Preface

The underlying motivation for this book is the study of the algebraic homotopy theory of nonsimply connected spaces; in the first instance, the algebraic classification of certain finite dimensional geometric complexes with nontrivial fundamental group G; more specifically, directed towards two basic problems, the $\mathcal{D}(2)$ and $\mathcal{R}(2)$ problems explained below.

The author's earlier book [52] demonstrated the equivalence of these two problems and developed algebraic techniques which were effective enough to solve them for some *finite* fundamental groups ([52], Chap. 12). However the theory developed there breaks down at a number of crucial points when the fundamental group G becomes infinite. In order to consider these problems for general finitely presented fundamental groups the foundations must first be re-built ab initio; in large part the aim of the present monograph is to do precisely that.

The $\mathcal{R}(2)$–$\mathcal{D}(2)$ Problem Having specified the fundamental group, the types of complex we aim to study are, from the point of view of homotopy theory, the simplest finite dimensional complexes which can then be envisaged; namely n-dimensional complexes X with $n \geq 2$ which satisfy

$$\pi_r(\widetilde{X}) = 0 \quad \text{for } r < n, \tag{$*$}$$

where $\widetilde{X}$ is the universal cover of X. These restrictions alone are not sufficient to specify the next homotopy group $\pi_n(\widetilde{X})$; nor, however, is the choice of $\pi_n(\widetilde{X})$ entirely arbitrary. We shall explain in detail throughout the book how to parametrize the possible choices for $\pi_n(\widetilde{X})$ as a module over the group ring $\mathbf{Z}[G]$ and the extent to which an admissible choice determines the homotopy type of X.

Given a complex X as above we can construct the cellular chain complex

$$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_2} C_0,$$

where $C_r = H_r(\widetilde{X}^r, \widetilde{X}^{r-1}; \mathbf{Z})$ is a free $\mathbf{Z}[G]$-module with basis the r-cells of X. By the Hurewicz theorem, the conditions $(*)$ above force

$$H_r(C_*) = \begin{cases} \mathbf{Z} & r = 0, \\ 0 & 1 \leq r < n, \\ \pi_n(X) & r = n, \end{cases}$$

so that we may extend the above chain complex to an exact sequence

$$C_*(X) = (0 \to \pi_n(\widetilde{X}) \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to \mathbf{Z} \to 0).$$

By an *algebraic n-complex* over $\mathbf{Z}[G]$ we mean an exact sequence of $\mathbf{Z}[G]$-modules

$$A_* = (0 \to J \to A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \to \mathbf{Z} \to 0)$$

in which each A_r is finitely generated and free over $\mathbf{Z}[G]$. An algebraic n-complex A_* is said to be *geometrically realizable* when there exists a geometric n-complex X of type $(*)$ such that $C_*(X) \simeq A_*$. One may then ask the obvious question:

$\mathcal{R}(n)$: Is every algebraic n-complex geometrically realizable?

For $n \geq 3$ the $\mathcal{R}(n)$ problem is answered in the affirmative in Chap. 9. In fact, this is a special case of an older and much more general result of Wall [98]. The question that remains is genuinely problematic:

$\mathcal{R}(2)$: Is every algebraic 2-complex geometrically realizable?

Whilst important in its own right, the $\mathcal{R}(2)$-problem is also of interest via its relation to a notorious and more obviously geometrical problem in low dimensional topology. First make a definition; say that a 3-dimensional cell complex X is *cohomologically 2-dimensional* when $H_3(\widetilde{X}; \mathbf{Z}) = H^3(X; \mathcal{B}) = 0$ for all coefficient systems $\mathcal{B}$ on X. The problem may then be stated as follows:

$\mathcal{D}(2)$: Let X be a finite connected cell complex of geometrical dimension 3 which is cohomologically 2-dimensional. Is X is homotopy equivalent to a finite complex of geometrical dimension 2?

Both $\mathcal{D}(2)$ and $\mathcal{R}(2)$ problems are parametrized by the fundamental group under discussion; each finitely presented group G has its own $\mathcal{D}(2)$ problem and its own $\mathcal{R}(2)$ problem. Moreover, for a given fundamental group G the $\mathcal{D}(2)$ problem is entirely equivalent to the $\mathcal{R}(2)$ problem; to solve one is to solve the other. This equivalence was shown by the present author in [51, 52], subject to a mild condition on G which was subsequently shown to be unnecessary by Mannan [71].

This book is in two parts, Theory and Practice. In this Preface we give a brief outline of the theory; a summary of the practical aspects is given in the Conclusion.

The Method of Syzygies The basic model in the theory of modules is the theory of vector spaces over a field. However, the modules encountered in this book are defined over more general rings and in dealing with them it is useful to keep in mind how far one is being forced to deviate from the basic paradigm.

Linear algebra over a field is rendered tractable by the fact that every module over a field is free; that is, has a spanning set of linearly independent vectors. General module theory takes as its point of departure the observation that when a module M is not free we may at least make a first approximation to its being free by taking a surjective homomorphism $\varphi : F_0 \to M$ where F_0 is free to obtain an exact sequence

$$0 \to K_1 \to F_0 \xrightarrow{\varphi} M \to 0.$$

We find it instructive to regard the kernel K_1 as *a first derivative* of M. Setting aside temporarily the question of uniqueness one may repeat the construction and approximate K_1 in turn by a free module to obtain an exact sequence

$$0 \to K_2 \to F_1 \to K_1 \to 0.$$

Iterating we obtain a long exact sequence

$$\begin{tikzcd}[column sep=small]
{} \arrow[r, "\partial_{n+1}"] & F_n \arrow[rr, "\partial_n"] \arrow[dr] & & F_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r, "\partial_3"] & F_2 \arrow[rr, "\partial_2"] \arrow[dr] & & F_1 \arrow[rr, "\partial_1"] \arrow[dr] & & F_0 \arrow[r] & \mathrm{M} \arrow[r] & 0 \\
& & K_n \arrow[ur] & & & & K_2 \arrow[ur] & & K_1 \arrow[ur] & & &
\end{tikzcd}$$

Thus arises the notion of *free resolution*, made famous by the work of Hilbert on Invariant Theory [43]. The intermediate modules K_n are called the *syzygies of* M. Indeed, the etymology ($\sigma\upsilon\zeta\upsilon\gamma o\zeta$ = yoke) is determined by the conventional view that the K_n are connections in this sense. Nevertheless, we prefer to regard them as objects in their own right, as *derivatives of* M. Before doing this, however, we must first answer the question we have avoided; to what extent are they unique?

At one level the most simple minded considerations show that they *cannot possibly* be unique; given an exact sequence

$$0 \to K_1 \to F_0 \xrightarrow{\varphi} M \to 0$$

then by stabilizing the middle term thus $0 \to K_1 \oplus \Lambda \to F_0 \oplus \Lambda \xrightarrow{\varphi} M \to 0$ it is clear that if K_1 is to be considered as a first derivative of M then $K_1 \oplus \Lambda$ must also be so considered. So much must have been apparent to Hilbert. Even so, it is clear that the pioneers of the subject considered that the syzygies *ought*, somehow, to be unique. In the original context of Invariant Theory [28] this can be made to work if the resolution is, in some sense, minimal. In our context, as we shall see, the notion of 'uniqueness via minimality' fails badly. However there is indeed a sense in which the syzygies are uniquely specified, and it is to this we now turn.

Stable Modules and Schanuel's Lemma According to legend, in the autumn of 1958, during a lecture of Kaplansky at the University of Chicago, Stephen Schanuel, then still an undergraduate, observed that if we are given exact sequences of modules over a ring Λ

$$0 \to K \to \Lambda^n \xrightarrow{\varphi} M \to 0;$$

$$0 \to K' \to \Lambda^m \xrightarrow{\varphi} M \to 0$$

then $K \oplus \Lambda^m \cong K' \oplus \Lambda^n$. In fact, Schanuel proved slightly more than this; however it suggests that given Λ-modules K, K' we should write:

$$K \sim K' \iff K \oplus \Lambda^m \cong K' \oplus \Lambda^n \quad \text{for some positive integers } m, n.$$

When this happens we say that K, K' are *stably equivalent*. The relation '$\sim$' is an equivalence relation on Λ modules and, applied to the above exact sequences, Schanuel's Lemma shows that $K \sim K'$; it is in this sense that syzygies are unique.

Schanuel's Lemma explains neatly why the attempt to force uniqueness of the syzygy modules by minimising the resolution is, in general, doomed to failure. Thus suppose that m is the minimum number of generators of the Λ-module M and suppose given exact sequences

$$0 \to K \to \Lambda^m \xrightarrow{\varphi} M \to 0;$$

$$0 \to K' \to \Lambda^m \xrightarrow{\varphi} M \to 0.$$

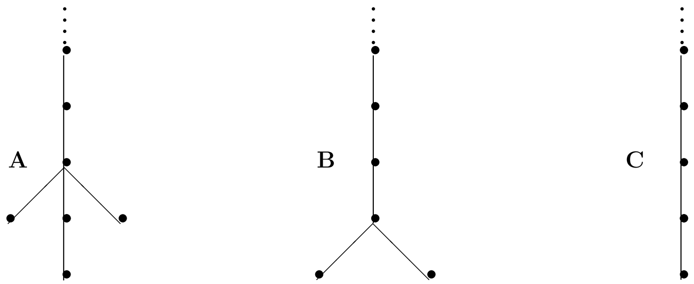
Schanuel's Lemma then tells us that $K \oplus \Lambda^m \cong K' \oplus \Lambda^m$. We are left to solve the following:

Cancellation Problem Does $K \oplus \Lambda^m \cong K' \oplus \Lambda^m$ imply that $K \cong K'$?

In dealing with modules over integral group rings the expected answer is 'No'; as we shall see, cancellation is the exception not the rule. The failure of cancellation may be starkly portrayed by representing the stable module $[K]$ as a graph.

When M is a finitely generated Λ-module, the stable module $[M]$ has the structure of a directed graph in which the vertices are the isomorphism classes of modules $N \in [M]$ and where we draw an edge $N_1 \to N_2$ when $N_2 \cong N_1 \oplus \Lambda$. We will show, in Chap. 1, that $[M]$ is a 'tree with roots that do not extend infinitely downwards'. This graphical method of representing stable modules is due to Dyer and Sieradski [24].

The extent to which cancellation fails in $[M]$ is captured by the amount of branching. We illustrate the point with some examples; **A** below represents a tree with a single root and no branching above level two; **B** represents a tree with two roots but with no branching above level one; **C** represents a tree with a single root and no branching whatsoever. Cancellation holds in **C** but fails in both **A** and **B**.



A significant difference between finite and infinite groups is the extent of our knowledge of the branching behaviour in stable modules over $\mathbf{Z}[G]$. When G is finite, the Swan-Jacobinski Theorem [46, 93] imposes severe restrictions on the type of branching that may occur; for example, the odd syzygies $\Omega_{2n+1}(\mathbf{Z})$ can behave only like **B** and **C** with possibly multiple roots but with no branching above level one; the even syzygies $\Omega_{2n}(\mathbf{Z})$ may resemble any of the three types but nothing worse. By contrast, when G is infinite very little is known in detail about the levels at which a stable module over $\mathbf{Z}[G]$ may branch.[1] We explore this question for some familiar infinite groups starting with the most basic case, namely the stable class of 0.

Iterated Fibre Squares and Stably Free Modules In passing from finite groups to infinite groups the first point of difference is the increased incidence of non-cancellation. For finite Φ non-cancellation over $\mathbf{Z}[\Phi]$ is comparatively rare. By the theorem of Swan and Jacobinski, it can only occur when the real group ring

$$\mathbf{R}[\Phi] \cong \prod_{i=1}^{m} M_{d_i}(\mathcal{D}_i)$$

fails the *Eichler condition*; that is when for some i, $d_i = 1$ and $\mathcal{D}_i = \mathbf{H}$ is the division ring of Hamiltonian quaternions. However, the proof of the Swan-Jacobinski theorem does not survive the passage to infinite groups and so we are forced to fall back on other methods.

The approach which has proved profitable is the method of iterated fibre squares which was used by Swan in [94] to consider the *extent* to which non-cancellation fails in finite groups which fail the Eichler condition. We elaborate the necessary theory of fibre squares in Chap. 3. As a working method it proceeds like this; take a convenient finite group Φ and establish the cancellation properties of $\mathbf{Z}[\Phi]$ from first principles by using the method of fibre squares. Now generalize the statement, replacing $\mathbf{Z}[\Phi]$ by $R[\Phi]$; on taking $R = \mathbf{Z}[G]$ where G is infinite one hopes to analyze the cancellation properties of $R[\Phi] \cong \mathbf{Z}[G \times \Phi]$. Some successful attempts are exhibited in Chaps. 10 through 12.

[1] Although over more general rings, for example the coordinate rings of spheres, the pattern of branching away from the main stem may be very complicated.

The Derived Module Category We have set ourselves the task of classifying algebraic complexes and, in particular, algebraic 2-complexes. To see the relevance of syzygies for this, suppose given a Λ-module M and write $\Omega_n(M)$ for the stable class any nth-syzygy of M; then we may portray an algebraic 2-complex formally as

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \Omega_3(\mathbf{Z}) \arrow[r] & F_2 \arrow[rr, "\partial_2"] \arrow[dr] & & F_1 \arrow[rr, "\partial_1"] \arrow[dr] & & F_0 \arrow[r] & \mathbf{Z} \arrow[r] & 0 \\ & & & \Omega_2(\mathbf{Z}) \arrow[ur] & & \Omega_1(\mathbf{Z}) \arrow[ur] & & & \end{tikzcd}$$

showing, in particular, that when X is a connected geometric 2-complex with $\pi_1(X) = G$ the $\mathbf{Z}[G]$-module $\pi_2(\widetilde{X})$ is constrained to lie in the third syzygy $\Omega_3(\mathbf{Z})$.

The Ω_n formalism was first introduced by Heller in the context of modular representations of finite groups [39]. In that restricted setting it is relatively easy, with suitable interpretations, to regard the correspondence $M \mapsto \Omega_n(M)$ as a functor. In more general contexts attempting to make Ω_n functorial involves additional technical complications.

The first question to be answered is '*In what category is* $\Omega_n(M)$ *supposed to live*?' As a first approximation we take the quotient of the category $\mathcal{M}\mathrm{od}_\Lambda$ of Λ-modules obtained by ignoring morphisms which factorize through a free module; more precisely, we equate morphisms whose difference factorizes through a free module; that is if $f, g : M \to N$ are Λ-homomorphisms we write '$f \approx g$' when $f - g$ can be written as a composite $f - g = \xi \circ \eta$ as below where F is a free module:

$$\begin{tikzcd} M \arrow[rr, "f-g"] \arrow[dr, "\eta"'] & & N \\ & F \arrow[ur, "\xi"'] & \end{tikzcd}$$

The quotient category $\mathcal{D}\mathrm{er}(\Lambda) = \mathcal{M}\mathrm{od}_\Lambda / \approx$ is called the *derived module category*. It is too crude an approximation, if only on the basis of size for, as we have imposed no size restrictions, our modules can be arbitrarily large. We can attempt to restrict all definitions to apply only to finitely generated modules; thus if N is a module we say that its stable class $[N]$ is finitely generated when N is finitely generated; in that case, *any* module in $[N]$ is also finitely generated. In the original context of modular representation theory, such size restriction causes no difficulty. In our more general context however, the difficulty arises that if M is finitely generated then $\Omega_n(M)$ need not be. To restrict attention to rings where this behaviour does not occur would exclude the integral group rings $\mathbf{Z}[G]$ of many interesting groups [53] (See Appendix D).

However, under a mild restriction on the ring,[2] if M is countably generated so also is $\Omega_n(M)$; then restricting all definitions to apply only to countably generated modules yields a derived module category $\mathcal{D}\mathrm{er}_\infty(\Lambda)$ of realistic size.

[2]Weak coherence. See Chap. 1.

There is, however, a complication more subtle than mere size. Recall that any projective module is a direct summand of a free module. Thus the above condition '$f \approx g$' is equivalent to the requirement that $f - g$ factors through a projective. This has the eventual consequence for modules K, K' over Λ that

$$K \cong_{\mathcal{D}\mathrm{er}} K' \iff K \oplus P \cong_{\Lambda} K' \oplus P'$$

for some projective modules P, P'; that is, isomorphism classes in $\mathcal{D}$er correspond not to stability classes of modules but, in MacLane's terminology, to *projective equivalence classes*[3] ([68], p. 101). Moreover, this applies even when all modules under consideration are finitely generated. In the original context of modular representation theory all projective modules are free, there is no distinction between stability and projective equivalence and Ω_n defines a functor on the derived module category. However, in general, to obtain functoriality one must consider not Ω_n but rather its analogue using the appropriate notion of *generalized syzygy*; disregarding finiteness restrictions and taking the successive kernels in a projective resolution $\mathcal{P}$

$$\begin{tikzcd}
{} \arrow[r, "\partial_{n+1}"] & P_n \arrow[rr, "\partial_n"] \arrow[dr] & & P_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r, "\partial_3"] & P_2 \arrow[rr, "\partial_2"] \arrow[dr] & & P_1 \arrow[rr, "\partial_1"] \arrow[dr] & & P_0 \arrow[r] & \mathrm{M} \arrow[r] & 0 \\
& & D_n \arrow[ur] & & & & D_2 \arrow[ur] & & D_1 \arrow[ur] & & &
\end{tikzcd}$$

the correspondence $M \mapsto D_n$ gives a functor $D_n : \mathcal{D}\mathrm{er}_\infty \to \mathcal{D}\mathrm{er}_\infty$. As classes of modules $\Omega_n(M) \subset D_n(M)$ and we may regard $\Omega_n(M)$ as a sort of *polarization state* of $D_n(M)$. We note that for most computational purposes we may legitimately revert to $\Omega_n(M)$ as $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_n(M), N) \equiv \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(D_n(M), N)$.

Eliminating Injectives In the late 1940s the introduction of Eilenberg-Maclane cohomology as the *derived functors* of Hom completely transformed module theory. The indeterminate nature of syzygies was replaced by the definiteness of computable invariants. In the aftermath the syzygetic method, insofar as it was still pursued, was regarded as an unwelcome reminder of a more primitive past. For us now, however, its rehabilitation via the derived module category raises the question of relating syzygies directly to cohomology.

Here we encounter a difficulty which is inherent in the cohomological method itself. In the standard treatments it is shown that one may compute the derived functor of $\mathrm{Hom}(-, -)$ *either* by taking a projective resolution in the first variable *or*, *equally*, by taking an injective co-resolution in the second. Moreover, this symmetry is not a point of esoteric scholarship, or at least, not merely so. With each variable one has a long exact sequence obtained by systematic appeal to the properties of the appropriate type of module. Which leads us back to the two sorts of modules themselves.

[3] For countably generated modules it is technically more convenient to replace the relation of projective equivalence by the equivalent notion of *hyperstable equivalence*, which is to say that $K \oplus \Lambda^\infty \cong_\Lambda K' \oplus \Lambda^\infty$. But again, see Chap. 1.

Projective modules, as direct summands of free modules, were in common use[4] before the name was ever applied to them; however the history and nature of injective modules is entirely different. Whereas projective modules are unavoidable, injective modules are a deliberate contrivance, only introduced to have arrow-theoretic properties dual to those of projectives [6]. Whereas projective modules are natural, injective modules are formal. Whereas projective modules are constructible (and we shall show how to construct some of them) injective modules are essentially non-constructible. One needs a theorem to show they exist. Except in the most elementary cases, where the point is irrelevant, they are not describable by any *effective* process. In our context this last point is the most pressing; injectives are so different from the objects with which we must deal that, arguments of formal simplicity notwithstanding, the need to dispense with them becomes insistent.[5]

The elimination of magic from homological algebra, in this case the avoidance of injective modules, forces us in every case to use projective resolutions. Whilst dispensing with the dualising services of injectives it is nevertheless essential to employ some form of homological duality which, however weak, can be confined entirely within the 'projective quotient' category. In fact, this requirement has a precedent as does the remedy; in the cohomology of lattices over finite groups the dual arrow theoretic properties of projectives are possessed by projectives themselves. Thus one may dispense with injectives entirely and describe the theory solely in terms of projectives. This is *Tate cohomology*, a point to which we will return. Our solution is comparable but not quite so convenient.

Corepresentability of Cohomology The appropriate notion, which we shall use systematically, is that of 'coprojectivity'; a module M is said to be *coprojective* when $\mathrm{Ext}^1(M, \Lambda) = 0$. To see how coprojectivity works take an exact sequence $\mathcal{E} = (0 \to K \stackrel{i}{\to} F \stackrel{\varphi}{\to} M \to 0)$ where F is free so that K is a first syzygy of M; if $\alpha : K \to N$ is a Λ-homomorphism one may form the pushout diagram

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ \alpha_*(\mathcal{E}) \end{array} \quad c = \begin{pmatrix} 0 \to K \stackrel{i}{\to} & F \stackrel{\varphi}{\to} & M \to 0 \\ \downarrow \alpha & \downarrow \nu & \downarrow \mathrm{Id} \\ 0 \to N \to & \varinjlim(\alpha, i) \to & M \to 0 \end{pmatrix}$$

from which we obtain the *connecting homomorphism* $\delta : \mathrm{Hom}_\Lambda(K, N) \to \mathrm{Ext}^1(M, N)$ by means of $\delta([\mathcal{E}]) = [\alpha_*(\mathcal{E})]$. When M is coprojective (and not otherwise) δ descends to give a natural equivalence $\delta : \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(K, -) \to \mathrm{Ext}^1(M, -)$ so that we may write

$$\mathrm{Ext}^1(M, -) \cong \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_1(M), -).$$

[4]For example in Wedderburn theory.

[5]The disadvantages, *for any practical purpose*, of an object about which one has to think hard before even being able to admit its existence ought to be obvious. Doubtless some will regret this as yet another instance of a depressing but universal trend; in Weber's succinct phrase 'The elimination of Magic from the World' ([99], p. 105).

In other-words, when M is coprojective, $\Omega_1(M)$ is a *corepresenting object* for $\mathrm{Ext}^1(M,-)$[6] considered as a functor on the derived module category. More generally, in higher dimensions there is a corresponding corepresentation theorem

$$H^n(M,-) \cong \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_n(M),-)$$

which holds provided that $H^n(M,\Lambda) = 0$. That is, we have replaced the *derived functor H^n* by the *derived object Ω_n*. Corepresenting cohomology in this way is the first step towards geometrizing extension theory so as to be able to apply it to the question of realizing algebraic complexes. Moreover, the groups $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_n(M), N)$ are then natural generalizations of the Tate cohomology groups defined for modules over finite groups.

Homotopy Classification and the Swan Homomorphism The problem of classifying algebraic complexes up to homotopy equivalence may be compared with the simpler Yoneda theory of module extensions up to congruence [68, 101]. For a specified fundamental group G let $\mathbf{Alg}_n(\mathbf{Z})$ denote the set of homotopy types of algebraic n-complexes of the form

$$A_* = (0 \to J \to A_n \to A_{n-1} \to \cdots \to A_0 \to \mathbf{Z} \to 0).$$

The stabilization $\Sigma_+(A_*)$ is obtained by adding $\Lambda = \mathbf{Z}[G]$ to the final two terms thus

$$\Sigma_+(A_*) = (0 \to J \oplus \Lambda \to A_n \oplus \Lambda \to A_{n-1} \to \cdots \to A_0 \to \mathbf{Z} \to 0)$$

and $\mathbf{Alg}_n(\mathbf{Z})$ also acquires a tree structure by drawing arrows $A_* \to \Sigma_+(A_*)$. Moreover the correspondence $A_* \mapsto J$ defines a mapping of trees, 'algebraic π_n',

$$\pi_n : \mathbf{Alg}_n(\mathbf{Z}) \to \Omega_{n+1}(\mathbf{Z}).$$

In his unpublished paper [12] Browning described the fibres $\pi_2 : \mathbf{Alg}_2(\mathbf{Z}) \to \Omega_3(\mathbf{Z})$ for those finite groups G which satisfy the Eichler condition. In [52], generalizing a criterion of Swan [91], we showed, still within the confines of finite groups, how to circumvent dependence on the Eichler condition and gave a rather different description of the fibres of π_2. Here we show how to extend the description of [52] to a much wider class of rings.[7]

A significant difficulty lies in being able to generalize the Swan mapping. In the original version [91] the homomorphism property of the Swan mapping is an easy consequence of special circumstances; in the wider context it is less obvious. Again

[6]Notice that the blank space would normally have to be co-resolved by means of injectives; the coprojectivity hypothesis removes this necessity.

[7]We note that a very special case of our classification theorem, for algebraic n-complexes over the group rings of n-dimensional Poincaré Duality groups ($n \geq 4$), was given by Dyer in [23].

take an exact sequence $\mathcal{E} = (0 \to J \xrightarrow{i} F \xrightarrow{\varphi} M \to 0)$ where F is free; if $\alpha : J \to J$ is a Λ-homomorphism one may again form the pushout diagram

$$\begin{array}{ccc} J & \xrightarrow{i} & F \\ \downarrow \alpha & & \downarrow \nu \\ J & \to & \varinjlim(\alpha, i) \end{array}$$

It turns out (*Swan's projectivity criterion*) that $\varinjlim(\alpha, i)$ is projective precisely when α is an isomorphism in $\mathcal{D}\mathrm{er}$. When M and J are finitely generated one obtains a mapping

$$S : \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J) \to \widetilde{K_0}(\Lambda)$$

to the reduced projective class group of Λ. This is the generalized Swan mapping and is, nontrivially, a homomorphism. This result was first shown in [56]. Moreover, despite the apparent dependence upon J, when M is coprojective it depends only upon M and is independent of the sequence $\mathcal{E}$ used to produce it. More generally, if

$$0 \to J \to A_n \to A_{n-1} \to \cdots \to A_0 \to \mathbf{Z} \to 0$$

is an algebraic n-complex and $H^{n+1}(M, \Lambda) = 0$ the same mapping $S : \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J) \to \widetilde{K_0}(\Lambda)$ again reappears independently of the sequence used to produce it. By contrast, however, the natural mapping $\nu_J : \mathrm{Aut}_\Lambda(J) \to \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J)$ is heavily dependent on J. The detailed homotopy classification of algebraic n-complexes over M requires a knowledge of the cosets $\mathrm{Ker}(S)/\mathrm{Im}(\nu_J)$ as J runs through $\Omega_{n+1}(M)$.

Imposing the coprojectivity condition or its higher dimensional analogues does, of course, restrict the range of applicability of the theory. In practice it is not too serious; for example, the classification of algebraic 2-complexes over $\mathbf{Z}[G]$ requires us to impose the condition

$$H^3(\mathbf{Z}, \mathbf{Z}[G]) = 0.$$

This condition is satisfied in many familiar cases; in particular, when G is a virtual duality group of virtual dimension n it is satisfied whenever $n \neq 3$.

Parametrizing the First Syzygy In applying the classification theorem to our original problem one needs specific information about the syzygies $\Omega_n(\mathbf{Z})$. In practice, this is a matter of severe computational difficulty. At the time of writing, the only finite fundamental groups for which there are complete descriptions for *all* $\Omega_n(\mathbf{Z})$ are certain groups of periodic cohomology. For infinite fundamental groups the situation is far worse.

In the first instance we are content to study $\Omega_1(\mathbf{Z})$. Here we find that the branching properties at the minimal level are intimately related to the existence of stably free modules; that is, to the stable class of the zero module. When G is infinite and $\mathrm{Ext}^1(\mathbf{Z}, \mathbf{Z}[G]) = 0$ we show that the stably free modules describe a lower bound for the branching behaviour in $\Omega_1(\mathbf{Z})$ and give a complete description of the minimal level $\Omega_1^{\min}(\mathbf{Z})$. This is done in Chap. 13.

Finally, in the most familiar case where $\mathrm{Ext}^1(\mathbf{Z}, \mathbf{Z}[G]) \neq 0$, namely when $G \cong F_n \times C_m$, we give a complete description of all the *odd* syzygies $\Omega_{2n+1}(\mathbf{Z})$. By way of illustration we conclude the book with Edwards' solution [25, 26] of the $\mathcal{R}(2)$ problem for the groups $C_\infty \times C_m$.

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