Chapter 2
Basics of Vibration Dynamics

Vibrations are mechanical oscillations about an equilibrium position. There are cases when vibrations are desirable, such as in certain types of machine tools or production lines. Most of the time, however, the vibration of mechanical systems is undesirable as it wastes energy, reduces efficiency and may be harmful or even dangerous. For example, passenger ride comfort in aircraft or automobiles is greatly affected by the vibrations caused by outside disturbances, such as aeroelastic effects or rough road conditions. In other cases, eliminating vibrations may save human lives, a good example is the vibration control of civil engineering structures in an earthquake scenario.

All types of vibration control approaches—passive, semi-active and active—require analyzing the dynamics of vibrating systems. Moreover, all active approaches, such as the model predictive control (MPC) of vibrations considered in this book require simplified mathematical models to function properly. We may acquire such mathematical models based on a first principle analysis, from FEM models and from experimental identification. To introduce the reader into the theoretical basics of vibration dynamics, this chapter gives a basic account of engineering vibration analysis.

There are numerous excellent books available on the topic of analyzing and solving problems of vibration dynamics. This chapter gives only an outline of the usual problems encountered in vibration engineering and sets the ground for the upcoming discussion. For those seeking a more solid ground in vibration mechanics, works concentrating rather on the mechanical view can be very valuable such as the work of de Silva [10] and others [4, 22]. On the other hand, the reader may get a very good grip of engineering vibrations from the books discussing active vibration control such as the work of Inman [21] and others [15, 18, 37, 38].

The vibration of a point mass may be a simple phenomenon from the physical viewpoint. Still, it is important to review the dynamic analysis beyond this phenomenon, as the vibration of a mass-spring-damper system acts as a basis to understand more complex systems. A system consisting of one vibrating mass has one natural frequency, but in many cases, in a controller it is sufficient to replace a continuous

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structure with complex geometry. The vibration dynamics of point mass and other comparably simple models may represent a surprisingly large portion of real-life mechanical systems [8, 21]. We will begin our analysis in the first section with a case in which damping is not considered, then gradually build a more detailed representation of the physics of vibrations. Section 2.2 will introduce damping to the simple vibrating point mass, and following this, Sect. 2.3 considers the forced vibration of this essential mechanical system.

Multiple degree of freedom systems will be introduced in Sect. 2.4 including a concise treatment of the eigenvalue problem and modal decomposition. Since vibration dynamics of the continuum is a complex and broad topic, Sect. 2.5 will only make a brief excursion to distributed parameter systems. The transversal vibration of cantilever beams will be covered, as in upcoming chapters such a demonstration system will be utilized to test the implementation of model predictive controllers. Finally, this chapter ends with a discussion on the models used in vibration control in Sect. 2.6. This section covers transfer function models, state-space models, identification from FEM models and experimental identification. The aim of this chapter is to briefly introduce the mathematical description of vibration phenomena, in order to characterize the nature of the mechanical systems to be controlled by the model predictive control strategy presented in the upcoming chapters of this book.

Figure 2.1 illustrates the Venus Express spacecraft[1] under preparation for experimental vibration dynamics analysis [12]. The body of the spacecraft is equipped with accelerometers while outside disturbance is supplied to the structure via a shake table. Gaining knowledge on the vibration properties of mechanical systems is essential for both active and passive vibration engineering, as unexpected vibratory response may jeopardize mission critical performance or structural integrity.

2.1 Free Vibration Without Damping

The simplest possible example that may help understand the dynamics of vibrations is the oscillating point mass, which has one degree of freedom. Such a system is schematically illustrated in Fig. 2.2. Let us assume for now that damping is negligible and there is no external force acting on the system. The vibrating mass, often referred to as the simple harmonic oscillator, has a mass of m and is sliding on a frictionless surface. The position of the mass is denoted by one time-dependent coordinate, $q(t)$. The mass is connected to a surface with a linear spring, having the spring constant k.

According to Newton's second law of motion, there is an inertial force generated by the mass, which is proportional to its acceleration [10]:

$$F_m = m\frac{d^2q}{dt^2} = m\ddot{q}(t) \tag{2.1}$$

[1] Courtesy of the European Space Agency (ESA) and European Aeronautic Defence and Space Company (EADS)-Astrium.

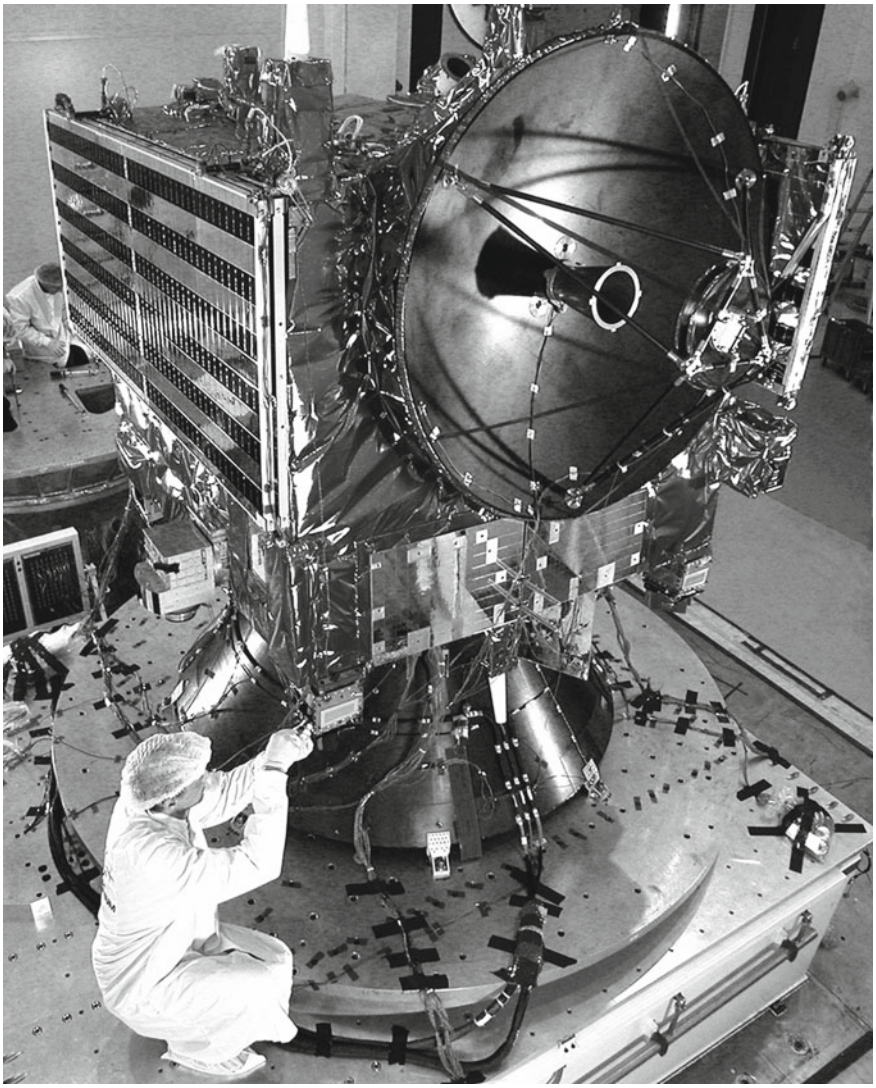


Fig. 2.1 Venus Express spacecraft is under preparation for experimental vibration dynamics analysis on a shake table [12]

where $\ddot{q}(t)$ is the acceleration and F_m is the inertial force of the mass. There is another force acting against this, which is proportional to the spring constant k [4, 22, 41]:

$$F_k = kq(t) \tag{2.2}$$

Because there is no other energy source or sink, the sum of these two forces will be zero. We can now assemble our equation of motion, which is an ordinary differential equation (ODE) given by [4, 10, 21, 49, 51]:

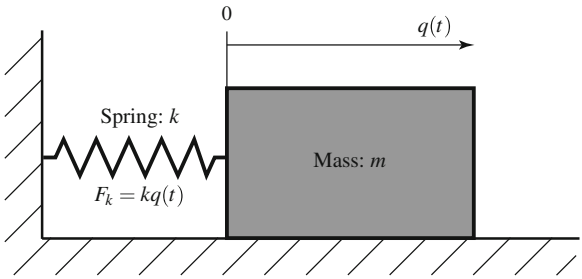


Fig. 2.2 Free vibration of a point mass without damping

$$m\ddot{q}(t) + kq(t) = 0 \tag{2.3}$$

One may classify mechanical vibrations according to whether an outside force is present:

- free vibration
- forced vibration

In free vibration, a mechanical system is excited by an initial condition, such as a displacement, velocity or acceleration and then is allowed to vibrate freely without further force interaction. A mechanical system in free vibration will oscillate with its natural frequency and eventually settle down to zero due to damping effects. In forced vibration, an external force is supplied to the system.

We may inspect how the moving mass will physically behave by imagining that we deflect our spring and move the mass to an initial position of $q(0)$, then let it go without inducing an initial velocity or acceleration. The mass will start to vibrate back and forth, and since there is no energy dissipation, its position will oscillate between $q(0) = \pm\tilde{q}$. If we plot its position in relation with time, we will get harmonic motion, which can be described by a trigonometric function. If we multiply a sine function shifted from zero by ϕ radians by our amplitude, we have an oscillating harmonic motion between $\pm\tilde{q}$. In addition to the amplitude, this function has a period of oscillations as well. Let us denote the angular frequency by ω_n, which in fact expresses the frequency of oscillations. Now we have a full mathematical description of the assumed motion of the mass [15, 21]:

$$q(t) = \tilde{q}\sin(\omega_n t + \phi) \tag{2.4}$$

where t is the progressing time, $\tilde{q}$ is the amplitude and ω_n is the angular velocity expressing the period of the oscillations. The constant amplitude $\tilde{q}$ and phase shift ϕ are constants that can be uniquely determined based on the initial conditions. We can substitute this trial solution in (2.4) back into the equation of motion in (2.3) and get:

$$m\frac{d^2\left(\tilde{q}\sin(\omega_n t + \phi)\right)}{dt^2} + k\left(\tilde{q}\sin(\omega_n t + \phi)\right) = 0 \tag{2.5}$$

after double differentiating the first term we will get

$$-m\omega_n^2\left(\tilde{q}\sin(\omega_n t+\phi)\right)+k\left(\tilde{q}\sin(\omega_n t+\phi)\right)=0 \tag{2.6}$$

we can further simplify this to calculate ω_n and get [41, 49]:

$$\omega_n=\sqrt{\frac{k}{m}} \tag{2.7}$$

Substituting this back to the original trial solution (2.4) and using the initial conditions, we get a solution of our ODE. We can convert the angular or circular period of vibration ω_n expressed in rad/sec into more familiar units [15, 51, 49]:

$$f_n=\frac{1}{2\pi}\omega_n=\frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{2.8}$$

where f_n gives oscillations per second or Hz (Hertz), or [49]

$$T_n=\frac{1}{f_n}=\frac{2\pi}{\omega_n}=2\pi\sqrt{\frac{m}{k}} \tag{2.9}$$

which gives us the period of one oscillation in seconds.

If we divide the equation of motion in (2.3) by the mass, we can express it in terms of the angular natural frequency ω_n:

$$\ddot{q}(t)+\omega_n^2 q(t)=0 \tag{2.10}$$

The solution in (2.4) is in fact a *trial solution*, which is a type of educated engineering guess; nevertheless if it works then it is the solution itself [21]. In our case, the trial solution in Eq. (2.4) works and it is a valid solution of the vibrating point mass. Although it is a product of a logical deduction, there are other ways to express the expected solution of the ODE describing the equation of motion of the point mass. A common alternative way to express the displacement of the point mass is to use an exponential function of time. This is a more mathematical representation of the same concept [21]:

$$q(t)=\tilde{q}e^{\kappa t} \tag{2.11}$$

where $\tilde{q}$ is the complex vibration amplitude. This representation is called the *phasor* representation, where a phasor can be simply understood as a rotating vector [15]. Note that in the field of vibration mechanics instead of κ the phasor representation uses λ. In order to keep the notation consistent throughout the book, this custom has been changed in order to reserve λ for concepts used in predictive control. The rotation velocity of the vector is included in the complex variable κ, which is essentially an eigenvalue. The real part of the amplitude $\tilde{q}$ is the physical amplitude, while the real component of the phasor describes the harmonic motion of the point mass.

Substituting the trial solution in (2.11) to the original equation of motion and differentiating results the same expressions for ω_n, however κ can assume both negative and positive values. The undamped natural frequency ω_n is the positive of the two κ solutions.

$$\kappa = \pm j\omega_n t \tag{2.12}$$

The reason that $e(x)$ or the natural exponent is often used in vibration analysis instead of simple trigonometric functions comes from the fact that it is mathematically easier to manipulate the exponential function and solve differential equations by expecting solutions in this form. The complex exponential is simply an eigenfunction of differentiation. Although trigonometric functions naturally come to mind when describing oscillatory motion, the natural exponential function $e(x)$ appears commonly in trial solutions of ODE describing vibration phenomena. The equivalence between trigonometric functions and the exponential function is given by Euler's formula. The general solution of the equation of motion after substituting (2.11) and solving for κ will be

$$q(t) = Ae^{-j\omega_n t} + Be^{j\omega_n t} \tag{2.13}$$

where A and B are integration constants determined by the initial conditions. The general solution can be equivalently described by an equation using trigonometric functions [4, 22]:

$$q(t) = A\cos\omega_n t + B\sin\omega_n t \tag{2.14}$$

where A and B are again integration constants to be determined based on the initial conditions.

2.2 Free Vibration with Damping

The previous section discussed free vibration of a point mass without damping. This means that a mass connected to a spring and deflected to the initial position of $q(0) = \tilde{q}$ would oscillate with the same amplitude indefinitely. As we can see, this is a very unrealistic model—we have to add some sort of energy dissipation mechanism, or in other words, damping.

Damping is a complex phenomenon, not very well understood and modeled in science. Different damping models exist; however, these represent reality only under certain conditions and assumptions. Probably the most popular damping model is *viscous damping*, which expresses the damping force that is proportional to velocity by a constant b. This force can be expressed by [21, 50]:

$$F_b = b\frac{d\dot{q}(t)}{dt} = b\dot{q}(t) \tag{2.15}$$

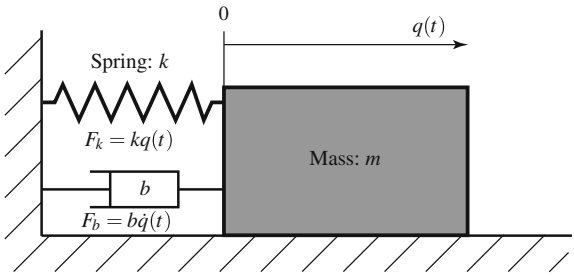


Fig. 2.3 Free vibration of a point mass with viscous damping

We can improve our previous model by adding this damping force to our system. Let us have the same vibrating mass m connected by a spring with the linear spring constant k to a fixed surface. Displacement is measured by $q(t)$ and let us add a viscous damper with the constant b to our representation. This is represented schematically by Fig. 2.3.

Now that we know the viscous damping force is expressed by (2.15), we can add it to the original equation of motion. The spring force F_k and the damping force F_b act against the inertial force F_m. The sum of these forces is zero, if we express this, we obtain an ODE again [4, 10, 21, 22]:

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = 0 \tag{2.16}$$

Dividing the whole equation of motion by m results in the following term:

$$\ddot{q}(t) + \frac{b}{m}\dot{q}(t) + \frac{k}{m}q(t) = 0 \tag{2.17}$$

Let us call half of the ratio of the viscous damping constant b and mass m as δ_d or [49]:

$$\delta_d = \frac{1}{2}\frac{b}{m} \tag{2.18}$$

and use (2.7) to substitute for k/m with yields [51, 52]:

$$\ddot{q}(t) + 2\delta_d\dot{q}(t) + \omega_n^2 q(t) = 0 \tag{2.19}$$

Another common representation of the damping both in mechanical vibration analysis and vibration control is proportional damping ζ [2] which is expressed as a percentage of critical damping [18, 51, 52]:

$$\frac{b}{m} = 2\zeta\omega_n = 2\frac{b}{b_c}\sqrt{\frac{k}{m}} \tag{2.20}$$

[2] Note that ζ is not the same as δ_d.

where critical damping is denoted by $b_c = 2\sqrt{km}$. Instead of expressing the simplified differential equation in the terms of the damping coefficient δ_d as in Eq. (2.19) we may express it using proportional damping ζ and get [4, 10, 21]:

$$\ddot{q}(t) + 2\zeta\omega_n\dot{q}(t) + \omega_n^2 q(t) = 0 \tag{2.21}$$

Let us now assume that the trial solution to the ODE will come in the form of (2.11). Remember, this is the more mathematical representation of our solution but the differentiation of an exponential function is the exponential itself which makes the evaluation process a little simpler. After substituting the trial solution into (2.21) we obtain the following equation:

$$\frac{d^2(e^{\kappa t})}{dt^2} + 2\zeta\omega_n\frac{d(e^{\kappa t})}{dt} + \omega_n^2(e^{\kappa t}) = 0 \tag{2.22}$$

and after differentiating this will be reduced to

$$\kappa^2 + 2\zeta\omega_n\kappa + \omega_n^2 = 0 \tag{2.23}$$

Solving this using simple algebra will yield the solution for κ. The roots of this equation will be [49]:

$$\kappa_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm j\omega_d \tag{2.24}$$

The damped natural frequency in terms of proportional damping ζ will be then:

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} \tag{2.25}$$

In this interpretation the overdamped, underdamped and critically damped oscillations are defined by the magnitude of ζ. As ζ is a percentage of critical damping, $\zeta < 1$ will result in an underdamped decaying periodic motion, $\zeta > 1$ will result in an overdamped aperiodic motion, while $\zeta = 1$ will result in a periodic and critically damped motion. Similarly, by substituting the same trial solution into Eq. (2.19) will yield:

$$\frac{d^2(e^{\kappa t})}{dt^2} + 2\delta_d\frac{d(e^{\kappa t})}{dt} + \omega_n^2(e^{\kappa t}) = 0 \tag{2.26}$$

which after differentiation will be reduced to

$$\kappa^2 + 2\delta_d\kappa + \omega_n^2 = 0 \tag{2.27}$$

in the terms of δ_d expressing the amount of damping in the system. The roots $\kappa_{1,2}$ of this characteristic equation are expressed by:

$$\kappa_{1,2} = -\delta_d \pm \sqrt{\delta_d^2 - \omega_n^2} \tag{2.28}$$

The second term is the damped natural frequency ω_d or

$$\omega_d = \sqrt{\delta_d^2 - \omega_n^2} \tag{2.29}$$

Depending on the magnitude of the damped natural frequency ω_d we may have overdamping, critical damping or underdamping. Overdamping is the case, when the initial conditions such as initial displacement or velocity result in an aperiodic motion. From now on we will assume that $\zeta < 1$ or equivalently $\omega_d^2 > \delta_d^2$ which results in a periodic vibration with a constantly decaying amplitude.

Let us now interpret this in the physical sense: we cannot express the solution as a simple harmonic function anymore. Because of the energy dissipation, for an underdamped system, the vibration amplitudes will gradually decay and the system will settle at equilibrium. We have to introduce an exponential term, to simulate decay caused by the damping. Our previously assumed solution general solution in (2.4) will be changed to [21]:

$$q(t) = \tilde{q} e^{-\zeta \omega_n t} \sin(\omega_d t + \phi) \tag{2.30}$$

The first exponential term $e^{-\delta_d \omega_n t}$ introduces the exponential decay and emulates the damping effect. Using trigonometric identities, this can also be written as [22]:

$$q(t) = e^{-\zeta \omega_n t} (A \cos(\omega_d t) + B \sin(\omega_d t)) \tag{2.31}$$

where A and B are integration constants which can be determined from the initial conditions. The general solution of the free vibration of the underdamped point mass can also be written in terms of δ_d by stating that [51, 52]:

$$q(t) = \tilde{q} e^{-\delta_d t} \sin(\omega_d t + \phi) \tag{2.32}$$

or equivalently as

$$q(t) = e^{-\delta_d t} (A \sin(\omega_d t) + B \cos(\omega_d t)) \tag{2.33}$$

2.3 Forced Vibration of a Point Mass

The damped vibration of a point mass is a passive representation of dynamic systems. Its motion is only controlled by initial conditions such as deflecting the spring into an initial displacement $q(0)$ or adding an initial velocity $\dot{q}(0)$ to the vibrating mass. We have to introduce an outside force in order to model a controllable active vibrating system. Let us assume that—in addition to the spring, damping and inertial forces—an external force $f_e(t)$ can also supply energy to the system. Combining the equation of motion in (2.16) for the damped vibrating point mass with this external force $f_e(t)$ we will get the following new equation of motion incorporating an outside force effect, for example an actuator or a disturbance [49]:

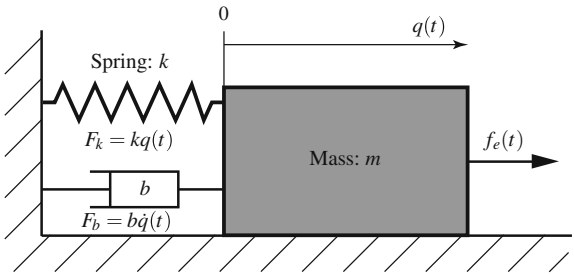


Fig. 2.4 Forced vibration of a point mass with viscous damping

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = f_e(t) \tag{2.34}$$

This is a second order ordinary differential equation, just like in the previous cases.

The type of the outside excitation force can be arbitrary and its course may be mathematically described by a step, impulse, harmonic, random or any other functions. To evaluate an analytic solution of the forced equation of motion, we have to have at least some sort of knowledge about the excitation. Let us therefore assume that our excitation force is harmonic, generated for example by a rotating imbalance so we may describe it by [4, 21]:

$$f_e(t) = \tilde{f}_e \sin \omega_f t \tag{2.35}$$

where $\tilde{f}_e$ is the amplitude of the excitation force and ω_f is the angular frequency of the outside disturbance. If we substitute this back into our original equation of motion for the forced response of the point mass in (2.34) we will get

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = \tilde{f}_e \sin \omega_f t \tag{2.36}$$

It would be natural to assume that, after an initial transient phase, the vibrating point mass will start to copy the harmonic motion of the outside excitation. This physical assumption can be translated into an expected trial solution of the time response given by:

$$q(t) = \tilde{q} e^{j(\omega_f t + \phi)} \tag{2.37}$$

where $\tilde{q}$ is the amplitude of our vibrations. Substitute this back into the ODE expressing the forced response of the point mass in (2.36), differentiate and simplify to get the amplitude [21]:

$$\tilde{q} = \tilde{f}_e \frac{1}{-m\omega_f^2 + j\omega_f b + k} \tag{2.38}$$

This is of course just a solution for one type of external force. This representation looks much like a transfer function, and in fact, it is easy to apply Laplace transformation to get transfer functions for controller design. For control purposes, it is also

possible to transform our ODE into a decoupled form, which is referred to as the state-space representation.

The solution of the equation of motion consists of two parts. The transient response describes the passing effects, while the steady-state response will characterize the response after the initial effects have settled. The total time response of an underdamped system with $\zeta < 1$ will be [21]:

$$q(t) = e^{-\zeta\omega_n t}(A \sin \omega_d t + B \cos \omega_d t) + \tilde{q}(\sin \omega_f t - \phi) \tag{2.39}$$

which is a sum of the steady state and the transient solution. Note that this general solution contains integration constants A and B which in general are not the same as the ones for free vibration. Furthermore, note the three angular frequencies in this equation: the angular natural frequency ω_n, the damped angular natural frequency ω_d and the frequency of the periodic excitation ω_n.

The analytic solution for other types of excitation is analogous to the periodic case. As this work is interested rather in the control aspects of (forced) vibrations, tools known from control engineering such as transfer function and state-space representations will be used to evaluate the response of a system to an outside excitation. To those interested in more details on the analytic representation of forced vibrations the books by de Silva [10] and others [4, 22] may be recommended.

2.4 Multiple Degree of Freedom Systems

The very simple single degree of freedom mass-spring-damper system introduced in the previous sections gives a good foundation for the analysis of vibrating systems. It is possible to simplify the essential dynamic behavior of many real mechanical systems to SDOF and replace it with an analysis procedure similar to the one introduced previously [8, 21].

In the vibration analysis of mechanical systems with multiple degrees of freedom (MDOF), instead of one vibrating mass, we replace our real structure with two or more oscillating masses. If the real system has well-defined separate moving parts, we can consider it as a lumped interconnected parameter system. The degrees of freedom of a lumpedparameter system are equal to the number of vibrating mass points and this is also true for the number of resonant frequencies. A mechanical system or structure which does not have well-defined separately oscillating parts but consists of a continuously spread mass and geometry is a distributed system. Continuum or distributed parameter systems have an infinite amount of resonant frequencies and corresponding vibration shapes. It is, however, possible to discretize the system into large amounts of lumped interconnected parameters and approximate its behavior with methods commonly used for lumped parameter systems. This idea is in fact used in FEM software to extract the vibration dynamics of distributed mechanical systems defined with complex three-dimensional geometry.

Let us choose a very simple example, which has more than one degree of freedom and therefore may be considered as a MDOF system. The system of connected

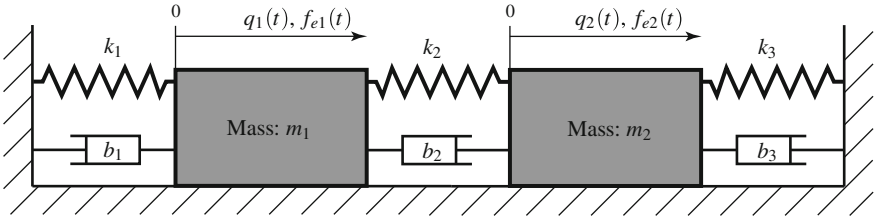


Fig. 2.5 Multiple degrees of freedom system: connected set of two masses

moving masses illustrated in Fig. 2.5 is sliding on a frictionless surface. Now instead of one coordinate defining the movement, we have two for each moving mass: $q_1(t)$ and $q_2(t)$. The two moving masses m_1 and m_2 are connected to each other and the surrounding fixed wall through a series of springs and dampers with the spring and damping coefficients k_1, k_2, k_3 and b_1, b_2, b_3. There are external force inputs associated with individual masses denoted by f_{e1} and f_{e2} .

Using a simple mechanical analysis, we may create a free body diagram for each mass and analyze the forces acting on them. After assembling the equations of motion, we obtain the following set of equations for our two masses [4, 10, 18]:

$$m_1\ddot{q_1} + (b_1 + b_2)\dot{q_1} - b_2\dot{q_2} + (k_1 + k_2)q_1 - k_2q_2 = f_1 \tag{2.40}$$

$$m_2\ddot{q_2} - b_2\dot{q_1} + (b_2 + b_3)\dot{q_2} - k_2q_1 + (k_2 + k_3)q_2 = f_2 \tag{2.41}$$

It is possible to rewrite the one equation for motion per moving mass into a compact set, using matrix notation [10, 37, 52]:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{q_1} \\ \ddot{q_2} \end{bmatrix} + \begin{bmatrix} b_1 + b_2 & -b_2 \\ -b_2 & b_2 + b_3 \end{bmatrix}\begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} f_{e1} \\ f_{e2} \end{bmatrix} \tag{2.42}$$

Note the similarity between this equation of motion and the SDOF forced equation motion in (2.34). We have a matrix containing the masses, which is multiplied by a vector of accelerations. Similarly, we have matrices containing damping elements and spring constants. We can in fact use a matrix notation to create from this [21]:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B_d}\dot{\mathbf{q}} + \mathbf{K_s}\mathbf{q} = \mathbf{f_e} \tag{2.43}$$

where matrix $\mathbf{M}$ is the mass matrix,[3] $\mathbf{B_d}$ is the structural damping matrix and $\mathbf{K_s}$ is the stiffness matrix. Vector $\mathbf{q}$ contains the displacement coordinates for each degree of freedom. For an N degree of freedom system the constant matrices $\mathbf{M}$, $\mathbf{B_d}$ and $\mathbf{K_s}$ will all have $N \times N$ elements.

[3] It is customary to denote the mass matrix with $\mathbf{M}$ however in the upcoming chapter this symbol will be reserved for an entirely different concept.

The solution of such systems is in fact very similar to the solution of SDOF systems. To illustrate this, let us consider a case without damping and with no outside force. Removing these effects from the equation of motion in matrix from (2.43), we get [38]:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K_s}\mathbf{q} = 0 \tag{2.44}$$

for which we have to find a solution. Similar to the SDOF systems, our solution can be expected in a form of a set of harmonic functions, which mathematically is simply an amplitude multiplied by a complex exponential:

$$\mathbf{q} = \tilde{\mathbf{q}}e^{j\omega_n t} = \tilde{\mathbf{q}}e^{\kappa t} \tag{2.45}$$

As introduced previously, the term $e^{j\omega_n t}$ or analogously $e^{\kappa t}$ is just a mathematical trick to solve differential equations by using the so-called phasor form for the solution. If we take the real part of Euler's formula, we essentially expect a cosine function. Let us substitute this solution to our matrix equation of motion, and differentiate to get:

$$\left(-\omega_n^2\mathbf{M} + \mathbf{K_s}\right)\tilde{\mathbf{q}}e^{j\omega_n t} = 0 \tag{2.46}$$

2.4.1 The Eigenvalue Problem

To solve the equation expressed by (2.46) we can assume that the exponential part $e^{j\omega t}$ cannot be zero, therefore we will reduce our expression to [37, 38]:

$$\left(-{\omega_n}^2\mathbf{M} + \mathbf{K_s}\right)\tilde{\mathbf{q}} = 0 \tag{2.47}$$

This is a problem often encountered in mathematics, called the *eigenvalue problem* which in general mathematics assumes the form [22]:

$$(\mathbf{A} - \kappa\mathbf{I})\,\delta = 0 \tag{2.48}$$

where κ contains the eigenvalues of the system while δ is the eigenvector. To get our problem (2.47) into a similar form, we have to multiply it by the inverse of the mass matrix $\mathbf{M}^{-1}$ to get

$$\left(\mathbf{M}^{-1}\mathbf{K_s} - \omega_n^2\mathbf{M}^{-1}\mathbf{M}\right)\tilde{\mathbf{q}} = 0 \tag{2.49}$$

which is essentially just given by [10, 52]

$$\left(\mathbf{M}^{-1}\mathbf{K_s} - \omega_n^2\mathbf{I}\right)\tilde{\mathbf{q}} = 0 \tag{2.50}$$

The solution to the eigenvalue problem expressed by our physical vibrating system is a set of N *eigenvalues* ranging from $\omega_1^2, \omega_2^2 \ldots \omega_N^2$, where N is the number of degrees of freedom. These eigenvalues have a very well-defined physical meaning: they contain the (square of the) angular natural frequencies associated with the individual masses. Substituting these eigenvalues back into the original equation, we get a set of amplitudes $\tilde{\mathbf{q}}$ called the *eigenvectors*. Each eigenvalue or natural frequency has an associated eigenvector. This eigenvector expresses the mode shapes of the system, in other words, the geometrical shape of the vibration within a given resonant frequency. The magnitude of the eigenvectors is not expressed in physical coordinates—instead, modal shapes are scaled by a method of choice. To avoid confusion, we will substitute the notation $\mathbf{q}_i$ with δ_i referring to the fact that these amplitudes have a physically valid magnitude only in relation to each other but not globally.

Eigenvalues and eigenvectors expressing the angular natural vibration frequency of individual masses and the vectors of modal shape associated with those frequencies can be assembled in a compact notation:

$$\Lambda = \text{diag}(\omega_i^2) = \text{diag}(\kappa_i) = \begin{bmatrix} \omega_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_N^2 \end{bmatrix} = \begin{bmatrix} \kappa_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_N \end{bmatrix} \tag{2.51}$$

$$\Delta = \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & \ldots & \delta_N \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{q}}_1 & \tilde{\mathbf{q}}_2 & \tilde{\mathbf{q}}_3 & \ldots & \tilde{\mathbf{q}}_N \end{bmatrix} \tag{2.52}$$

where Λ is a diagonal matrix with the square of the individual eigenfrequencies ω_i^2 on its main diagonal. Solving the eigenvalue problem, we get the modal shapes which are expressed by the amplitudes δ_i associated with the eigenfrequencies.

2.4.2 Modal Decomposition

It is possible to simplify the solution of a multi-degree freedom system by substituting it with a set of single degree freedom systems. Eigenvectors have a mathematical property called *orthogonality*, which is the basis of this simplification. If Δ is the set of eigenvectors, it can be shown that when we use it to multiply the mass matrix from both sides we obtain [22, 38, 52]:

$$\begin{aligned} \Delta^T \mathbf{M} \Delta &= \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & \ldots & \delta_N \end{bmatrix}^T \mathbf{M} \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & \ldots & \delta_N \end{bmatrix} \\ &= \begin{bmatrix} \delta_1^T \mathbf{M} \delta_1 & \delta_1^T \mathbf{M} \delta_2 & \ldots & \delta_1^T \mathbf{M} \delta_N \\ \delta_2^T \mathbf{M} \delta_1 & \delta_2^T \mathbf{M} \delta_2 & \ldots & \delta_2^T \mathbf{M} \delta_N \\ \vdots & \ddots & \ldots & \vdots \\ \delta_N^T \mathbf{M} \delta_1 & \delta_N^T \mathbf{M} \delta_2 & \ldots & \delta_N^T \mathbf{M} \delta_N \end{bmatrix} = \mathbf{I} \end{aligned} \tag{2.53}$$

while multiplying the stiffness matrix by Δ from both sides we get

$$\begin{aligned}\Delta^T \mathbf{K_s} \Delta &= \left[\, \delta_1 \;\; \delta_2 \;\; \delta_3 \;\ldots\; \delta_N \,\right]^T \;\; \mathbf{K_s} \left[\delta_1 \delta_2 \delta_3 \ldots \delta_N\right] \\ &= \begin{bmatrix} \delta_1^T \mathbf{K_s} \delta_1 & \delta_1^T \mathbf{K_s} \delta_2 & \ldots & \delta_1^T \mathbf{K_s} \delta_N \\ \delta_2^T \mathbf{K_s} \delta_1 & \delta_2^T \mathbf{K_s} \delta_2 & \ldots & \delta_2^T \mathbf{K_s} \delta_N \\ \vdots & \ddots & \ldots & \vdots \\ \delta_N^T \mathbf{K_s} \delta_1 & \delta_N^T \mathbf{K_s} \delta_2 & \ldots & \delta_N^T \mathbf{K_s} \delta_N \end{bmatrix} = \Lambda \end{aligned} \tag{2.54}$$

Keeping in mind the orthogonality properties of the modal matrices, we can introduce a coordinate transformation, which changes the original displacement coordinates into the so-called *modal coordinates* or *modal participation factors*:

$$\mathbf{q} = \Delta \xi \tag{2.55}$$

This transformation may be interpreted in a way that we take the vibration amplitude in physical coordinates as a linear sum of modal shapes. The coordinate ξ is also called the modal participation factor, because it determines how each mode participates in the final vibration. Mathematically this is:

$$\mathbf{q} = \xi_1 \delta_1 + \xi_2 \delta_2 + \xi_3 \delta_3 + \cdots + \xi_i \delta_i + \cdots + \xi_N \delta_N \tag{2.56}$$

We can use (2.55) substitute for $\mathbf{q}$ in the matrix equation of motion for free, undamped systems to get [22, 38]:

$$\mathbf{M} \Delta \ddot{\xi} + \mathbf{K_s} \Delta \xi = 0 \tag{2.57}$$

Let us now multiply the equation by Δ^T from the left to get

$$\Delta^T \mathbf{M} \Delta \ddot{\xi} + \Delta^T \mathbf{K_s} \Delta \xi = 0 \tag{2.58}$$

Using the orthogonality properties introduced in (2.53) and (2.54) we can simplify this equation to get

$$\ddot{\xi} + \Lambda \xi = 0 \tag{2.59}$$

Instead of having a large coupled multiple degree of freedom, this decomposes the original system into a set of several single degree of freedom systems [22]:

$$\xi_i + \omega_i^2 \xi_i = 0 \tag{2.60}$$

where ξ_i are the individual modal participation factors associated with the given mass m_i and ω_i is the angular natural frequency.

Solutions for the free and forced vibration for both damped and undamped systems can be developed using similar methods.

2.5 Distributed Parameter Systems

In practice, the vibration of continuously distributed parameter systems is solved and analyzed through the finite element method. As in the case of other fields of science, in FEM vibration analysis the continuous structure and its geometry are discretized into finite portions, the *elements*. The continuously distributed structure is then considered a large lumped parameter system with hundreds, thousands and even millions of degrees of freedom. If one aims to perform a modal analysis on a continuous system with complex geometry, the FEM software first creates a discretized version of the original structure. The equation of motion is then expressed in the matrix form of (2.43) and then the eigenvalue problem is solved. The solution of the eigenvalue problem with large matrices is not a trivial task, fortunately numerical mathematics have provided us with tools to speed up this process.

2.5.1 Exact Solution

Exact analytical solution for the vibration response of distributed parameter systems is complex and it is worked out only for certain shapes with primitive geometry, such as beams, bars, plates or discs. As we will be using a clamped cantilever beam with a fixed and free end in our examples, the foundations and basics for formulating the exact solution of this problem will be introduced here.

Let us therefore consider a clamped cantilever beam according to Fig. 2.6. The beam is vibrating in a transversal fashion, up and down in the y direction. The vibration is predominant in the y direction and we will neglect vibration in the x direction. The vibration in the direction perpendicular to the beam length is commonly referred to as *flexural* or *transverse* vibration. We may separate the beam to create infinitesimal slices dx. The position of this element is denoted by the coordinate x. The beam is made of the homogeneous material with the density ρ, its constant cross section is given by A, Young's modulus by E and its second moment of area by I.

2.5.1.1 Equation of Motion for the Transversal Movement of a Beam

Figure 2.7 denotes a separate element of this beam. The beam element has a centerline known as the neutral axis marked by a dashed line. The horizontal position of the beginning of the center line is marked by the coordinate x, while the vertical is marked by y. As it has been noted before, the width of this slice is dx. There are two types of force effects acting on the right side of this slice: a shear force T and a bending moment M. The left size is subjected to an infinitesimally larger shear force and bending moment.

As the element is vibrating in the transversal direction, its vertical position is expressed by y. There is also rotation involved in the element, which can be described by $\phi(x, t)$ or [10]:

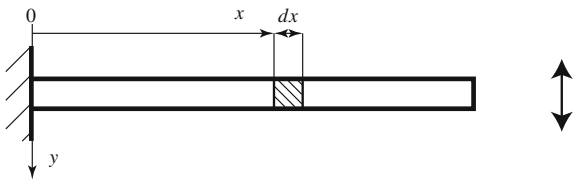


Fig. 2.6 Schematic representation of a clamped beam under transversal vibration

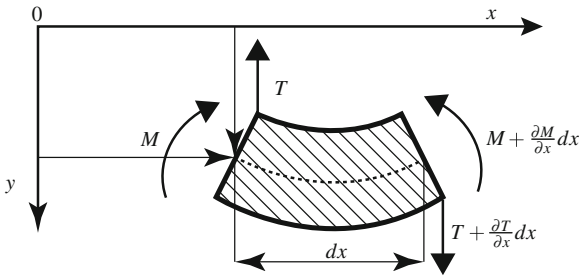


Fig. 2.7 Forces and moments acting on the infinitesimally small portion of the clamped beam under transversal vibration

$$\theta(x,t) = \frac{dy(x,t)}{dx} \tag{2.61}$$

therefore both the position of the element and its rotation is expressed by just one coordinate and its derivative with respect to x. Let us now express the second moment of inertia I of the element dx defined in the plane perpendicular to the direction of the transversal motion using the polar moment of inertia defined at the neutral axis:

$$I = \rho J \tag{2.62}$$

If we look at the forces and moments acting on the element, we can write the equation of motion for the beam as [22, 10, 51, 52]:

$$\rho A \frac{\partial^2 y(x,t)}{\partial t^2} = -T + T + \frac{\partial T}{\partial x} dx \tag{2.63}$$

$$\rho J \frac{\partial^2}{\partial t^2} \left(\frac{\partial y(x,t)}{\partial x} \right) = T \frac{dx}{2} + M + \left(T + \frac{\partial T}{\partial x} dx \right) \frac{dx}{2} - M - \frac{\partial M}{\partial x} dx \tag{2.64}$$

where the first equation describes the effects of the shear force T and the second equation describes the moment effects. If we take a close look at the first equation, the term on the left is nothing more than the mass of the element (ρA) multiplied by the transversal acceleration expressed as the second derivative of the y coordinate, thus creating an inertial force. The right side of the equation is merely a sum of shear forces acting on both sides of this element. The second equation is very similar, with the inertial moment on the left and the sum of all moments acting on the elements on the right.

We may discard some of the second order terms that do not contribute to the solution significantly and after rearranging the equations we finally get [21, 52]:

$$\rho A \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial T}{\partial x} \tag{2.65}$$

$$\rho J \frac{\partial^3 y(x,t)}{\partial t^2 \partial x} = T - \frac{\partial M}{\partial x} \tag{2.66}$$

2.5.1.2 Simplifying the Equation of Motion

It is possible to collect these two equations into one, by expressing T from the second equation (2.66) and substituting it back into the first one after differentiation. We will get one equation of motion given by

$$\rho A \frac{\partial^2 y(x,t)}{\partial t^2} = \frac{\partial^2 M}{\partial x^2} + \rho J \frac{\partial}{\partial x}\left(\frac{\partial^3 y(x,t)}{\partial t^2 \partial x}\right) \tag{2.67}$$

In statics, the curvature r of the deflection curve marked by the dotted line in the middle of Fig. 2.7 can be expressed by:

$$\frac{1}{r} \approx \frac{d^2 y}{dx^2} = -\frac{M}{EJ} \tag{2.68}$$

We can express M from this and substitute it back into our simplified equation of motion in (2.67) so we get [22]:

$$\rho A \frac{\partial^2 y(x,t)}{\partial t^2} = -EJ \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 y(x,t)}{\partial t^2}\right) + \rho J \frac{\partial}{\partial x}\left(\frac{\partial^3 y(x,t)}{\partial t^2 \partial x}\right) \tag{2.69}$$

This is the equation of motion for a beam vibrating in the transversal direction. To simplify notation, let us mark the time differentiation of $y(x,t)$ with respect to t by dots as in $\dot{y}$ and the position differentiation with respect to x by Roman numerals as in y^{ii}. If we ignore the effects of rotational inertia, we may denote the simplified equation motion for the free transversal vibration of a beam with constant cross section by [4, 52]:

$$\rho A \ddot{y} + EJ y^{iv} = 0 \tag{2.70}$$

We may further simplify this by dividing the whole equation by ρA and introducing

$$c = \frac{EJ}{\rho A} \tag{2.71}$$

where c is a constant[4] encompassing the square of the longitudinal wave and the square of the radius of quadratic moment of inertia. We finally arrive at the following equation of motion [3, 10, 21]:

[4] Certain literature divides this constant to a $c = c_0^2 i^2$, where $c_0 = \frac{E}{\rho}$ is the speed of the longitudinal wave and i is the radius of quadratic moment of inertia given by $i = \frac{J}{A}$.

$$\ddot{y} + c^2 y^{iv} = 0 \tag{2.72}$$

This equation expresses the free transversal vibration of a beam, neglecting the dynamic effects of the longitudinal forces and rotational inertia. Clearly, there is a lot of simplification assumed in this representation. If the above equation of motion would also include the effects of the rotational inertia, it would be according to Rayleigh's beam theorem. On the other hand, if it would include both the effects of the rotational inertia and the longitudinal forces, it would be Timoshenko's beam theorem [52]. The equation presented here is thus a special version of the Timoshenko beam theory also called the Euler–Bernoulli beam equation—or the classical beam theory.

2.5.1.3 Solving the Equation of Motion

The equation of motion in (2.72) merely gives a simplified representation of beam dynamics. The solution depends on the problem, as we also have to introduce boundary conditions and initial conditions. As the position differentiation of y is of the fourth degree, we can have four types of boundary conditions at the beginning and at the end [51]:

$$\begin{aligned} y(0,t) &= \Xi_1(t) & y(l,t) &= \Xi_5(t) \\ y^{i}(0,t) &= \Xi_2(t) & y^{i}(l,t) &= \Xi_6(t) \\ y^{ii}(0,t) &= \Xi_3(t) & y^{ii}(l,t) &= \Xi_7(t) \\ y^{iii}(0,t) &= \Xi_4(t) & y^{iii}(l,t) &= \Xi_8(t) \end{aligned} \tag{2.73}$$

These boundary conditions express the position of the beam (and its derivatives) at any given time. In more practical terms, the zeroth derivation is a deflection position, while the first is the angle of the tangent line to the neutral axis. Moreover, the second and third derivatives can be expressed using the moment and shear force as:

$$y^{ii}(0,t) = -\frac{M}{EJ} \tag{2.74}$$

$$y^{iii}(0,t) = -\frac{T}{EJ} \tag{2.75}$$

In addition to the boundary condition, we also have initial conditions, expressing geometrical configuration at zero time:

$$y(x,0) = \Psi_1(x) \quad \dot{y}(x,0) = \Psi_2(x) \tag{2.76}$$

As our main interest is vibration dynamics, let us assume that we want to find the resonant frequencies and mode shapes of a beam under free transversal vibration. Furthermore, let us now expect to arrive at a solution in the following form [21, 22]:

$$y(x,t) = a(x)V(t) \tag{2.77}$$

meaning that the position at a given place and time is composed of a combination of function $a(x)$, which is only dependent on the horizontal position and $V(t)$ which is only dependent on time. We may substitute this expected form back to (2.72) and get a new equation of motion:

$$\frac{a^{iv}}{a} = -c\frac{\ddot{V}}{V} \tag{2.78}$$

Since the right-hand side of the equation is only a function of time and the left-hand side of the equation is only the function of position x, each side of the equation must be equal to a constant. Let us call this constant Ω^2, which will help us separate the partial differential equation in (2.78) into two ordinary differential equations [22]:

$$a^{iv} - \frac{\Omega^2}{c}a = 0 \tag{2.79}$$

$$\ddot{V} + \Omega^2 V = 0 \tag{2.80}$$

In order to keep the notation simple, let us introduce the new constant υ^4 as:

$$\upsilon^4 = \frac{\Omega^2}{c} \tag{2.81}$$

Now we can expect the solution of Eq. (2.79) in the following form, with a constant A and an exponential term:

$$a(x) = Ae^{rx} \tag{2.82}$$

Substituting this back into (2.79) will yield a characteristic equation with the following roots:

$$r_{1,2} = \pm\upsilon \quad r_{3,4} = \pm j\upsilon \tag{2.83}$$

The solution of (2.79) will now assume the form [10, 22]:

$$a(x) = A_1e^{\upsilon x} + A_2e^{-\upsilon x} + A_3e^{j\upsilon x} + A_3e^{-j\upsilon x} \tag{2.84}$$

Utilizing the well-known Euler's formula establishing a relationship between trigonometric and complex exponential functions

$$e^{\pm j\upsilon x} = \cos\upsilon x \pm j\sin\upsilon x \tag{2.85}$$

$$e^{\pm\upsilon x} = \cosh\upsilon x \pm \sinh\upsilon x \tag{2.86}$$

and substituting this back into (2.84) we will get the solution in the following form [4, 10]:

$$a(x) = C_1 \cosh \upsilon x + C_2 \sinh \upsilon x + C_3 \cos \upsilon x + C_4 \sin \upsilon x \tag{2.87}$$

Constants C_1, C_2, C_3 and C_4 are integration constants and can be uniquely determined from the boundary conditions.

As different beam setups have different boundary conditions, we will pick a clamped cantilever beam, which is fixed at one end and free to vibrate at the other, as an example. For this clamped cantilever beam, the boundary conditions are given by [10, 21]:

$$\begin{aligned} y(0,t) &= 0 \qquad y^{ii}(l,t) = 0 \\ y^{i}(0,t) &= 0 \qquad y^{iii}(l,t) = 0 \end{aligned} \tag{2.88}$$

or, in other words, the beam cannot move at the clamped end and there are no shear forces or moments at the free end. Substituting these boundary conditions into (2.87), we will get a set of equations for the integration constants [51, 52]:

$$C_1 + C_3 = 0 \tag{2.89}$$

$$\upsilon(C_2 + C_4) = 0 \tag{2.90}$$

$$\upsilon^2(C_1 \cosh \upsilon x + C_2 \sinh \upsilon x - C_3 \cos \upsilon x - C_3 \sin \upsilon x) = 0 \tag{2.91}$$

$$\upsilon^2(C_1 \cosh \upsilon x + C_2 \sinh \upsilon x + C_3 \cos \upsilon x - C_3 \sin \upsilon x) = 0 \tag{2.92}$$

where l is the overall length of the beam implied by the boundary conditions. Moreover, for a nonzero υ the first two equations also imply that

$$C_3 = -C_1 \quad C_4 = C_2 \tag{2.94}$$

Substituting for C_3 and C_4 in the remaining two equations yields a set of two homogeneous equations with C_1 and C_2 as unknowns:

$$(\cosh \upsilon l + \cos \upsilon l)C_1 + (\sinh \upsilon l + \sin \upsilon l)C_2 = 0 \tag{2.95}$$

$$(\sinh \upsilon l - \sin \upsilon l)C_1 + (\cosh \upsilon l + \cos \upsilon l)C_2 = 0 \tag{2.96}$$

In order for Eq. (2.95) to have nontrivial solutions, its determinant has to equal zero [21, 51]. Computing this will yield the following frequency equation [21]:

$$\cos \upsilon l \cosh \upsilon l + 1 = 0 \tag{2.97}$$

The resonant frequencies of the beam can be calculated based on (2.81) by computing the roots of (2.97) and substituting into [22]:

$$\Omega_n = (\upsilon_n l)^2 \sqrt{\frac{EJ}{\rho A l^4}} \tag{2.98}$$

The equations given by (2.89) will not make it possible to compute the integrating constants, though it is possible to compute their ratios and substitute it back into (2.87) to get $a_n(x)$. The general solution describing the mode shapes of vibration will be finally given by [22, 52]:

$$y(x,t) = \sum_{n=1}^{\infty} (A_n \cos \Omega_n t + B_n \sin \Omega_n t) a_n x \tag{2.99}$$

where the integration constants A_n and B_n are given by

$$A_n = \frac{2}{l} \int_0^l \Psi_1(x) a_n(x) dx \tag{2.100}$$

$$B_n = \frac{2}{l\Omega_n} \int_0^l \Psi_2(x) a_n(x) dx \tag{2.101}$$

The process of obtaining the resonant frequency and mode shapes for other types of distributed parameter systems with simple geometry is analogous to the above introduced process. The interested reader is kindly referred to works discussing this topic in more depth [10, 21]. It is easy to see that working out a solution is a fairly time-consuming and complicated process, even for systems with simple geometry. This is why most practitioners prefer to utilize finite element analysis or experimental procedures to assess the vibration properties of such systems.

2.5.2 Damping in Distributed Systems Simulated by FEA

The damping of distributed mechanical systems is a very complex phenomenon. Unfortunately, energy dissipation in materials is not entirely explored by science at present. One of the simplest methods to approximate damping is to use the so-called Rayleigh damping which is often utilized in FEM simulations. This involves calculating the damping matrix as a sum of the mass and stiffness matrices, multiplied by the damping constants α and β:

$$\mathbf{B_d} = \alpha \mathbf{M} + \beta \mathbf{K_s} \tag{2.102}$$

The damping constants α and β are not known directly, instead they are calculated from the modal damping ratios ζ_i. This is actually the ratio of actual damping to critical damping for a particular mode of vibration. In case ω_i is the natural circular frequency for a given mode of vibration, we can calculate the constants utilizing [51, 52]:

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2} \tag{2.103}$$

It is often assumed that α and β are constant over a range of frequencies. For a given ζ_i and a frequency range, two simultaneous equations may be solved to obtain the Rayleigh damping constants. In most real-life structural applications mass damping can be neglected, therefore setting constant $\alpha = 0$. In these cases, β may be evaluated from the known values of ζ_i and ω_i:

$$\beta = \frac{2\zeta_i}{\omega_i} \tag{2.104}$$

2.6 Creating Models for Vibration Control

Different control systems call for different models and representations. There are numerous popular methods suitable to create a mathematical representation of the real system such as transfer functions or the state-space representation. We may choose to begin with a completely analytical fashion, and build our model based on the exact underlying physics. A mathematical model created on such first principle models using the underlying physics and equations is known as a white-box model. Models of vibrating structures are derived on a phenomenological basis for example in [26, 36, 48].

Active vibration control often utilizes advanced materials bonded into the structure. Moreover, the materials can have a coupled nature, having an intense electromechanical interaction with the structure—for example piezoceramics. If the underlying model is too complicated or it is too formidable to create a first principles model we may use experimental identification. If one is aware of the structure of the underlying physics, they may choose a given model and fit its parameters on an experimental measurement. This is known as a grey-box model. If nothing is known about the physics or one does not choose a model with a fixed structure, black-box identification is carried out.

In active vibration control there is an outside energy source acting on our vibrating mechanical system, such as a piezoelectric actuator. The model representing the behavior of the system therefore must represent forced vibration, regardless of whether it is created based on first principles or experimental methods.

2.6.1 Transfer Function Models

We will introduce a transfer function representation based on the dynamics analysis presented previously in Sect. 2.3 on the forced vibration of a point mass. Recall that the equation of motion of a point mass system with damping and a generic outside force is

$$m\ddot{q}(t) + b\dot{q}(t) + kq(t) = f_e \tag{2.105}$$

Now in order to create a transfer function from this, we must apply Laplace transformation to our equation. Let us denote the Laplace operator with s. The position coordinate $q(t)$ will be now denoted instead with $Q(s)$ and our external force $f_e t$ with $F_e(s)$. Differentiation in the Laplace domain is simply multiplication with the Laplace operator, therefore we will get [22]:

$$ms^2 Q(s) + bs\,Q(s) + k\,Q(s) = F_e(s) \tag{2.106}$$

The transfer function $H(s)$ will be simply a ratio of the Laplace transform of the output divided by the Laplace transform of the input. In this case, we may state that our transfer function expressing the effect to a force input is simply

$$H(s) = \frac{\mathscr{L}\{q(t)\}}{\mathscr{L}\{f_e(t)\}} = \frac{Q(s)}{F_e(s)} = \frac{1}{ms^2 + bs + k} \tag{2.107}$$

In the Laplace domain, we can compute the output of the mass-spring-damper system by the linear combination of the transfer function and an arbitrary force signal by evaluating

$$Q(s) = H(s)F(s) \tag{2.108}$$

Another valid representation of the transfer function is given in the term of the natural frequency ω_n and the proportional damping ζ [18]. For this, we divide the transfer function in (2.107) by the mass to get

$$H(s) = \frac{\frac{1}{m}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2.109}$$

where $c/m = 2\zeta\omega_n$ and ζ is proportional damping given as a percentage of critical damping.

2.6.1.1 Multiple DOF Transfer Functions

Lumped parameter systems with several vibrating masses or equivalent representations of distributed systems can be expressed using a transfer function representation as well. Let us consider a matrix representation of the forced equation of motion for an N degrees of freedom system as in (2.43) and perform a Laplace transform on the equation in the following sense:

$$\mathbf{M}\mathbf{s}^2\mathscr{Q} + \mathbf{B_d}\mathbf{s}^2\mathscr{Q} + \mathbf{K_s}\mathscr{Q} = \mathbf{F_e} \tag{2.110}$$

where $\mathbf{s}\mathscr{Q}$ denotes an n elements long vector of Laplace operators and output displacements, $\mathbf{s}^2\mathscr{Q}$ expresses the elementwise square of the Laplace operators multi-

plied by the output displacements $\mathscr{Q}$[5] in the Laplace domain. After the transformation and rearranging the terms, the equation of motion will be given in a matrix form in the s-domain:

$$\mathscr{H}(s)\mathscr{Q} = \mathbf{F_e} \tag{2.111}$$

where $\mathscr{H}(s)$ is a matrix of partial dynamics contributions. To get the transfer function for particular input and output points, one must rearrange this term. It is highly recommended to use computer symbolic algebra to arrive at the final transfer functions given large systems with many DOFs [18]. The transfer function representation of systems with several inputs or outputs (MIMO, SIMO, MISO) is not common, most engineering practitioners prefer to use the state-space representation to model the behavior of more complex systems.

2.6.1.2 Poles and Zeros

From a control viewpoint, we may further analyze the transfer function representation in (2.109), to see where the poles and zeros of lightly damped systems are located and what the physical interpretation of the poles and zeros of the transfer functions is. For this, let us assume that the proportional damping is approaching zero $\zeta \to 0$ to get

$$H(s) = \frac{\frac{1}{m}}{s^2 + \omega_n^2} \tag{2.112}$$

As it is usual in control, the denominator of the transfer function is extremely important. By setting the denominator equal to zero, we will get the *characteristic equation*. The roots of this equation are the so-called *poles* of transfer function, which amongst others affect the dynamic behavior and stability of the system. The poles represent the resonant frequencies of the vibrating system, in other words, the frequency values at which the inputs will be amplified. The characteristic equation of the vibrating point mass is

$$s^2 + \omega_n^2 = 0 \tag{2.113}$$

from which we may calculate the poles

$$s = \pm j\omega_n \tag{2.114}$$

It is clear that the poles of an undamped system will be located only on the imaginary axis (no real part) on complex plane. Similarly, the poles for a lightly damped system will be located close to the imaginary axis with very small negative real parts.

[5] In order to avoid confusion with **Q** used in later chapters as *input penalty*, the multiple DOF displacements transformed into the Laplace domain are denoted as $\mathscr{Q}$ here.

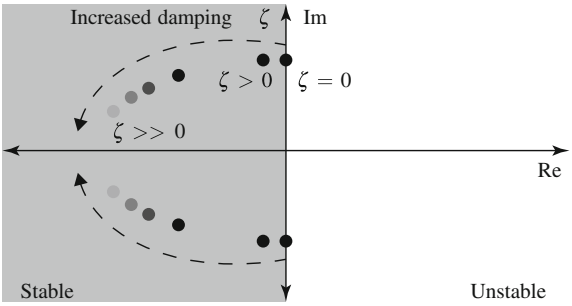


Fig. 2.8 Location of the poles for the vibrating point mass system depending on the amount of damping

Vibrating mechanical systems are inherently open-loop stable, therefore the poles of the transfer function must have negative real parts. Poles, and therefore the resonant frequency of a system, only depend on the distribution of mass, damping and stiffness and are independent on the excitation point or the measurement of the output [18].

Setting the numerator of a transfer function equal to zero and calculating roots will yield the *zeros* of the system. Zeros of the transfer function do depend on the point and type of excitation and the output. Some transfer functions may not even have zeros. In a physical interpretation, at zeros the vibrating system will attenuate the input. Similar to the poles, the zeros would also have no real part in the absence of damping. A pair of poles located at zero indicates a so-called *rigid body mode* [18]. A rigid body mode is the case when in the presence of a static or low frequency outside force the whole mechanical system moves without moving the parts relative to each other. In the case of the two vibrating masses example presented earlier in Fig. 2.5 a rigid body mode would occur if none of the masses were connected to a ground, but just to each other. At low frequencies or quasi-static forces, the two masses would move as one without considering the spring and damper in between them.

The location of the poles in the complex plane for the continuous transfer function of a simple second order vibrating system is demonstrated in Fig. 2.8. The left half of the plane is stable; this is where open-loop physical vibrating system poles are located. Without damping, the poles are on the imaginary axis, on the verge of stability (*right*). Lightly damped systems have poles very close to the imaginary axis (*middle*) while a more considerable amount of proportional damping would place the poles further away (*left*). Generally speaking, this is also true for the zeros of the system. The poles tend to be further away from each other in a lightly damped system, while increasing damping values will bring them closer together in the direction of the imaginary axis.

2.6.1.3 Discrete Transfer Functions

As digital control systems work in a sampled fashion instead of a continuous-time transfer function, we often need to use a discrete transfer function. In a discrete-time

system, the transfer function is created using the *Z-transform*, which is an analogy to the Laplace transform. For our case, a discrete time transfer function would be defined by:

$$H(z) = \frac{\mathscr{Z}\{q(kT)\}}{\mathscr{Z}\{f_e(kT)\}} = \frac{Q(z)}{F_e(z)} \tag{2.115}$$

A continuous time transfer function may be converted into discrete time using various algorithms using zero or first order hold to extract the function values in the time domain. The definition of the Z-transform essentially sums the pairs of the function values multiplied by a corresponding delay operator z^{-k}. Numerous numerical algorithms implement this procedure very effectively. Another example of converting a continuous transfer function into discrete is the so-called *bilinear transformation*. If T is the sampling time, to convert from continuous into discrete known as the Tustin transformation we can use: A continuous time transfer function may be converted into discrete time using various algorithms using zero or first order hold to extract the function values in the time domain. The definition of the Z-transform essentially sums the pairs of the function values multiplied by a corresponding delay operator z^{-k}. Numerous numerical algorithms implement this procedure very effectively. Another example of converting a continuous transfer function into discrete is the so-called *bilinear transformation*. If T is the sampling time, to convert from continuous into discrete known as the Tustin transformation we can use:

$$s = \frac{2}{T}\frac{z-1}{z+1} \tag{2.116}$$

while the reverse operation into convert from discrete to continuous is

$$z = \frac{2+sT}{2-sT} \tag{2.117}$$

Software tools such as for example Matlab also aid the conversion between continuous and discrete time models. In Matlab, this can be carried out using the *c2d* command.

A transfer function representation can be used to design or tune a controller for the vibrating system. A common approach used in practice is to utilize a software tool such as Matlab or Simulink to simulate the response of a system and the controller.

2.6.1.4 Experimental Identification

We may create a transfer function following a dynamical analysis and fill in the parameters such as m, b and k from direct measurements and consider them as physical quantities. However, in experimental identification, we may choose a given generic form of a transfer function and not expect to use parameters, which have a physical sense. Instead, we can use system identification algorithms, which compute

these parameters so that a test signal input matches a measured output. This makes sense for example when we would like to replace the dynamics of a continuous clamped beam with a single mass-spring-damper model. We cannot directly measure these quantities, rather we should find a generic second order transfer function that properly describes the first resonant mode and is given by

$$H(s) = \frac{\mathscr{L}\{q(t)\}}{\mathscr{L}\{f_e(t)\}} = \frac{Q(s)}{F_e(s)} = \frac{1}{a_1 s^2 + a_2 s + a_3} \tag{2.118}$$

where parameters a_1, a_2 and a_3 are found experimentally using an identification algorithm.

2.6.1.5 State-Space Representation

A *state-space* representation of a dynamic system may be perceived as a way to decouple high order differential equations into simpler low order equivalents, collected in a matrix representation. The set of first order differential equations then relates the inputs and outputs of a system by the aid of an intermediate variable called the *state*. If for example a single DOF vibrating system is described by a second order differential equation, we may use a state-space equivalent consisting of two first order differential equations. Analogously, if an MDOF vibrating system has its modes decomposed according to the guidelines presented previously in Sect. 2.4.2, then the behavior of each mode is described by N second order differential equations, which may be changed into $2N$ first order equations in the state-space representation. Most modern optimization-based control algorithms such as MPC utilize this type of mathematical model, and the numerical simulation of time responses is more convenient as well. To demonstrate the state-space representation of dynamic systems, we may divide the second order differential equation describing the forced response of the spring-mass-damper system in (2.34) into two first order ones.

2.6.1.6 State-Space Representation for a Single DOF System

Let us begin the rewriting of our second order differential equation by leaving the inertial force at the left side of our equation of motion (2.34) and moving the rest to the right which will result in:

$$m\ddot{q}(t) = -b\dot{q}(t) - kq(t) + f_e \tag{2.119}$$

while dividing the whole by m is

$$\ddot{q}(t) = -\frac{b}{m}\dot{q}(t) - \frac{k}{m}q(t) + \frac{1}{m}f_e \tag{2.120}$$

Because our original ODE is second order, let us choose two new variables $x_1(t)$ and $x_2(t)$. These two variables are the so-called state variables and we may combine

them to create a column state vector $\mathbf{x}(t)$. We may also choose that $x_1(t)$ is equivalent to our original position variable q, while $x_2(t)$ is simply a differentiation of it.

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \tag{2.121}$$

Let us now write a set of two first order ordinary differential equations with the help of our state variables $x_1(t)$ and $x_2(t)$. Substitute these into the transformed equation of motion and simply state that $\dot{x}_1(t) = x_2(t)$ [5]:

$$\dot{x}_1(t) = x_2(t) \tag{2.122}$$

$$\dot{x}_2(t) = -\frac{b}{m}x_2(t) - \frac{k}{m}x_1(t) + \frac{1}{m}f_e \tag{2.123}$$

which can be equivalently written in the following matrix form [5, 10]:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{b}{m} & -\frac{k}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \begin{bmatrix} 0 & f_e \end{bmatrix} \tag{2.124}$$

We may use a more compact matrix notation for the state-space variables $\mathbf{x}(t)$; moreover, let us define an input vector $\mathbf{u}(t)$ which contains the outside excitation force f_e as its first element: $\mathbf{u} = \begin{bmatrix} f_e & 0 \end{bmatrix}$. According to this, the matrix form of the state-space equation can be written as follows [5, 10]:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{2.125}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{2.126}$$

where $\mathbf{A}$ is the state transition matrix, $\mathbf{B}$ is the input matrix and $\mathbf{C}$ is the output matrix. Figure 2.9 illustrates the matrix block algebra of the continuous state-space dynamic system representation. The output vector $\mathbf{y}$(t) may contain both the position and velocity elements of the state or one of them according to the output matrix $\mathbf{C}$. For example a valid choice of $\mathbf{C}$ is $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ which would result in a scalar displacement output.

$$\mathbf{y}(t) = q(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \tag{2.127}$$

2.6.1.7 Discrete State-Space Representation

In the discrete time equivalent of the state-space equation we replace the continuous time variable t by its sampled analogy $t = kT$, where T is the sample time:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \tag{2.128}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \tag{2.129}$$

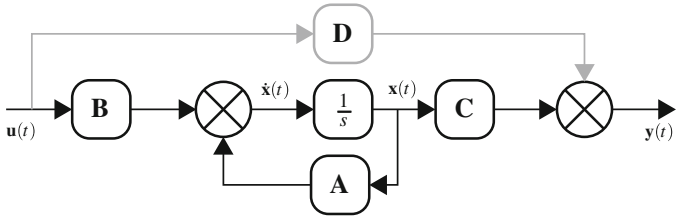


Fig. 2.9 Matrix block algebra of the continuous state-space dynamic system representation

Note that the contents of the matrices **A**, **B** and **C** also change if a system representation is converted from continuous into discrete. The Matlab command *c2d* is useful for the state-space representation as well.

Just as in the case of transfer functions, we may choose to apply an analytic approach and substitute for model parameters in a physical sense. Replacing complex vibrating systems with a mass-spring-damper equivalent, however, calls for parameters, which do not necessarily reflect physical quantities. In that case, we just choose our model, for example a second order state-space representation:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \mathbf{f}_e \tag{2.130}$$

$$\mathbf{y}(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \mathbf{x}(t) \tag{2.131}$$

and fill in the parameters using experimental identification procedures, based on test data.

2.6.1.8 State-Space Representation of Multiple DOF Systems

The previously introduced approach illustrated the process of obtaining a second order state-space model for a single degree of freedom system. As shown earlier, the equation of motion for multiple degree of freedom systems can be compactly written as:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{K_s}\mathbf{q} = \mathbf{f_e} \tag{2.132}$$

The order of systems expressed by this compact matrix notation can range from lumped parameter systems with model orders under ten, to several thousands and even millions for finite element models after discretization. To obtain a full order state-space model from the equation of motion in (2.132), at first one needs to transfer the terms $\mathbf{B}\dot{\mathbf{q}} + \mathbf{K_s}\mathbf{q}$ to the right side, while leaving $\mathbf{M}\ddot{\mathbf{q}}$:

$$\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{B}\dot{\mathbf{q}} - \mathbf{K_s}\mathbf{q} + \mathbf{f_e} \tag{2.133}$$

The equation of motion is then multiplied by the inverse of the mass matrix $\mathbf{M}^{-1}$ according to [10, 21]:

$$\ddot{\mathbf{q}} = -\mathbf{M}^{-1}\mathbf{B}\dot{\mathbf{q}} - \mathbf{M}^{-1}\mathbf{K_s}\mathbf{q} + \mathbf{M}^{-1}\mathbf{f_e} \tag{2.134}$$

In the next step, we will re-define the position output $\mathbf{q}$ and the outside excitation force $\mathbf{f_e}$ in terms of state variables and input according to:

$$\begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} = \mathbf{q} \quad \mathbf{u} = \mathbf{f_e} \tag{2.135}$$

Substituting our new state variables and input into the modified equation of motion in (2.134) will result in:

$$\ddot{\mathbf{x}} = -\mathbf{M}^{-1}\mathbf{B}\dot{\mathbf{x}} - \mathbf{M}^{-1}\mathbf{K_s}\mathbf{x} + \mathbf{M}^{-1}\mathbf{u} \tag{2.136}$$

which may be equivalently stated using the matrix notation

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{B} \\ -\mathbf{M}^{-1}\mathbf{K_s} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} + \mathbf{M}^{-1}\mathbf{u} \tag{2.137}$$

further, this is equivalent to stating that

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{B} & -\mathbf{M}^{-1}\mathbf{K_s} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{u} \tag{2.138}$$

where $\mathbf{I}$ is an identity matrix and $\mathbf{0}$ is a zero matrix of conforming dimensions. The state-space matrices $\mathbf{A}$ and $\mathbf{B}$ from this are:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{B} & -\mathbf{M}^{-1}\mathbf{K_s} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} \tag{2.139}$$

The resulting state-space system will have an order n_x, twice the DOF of the original system. In a small lumped parameter system, this will create a system model of manageable dimensions.

Typical FEM models of complex vibrating systems with ten to hundred thousand nodes can be decoupled and directly transformed into the state-space representation. Nevertheless, this results in extremely large state-space models not suitable for direct controller design. The order of these state-space systems needs to be reduced through the method of *model reduction*. State-space systems commonly used in control engineering have an order of 10–100 states, which may be even lower for the computationally intensive model predictive control approach. Although the order of the new and reduced state-space system will be severely truncated, it may still fully represent the dynamic behavior of the original system described by the much larger FEM model.

2.6.2 Experimental Identification Procedures

In case the modeling procedure of a controlled system is irrelevant or a first principles model would be too complex to develop, experimental identification may be used.

Experimental identification procedures are often utilized in order to create models for control design [7, 9, 54, 56, 63]. Control engineering uses system identification to build mathematical models of dynamical systems using statistical methods to find parameters in a set of experimental data.

2.6.2.1 Experiment Design

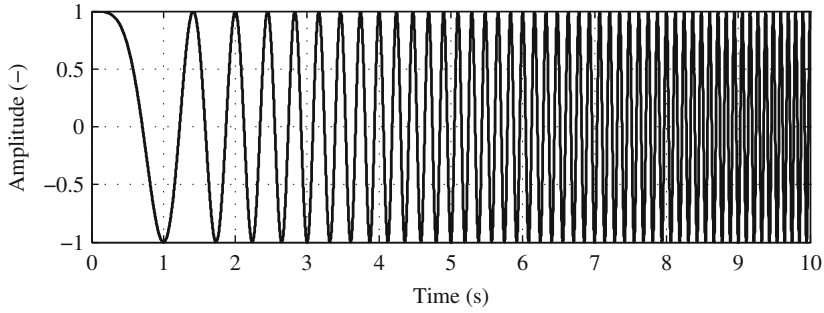
The quality of the identified model greatly depends on the quality of the experiment. In this relation, we have to emphasize the need for a properly designed input excitation signal. For example, in control engineering it is entirely acceptable to utilize a step change of temperature levels in a thermal related system—such as in heating. The resulting change of output carries enough information, so that an identification algorithm can extract a simple first or second order transfer function. In this case, the output levels are satisfactory and there are no resonant phenomena.

However, in the case of vibrating systems the previously considered step or impulse change at the actuators would not produce a satisfactory output signal. Due to its nature, one of the most important aspects of vibrational systems are not the precision of a static deflection after a step change in input, rather the dynamic response in and around the resonant frequency. A step or impulse input into a vibration attenuation system with bonded piezoelectrics would only result in vibrations with small amplitudes. In reality, the piezoelectric actuators may seriously affect and amplify vibration amplitudes in resonance. Therefore, it is wiser to use test signals that excite the vibrating system around the resonance frequencies of interest.

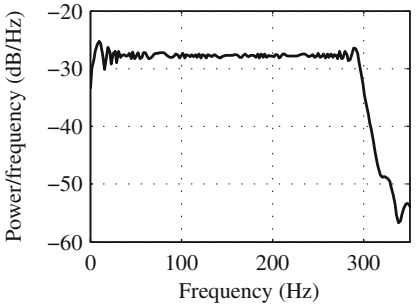
Such signals can be generated through the so-called *swept sine* or *chirp* function. The chirp signal is a type of signal with a constant amplitude, while the frequency changes from a low to a high limit in a given time. The frequency content of the signal covers the bandwidth of chirp signal—resulting in an ideally flat response in the frequency domain. An example of a chirp signal in the time domain is given in Fig. 2.10a. The frequency content of a different signal is shown in Fig. 2.10b, which indicates that the bandwidth of the chirp test signal is evenly spread out through the range of interest. A spectrogram featured in Fig. 2.10c relates the progression of time to the frequency content of the same signal. Chirp signals are commonly used to excite vibrating systems in academic works [23, 25, 28, 30, 31, 53, 57].

Another popular signal choice to excite vibrating systems in order to extract dynamic models is a pseudo-random binary signal [6, 25, 32, 39, 44, 55]. A pseudo-random signal has two levels changing in a random but repeatable fashion. The frequency content of the signal can also be influenced to concentrate on the bandwidth of interest.

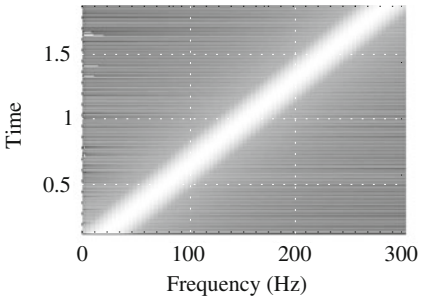
It is possible to supply time-domain measurement data directly to identification software, but by transforming it using fast Fourier transform (FFT) the identification procedure may be carried out in the frequency domain as well. Therefore, if the system identification software allows it, the mathematical model may be created based on fitting it to either the time or the frequency domain data set. As the FFT transform is a unitary transform, the frequency domain data set will contain exactly the same



(a) Chirp in the time domain



(b) Chirp in the frequency domain



(c) Spectrogram of a chirp signal

Fig. 2.10 An example chirp signal with a ± 1 (-) amplitude ranging from 0 to 10 Hz in 10 s is shown plotted in the time domain in (**a**). A different signal ranging from DC to 300 Hz in 2 s is shown in the frequency domain on a periodogram in (**b**) while the spectrogram of the same signal is featured in (**c**)

amount of data as the time domain, implying the same computational load for both domains. In case one requires to identify a very large time domain data set with clear resonance peaks in the frequency domain, it is advised to use a frequency domain data with non-equidistant frequency resolution. Leaving more data points in the vicinity of the peaks of the frequency domain data while reducing the number of data points elsewhere may significantly reduce the computational load and still produce high quality models. A common practice is to take the low and high frequency of interest and divide the region in between by logarithmically spaced frequencies.

2.7 Identification via Software Packages

There are off-the-shelf solutions for identification of mathematical models based on experimental test procedures. One of the most convenient and accessible is the System Identification Toolbox [59], a part of the Matlab software suite. In addition

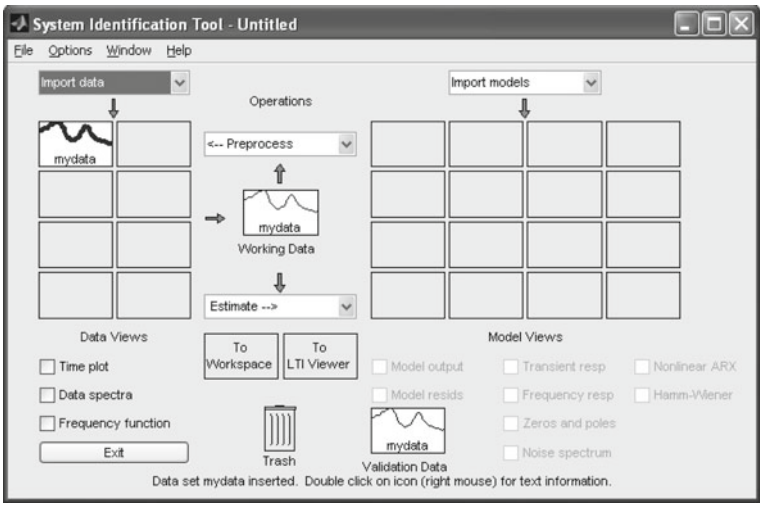


Fig. 2.11 The graphical user interface frontend of the Matlab System Identification Toolbox

to the general use, the System Identification Toolbox is also commonly used for creating models of vibrating mechanical systems [3, 14, 24, 27, 40, 47, 58, 60]. The System Identification Toolbox is largely based on the work of Ljung [29] and implements common techniques used in system identification. The toolbox aids the user to fit both linear and nonlinear models to measured data sets known as black box modeling [20]. Gray box modeling which tunes the parameters of a model with known structure is also offered by the suite. The types of available models are low order process models, transfer functions, state-space models, linear models with static nonlinearities, nonlinear autoregressive models, etc. The toolbox is made use of in the upcoming chapters of this book; see the system identification process in Sect. 5.2.

In addition to the usual Matlab command line interface, identification in the System Identification Toolbox may be carried out via the graphical user interface (GUI). A screenshot of the GUI frontend is featured in Fig. 2.11. The identification tasks are divided into separate parts. After creating an identification and validation data set, the data is pre-processed. Identification is initialized by selecting and setting up the proper model type. Finally the models can be validated using numerous techniques: comparing model response with measurement data, step response, a pole-zero plot, etc.

Transfer functions and low order process models are suitable for many controller types. It is also possible to create an MPC controller based on transfer functions, although state-space models are used more often in advanced control schemes. The MPC controller considered in this work utilizes a state-space representation as well. The aim of the identification process is therefore: given the input and output data set

one needs to identify the contents of matrices **A**, **B**, **C**. The Matlab System Identification toolbox offers two estimation methods for state-space models:

- subspace identification
- iterative prediction-error minimization method (PEM).

The order of a state-space system depends on many factors and it has to be determined by the control engineer at the time of the design of the representative mathematical model. It is always favorable to use the simplest possible system, which still describes the identified phenomena on a satisfactory level. MPC can be a computationally intensive operation in real-time, one must keep this in mind when creating a state-space model. In other words, the larger the model dimensionality or order n_x is, the more time it takes to perform the MPC optimization algorithm at each time step k. Surprisingly many practicing control engineers believe that most real phenomena involving single-input single-output (SISO) control can be approximated by simple second order state-space systems. For vibrating systems, this can be true mainly when one vibration mode is dominating over the others. Good examples are lightly damped vibrating systems where the first vibration mode is much more dominant than the others. In case the response of a second order system is unsatisfactory, one needs to increase the system order and inspect whether the response characteristics improve. In controlled mechanical vibrations the order of the identified system n_x should be an even number[6] and will contain an $f_i = n_x/2$ resonant frequencies. Given that one attempts to control a nonlinear vibrating system, there are ways to describe the phenomena at certain working points by several models. This however increases both the level of complexity of the linear MPC controller and the expected computational load. The application of nonlinear MPC (NMPC) in fast sampling application such as vibration control is still under development, as both the theoretical basis of NMPC and the computational hardware requires significant improvement for practical deployment.

One shall always verify whether the model produced by the experimental identification procedure is stable. In discrete systems, the poles of the transfer function must always reside within the unit circle. In other words, the absolute value or length of the vector denoting the pole position must be smaller than one. This condition can be restated for the case of state-space systems by saying that the absolute value of the eigenvalues of the matrix **A** shall be smaller than one. In practice, for vibrating mechanical systems, the absolute value of the eigenvalues will be a number which is smaller than one, albeit not by too much. This is understandable, since if the physical vibrating system is very lightly damped its behavior closely resembles that of a marginally stable system.[7]

Other system identification software is for example the System Identification Toolbox (or ID Toolbox)[8] [35], which is named identically to the official Mathworks

[6] Unless of course one uses an augmented system model with filters, observers etc.

[7] A vibrating system without outside energy cannot be marginally stable, since that would create a system without energy dissipation and without damping.

[8] Also known as The University of Newcastle Identification Toolbox (UNIT).

supported identification tool [33, 34]. This is however a free and entirely different software supporting a wide range of standard identification techniques. The Matlab suite developed by Ninness et al. also contains support for novel system identification algorithms [16, 43, 61, 62]. Similar to the System Identification Toolbox, the SMI Toolbox [19] is based on subspace identification techniques as well. Although this package has proven its worth over the years, its development has been on halt for a very long time and is slightly outdated. SLIDENT, which is incorporated into the Fortran 77 Subroutine Library in Control Theory (SLICOT) is suited for the identification of large-scale problems [45, 46]. ITSIE (Interactive Software Tool for System Identification Education) [17] is rather suited for the education process than research work or practical engineering deployment. SOCIT developed by NASA Langley Research Center [13] uses an eigensystem realization algorithm method based on Markov parameters and singular value decomposition techniques.

2.8 FEM-Based Identification

The study of physical systems described by ordinary or partial differential equations established through mathematical–physical analysis is not always possible. The reason for this is that there may be a lack of exact analytical solution, or boundary conditions can be too complex for a realistic solution. In some cases, analytical solution is possible, but a finite element (FE) model seems to be faster and more practical.

Models based on a finite element method (FEM) analysis may be used to create simplified mathematical representations of physical systems. Transient simulation results from the commercial FEM software package ANSYS were for example identified by Dong et al. in [11]. One may choose to extract and simplify the dynamic matrices assembled by the FEM software directly, or it is also possible to perform simulation analyses to get responses ready for an experimental identification procedure.

Direct output from an ANSYS harmonic analysis contains frequency data and the real and imaginary part of the amplitude response to a sinusoidal excitation in the frequency domain. First this data file is read into Matlab. The real and imaginary parts of the input can be generally described by

$$\begin{aligned} F_r &= F_0 \cos \Psi \\ F_i &= F_0 \sin \Psi \end{aligned} \tag{2.140}$$

where F_0 is the amplitude component of the signal, F_r is the real part of the response and F_i is the imaginary. The angle Ψ includes phase information. This formulation produced by ANSYS is not suitable for direct processing, therefore it has to be converted into amplitude and phase form. This may be done utilizing the following relation [1, 2]:

$$F = \sqrt{F_r + F_i}$$
$$\Psi = \tan^{-1}\left(\frac{F_i}{F_r}\right) \tag{2.141}$$

The conversion also involves manipulation with the direct results to ensure correct identification functionality. The converted raw data is made into a data object suitable for the System Identification package using the frequency function option [59]. Frequencies are input in rad/s, phase angles in degrees. The raw data from ANSYS must be also subjected to pre-processing such as filtering the bandwidth of interest.

The technique described above can be viable for system identification, given a good quality FEM model that has also been verified experimentally. In practice, however, the physical properties of mechanical systems (such as damping) are not always exactly known or cannot be directly measured. Moreover, a FEM model assumes Rayleigh damping, which distorts the amplitude levels especially at higher frequencies. According to the experience of the authors, if a structure with bonded piezoelectric transducers is modeled including the bonding glue or resin layer, the FEM model and measured frequency response may show a wide variation in the harmonic analysis results when properties of the explicitly modeled glue layer are adjusted. Using a control model based on a FEM model certainly cannot substitute a well-designed experiment; it may be however used at early stages of control system design-before the controlled plant is physically realized.

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