Chapter 2
Rank, Inner Product and Nonsingularity

2.1 Rank

Let A be an $m \times n$ matrix. The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the *column space* or the *column span* of A and is denoted by $\mathscr{C}(A)$. Similarly the subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called the *row space* of A, denoted by $\mathscr{R}(A)$. Clearly $\mathscr{R}(A)$ is isomorphic to $\mathscr{C}(A')$. The dimension of the column space is called the *column rank* whereas the dimension of the row space is called the *row rank* of the matrix. These two definitions turn out to be very short-lived in any linear algebra book since the two ranks are always equal as we show in the next result.

2.1 *The column rank of a matrix equals its row rank.*

Proof Let A be an $m \times n$ matrix with column rank r. Then $\mathscr{C}(A)$ has a basis of r vectors, say $b_1, \dots, b_r$. Let B be the $m \times r$ matrix $[b_1, \dots, b_r]$. Since every column of A is a linear combination of $b_1, \dots, b_r$, we can write $A = BC$ for some $r \times n$ matrix C. Then every row of A is a linear combination of the rows of C and therefore $\mathscr{R}(A) \subset \mathscr{R}(C)$. It follows by **1.7** that the dimension of $\mathscr{R}(A)$, which is the row rank of A, is at most r. We can similarly show that the column rank does not exceed the row rank and therefore the two must be equal. □

The common value of the column rank and the row rank of A will henceforth be called the *rank* of A and we will denote it by rank A. It is obvious that rank $A =$ rank A'. The rank of A is zero if and only if A is the zero matrix.

2.2 *Let A, B be matrices such that AB is defined. Then*

$$\operatorname{rank}(AB) \le \min\{\operatorname{rank} A, \operatorname{rank} B\}.$$

Proof A vector in $\mathscr{C}(AB)$ is of the form ABx for some vector x, and therefore it belongs to $\mathscr{C}(A)$. Thus $\mathscr{C}(AB) \subset \mathscr{C}(A)$ and hence by **1.7**,

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$$\operatorname{rank}(AB) = \dim \mathscr{C}(AB) \le \dim \mathscr{C}(A) = \operatorname{rank} A.$$

Now using this fact we have

$$\operatorname{rank}(AB) = \operatorname{rank}\bigl(B'A'\bigr) \le \operatorname{rank} B' = \operatorname{rank} B. \qquad \square$$

2.3 *Let A be an $m \times n$ matrix of rank r, $r \ne 0$. Then there exist matrices B, C of order $m \times r$, $r \times n$ respectively such that* $\operatorname{rank} B = \operatorname{rank} C = r$ *and* $A = BC$. *This decomposition is called a rank factorization of A.*

Proof The proof proceeds along the same lines as that of 2.1 so that we can write $A = BC$ where B is $m \times r$ and C is $r \times n$. Since the columns of B are linearly independent, $\operatorname{rank} B = r$. Since C has r rows, $\operatorname{rank} C \le r$. However, by 2.2, $r = \operatorname{rank} A \le \operatorname{rank} C$ and hence $\operatorname{rank} C = r$. $\square$

Throughout this monograph, whenever we talk of rank factorization of a matrix it is implicitly assumed that the matrix is nonzero.

2.4 *Let A, B be $m \times n$ matrices. Then* $\operatorname{rank}(A + B) \le \operatorname{rank} A + \operatorname{rank} B$.

Proof Let $A = XY$, $B = UV$ be rank factorizations of A, B. Then

$$A + B = XY + UV = [X, U]\begin{bmatrix} Y \\ V \end{bmatrix}.$$

Therefore, by 2.2,

$$\operatorname{rank}(A + B) \le \operatorname{rank}[X, U].$$

Let $x_1, \dots, x_p$ and $u_1, \dots, u_q$ be bases for $\mathscr{C}(X)$, $\mathscr{C}(U)$ respectively. Any vector in the column space of $[X, U]$ can be expressed as a linear combination of these $p + q$ vectors. Thus

$$\operatorname{rank}[X, U] \le \operatorname{rank} X + \operatorname{rank} U = \operatorname{rank} A + \operatorname{rank} B,$$

and the proof is complete. $\square$

The following operations performed on a matrix A are called *elementary column operations.*

(i) Interchange two columns of A.
(ii) Multiply a column of A by a nonzero scalar.
(iii) Add a scalar multiple of one column to another column.

These operations clearly leave $\mathscr{C}(A)$ unaffected and therefore they do not change the rank of the matrix. We may define elementary row operations similarly. The elementary row and column operations are particularly useful in computations. Thus to find the rank of a matrix we first reduce it to a matrix with several zeros by these operations and then compute the rank of the resulting matrix.

2.2 Inner Product

Let S be a vector space. A function which assigns a real number $\langle x, y\rangle$ to every pair of vectors x, y in S is said to be an *inner product* if it satisfies the following conditions:

(i) $\langle x, y\rangle = \langle y, x\rangle$
(ii) $\langle x, x\rangle \geq 0$ and equality holds if and only if $x = 0$
(iii) $\langle cx, y\rangle = c\langle x, y\rangle$
(iv) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

In $\mathbb{R}^n$, $\langle x, y\rangle = x'y = x_1y_1 + \cdots + x_ny_n$ is easily seen to be an inner product. We will work with this inner product while dealing with $\mathbb{R}^n$ and its subspaces, unless indicated otherwise.

For a vector x, the positive square root of the inner product $\langle x, x\rangle$ is called the *norm* of x, denoted by $\|x\|$. Vectors x, y are said to be *orthogonal* or *perpendicular* if $\langle x, y\rangle = 0$, in which case we write $x \perp y$.

2.5 *If $x_1, \dots, x_m$ are pairwise orthogonal nonzero vectors then they are linearly independent.*

Proof Suppose $c_1x_1 + \cdots + c_mx_m = 0$. Then

$$\langle c_1x_1 + \cdots + c_mx_m, x_1\rangle = 0$$

and hence

$$\sum_{i=1}^{m} c_i\langle x_i, x_1\rangle = 0.$$

Since the vectors $x_1, \dots, x_m$ are pairwise orthogonal, it follows that $c_1\langle x_1, x_1\rangle = 0$ and since x_1 is nonzero, $c_1 = 0$. Similarly we can show that each c_i is zero. Therefore the vectors are linearly independent. $\square$

A set of vectors $x_1, \dots, x_m$ is said to form an *orthonormal basis* for the vector space S if the set is a basis for S and furthermore, $\langle x_i, x_j\rangle$ is 0 if $i \neq j$ and 1 if $i = j$.

We now describe the *Gram–Schmidt procedure* which produces an orthonormal basis starting with a given basis, $x_1, \dots, x_n$.

Set $y_1 = x_1$. Having defined $y_1, \dots, y_{i-1}$, we define

$$y_i = x_i - a_{i,i-1}y_{i-1} - \cdots - a_{i1}y_1$$

where $a_{i,i-1}, \dots, a_{i1}$ are chosen so that y_i is orthogonal to $y_1, \dots, y_{i-1}$. Thus we must solve $\langle y_i, y_j\rangle = 0$, $j = 1, \dots, i-1$. This leads to

$$\langle x_i - a_{i,i-1}y_{i-1} - \cdots - a_{i1}y_1, y_j\rangle = 0, \quad j = 1, \dots, i-1$$

which gives

$$\langle x_i, y_j\rangle - \sum_{k=1}^{i-1} a_{ik}\langle y_k, y_j\rangle = 0, \quad j = 1, \dots, i-1.$$

Now since $y_1, \dots, y_{i-1}$, is an orthogonal set, we get

$$\langle x_i, y_j \rangle - a_{ij} \langle y_j, y_j \rangle = 0$$

and hence,

$$a_{ij} = \frac{\langle x_i, y_j \rangle}{\langle y_j, y_j \rangle}; \quad j = 1, \dots, i-1.$$

The process is continued to obtain the basis $y_1, \dots, y_n$ of pairwise orthogonal vectors. Since $x_1, \dots, x_n$ are linearly independent, each y_i is nonzero. Now if we set $z_i = \frac{y_i}{\|y_i\|}$, then $z_1, \dots, z_n$ is an orthonormal basis. Note that the linear span of $z_1, \dots, z_i$ equals the linear span of $x_1, \dots, x_i$ for each i.

We remark that given a set of linearly independent vectors $x_1, \dots, x_m$, the Gram–Schmidt procedure described above can be used to produce a pairwise orthogonal set $y_1, \dots, y_m$, such that y_i is a linear combination of $x_1, \dots, x_{i-1}$, $i = 1, \dots, m$. This fact is used in the proof of the next result.

Let W be a set (not necessarily a subspace) of vectors in a vector space S. We define

$$W^{\perp} = \{x : x \in S,\ \langle x, y \rangle = 0 \text{ for all } y \in W\}.$$

It follows from the definitions that $W^{\perp}$ is a subspace of S.

2.6 *Let S be a subspace of the vector space T and let $x \in T$. Then there exists a unique decomposition $x = u + v$ such that $u \in S$ and $v \in S^{\perp}$. The vector u is called the orthogonal projection of x on the vector space S.*

Proof If $x \in S$ then $x = x + 0$ is the required decomposition. Otherwise, let $x_1, \dots, x_m$ be a basis for S. Use the Gram–Schmidt process on the set $x_1, \dots, x_m, x$ to obtain the sequence $y_1, \dots, y_m, v$ of pairwise orthogonal vectors. Since v is perpendicular to each y_i and since the linear span of $y_1, \dots, y_m$ equals that of $x_1, \dots, x_m$, then $v \in S^{\perp}$. Also, according to the Gram–Schmidt process, $x - v$ is a linear combination of $y_1, \dots, y_m$ and hence $x - v \in S$. Now $x = (x - v) + v$ is the required decomposition. It remains to show the uniqueness.

If $x = u_1 + v_1 = u_2 + v_2$ are two decompositions satisfying $u_1 \in S$, $u_2 \in S$, $v_1 \in S^{\perp}$, $v_2 \in S^{\perp}$; then

$$(u_1 - u_2) + (v_1 - v_2) = 0.$$

Since $\langle u_1 - u_2, v_1 - v_2 \rangle = 0$, it follows from the preceding equation that $\langle u_1 - u_2, u_1 - u_2 \rangle = 0$. Then $u_1 - u_2 = 0$ and hence $u_1 = u_2$. It easily follows that $v_1 = v_2$. Thus the decomposition is unique. □

2.7 *Let W be a subset of the vector space T and let S be the linear span of W. Then*

$$\dim(S) + \dim(W^{\perp}) = \dim(T).$$

Proof Suppose $\dim(S) = m$, $\dim(W^{\perp}) = n$ and $\dim(T) = p$. Let $x_1, \dots, x_m$ and $y_1, \dots, y_n$ be bases for S, $W^{\perp}$ respectively. Suppose

$$c_1 x_1 + \cdots + c_m x_m + d_1 y_1 + \cdots + d_n y_n = 0.$$

Let $u = c_1 x_1 + \cdots + c_m x_m$, $v = d_1 y_1 + \cdots + d_n y_n$. Since x_i, y_j are orthogonal for each i, j; u and v are orthogonal. However $u + v = 0$ and hence $u = v = 0$. It follows that $c_i = 0$, $d_j = 0$ for each i, j and hence $x_1, \dots, x_m, y_1, \dots, y_n$ is a linearly independent set. Therefore $m + n \leq p$. If $m + n < p$, then there exists a vector $z \in T$ such that $x_1, \dots, x_m, y_1, \dots, y_n, z$ is a linearly independent set. Let M be the linear span of $x_1, \dots, x_m, y_1, \dots, y_n$. By **2.6** there exists a decomposition $z = u + v$ such that $u \in M$, $v \in M^{\perp}$. Then v is orthogonal to x_i for every i and hence $v \in W^{\perp}$. Also, v is orthogonal to y_i for every i and hence $\langle v, v\rangle = 0$ and therefore $v = 0$. It follows that $z = u$. This contradicts the fact that z is linearly independent of $x_1, \dots, x_m, y_1, \dots, y_n$. Therefore $m + n = p$. □

The proof of the next result is left as an exercise.

2.8 *If* $S_1 \subset S_2 \subset T$ *are vector spaces, then*: (i) $(S_2)^{\perp} \subset (S_1)^{\perp}$. (ii) $(S_1^{\perp})^{\perp} = S_1$.

Let A be an $m \times n$ matrix. The set of all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$ is easily seen to be a subspace of $\mathbb{R}^n$. This subspace is called the *null space* of A, and we denote it by $\mathscr{N}(A)$.

2.9 *Let A be an $m \times n$ matrix. Then* $\mathscr{N}(A) = \mathscr{C}(A')^{\perp}$.

Proof If $x \in \mathscr{N}(A)$ then $Ax = 0$ and hence $y'Ax = 0$ for all $y \in \mathbb{R}^m$. Thus x is orthogonal to any vector in $\mathscr{C}(A')$. Conversely, if $x \in \mathscr{C}(A')^{\perp}$, then x is orthogonal to every column of A' and therefore $Ax = 0$. □

2.10 *Let A be an $m \times n$ matrix of rank r. Then* $\dim(\mathscr{N}(A)) = n - r$.

Proof We have

$$\begin{aligned} \dim\bigl(\mathscr{N}(A)\bigr) &= \dim\bigl(\mathscr{C}\bigl(A'\bigr)^{\perp}\bigr) \quad \text{by } \mathbf{5.5} \\ &= n - \dim\bigl(\mathscr{C}\bigl(A'\bigr)\bigr) \quad \text{by } \mathbf{2.7} \\ &= n - r. \end{aligned}$$

That completes the proof. □

The dimension of the null space of A is called the *nullity* of A. Thus **2.10** says that *the rank plus the nullity equals the number of columns*. For this reason we will refer to **2.10** as the "rank plus nullity" theorem.

2.3 Nonsingularity

Suppose we have m linear equations in the n unknowns $x_1, \dots, x_n$. The equations can conveniently be expressed as a single matrix equation $Ax = b$, where A is the $m \times n$ matrix of coefficients. The equation $Ax = b$ is said to be *consistent* if it has at least one solution, otherwise it is *inconsistent.* The equation is *homogeneous* if $b = 0$. The set of solutions of the homogeneous equation $Ax = 0$ is clearly the null space of A.

If the equation $Ax = b$ is consistent then we can write

$$b = x_1^0 a_1 + \cdots + x_n^0 a_n$$

for some $x_1^0, \dots, x_n^0$ where $a_1, \dots, a_n$ are the columns of A. Thus $b \in \mathscr{C}(A)$. Conversely, if $b \in \mathscr{C}(A)$ then $Ax = b$ must be consistent. If the equation is consistent and if x^0 is a solution of the equation then the set of all solutions of the equation is given by

$$\left\{x^0 + x : x \in \mathscr{N}(A)\right\}.$$

Clearly, the equation $Ax = b$ has either no solution, a unique solution or infinitely many solutions.

A matrix A of order $n \times n$ is said to be *nonsingular* if $\operatorname{rank} A = n$, otherwise the matrix is *singular*.

2.11 *Let A be an $n \times n$ matrix. Then the following conditions are equivalent*:

(i) *A is nonsingular, i.e.*, $\operatorname{rank} A = n$.
(ii) *For any $b \in R^n$, $Ax = b$ has a unique solution.*
(iii) *There exists a unique matrix B such that $AB = BA = I$.*

Proof (i) $\Rightarrow$ (ii). Since $\operatorname{rank} A = n$ we have $\mathscr{C}(A) = \mathbb{R}^n$ and therefore $Ax = b$ has a solution. If $Ax = b$ and $Ay = b$ then $A(x - y) = 0$. By 2.10, $\dim(\mathscr{N}(A)) = 0$ and therefore $x = y$. This proves the uniqueness.

(ii) $\Rightarrow$ (iii). By (ii), $Ax = e_i$ has a unique solution, say b_i, where e_i is the i-th column of the identity matrix. Then $B = (b_1, \dots, b_n)$ is a unique matrix satisfying $AB = I$. Applying the same argument to A' we conclude the existence of a unique matrix C such that $CA = I$. Now $B = (CA)B = C(AB) = C$.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Then any $x \in \mathbb{R}^n$ can be expressed as $x = A(Bx)$ and hence $\mathscr{C}(A) = \mathbb{R}^n$. Thus $\operatorname{rank} A$, which by definition is $\dim(\mathscr{C}(A))$ must be n. $\square$

The matrix B of (ii) of 2.11 is called the *inverse* of A and is denoted by A^{-1}.

If A, B are $n \times n$ matrices, then $(AB)(B^{-1}A^{-1}) = I$ and therefore $(AB)^{-1} = B^{-1}A^{-1}$. In particular, the product of two nonsingular matrices is nonsingular.

Let A be an $n \times n$ matrix. We will denote by A_{ij} the submatrix of A obtained by deleting row i and column j. The *cofactor* of a_{ij} is defined to be $(-1)^{i+j}|A_{ij}|$. The *adjoint* of A, denoted by $\operatorname{adj} A$, is the $n \times n$ matrix whose (i, j)-entry is the cofactor of a_{ji}.

From the theory of determinants we have

$$\sum_{j=1}^{n} a_{ij}(-1)^{i+j}|A_{ij}| = |A|$$

and for $i \neq k$,

$$\sum_{j=1}^{n} a_{ij}(-1)^{i+k}|A_{kj}| = 0.$$

These equations can be interpreted as

$$A \operatorname{adj} A = |A|I.$$

Thus if $|A| \neq 0$, then A^{-1} exists and

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A.$$

Conversely if A is nonsingular, then from $AA^{-1} = I$, we conclude that $|AA^{-1}| = |A||A^{-1}| = 1$ and therefore $|A| \neq 0$. We have therefore proved the following result.

2.12 *A square matrix is nonsingular if and only if its determinant is nonzero.*

An $r \times r$ *minor* of a matrix is defined to be the determinant of an $r \times r$ submatrix of A.

Let A be an $m \times n$ matrix of rank r, let $s > r$, and consider an $s \times s$ minor of A, say the one formed by rows $i_1, \dots, i_s$ and columns $j_1, \dots, j_s$. Since the columns $j_1, \dots, j_s$ must be linearly dependent then by **2.12** the minor must be zero.

Conversely, if A is of rank r then A has r linearly independent rows, say the rows $i_1, \dots, i_r$. Let B be the submatrix formed by these r rows. Then B has rank r and hence B has column rank r. Thus there is an $r \times r$ submatrix C of B, and hence of A, of rank r. By **2.12**, C has a nonzero determinant.

We therefore have the following definition of rank in terms of minors: The rank of the matrix A is r if (i) there is a nonzero $r \times r$ minor and (ii) every $s \times s$ minor, $s > r$, is zero. As remarked earlier, the rank is zero if and only if A is the zero matrix.

2.4 Frobenius Inequality

2.13 *Let B be an $m \times r$ matrix of rank r. Then there exists a matrix X (called the* left inverse *of B), such that $XB = I$.*

Proof If $m = r$ then B is nonsingular and admits an inverse. So suppose $r < m$. The columns of B are linearly independent. Thus we can find a set of $m - r$ columns, which, together with the columns of B, form a basis for $\mathbb{R}^m$. In other words, we can find a matrix U of order $m \times (m-r)$ such that $[B, U]$ is nonsingular. Let the inverse of $[B, U]$ be partitioned as $\left[\begin{smallmatrix} X \\ V \end{smallmatrix}\right]$ where X is $r \times m$. Since

$$\begin{bmatrix} X \\ V \end{bmatrix}[B, U] = I,$$

we have $XB = I$. $\square$

We can similarly show that an $r \times n$ matrix C of rank r has a *right inverse,* i.e., a matrix Y such that $CY = I$. Note that a left inverse or a right inverse is not unique, unless the matrix is square and nonsingular.

2.14 *Let B be an $m \times r$ matrix of rank r. Then there exists a nonsingular matrix P such that*

$$PB = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Proof The proof is the same as that of **2.13**. If we set $P = \left[\begin{smallmatrix} X \\ V \end{smallmatrix}\right]$ then P satisfies the required condition. $\square$

Similarly, if C is $r \times n$ of rank r then there exists a nonsingular matrix Q such that $CQ = [I, 0]$. These two results and the rank factorization (see **2.3**) immediately lead to the following.

2.15 *Let A be an $m \times n$ matrix of rank r. Then there exist nonsingular matrices P, Q such that*

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Rank is not affected upon multiplying by a nonsingular matrix. For, if A is $m \times n$ and P is nonsingular of order m then

$$\begin{aligned} \operatorname{rank} A &= \operatorname{rank}\bigl(P^{-1}PA\bigr) \\ &\le \operatorname{rank}(PA) \\ &\le \operatorname{rank} A. \end{aligned}$$

Hence $\operatorname{rank}(PA) = \operatorname{rank} A$. A similar result holds for post-multiplication by a nonsingular matrix.

2.16 *If A is an $n \times n$ matrix of rank r then there exists an $n \times n$ matrix Z of rank $n - r$ such that $A + Z$ is nonsingular.*

Proof By **2.15** there exist nonsingular matrices P, Q such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Set

$$Z = P^{-1}\begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^{-1}.$$

Then $A + Z = P^{-1}Q^{-1}$ which is nonsingular. □

Observe that 2.16 may also be proved using rank factorization; we leave this as an exercise.

2.17 The Frobenius Inequality *Let A, B be $n \times n$ matrices. Then*

$$\operatorname{rank}(AB) \geq \operatorname{rank} A + \operatorname{rank} B - n.$$

Proof By 2.16 there exists a matrix Z of rank $n - \operatorname{rank} A$ such that $A + Z$ is nonsingular. We have

$$\begin{aligned} \operatorname{rank} B &= \operatorname{rank}\bigl((A+Z)B\bigr) \\ &= \operatorname{rank}(AB + ZB) \\ &\leq \operatorname{rank}(AB) + \operatorname{rank}(ZB) \quad \text{by } 2.4 \\ &\leq \operatorname{rank}(AB) + \operatorname{rank}(Z) \\ &= \operatorname{rank}(AB) + n - \operatorname{rank} A. \end{aligned}$$

Hence $\operatorname{rank}(AB) \geq \operatorname{rank} A + \operatorname{rank} B - n$. □

2.5 Exercises

1. Find the rank of the following matrix for each real number α:
$$\begin{bmatrix} 1 & 4 & \alpha & 4 \\ 2 & -6 & 7 & 1 \\ 3 & 2 & -6 & 7 \\ 2 & 2 & -5 & 5 \end{bmatrix}.$$
2. Let $\{x_1, \ldots, x_p\}$, $\{y_1, \ldots, y_q\}$ be linearly independent sets in $\mathbb{R}^n$, where $p < q \leq n$. Show that there exists $i \in \{1, \ldots, q\}$ such that $\{x_1, \ldots, x_p, y_i\}$ is linearly independent.
3. Let $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ be bases for $\mathbb{R}^n$ and let $S \subset X$ be a set of cardinality r, $1 \leq r \leq n$. Show that there exists $T \subset Y$ of cardinality r such that $(X \setminus S) \cup T$ is a basis for $\mathbb{R}^n$.
4. Let A be an $m \times n$ matrix and let B be obtained by changing any k entries of A. Show that
$$\operatorname{rank} A - k \leq \operatorname{rank} B \leq \operatorname{rank} A + k.$$
5. Let A, B, C be $n \times n$ matrices. Is it always true that $\operatorname{rank}(ABC) \leq \operatorname{rank}(AC)$?

6. Find two different rank factorizations of the matrix

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & -3 & 1 & 1 \\ 3 & -2 & 3 & 1 \\ 5 & -5 & 4 & 2 \end{bmatrix}.$$

7. Let A, B, C, D be $n \times n$ matrices such that the matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

has rank n. Show that $|AD| = |BC|$.

8. Which of the following functions define an inner product on R^3?
   (i) $f(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3 + 1$
   (ii) $f(x, y) = 2x_1 y_1 + 3x_2 y_2 + x_3 y_3 - x_1 y_2 - x_2 y_1$
   (iii) $f(x, y) = x_1 y_1 + 2x_2 y_2 + x_3 y_3 + 2x_1 y_2 + 2x_2 y_1$
   (iv) $f(x, y) = x_1 y_1 + x_2 y_2$
   (v) $f(x, y) = x_1^3 y_1^3 + x_2^3 y_2^3 + x_3^3 y_3^3$.
9. Find the orthogonal projection of [2, 1, 0] on the space spanned by $[1, -1, 1]$, [0, 1, 1].
10. The following vectors form a basis for $\mathbb{R}^3$. Use the Gram–Schmidt procedure to convert it into an orthonormal basis.

$$x = \begin{bmatrix} 2 & 3 & -1 \end{bmatrix}, \qquad y = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}, \qquad z = \begin{bmatrix} 4 & -1 & 2 \end{bmatrix}.$$

11. Let A be an $n \times n$ matrix. Show that A is nonsingular if and only if $Ax = 0$ has no nonzero solution.
12. Let A be a nonsingular matrix, let $B = A^{-1}$, and suppose A, B are conformally partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

Then assuming the inverses exist, show that

$$B_{11} = \left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1} = A_{11}^{-1} + A_{11}^{-1} A_{12} B_{22} A_{21} A_{11}^{-1},$$
$$B_{22} = \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} = A_{22}^{-1} + A_{22}^{-1} A_{21} B_{11} A_{12} A_{22}^{-1},$$
$$B_{12} = -A_{11}^{-1} A_{12} B_{22} = -B_{11} A_{12} A_{22}^{-1}.$$

13. Let A be an $n \times n$ matrix and let $b \in \mathbb{R}^n$. Show that A is nonsingular if and only if $Ax = b$ has a unique solution.
14. Let A be an $n \times n$ matrix with only integer entries. Show that A^{-1} exists and has only integer entries if and only if $|A| = \pm 1$.
15. Compute the inverses of the following matrices:
   (i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $ad - bc \neq 0$
   (ii) $\begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & -1 \\ 1 & 0 & 4 \end{bmatrix}$.

16. Let A, B be matrices of order 9×7 and 4×3 respectively. Show that there exists a nonzero 7×4 matrix X such that $AXB = 0$.
17. Let A, X, B be $n \times n$ matrices. Prove the following generalization of the Frobenius Inequality:

$$\operatorname{rank}(AXB) \geq \operatorname{rank}(AX) + \operatorname{rank}(XB) - \operatorname{rank}(X).$$

18. Let A, B, C, D be $n \times n$ matrices such that A is nonsingular and suppose $AC = CA$. Then show that

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |AD - CB|.$$

19. Let P be an orthogonal matrix and let Q be obtained by deleting the first row and column of P. Show that p_{11} and $|Q|$ are equal in absolute value.

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