)/E_n(\Lambda) \to (\Lambda^*)^{ab}$. Thus consider the diagram

$$\begin{tikzcd}
(\Lambda^*)^{ab} \arrow[r, "\natural_n"] & GE_n(\Lambda)/E_n(\Lambda) \arrow[r, "(\det_n)_*"] & (\Lambda^*)^{ab} \\
& \Lambda^* \arrow[ul, "{[\,]}"] \arrow[u, "i"] \arrow[ur, "{[\,]}"'] &
\end{tikzcd}$$

where $\natural_n$ is the homomorphism of (2.66) and $i : \Lambda^* \to GE_n(\Lambda)/E_n(\Lambda)$ is the homomorphism $i(\delta) = D_n(1, \delta) \cdot E_n(\Lambda)$. Computing we see that

$$\natural_n([\delta]) = D_n(1, \delta) \cdot E_n(\Lambda) \quad \text{and} \quad (\det_n)_*(D_n(1\delta) \cdot E_n(\Lambda)) = [\delta]$$

so that $(\det_n)_* \circ \natural_n = \mathrm{Id}_{|(\Lambda^*)^{ab}}$. In particular, $\natural_n$ is injective. However, by (2.66),

$$\mathrm{Ker}(\natural_n) = \Lambda^* \cap E_n(\Lambda)/[\Lambda^*, \Lambda^*]$$

and so $\Lambda^* \cap E_n(\Lambda) = [\Lambda^*, \Lambda^*]$. This proves (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i) From the filtration

$$[\Lambda^*, \Lambda^*] \subset \Lambda^* \cap E_k(\Lambda) \subset \Lambda^* \cap E_n(\Lambda) \subset \Lambda^* \tag{I}$$

we obtain a commutative diagram with exact rows

$$\left\{\begin{array}{ccccccccc}
1 \to & E_n(\Lambda) & \to & GE_n(\Lambda) & \xrightarrow{\gamma_n} & \Lambda^*/\Lambda^* \cap E_n(\Lambda) & \to 1 \\
& \uparrow s_{n,k} & & \uparrow s_{n,k} & & \uparrow \sigma_{n,k} & \\
1 \to & E_k(\Lambda) & \to & GE_k(\Lambda) & \xrightarrow{\gamma_k} & \Lambda^*/\Lambda^* \cap E_k(\Lambda) & \to 1 \\
& \uparrow s_{k,1} & & \uparrow s_{k,1} & & \uparrow \sigma_{k,1} & \\
1 \to & [\Lambda^*, \Lambda^*] & \to & \Lambda^* & \xrightarrow{[\,]} & \Lambda^*/[\Lambda^*, \Lambda^*] & \to 1
\end{array}\right. \tag{II}$$

in which γ_n is the composition $\gamma_n = \nu^{-1} \circ \langle\,\rangle_n$ where $\langle\,\rangle_n : GE_n(\Lambda) \to GE_n(\Lambda)/E_n(\Lambda)$ is the canonical mapping and $\nu^{-1} : GE_n(\Lambda)/E_n(\Lambda) \to \Lambda^*/\Lambda^* \cap E_n(\Lambda)$

is the inverse of the Noether isomorphism already considered. Here the mappings $s_{n,k}$ are induced by stabilization and the $\sigma_{n,k}$ are the appropriate quotient mappings from the filtration (i). By hypothesis $\Lambda^* \cap E_n(\Lambda) = [\Lambda^*, \Lambda^*]$ so that each $\sigma_{n,k}$ is the identity and (II) becomes

$$\begin{cases} 1 \to \quad E_n(\Lambda) \quad \to GE_n(\Lambda) \overset{\gamma_n}{\to} (\Lambda^*)^{ab} \to 1 \\ \qquad\quad \uparrow s_{n,k} \qquad\quad \uparrow s_{n,k} \qquad\quad \|\mathrm{Id} \\ 1 \to \quad E_k(\Lambda) \quad \to GE_k(\Lambda) \overset{\gamma_k}{\to} (\Lambda^*)^{ab} \to 1 \\ \qquad\quad \uparrow s_{k,1} \qquad\quad \uparrow s_{k,1} \qquad\quad \|\mathrm{Id} \\ 1 \to [\Lambda^*, \Lambda^*] \to \quad \Lambda^* \quad \overset{[\,]}{\to} (\Lambda^*)^{ab} \to 1 \end{cases} \tag{III}$$

Evidently γ_k is a homomorphism and we claim it extends prot_k. To see this, first note that $s_{k,1}(\delta) = D_k(1, \delta)$ so that, by commutativity, $\gamma_k(s_{k,1}(\delta)) = [\delta]$. By (2.11) we have

$$P(i, 1) = s_{k,1}(-1)E(1, 1; 1)E(1, i; -1)E(i, 1; 1) \quad \text{and hence}$$

$$\begin{aligned} \gamma_k(P(i, 1)) &= \gamma_k(s_{k,1}(-1))\gamma_k(E(i, 1; 1)E(1, i; -1)E(i, 1; 1)) \\ &= [-1]\gamma_k(E(i, 1; 1)E(1, i; -1)E(i, 1; 1)). \end{aligned}$$

However $E(i, 1; 1)E(1, i; -1)E(i, 1; 1) \in \mathrm{Ker}(\gamma_k)$ so that $\gamma_k(P(i, 1) = [-1]$. Now $P(i, j) = P(i, 1)P(j, 1)P(i, 1)$ so that

$$\gamma_k(P(i, j)) = \gamma_k(P(i, 1))\gamma_k(P(j, 1))\gamma_k(P(i, 1)) = [-1][-1][-1] = [-1].$$

It follows by induction that $\gamma_k(P(\sigma) = [\mathrm{sign}\,\sigma]$. Also $D_k(r, \delta) = P(1, r)D_k(1, \delta) \times P(1, r)$ so that

$$\gamma_k(D_k(r, \delta)) = \gamma_k(P(1, r))\gamma_k(D_k(1, \delta))\gamma_k(P(1, r)) = [-1][\delta][-1] = [\delta].$$

However $\Delta(\delta_1, \dots, \delta_k) = \prod_{r=1}^{k} D_k(r, \delta_r)$, hence $\gamma_k(\Delta(\delta_1, \dots, \delta_k)) = [\delta_1][\delta_2] \cdots [\delta_k]$. Thus

$$\gamma_k(\Delta(\delta_1, \dots, \delta_n)P(\sigma)) = [\mathrm{sign}(\sigma)][\delta_1][\delta_2] \cdots [\delta_k]$$

and so γ_k is a homomorphism extending prot_k. Moreover, as is clear from (III) above, $\{\gamma_n\}_{2 \le n}$ is compatible with stabilization; that is $\{\gamma_n\}_{2 \le n}$ is a weak determinant for Λ and so (ii) $\Longrightarrow$ (i).

(iii) $\Longleftrightarrow$ (vi) The canonical mapping $i : (\Lambda^*)^{ab} \to K_1(\Lambda)$ is simply the composition

$$(\Lambda^*)^{ab} \xrightarrow{\natural_\infty} GE(\Lambda)/E(\Lambda) \subset GL(\Lambda)/E(\Lambda)$$

so that, by exactness of the rows in (2.66), $\mathrm{Ker}(i) = \mathrm{Ker}(\natural_\infty) = \Lambda^* \cap E(\Lambda)/[\Lambda^*, \Lambda^*]$. In particular, $\Lambda^* \cap E(\Lambda) = [\Lambda^*, \Lambda^*] \Longleftrightarrow i$ is injective. This proves (iii) $\Longleftrightarrow$ (vi) and completes the proof. □

The proof of Theorem 2.64 shows more than the formal statement. As there is at most one weak determinant for Λ, the proof that (ii) $\Longrightarrow$ (i) and, in particular, the diagram (III) shows that:

Corollary 2.65 *If* $\{\det_n\}_{2\geq n}$ *is a weak determinant for* Λ *then for each* n *the sequence* $1 \to E_n(\Lambda) \to GE_n(\Lambda) \xrightarrow{\det_n} (\Lambda^*)^{ab} \to 1$ *is exact.*

We note that the condition '$\Lambda^* \cap E_2(\Lambda) = [\Lambda^*, \Lambda^*]$' occurs, albeit obliquely, in Cohn's study of GL_2 [16]. Cohn shows that for those rings Λ which are 'universal for GE_2' there is an isomorphism $GE_2(\Lambda)/E_2(\Lambda) \cong (\Lambda^*)^{ab}$ ([16], Theorem (9.1)). Although not expressed as such, his proof may be re-arranged to give the condition '$\Lambda^* \cap E_2(\Lambda) = [\Lambda^*, \Lambda^*]$' directly.

2.8 A Recognition Criterion for Dieudonné Rings

We begin with a straightforward group-theoretic observation:

Proposition 2.66 *Let* $\varphi : G \to H$ *be a surjective group homomorphism; then*

$$\varphi^{ab} : G^{ab} \to H^{ab} \text{ is an isomorphism} \quad \Longleftrightarrow \quad \mathrm{Ker}(\varphi) \subset [G, G].$$

Proof ($\Longrightarrow$) First note the inclusions

$$[G, G] \subset \varphi^{-1}([H, H]) \subset G, \qquad (*)$$

$$\mathrm{Ker}(\varphi) \subset \varphi^{-1}([H, H]) \subset G. \qquad (**)$$

We have a Noether isomorphism $\widehat{\varphi} : G/\varphi^{-1}([H, H]) \to H/[H, H] = H^{ab}$ whilst $(**)$ gives an exact sequence

$$1 \to \varphi^{-1}([H, H])/[G, G] \to G/[G, G] \xrightarrow{\nu} G/\varphi^{-1}([H, H]) \to 1.$$

Moreover, the induced map $\varphi^{ab} : G^{ab} \to H^{ab}$ is the composition $\varphi^{ab} = \widehat{\varphi} \circ \nu$. Thus we have an exact sequence

$$1 \to \varphi^{-1}([H, H])/[G, G] \to G^{ab} \xrightarrow{\varphi^{ab}} H^{ab} \to 1$$

in which, by hypothesis, φ^{ab} is an isomorphism. Hence $[G, G] = \varphi^{-1}([H, H])$ and so $\mathrm{Ker}(\varphi) \subset [G, G]$ by $(**)$. This proves ($\Longrightarrow$).

($\Longleftarrow$) Conversely suppose that $\mathrm{Ker}(\varphi) \subset [G, G]$. Then we have an exact sequence

$$1 \to [G, G]/\mathrm{Ker}(\varphi) \to G/\mathrm{Ker}(\varphi) \xrightarrow{\mu} G^{ab} \to 1.$$

As φ is surjective there is a Noether isomorphism $\varphi_* : G/\mathrm{Ker}(\varphi) \to H$. Define

$$\nu = \mu \circ \varphi_*^{-1} : H \to G^{ab}.$$

Then ν is surjective and the diagram below commutes where $\natural_G$, $\natural_H$ are the canonical homomorphisms:

$$\begin{array}{ccc} G & \overset{\natural_G}{\to} & G^{ab} \\ \varphi \downarrow & \nu \nearrow & \downarrow \varphi^{ab} \\ H & \overset{\natural_H}{\to} & H^{ab} \end{array}$$

Let $\nu^{ab} : H^{ab} \to G^{ab}$ be the homomorphism induced from ν via the universal property for abelianization. Then ν^{ab} is surjective. Moreover $\varphi^{ab} \circ \nu = \natural_H \Longrightarrow \varphi^{ab} \circ \nu^{ab} = \mathrm{Id}$ so that ν^{ab} is also injective and hence is an isomorphism with $(\nu^{ab})^{-1} = \varphi^{ab}$. In particular, φ^{ab} is an isomorphism. This proves ($\Longleftarrow$) and completes the proof. □

In what follows we adopt the convention that $\prod_{r=1}^{N} \lambda_r$ means $\lambda_1 \cdots \lambda_N$; moreover, for a ring homomorphism $\varphi : A \to B$, φ_u will denote the induced map on units

$$\varphi_u = \varphi_{|A^*} : A^* \to B^*.$$

We have a Recognition Criterion for Dieudonné rings.

Theorem 2.67 *Let $\varphi : A \to B$ be a ring homomorphism such that*

(i) *$\varphi_u : A^* \to B^*$ is surjective and*
(ii) *$\varphi_u^{ab} : (A^*)^{ab} \to (B^*)^{ab}$ is an isomorphism.*

If B is a Dieudonné ring then so also is A.

Proof By Theorem 2.64 we must show $[A^*, A^*] = A^* \cap E(A)$. However, $[A^*, A^*] \subset A^* \cap E(A)$ by Proposition 2.62 so it suffices to show that $A^* \cap E(A) \subset [A^*, A^*]$. Thus let $\delta \in A^* \cap E(A)$. Then $\varphi(\delta) \in B^* \cap E(B)$. However, B is a Dieudonné ring so that, by Theorem 2.64, we may write

$$\varphi(\delta) = \prod_{r=1}^{N} [\alpha_r, \beta_r]$$

with $\alpha_r, \beta_r \in B^*$. Now $\varphi_u : A^* \to B^*$ is surjective so that we may choose $\widehat{\alpha_r}, \widehat{\beta_b} \in A^*$ such that $\varphi(\widehat{\alpha_r}) = \alpha_r$ and $\varphi(\widehat{\beta_r}) = \beta_r$. Put

$$\gamma = \prod_{r=1}^{N} [\widehat{\alpha_r}, \widehat{\beta_r}] \in [A^*, A^*].$$

Then $\varphi_u(\gamma^{-1}\delta) = 1$; that is, $\gamma^{-1}\delta \in \mathrm{Ker}(\varphi_u)$. However, φ_u^{ab} is an isomorphism so that, by Proposition 2.66, $\mathrm{Ker}(\varphi_u) \subset [A^*, A^*]$. Hence we may write $\delta = \gamma\eta$ for some $\eta \in [A^*, A^*]$. Thus $\delta \in [A^*, A^*]$ as $\gamma, \eta \in [A^*, A^*]$. □

The standard theory of the determinant over a commutative ring shows that:

$$\text{Any commutative ring is a Dieudonné ring.} \tag{2.68}$$

The theorem of Dieudonné [21] shows:

$$\text{Any division ring is a Dieudonné ring.} \tag{2.69}$$

We give a complete proof of (2.69) in Appendix A. More interestingly, the group rings of certain infinite groups satisfy the Dieudonné condition. For example:

Proposition 2.70 *Let G be a finitely generated $\mathcal{TUP}$ group such that $H_1(G, \mathbf{Z})$ is torsion free. Then for any commutative integral domain A the group ring $A[G]$ satisfies the Dieudonné condition.*

Proof As noted in Theorem 2.51, the $\mathcal{TUP}$ condition guarantees that $A[G]$ has only trivial units; that is

$$A[G]^* \cong A^* \times G.$$

Let $\nu : A[G] \to A[G^{ab}]$ be the canonical mapping. As $G^{ab} \cong H_1(G; \mathbf{Z})$ is finitely generated and torsion free then G^{ab} is a free abelian group of finite rank and so also satisfies the $\mathcal{TUP}$ condition (see Appendix C). In particular, $A[G^{ab}]$ also has only trivial units; that is

$$A[G^{ab}]^* \cong A^* \times G^{ab}.$$

The canonical mapping ν thus induces a surjection $\nu : A[G]^* \to A[G^{ab}]^*$ and an isomorphism

$$\nu : (A[G]^*)^{ab} \xrightarrow{\simeq} A[G^{ab}]^*.$$

Moreover, $A[G^{ab}]^*$ is its own abelianization. As $A[G^{ab}]$ is commutative it satisfies the Dieudonné condition. Thus $A[G]$ also satisfies the Dieudonné condition by Theorem 2.67. □

As examples of groups G satisfying the hypotheses of Proposition 2.70 we may take, for example, any finite product $G = G_1 \times \cdots \times G_N$ where G_i is either a free group or the fundamental group of an orientable surface (cf. Appendix C).

2.9 Fully Determinantal Rings

To a ring Λ is associated a canonical homomorphism $i : (\Lambda^*)^{ab} \to K_1(\Lambda)$. We shall say that Λ is *fully determinantal* when i admits a left inverse. Given such a left inverse $\delta : K_1(\Lambda) \to (\Lambda^*)^{ab}$ then composition with the canonical mappings

$\nu_n : GL_n(\Lambda) \to K_1(\Lambda)$ gives a family $\{\delta_n\}_{2\le n}$ of group homomorphisms

$$\delta_n = \delta \circ \nu_n : GL_n(\Lambda) \to (\Lambda^*)^{ab}$$

making the following diagram commute for each k, n with $1 \le k$:

$$\begin{array}{ccc} GL_{n+k}(\Lambda) & \stackrel{\delta_{n+k}}{\to} & (\Lambda^*)^{ab} \\ \uparrow s_{n,k} & & \uparrow \mathrm{Id} \\ GL_n(\Lambda) & \stackrel{\delta_n}{\to} & (\Lambda^*)^{ab} \end{array}$$

The family $\{\delta_n\}_{2\le n}$ is then said to be a *full determinant* for Λ. As i has a left inverse it is injective. The restriction $\delta_n : GE_n(\Lambda) \to (\Lambda^*)^{ab}$ is then the (unique) weak determinant of Λ which exists by virtue of the injectivity of i; hence δ_n extends $\mathrm{prot}_n : \mathcal{C}_n(\Lambda) \to (\Lambda^*)^{ab}$. Clearly one has:

If Λ is Dieudonné and weakly Euclidean then Λ is fully determinantal. (2.71)

In particular, as division rings are both Dieudonné and weakly Euclidean then

Any division ring is fully determinantal. (2.72)

We note that a commutative ring Λ is fully determinantal as there is a natural choice of left inverse for i induced from the standard determinant construction. However, in contrast to weak determinants, full determinants, where they exist, need not necessarily be unique. This may be true even in the commutative case. In general, if $i : (\Lambda^*)^{ab} \to K_1(\Lambda)$ has a left inverse then any full determinant of Λ is unique only when there is no nontrivial group homomorphism $\mathrm{Coker}(i) \to (\Lambda^*)^{ab}$.

In conjunction with the Dieudonné condition the additional condition of being weakly Euclidean is sufficient but not necessary for the possession of a full determinant. Every commutative ring is fully determinantal but very few are weakly Euclidean. Less trivially, as the following shows, there are examples of noncommutative rings which are fully determinantal and, in general, very far from being weakly Euclidean.

Theorem 2.73 *If A is a commutative integral domain then the group ring $A[F_N]$ is fully determinantal.*

Proof Let k be the field of fractions of A. As both F_N and C_∞^N satisfy the $\mathcal{TUP}$ condition then $k[F_N]^* \cong k^* \times F_N$ and $k[C_\infty^N]^* \cong k^* \times C_\infty^N$ so that $(k[F_N]^*)^{ab} \cong k^* \times C_\infty^N$. We saw in Proposition 2.70 that $k[F_N]$ satisfies the Dieudonné condition. Moreover, by the theorem of Cohn Theorem 2.48 $k[F_N]$ is weakly Euclidean. Thus

$k[F_N]$ is fully determinantal; moreover, Proposition 2.61 and the weakly Euclidean condition guarantee the determinant is unique.

Let $\delta_m : GL_m(k[F_N]) \to (k[F_N]^*)^{ab}$ be the determinant; there is a commutative diagram

$$\begin{tikzcd} GL_m(k[F_N]) \arrow[rr, "\natural"] \arrow[dr, "\delta_m"'] & & GL_m(k[C_\infty^N]) \arrow[dl, "\det"] \\ & k^* \times C_\infty^N & \end{tikzcd}$$

where $\natural : GL_m(k[F_N]) \to GL_m(k[C_\infty^N])$ be the homomorphism induced from the abelianization $F_N \to C_\infty^N$. Evidently this imbeds in a larger commutative diagram

$$\begin{tikzcd} GL_m(A[F_N]) \arrow[rr, "\natural"] \arrow[d, "j"] & & GL_m(A[C_\infty^N]) \arrow[d, "i"] \\ GL_m(k[F_N]) \arrow[rr, "\natural"] \arrow[dr, "\delta_m"'] & & GL_m(k[C_\infty^N]) \arrow[dl, "\det"] \\ & k^* \times C_\infty^N & \end{tikzcd}$$

As $A[C_\infty^N]$ is commutative, if $X \in GL_m(A[C_\infty^N])$ then $\det(X) \in A[C_\infty^N]^*$. However, again by the $\mathcal{TUP}$ condition, $A[C_\infty^N]^* \cong A^* \times C_\infty^N$. From the commutativity of the above diagram, if $X \in GL_m(A[F_N])$ then $\delta_m(j(X)) \in A[C_\infty^N]^* = A^* \times C_\infty^N$. Once more by the $\mathcal{TUP}$ condition $(A[F_N]^*)^{ab} \cong A^* \times C_\infty^N$. We define a homomorphism $\delta'_m : GL_m(A[F_N]) \to (A[F_N]^*)^{ab}$ by

$$\delta'_m = \delta_m \circ j.$$

Again by Proposition 2.70, $A[F_N]$ satisfies the Dieudonné condition and it straightforward to see that δ'_m extends the weak determinant $d_m : GE_m(A[F_N]) \to (A[F_N]^*)^{ab}$. Finally the diagram below commutes

$$
\begin{tikzcd}
GL_n(A[F_N]) \arrow[rrr, "j"] \arrow[dr, "\delta'_n"'] & & & GL_n(k[F_N]) \arrow[dl, "\delta_n"] \\
& A^* \times C_\infty^N \arrow[r, phantom, "\subset"] & k^* \times C_\infty^N & \\
GL_m(A[F_N]) \arrow[uu, "s_{n,m}"] \arrow[ur, "\delta'_m"] \arrow[rrr, "j"] & & & GL_m(k[F_N]) \arrow[uu, "s_{n,m}"'] \arrow[ul, "\delta_m"']
\end{tikzcd}
$$

in consequence of which $A[F_N]$ is fully determinantal as claimed. $\square$

Syzygies and Homotopy Theory
Johnson, F.E.A.
2012, XXIV, 296 p., Hardcover
ISBN: 978-1-4471-2293-7