2

Sobolev Spaces and Embedding Theorems

2.1 Definitions and First Properties

Definition 2.1. Let Ω be an open subset of $\mathbb{R}^N$. For $m \in \mathbb{N}$ and $1 \leqslant p \leqslant +\infty$, the Sobolev space denoted by $W^{m,p}(\Omega)$ consists of the functions in $L^p(\Omega)$ whose partial derivatives up to order m, in the sense of distributions, can be identified with functions in $L^p(\Omega)$.

For these derivatives, we set $\alpha = (\alpha_1, \ldots, \alpha_N)$ and $|\alpha| = \sum_1^N \alpha_i$. Moreover, we use the notation

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_N} x_N}. \tag{2.2}$$

The definition above can now be written as

$$W^{m,p}(\Omega) = \big\{ u \in L^p(\Omega) \mid \forall\, \alpha \in \mathbb{N}^N,\ |\alpha| \leqslant m \Rightarrow D^\alpha u \in L^p(\Omega) \big\}. \tag{2.3}$$

Remark 2.4 (on the structure of the derivatives in $W^{1,p}(\Omega)$). We will use the notion of the derivative of an absolutely continuous function in the usual sense (cf. Exercise 2.3) to better understand what it means for u to belong to $W^{1,p}(\Omega)$.

Let $u \in W^{1,p}(\Omega)$; then for every i, the function u is absolutely continuous along almost all lines parallel to the vector $\overrightarrow{e_i}$ of the canonical basis of $\mathbb{R}^N$. Moreover, the derivative $\partial_i u$ of u in the usual sense, which exists almost everywhere on Ω, belongs to $L^p(\Omega)$ and is almost everywhere equal to the derivative in the sense of distributions. Conversely, if for every i, $u \in L^p(\Omega)$ is absolutely continuous along almost all lines parallel to e_i, with derivatives $\partial_i u$ in $L^p(\Omega)$, then $u \in W^{1,p}(\Omega)$.

It follows that if u is of class $\mathcal{C}^1$ on Ω, then we can verify that $u \in W^{1,p}(\Omega)$ by showing that the functions u and $\partial_i u$ belong to $L^p(\Omega)$. The following examples use this property.

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Remark 2.5. For $p = 2$, the notation $W^{m,2}(\Omega)$ is generally replaced by $H^m(\Omega)$.

Remark 2.6. When $\Omega = \mathbb{R}^N$, we can use the Fourier transform $\xi \mapsto \widehat{u}(\xi)$ of a function u in $L^2(\mathbb{R}^N)$ to give the following equivalent definition:

$$W^{m,2}(\mathbb{R}^N) = H^m(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) \mid \xi \mapsto (1+|\xi|^2)^{m/2}\,\widehat{u}(\xi) \in L^2(\mathbb{R}^N)\}.$$

Example 2.7. Consider the open unit ball $\Omega = B(0,1)$ in $\mathbb{R}^2$. Let us determine under which condition the function u on Ω defined by $u(x,y) = xy(x^2+y^2)^{-\beta}$ outside of the origin, with $\beta > 0$, is an element of $H^1(\Omega)$.

More precisely, let us show that $u \in H^1(\Omega)$ if and only if $\beta < 1$. The integral of $|u|^2$ on Ω exists if $5 - 4\beta > -1$ or, equivalently, if $\beta < 3/2$. Indeed, in polar coordinates, the integrand can be written as

$$|u|^2 r dr d\theta = r^{5-4\beta}(\sin\theta\cos\theta)^2 dr d\theta.$$

For the derivative in x in the usual sense, this gives

$$\partial_x u = y(x^2+y^2)^{-\beta} - 2\beta x^2 y(x^2+y^2)^{-\beta-1}.$$

This derivative is continuous outside of $(0,0)$. The integral of its square consists of three terms in which the exponent of r is equal to $3 - 4\beta$. These exponents are all greater than -1 if and only if the condition $\beta < 1$ is satisfied. Since the function is symmetric in x and y, it follows that if $\beta < 1$, then u and its derivatives belong to $L^2(B)$. By Remark 2.4 above, this implies that the latter are derivatives in the sense of distributions.

This concludes the proof of the necessity and sufficiency of the condition stated above.

Example 2.8. Consider the open unit ball $\Omega = B(0,1)$ in $\mathbb{R}^N$. Let $r^2 = \sum_1^N x_j^2$ and let u be defined on Ω by $u(x) = (1-r)^\beta(-\ln(1-r))^\alpha$, where α is an arbitrary real number and $\beta > 0$. We want to know under which conditions on α and β that u is an element of $W^{1,p}(\Omega)$.

The function u admits two singularities, at $r = 0$ and at $r = 1$. As the logarithm is equivalent to r^α at 0, the function $|u|^p$ is summable on Ω if $N - 1 + \alpha p > -1$, that is, if $\alpha > -N/p$. At $r = 1$, the function can be extended by continuity. The derivative in the usual sense, for example at x_1, is then

$$\partial_1 u(x) = \frac{x_1}{r}(1-r)^{\beta-1}|\ln(1-r)|^{\alpha-1}\big(\beta|\ln(1-r)| + \alpha\big).$$

At $r = 0$, as the first logarithm on the right-hand side is equivalent to $r^{\alpha-1}$, we find that u and its derivative both belong to L^p in a neighborhood of 0 if $1 - \alpha < N/p$.

At $r = 1$, the integral of $|\partial_1 u|^p$ converges if

- either $\beta > 1 - 1/p$, or
- $\beta = 1 - 1/p$ and $\alpha p < -1$.

Summarizing, $u \in W^{1,p}(B(0,1))$ if and only if either $\beta > 1 - 1/p$ and $\alpha > -N/p$, or $\beta = 1 - 1/p$ and $-N/p < \alpha < -1/p$.

Example 2.9. Given $k > 0$, consider the open subset

$$\Omega = \{(x,y) \mid 0 < x < 1,\, x^k < y < 2x^k\}$$

of $\mathbb{R}^2$. We will study for which $\alpha \in \mathbb{R}$, $(x,y) \mapsto u(x,y) = y^\alpha$ belongs to H^m, where $m \in \{1, 2, 3, \dots\}$.

For $\alpha > 0$, the function u admits a continuous extension to $\partial\Omega$, so that $u \in L^2(\Omega)$. The first derivative $\partial_y u(x,y) = \alpha y^{\alpha-1}$ cannot be extended by continuity to the point $x = 0$ if $\alpha < 1$. Nevertheless, it does belong to $L^2(\Omega)$ if the integral

$$\int_0^1 \left[\int_{x^k}^{2x^k} y^{2\alpha-2} dy\right] dx$$

exists, or, equivalently, if $(2\alpha - 1)k > -1$. We can deduce from this that for $k > 0$, we have $u \in H^1(\Omega)$ if $\alpha > 1/2 - 1/2k$.

The second derivative belongs to $L^2(\Omega)$ if $(2\alpha - 3)k > -1$, that is, if $\alpha > 3/2 - 1/2k$. Under this condition, $u \in H^2(\Omega)$. This holds, for example, when $k = 1/6$ (cf. Figure 2.1) and $\alpha > -3/2$, in which case u need not be bounded on Ω.

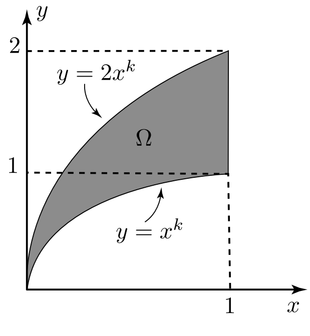

Fig. 2.1. An open subset Ω and elements of H^m.

Let us continue. We find that the condition under which u belongs to $H^m(\Omega)$ can be written as $(2\alpha - 2m + 1)k > -1$. Given m, we can choose α and k such that this necessary condition is satisfied.

Proposition 2.10. *The space $W^{m,p}(\Omega)$ endowed with the norm defined by*

$$\|u\|_{W^{m,p}(\Omega)} = \begin{cases} \left[\sum_{0\leqslant|\alpha|\leqslant m} \left\|D^\alpha u\right\|^p_{L^p(\Omega)}\right]^{1/p} & \text{if } 1 \leqslant p < +\infty; \\ \max_{0\leqslant|\alpha|\leqslant m} \left\|D^\alpha u\right\|_{L^\infty(\Omega)} & \text{if } p = +\infty, \end{cases}$$

is a Banach space. For $p \in\,]1, +\infty[$, this space is uniformly convex and therefore a reflexive space. The space $H^m(\Omega)$ endowed with the inner product

$$\langle u, v\rangle = \sum_{0\leqslant|\alpha|\leqslant m} \left(D^\alpha u,\, D^\alpha v\right)_{L^2(\Omega)}$$

is a Hilbert space.

Exercise 2.1 offers a proof of these statements. Many propositions in this chapter are concerned with the approximation of functions in $W^{1,p}(\Omega)$ or the density of certain subspaces. For such problems, we often use a cover of the open set Ω by a family of open subsets $\{A_j\}$. We admit (cf. Exercise 2.2) that to such a cover, we can associate a family of functions $\{\psi_j\}$ called a *partition of unity subordinate to the cover* $\{A_j\}$ of Ω.

Definition 2.11. A $\mathcal{C}^\infty$ partition of unity subordinate to an open cover $\{A_j\}_{j\in\mathbb{N}}$ of the open set Ω is a set of functions ψ_j with the following properties:

(1) For every j, the function ψ_j is a nonnegative element of $\mathcal{C}^\infty(\Omega)$ with support in A_j.
(2) For any compact subset K of Ω, only a finite number of the functions ψ_j are not zero on K.
(3) For all $x \in \Omega$, $\sum_{j\in\mathbb{N}} \psi_j(x) = 1$.

We use such a partition in the proposition below, where it allows us to approximate functions in $W^{m,p}(\Omega)$ *from the inside*, without any regularity assumption on Ω. The proposition makes it possible, for example, to replace functions that belong to $W^{m,p}(\Omega)$ by $\mathcal{C}^\infty(\Omega)$ functions during computations, in particular during the proof of the Sobolev embedding theorem.

Proposition 2.12. *Let Ω be an arbitrary open subset of $\mathbb{R}^N$. The subspace $\mathcal{C}^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.*

Proof of Proposition 2.12.

We begin with the case $\Omega = \mathbb{R}^N$. Let $u \in W^{m,p}(\mathbb{R}^N)$. Consider a regularizing sequence (cf. Section 1.4.2) $x \mapsto \rho_\varepsilon(x) = 1/\varepsilon^N \rho(x/\varepsilon)$ and a real number $\delta > 0$. In Section 1.4.2, and in particular in the proof of Theorem 1.91, we saw that the function $\rho_\varepsilon \star u \in \mathcal{C}^\infty(\mathbb{R}^N)$ and its derivatives, which satisfy

$D^\alpha(\rho_\varepsilon \star u) = \rho_\varepsilon \star D^\alpha u$, are elements of $L^p(\mathbb{R}^N)$. Moreover, we saw that there exists an ε_0 such that for all $\varepsilon < \varepsilon_0$, we have

$$\|u - \rho_\varepsilon \star u\|_{L^p} \leqslant \delta \quad \text{and} \quad \forall\, \alpha,\ |\alpha| \leqslant m, \quad \|D^\alpha u - \rho_\varepsilon \star D^\alpha u\|_{L^p} \leqslant \delta \tag{2.13}$$

(cf. (1.92)). It follows that $\rho_\varepsilon \star u \in W^{m,p}(\mathbb{R}^N)$ and that there exists a constant C_m such that

$$\|u - \rho_\varepsilon \star u\|_{W^{m,p}} \leqslant C_m \delta, \tag{2.14}$$

which concludes the proof in the case of $\mathbb{R}^N$.

Next, consider an open subset $\Omega \neq \mathbb{R}^N$. We will use an open cover $\{\Omega_j\}_{j \in \mathbb{N}^*}$ of Ω defined by

$$\Omega_j = \bigl\{x \in \Omega \mid |x| \leqslant jC_1 \text{ and } d(x, \partial\Omega) > C_2/j + 1\bigr\}.$$

The constants C_1 and C_2 are chosen such that $\Omega_2 \neq \varnothing$. The resulting sequence of bounded open subsets is increasing and covers Ω. After setting $\Omega_0 = \Omega_{-1} = \varnothing$, we define the sequence of open subsets $\{A_j\}$ by setting $A_j = \Omega_{j+2} \smallsetminus \overline{\Omega_{j-1}}$ for $j > 1$ and $A_0 = \Omega_2$, $A_1 = \Omega_3$.

The family $\{A_j\}$ is again an open cover of Ω, and we can easily verify that if $|j - j'| \geqslant 3$, then $A_j \cap A_{j'} = \varnothing$. Let $\{\psi_j\}$ be a partition of unity associated with the cover $\{A_j\}$. Let ε_j be sufficiently small that for a given ε, we have

$$\begin{aligned} &\forall\, j \geqslant 2, \quad A_j + B(0, \varepsilon_j) \subset A_{j-1} \cup A_j \cup A_{j+1}, \\ &\forall\, j \geqslant 0, \quad \|\rho_{\varepsilon_j} \star (\psi_j u) - (\psi_j u)\|_{W^{m,p}} < \frac{\varepsilon}{2^{j+1}}. \end{aligned}$$

Next, consider the function $v^{(\varepsilon)}$ defined by

$$v^{(\varepsilon)} = \sum_0^{+\infty} \bigl(\rho_{\varepsilon_j} \star (\psi_j u)\bigr). \tag{2.15}$$

This function is well defined, as the sum on the right-hand side is locally finite. We can deduce from the inequalities above that $v^{(\varepsilon)} \in W^{m,p}(\Omega)$.

Setting $u = \sum_0^{+\infty} (\psi_j u)$, we can conclude the proof using the following inequality:

$$\begin{aligned} \|v^{(\varepsilon)} - u\|_{W^{m,p}(\Omega)} &\leqslant \sum_0^{+\infty} \|\rho_{\varepsilon_j} \star (\psi_j u) - (\psi_j u)\|_{W^{m,p}} \\ &\leqslant \sum_0^{+\infty} \frac{\varepsilon}{2^{j+1}} = \varepsilon. \end{aligned} \tag{2.16}$$

□

Corollary 2.17. (1) *Let* $u \in W^{1,p}(\Omega)$ *and let* $v \in W^{1,p'}(\Omega)$*, where* p *and* p' *satisfy* $1/p + 1/p' = 1$*. The product* uv *is then an element of* $W^{1,1}(\Omega)$*, and*

$$\forall\, i \in [1, N], \quad \partial_i(uv) = u\partial_i v + v\partial_i u,$$

where the expressions in the equality are all well defined under the assumptions.

(2) *Let* u *be an element of* $W^{1,N}(\Omega)$*; then* $|u|^{N-1}u$ *and* $|u|^N$ *both belong to* $W^{1,1}(\Omega)$*, while*

$$\nabla\big(|u|^{N-1}u\big) = N|u|^{N-1}\nabla u. \quad \textit{and} \quad \nabla\big(|u|^N\big) = N|u|^{N-2}u\nabla u.$$

Remark 2.18. In (2), $W^{1,N}(\Omega)$ may be replaced by $W^{1,q}(\Omega)$ for $q \in\,]1, \infty[$. The result is then

Let $u \in W^{1,q}(\Omega)$*; then* $|u|^{q-1}u$ *and* $|u|^q$ *both belong to* $W^{1,1}(\Omega)$*, while*

$$\nabla(|u|^{q-1}u) = q|u|^{q-1}\nabla u \quad \textit{and} \quad \nabla(|u|^q) = q|u|^{q-2}u\nabla u.$$

$$\nabla(|u|^{q-1}u) = q|u|^{q-1}\nabla u \quad \textit{and} \quad \nabla(|u|^q) = q|u|^{q-2}u\nabla u.$$

Proof of the Corollary. (1) By the proposition above, there exists a sequence $\{u_n\} \subset \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$ that converges to u in $W^{1,p}(\Omega)$. For this sequence, we have

$$\partial_i(u_n v) = \partial_i(u_n)v + u_n\partial_i v,$$

where each term is seen as a product of a $\mathcal{C}^\infty$ function and a distribution. Let us take the limit of the left-hand side in the sense of distributions. We have $u_n v \in L^1(\Omega)$ and $\|u_n v - uv\|_{L^1} \leqslant \|u_n - u\|_{L^p}\, \|v\|_{L^{p'}} \to 0$. It follows that $\{u_n v\} \to uv$ in L^1, and consequently also in the sense of distributions. By a property of distributions stated in Section (1.4.8), $\partial_i(u_n v) \to \partial_i(uv)$ in the sense of distributions. Likewise, as $u_n \to u$ and $\partial_i u_n \to \partial_i u$ in L^p, the right-hand side converges in $\mathcal{D}'(\Omega)$. Taking the limit therefore gives the desired equality and, moreover, shows that $\partial_i(uv) \in L^1$, whence $uv \in W^{1,1}(\Omega)$.

(2) Consider a sequence $u_n \in \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$ that converges to u in $W^{1,N}(\Omega)$. We can easily show that the gradient of $|u_n|^N$ is given by

$$N|u_n|^{N-2}u_n\,[\nabla u_n].$$

Since $|u_n|^{N-2}u_n$ converges to $|u|^{N-2}u$ in $L^{N/(N-1)}$ and ∇u_n converges to ∇u in L^N, it follows that $N|u_n|^{N-2}u_n\nabla u_n$ converges to $N|u|^{N-2}u\nabla u$ in L^1. Moreover, as $|u_n|^N \to |u|^N$ in L^1, the convergence also holds in $\mathcal{D}'(\Omega)$. Consequently, $\nabla(|u_n|^N)$ converges to $\nabla(|u|^N)$ in $\mathcal{D}'(\Omega)$. Taking the limit therefore provides us with the identity

$$\nabla(|u|^N) = N|u|^{N-2}u\nabla u.$$

Finally, using Hölder's inequality with the conjugate exponents $N/(N-1)$ and N, we have

$$\int_\Omega |\nabla(|u|^N)|dx \leqslant N\Big(\int_\Omega |u|^N|dx\Big)^{N-1/N}\Big(\int_\Omega |\nabla u|^N dx\Big)^{1/N}.$$

We have therefore proved that $|u|^{N-1}u \in W^{1,1}(\Omega)$.
The reasoning for the second statement concerning the gradient of $|u|^{N-1}u$ is similar. □

Corollary 2.19. *Let $u \in W^{1,p}_{\text{loc}}(\Omega)$. This means that for every function $\varphi \in \mathcal{D}(\Omega)$, we have $\varphi u \in W^{1,p}(\Omega)$. Let x_0 be the point $(x_0', t) \in \Omega$, where $x_0' \in \mathbb{R}^{N-1}$ and $t \in \mathbb{R}$. Let $B'(x_0', r)$ denote an open ball in $\mathbb{R}^{N-1}$, and let $B^*(x_0, r)$ denote the open cylinder $B'(x_0', r) \times]-r, r[$ whose closure, for r sufficiently small, is included in Ω. Then, for almost all pairs (x', t) and (x', t') of elements of $B^*(x_0, r)$, we have*

$$u(x', t) - u(x', t') = \int_{t'}^{t} \partial_N u(x', s)ds. \tag{2.20}$$

Proof of Corollary 2.19.
For $(t, t') \in (]-r, r[)^2$ and $x' \in B'(x_0', r)$, let

$$v(x') = \int_{t'}^{t} \partial_N u(x', s)ds.$$

Let us show that $v \in L^p(B'(x_0', r))$. The function $(x', s) \mapsto \partial_N u(x', s)$ is an element of $L^p(\Omega)$, as $\overline{B^*(x_0, r)} \subset \Omega$, and hence is summable in s on the interval $[t', t]$ in $]-r, r[$. It follows that v is defined almost everywhere on $B'(x_0', r)$. Next, by Hölder's inequality and Fubini's theorem, the following holds for almost every pair (t, t'):

$$\begin{aligned}\|v\|^p_{L^p(B')} &= \int_{B'}\Big|\int_t^{t'} \partial_N u(x', s)ds\Big|^p dx' \\ &\leqslant \int_{B'} |t-t'|^{p-1}\int_t^{t'} \big|\partial_N u(x', s)\big|^p ds dx' \\ &\leqslant |t-t'|^{p-1}\int_{B^*} \big|\partial_N u(x)\big|^p dx < +\infty.\end{aligned}$$

Let $\{u_n\}$ be a sequence of elements of $\mathcal{C}^\infty(B^*) \cap W^{1,p}(B^*)$ that converges to u (cf. Proposition 2.12). We define the sequence $\{v_n\}$ on B' by setting

$$v_n(x') = \int_{t'}^{t} \partial_N u_n(x', s)ds.$$

Replacing u by $u_n - u$ in the preceding computation, we see that $v_n \to v$ in $L^p(B')$. We can therefore extract a subsequence $\{v_{n_j}\}$ that converges almost everywhere to v on B'. Likewise, we can extract from $\{u_{n_j}\}$ a subsequence $\{u_{\sigma(n)}\}$ that converges almost everywhere to u on B^*. Since the functions $u_{\sigma(n)}$ are regular, we have

$$u_{\sigma(n)}(x',t) - u_{\sigma(n)}(x',t') = \int_{t'}^{t} \partial_N \, u_{\sigma(n)}(x',s)ds = v_{\sigma(n)}(x').$$

The corollary's formula follows from the almost everywhere convergence on both sides. □

Below we give another consequence of Theorem 2.12, which is very useful, in particular when extending a function in $W^{m,p}(\Omega)$ to a function in $W^{m,p}(\mathbb{R}^N)$ when Ω is a Lipschitz open set. For a function in $W^{m,p}(\Omega)$, such an extension requires a technical lemma about changes of variables.

Corollary 2.21. *Consider two bounded open subsets Ω and Ω' of $\mathbb{R}^N$. Let a be a function giving a bijection from Ω' to Ω, where a and a^{-1} are moreover both Lipschitz. Let $p \geqslant 1$ be given. If $u \in W^{1,p}(\Omega)$, then the composed function $v = u \circ a$ is an element of $W^{1,p}(\Omega')$ and the derivatives of v in the sense of distributions are given by the usual derivation formulas for composed functions. Moreover, there exists a constant $C(|\nabla a|_\infty)$ depending on $|\nabla a|_\infty$, such that*

$$\|u \circ a\|_{W^{1,p}(\Omega')} \leqslant C(|\nabla a|_\infty)\|u\|_{W^{1,p}(\Omega)}.$$

Proof of Corollary 2.21.

Let $\{u_n\}$ be a sequence in $W^{1,p}(\Omega) \cap \mathcal{C}^\infty(\Omega)$ that converges to u in $W^{1,p}(\Omega)$. The function $y \mapsto v_n(y) = u_n(a(y))$ is Lipschitz on Ω', and therefore on all lines parallel to any of the coordinate axes y_i. Since Lipschitz implies absolute continuity, it follows (cf. Remark 2.4) that v_n is almost everywhere derivable on Ω' and

$(*)$ for almost all $y \in \Omega'$, $\partial_i(v_n)(y) = \sum_1^N \partial_j(u_n)(a(y))\partial_i(a_j)(y)$.

We now need the following lemma.

Lemma 2.22. *Given bounded open sets Ω and Ω', let a be a continuous bijection from Ω' to Ω such that a^{-1} is Lipschitz. Then, if $u \in L^p(\Omega)$, we have $u \circ a \in L^p(\Omega')$ and there exists a constant c such that $\|u \circ a\|_{L^p(\Omega')} \leqslant c\|u\|_{L^p(\Omega)}$.*

Let us continue the proof of Corollary 2.21 using this result. Applying it to $\partial_i(u_n - u)$, the inequality of the lemma gives us

$$\|\partial_i(u_n) \circ a - \partial_i(u) \circ a\|_{L^p(\Omega')} \leqslant c\|\partial_i u_n - \partial_i u\|_{L^p(\Omega)}.$$

Since we know that $\partial_i(u_n) \to \partial_i u$ in $L^p(\Omega)$, we deduce that $\{\partial_i(u_n)\circ a\}$ converges to $\partial_i u \circ a$ in $L^p(\Omega')$. Consequently, we can use (*) and the assumptions of the corollary to show that the open sets and derivatives $\partial_i(a_j)$ are bounded, and that the sequence $\{\partial_i(v_n)\}$ converges in $L^p(\Omega')$ to the function $\sum_1^N (\partial_j u \circ a)\, \partial_i(a_j)$, which itself belongs to $L^p(\Omega')$. Taking the limit of a subsequence, the inequality $(*)$ then gives

$$\text{for almost all } y \in \Omega', \quad \partial_i(u \circ a)(y) = \textstyle\sum_1^N \partial_j(u)(a(y))\partial_i(a_j)(y).$$

Since these almost everywhere derivatives are in $L^p(\Omega')$, it follows from Remark 2.4 that they are derivatives in the sense of distributions. By the lemma, we have $u \circ a \in L^p(\Omega')$. Consequently, $u \circ a \in W^{1,p}(\Omega')$. Moreover, $\|u \circ a\|_{L^p(\Omega')} \leqslant c\|u\|_{L^p(\Omega)}$ and $\|\partial_i(u \circ a)\|_{L^p(\Omega')} \leqslant c'\|u\|_{W^{1,p}(\Omega)}\|\nabla(a)\|_{L^\infty(\Omega')}$. From this, we deduce the existence of a constant C that depends only on the Lipschitz constants of a and a^{-1}, such that $\|u \circ a\|_{W^{1,p}(\Omega')} \leqslant C\|u\|_{W^{1,p}(\Omega)}$. $\square$

Proof of Lemma 2.22. Let L denote the Lipschitz constant of a^{-1}. Let us take a sequence $\{u_n\}$ as in the proof of the corollary above. If we cover $\overline{\Omega'}$ by a finite number n_η of N*-hypercubes* C_k with edge of length 2η and extend $u_n \circ a$ by 0 outside of Ω', then the definition of the Riemann-integrability of $|u_n \circ a|^p$ gives

$$\int_{\Omega'} |u_n(a(y))|^p dy = \lim_{\eta \to 0} \sum_1^{n_\eta} (2\eta)^N \inf_{y \in C_k} |u_n(a(y))|^p.$$

We may, and do, assume that the hypercubes all satisfy $\overline{C_k} \subset \Omega'$. Let y_k be the center of C_k, so that $x_k = a(y_k) \in \Omega$. If $x \in \partial(a(C_k))$, then the properties of a imply that $y = a^{-1}(x) \in \partial C_k$. Hence, as $|y_k - y| \geqslant \eta$, we have the following inequalities for the distances in $\mathbb{R}^N$: $\eta \leqslant |y - y_k| = |a^{-1}(x) - a^{-1}(x_k)| \leqslant L|x - x_k|$. It follows that $a(C_k)$ contains the ball of radius η/L with center x_k, whence $\operatorname{mes}(a(C_k)) \geqslant \omega_N \eta^N / L^N \geqslant K \operatorname{mes}(C_k)$, where K depends only on N and L. We can now deduce the following upper bound:

$$\begin{aligned}\sum_1^{n_\eta} \operatorname{mes}(C_k) \inf_{y \in C_k} |u_n(a(y))|^p &\leqslant \frac{1}{K} \sum_1^{n_\eta} \operatorname{mes}(a(C_k)) \inf_{x \in a(C_k)} |u_n(x)|^p \\ &\leqslant \frac{1}{K} \int_\Omega |u_n(x)|^p dx.\end{aligned}$$

Taking the limit for $\eta \to 0$ gives

$$(**) \qquad \int_{\Omega'} |u_n(a(y))|^p dy \leqslant \frac{1}{K} \int_\Omega |u_n(x)|^p dx.$$

We can find a subsequence $u_{\sigma(n)}$ that converges almost everywhere to u. The result of the lemma then follows from $(**)$ using Fatou's lemma. $\square$

Let us now give a definition of $W^{1,p}$, using approximations of the derivatives by translation operators.

Proposition 2.23. *For $1 < p < \infty$, the following properties are equivalent:*

(1) $u \in W^{1,p}(\Omega)$.
(2) $u \in L^p(\Omega)$ *and there exists a constant* $C > 0$ *such that for any open set* ω *with closure contained in* Ω*, we have*

$$\forall\, h \in \mathbb{R}^N, \quad |h| \leqslant d(\omega, \partial\Omega) \Longrightarrow \|\tau_h u - u\|_{L^p(\omega)} \leqslant C|h|.$$

In the case $p = 1$, property (2) must be replaced by

(2′) *For every open set ω with closure contained in Ω, there exists a constant $c(\omega)$ such that $c(\omega) \leqslant C$, $c(\omega) \to 0$ when $|\omega| \to 0$, and $\|\tau_h u - u\|_{L^1(\omega)} \leqslant c(\omega)|h|$.*

Proof of Proposition 2.23.
Let us assume that $1 < p < +\infty$. We will first show that (1) $\Rightarrow$ (2) when the translation is parallel to a base vector.

Consider $u \in W^{1,p}(\Omega)$ and $\overline{\omega} \subset \Omega$. Let e_i be the ith vector of the canonical basis of $\mathbb{R}^N$, and let $h_0 = d(\omega, \partial\Omega)$. Then $\overline{\omega} \subset \Omega$ implies that $h_0 > 0$ and if $|h| < h_0$, we have $x \in \omega \Rightarrow x + he_i \in \Omega$. Corollary 2.19 subsequently tells us that for every h such that $|h| < h_0$ and that for almost all x in ω, we have

$$u(x + he_i) - u(x) = \int_0^h \partial_i u(x + se_i) ds. \tag{2.24}$$

Consequently, by Hölder's inequality,

$$|u(x + he_i) - u(x)|^p \leqslant |h|^{p-1} \int_0^h |\partial_i u(x + se_i)|^p ds. \tag{2.25}$$

Since $|u|^p \in L^1(\Omega)$, we can integrate this inequality over ω, whence, using Fubini and noting that $\omega + B(0, h) \subset \Omega$,

$$\begin{aligned} \int_\omega |\tau_{he_i} u - u|^p(x) dx &\leqslant |h|^{p-1} \int_0^h \int_\omega |\partial_i u(x + se_i)|^p dx\, ds \\ &\leqslant |h|^p \|\partial_i u\|^p_{L^p(\Omega)}. \end{aligned} \tag{2.26}$$

Taking the $1/p$th power of this inequality gives property (2) for the translation τ_{he_i}.

For $h \in \mathbb{R}^N$ such that $\omega + B(0, h) \subset \Omega$, it suffices to replace ∂_i by the derivative along h, namely $\partial_h u = \nabla u \cdot (h/|h|)$. This leads to property (2) with, for example, constant $C = (\sum_1^N \|\partial_i u\|^2_{L^p(\Omega)})^{1/2}$.

Let us now show the implication (2) $\Rightarrow$ (1).

Let u satisfy (2). We must prove that $\partial_i u \in L^p(\omega)$. Setting, for example, $h = 1/n$, consider the sequence $\{(\tau_{he_i} u - u)/h\}$ of distributions on ω. We know (Subsection 1.4.8) that this sequence converges in $\mathcal{D}'(\omega)$ to the distribution $\partial_i u$. Consequently,

$$(*) \qquad \forall\, \varphi \in \mathcal{D}(\omega), \quad \left\langle \frac{\tau_{he_i} u - u}{h}, \varphi \right\rangle \longrightarrow \langle \partial_i u, \varphi \rangle.$$

Now, by Hölder's inequality and property (2), we have

$$\left| \left\langle \frac{\tau_{he_i} u - u}{h}, \varphi \right\rangle \right| \leqslant C \, \|\varphi\|_{L^{p'}}.$$

Using $(*)$, taking the limit of this inequality for $h \to 0$ gives us the inequality $|\langle \partial_i u, \varphi \rangle| \leqslant C \|\varphi\|_{L^{p'}}$. Now, as $p' < \infty$, $\mathcal{D}(\omega)$ is dense in $L^{p'}(\omega)$ (cf. Theorem 1.91). The distribution $\partial_i u$ therefore defines a linear functional on $L^{p'}(\omega)$ and the previous inequality becomes

$$\forall\, g \in L^{p'}(\omega), \quad |\langle \partial_i u, g \rangle| \leqslant C \|g\|_{L^{p'}(\omega)},$$

proving that $\partial_i u$ can therefore be identified with a function in $L^p(\omega)$ whose norm moreover satisfies $\|\partial_i u\|_{L^p(\omega)} \leqslant C$. Since this is true for every relatively compact open subset ω of Ω, we can use an increasing sequence of such open subsets on which the L^p norms of $\partial_i u$ are uniformly bounded to show that $\partial_i u \in L^p(\Omega)$. Since this result holds for every i, it follows that $u \in W^{1,p}(\Omega)$, which concludes the proof.

Let us now consider the case $p = 1$. For the implication (1) $\Rightarrow$ (2′), the reasoning remains the same as above and we see in inequality (2.25) that we can use a constant $c(\omega)$ such that $c(\omega) \leqslant \int_{\omega + B(0,h)} |\nabla u(x)| dx$, which therefore tends to $\int_\omega |\nabla u(x)| dx$ when h tends to 0. In particular, as $\nabla u \in L^1$, this inequality tends to 0 when $\operatorname{mes}(\omega) \to 0$ (in the sense of Lebesgue).

Conversely, by an argument similar to that in the case $p > 1$, the inequality in (2′) implies that ∇u is in the dual of $\mathcal{C}_c(\Omega)$, which means that ∇u is a measure (cf. Chapter 6). Since this estimate does not depend on the support of φ, we deduce from it that ∇u is a bounded measure.

Moreover, the inequality $\int_\omega |\nabla u| \leqslant c(\omega)$ shows that the measure ∇u is absolutely continuous with respect to the Lebesgue measure (cf. Chapter 6), which proves that $\nabla u \in L^1(\omega)$. Since ω is arbitrary and $c(\omega)$ is bounded independently of ω, we conclude that $\nabla u \in L^1(\Omega)$. □

Remark 2.27. In the case $p = 1$, the above proof shows that property (2) for $p > 1$ only implies that $u \in BV(\Omega)$, the space of functions with bounded variation (cf. Chapter 6).

Definition 2.28. Let Ω be an open subset of $\mathbb{R}^N$, either bounded or not. We let $W_0^{m,p}(\Omega)$ denote the closure of the space $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$ for the norm $\|\cdot\|_{m,p}$.

In general, finding an intrinsic characterization of the functions in $W_0^{m,p}(\Omega)$ is not obvious and depends strongly on the structure of Ω. When $\Omega = \mathbb{R}^N$, a method involving truncation and regularization allows us to show the following result.

Proposition 2.29. *The space $\mathcal{D}(\mathbb{R}^N)$ is dense in $W^{m,p}(\mathbb{R}^N)$, so that*

$$W^{m,p}(\mathbb{R}^N) = W_0^{m,p}(\mathbb{R}^N).$$

Proof of Proposition 2.29.
Let $u \in W^{m,p}(\mathbb{R}^N)$ and let $n \in \mathbb{N}^*$. Let φ be a function in $\mathcal{D}(B(0,2))$ with value 1 on $B(0,1)$ and such that $0 \leqslant \varphi \leqslant 1$. Let $\varphi_n(x) = \varphi(x/n)$; then the sequence u_n defined by $u_n(x) = \varphi(x/n)u(x)$ converges to u in $W^{m,p}(\mathbb{R}^N)$. Indeed, as $|u|^p \in L^1$, we have

$$\|u - u_n\|_p^p = \|(1-\varphi_n)u\|_p^p \leqslant \int_{|x|\geqslant n} |u(x)|^p dx \longrightarrow 0.$$

On the other hand, the Leibniz formula for the derivative of the product of a $\mathcal{C}^\infty$ function and a distribution implies that if $|\alpha| = m$, then $D^\alpha(\varphi_n u)$ is the sum of $\varphi_n D^\alpha u$ and expressions of the form $(1/n)^j\, D^{\alpha_1}\varphi(x/n) D^{\alpha_2} u$, where $|\alpha_1| + |\alpha_2| = m$ and $|\alpha_1| = j \geqslant 1$. We can bound the L^p norm of these expressions from above by

$$\frac{1}{n^j} |D^{\alpha_1}\varphi|_\infty \Big(\int_{|x|\geqslant n} |D^{\alpha_2} u(x)|^p dx \Big)^{1/p},$$

which tends to 0 because $j \geqslant 1$. It follows that

$$|D^\alpha(\varphi_n u) - D^\alpha u|_p \leqslant |D^\alpha(\varphi_n u) - \varphi_n D^\alpha u|_p + |\varphi_n D^\alpha u - D^\alpha u|_p,$$

where the right-hand side is the sum of two quantities that both tend to 0.

We will now use regularization. Given a regularizing function ρ, we let $\rho_n(x) = n^N \rho(nx)$ and $u_n = \rho_n \star (\varphi_n u)$. The functions u_n then belong to $\mathcal{D}(\mathbb{R}^N)$, and the sequence $\{u_n\}$ converges to u in $W^{1,p}$. □

In general, we will see that under regularity conditions on Ω, a sufficient condition for the inclusion $u \in W_0^{m,p}(\Omega)$ is that the extension $\widetilde{u}$ of u by 0 outside of Ω belongs to $W^{m,p}(\mathbb{R}^N)$.

Remark 2.30. Later on, we will give a result concerning the density of $\mathcal{C}^1(\overline{\Omega})$ in $W^{m,p}(\Omega)$ when Ω is Lipschitz.

2.2 Sobolev Embeddings for $W^{m,p}(\mathbb{R}^N)$

2.2.1 Definitions of Functional Spaces

Given an integer $j \geqslant 0$, we define the family of spaces $\mathcal{C}_b^j(\mathbb{R}^N)$ by setting

$$\mathcal{C}_b^j(\mathbb{R}^N) = \left\{u \in C^j(\mathbb{R}^N) \mid \forall\, \alpha \in \mathbb{N}^N,\ |\alpha| \leqslant j,\ \exists\, K_\alpha,\ \left\|D^{(\alpha)}u\right\|_\infty \leqslant K_\alpha\right\}.$$

For a positive real number λ, the subspace $\mathcal{C}_b^{j,\,\lambda}(\mathbb{R}^N)$ consist of the functions in $\mathcal{C}_b^j(\mathbb{R}^N)$ such that if $|\alpha| \leqslant j$, then

$$\exists\, C_{\alpha,\,\lambda},\ \forall\, x, y \in \mathbb{R}^N, \quad \left|D^{(\alpha)}u(x) - D^{(\alpha)}u(y)\right| \leqslant C_{\alpha,\,\lambda}\,|x-y|^\lambda.$$

2.2.2 Statement of the Theorem and Preliminary Remarks

Theorem 2.31 (Sobolev embedding theorem). *For $p \geqslant 1$ and $m \in \mathbb{N}$, we have:*

(1) *If $N > mp$, then for every q satisfying $p \leqslant q \leqslant Np/(N-mp)$, we have $W^{m,\,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$. More precisely, under the given conditions, there exists a constant C such that*

$$\forall\, \varphi \in W^{m,\,p}(\mathbb{R}^N), \quad \|\varphi\|_q \leqslant C\|\varphi\|_{W^{m,p}(\mathbb{R}^N)}.$$

(2) *For $p = 1$, we have $W^{N,\,1}(\mathbb{R}^N) \hookrightarrow C_b(\mathbb{R}^N)$.*

(3) *If $N = mp$ and $p > 1$, then for every q satisfying $p \leqslant q < \infty$, we have $W^{m,\,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$.*

(4) *If $p > N$, then we have*

$$0 < \lambda \leqslant 1 - N/p \Longrightarrow W^{1,\,p}(\mathbb{R}^N) \longhookrightarrow C_b^{0,\,\lambda}(\mathbb{R}^N).$$

(5) *If $mp > N$, $N/p \notin \mathbb{N}$, and j satisfies $(j-1)p < N < jp$, then*

$$0 < \lambda \leqslant j - N/p \Longrightarrow W^{m,\,p}(\mathbb{R}^N) \longhookrightarrow C_b^{m-j,\,\lambda}(\mathbb{R}^N).$$

If $N/p \in \mathbb{N}$ and $m \geqslant j = N/p + 1$, then $W^{m,p}(\mathbb{R}^N) \hookrightarrow \mathcal{C}_b^{m-N/p-1,\lambda}(\mathbb{R}^N)$ for every $\lambda < 1$.

The following preliminary remarks allow us to better understand the proof of Theorem 2.31.

Remark 2.32 (reduction to functions in $\mathcal{D}(\mathbb{R}^N)$). By Proposition 2.29, it suffices to prove the statements of the theorem for functions in $\mathcal{D}(\mathbb{R}^N)$.

Let us, for example, assume that under the conditions of statement (1), we have proved the existence of a C depending on N, p, q, such that

$$(*) \qquad \forall\, \varphi \in \mathcal{D}(\mathbb{R}^N), \quad \|\varphi\|_q \leqslant C\|\varphi\|_{W^{m,p}(\mathbb{R}^N)}.$$

Consider $u \in W^{m,p}(\mathbb{R}^N)$ and a sequence $\{\varphi_n\}$ in $\mathcal{D}(\mathbb{R}^N)$ that converges to u in $W^{m,p}(\mathbb{R}^N)$. Since the inequality $(*)$ shows that this is a Cauchy sequence in $L^q(\mathbb{R}^N)$, we deduce that it converges to $v \in L^q(\mathbb{R}^N)$ in this space. As, moreover, it also converges to u in $L^p(\mathbb{R}^N)$, we conclude that $u = v$ and $u \in L^q(\mathbb{R}^N)$. Furthermore, by taking the limit in $(*)$, we obtain the existence of a constant C depending on N, p, q, such that

$$\forall\, u \in W^{m,p}(\mathbb{R}^N), \quad \|u\|_q \leqslant C\,\|u\|_{W^{m,p}(\mathbb{R}^N)},$$

which shows that the injection is continuous.

The reasoning for the other types of injections is similar.

Remark 2.33 (reduction to the case of critical injections). To prove statements (1), (4) and (5) of Theorem 2.31, it suffices to prove them in the critical cases, namely, for $q = Np/(N-mp)$ for statement (1), for $\lambda = 1 - N/p$ for statement (4), and for $\lambda = j - N/p$ for statement (5).

Indeed, let us suppose that statement (1) has been proved for $q = p^* = Np/(N-mp)$. Let $q \in\,]p, p^*[$ and $\theta \in\,]0,1[$ satisfy $q = \theta p + (1-\theta)p^*$. Hölder's inequality with conjugate exponents $1/\theta$ and $1/(1-\theta)$ gives

$$\begin{aligned}\int_{\mathbb{R}^N} |u(x)|^q dx &= \int_{\mathbb{R}^N} |u(x)|^{\theta\, p} |u(x)|^{(1-\theta)\, p^*} dx \\ &\leqslant \left[\int_{\mathbb{R}^N} |u(x)|^{p\,\theta/\theta} dx\right]^{\theta} \left[\int_{\mathbb{R}^N} |u(x)|^{p^*\,(1-\theta)/(1-\theta)} dx\right]^{1-\theta} \\ &\leqslant \|u\|_{L^p}^{p\,\theta}\, \|u\|_{L^{p^*}}^{p^*(1-\theta)}.\end{aligned}$$

We know that $u \in L^p$, $u \in L^{p^*}$, and that there exists a C such that $\|u\|_{L^{p^*}} \leqslant C\,\|u\|_{W^{m,p}}$. Consequently, the previous inequality shows that $u \in L^q$ and $\|u\|_{L^q}^q \leqslant C\,\|u\|_{W^{m,p}}^{p\theta+(1-\theta)p^*} = C\|u\|_{W^{m,p}}^q$, which implies the continuity of the injection into L^q.

A similar reasoning makes it possible to reduce the proof of statements (4) and (5) to the critical cases mentioned above.

Remark 2.34 (on the impossibility of improving (1)). A simple scaling argument shows that when $N > p$, there cannot exist an embedding from $W^{1,p}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ for $q < p$ or $q > p^*$, where $p^* = Np/(N-mp)$.

Indeed, let us assume, in either case, the existence of a C such that for every $u \in W^{1,p}(\mathbb{R}^N)$, $\|u\|_{L^q} \leqslant C\|u\|_{W^{1,p}}$. Applying this inequality to the family defined by $u_\lambda(x) = u\big(x/\lambda\big)$ gives

$$\left(\int_{\mathbb{R}^N} \left|u\Big(\frac{x}{\lambda}\Big)\right|^q dx\right)^{1/q} \leqslant C\left[\int_{\mathbb{R}^N} \left|u\Big(\frac{x}{\lambda}\Big)\right|^p dx + \sum_1^N \int_{\mathbb{R}^N} \frac{1}{\lambda^p}\left|\partial_i u\Big(\frac{x}{\lambda}\Big)\right|^p dx\right]^{1/p}.$$

Substituting the variable $y = x/\lambda$ and using Minkowski's inequality, this becomes

$$\|u\|_q \lambda^{N/q} \leqslant C\Big[\|u\|_p \lambda^{N/p} + \|\nabla u\|_p \lambda^{-1+N/p}\Big],$$

or an inequality of the form

$$C_1 \leqslant C_2 \lambda^{N(1/p-1/q)} + C_3 \lambda^{-1+N(1/p-1/q)},$$

where C_1, C_2, C_3 are three fixed nonnegative numbers.

The hypothesis implies, when $q < p$, that the exponents on the right-hand side are negative, giving a contradiction when $\lambda \to +\infty$. Likewise, we see that the hypothesis $q > p^*$ implies that the exponents are positive, giving a contradiction when $\lambda \to 0$.

Remark 2.35 (reasoning in Sobolev's proof). The idea Sobolev originally used to show the embedding consists in writing u formally as $u = u \star \delta = u \star \Delta E$, where E, a fundamental solution of the Laplacian, is defined as follows (cf. Exercise 2.19).

For $N > 2$, it is the function $E = k_N r^{2-N}$ with $k_N = 1/((2-N)\omega_{N-1})$, where ω_{N-1} denotes the $(N-1)$-dimensional surface area of the unit sphere in $\mathbb{R}^N$.

For $N = 2$, it is the function $E = k_2 \ln(r)$ with $k_2 = 1/(2\pi)$. More precisely, if ζ is a function in $\mathcal{D}(\mathbb{R}^N)$ equal to 1 in a neighborhood of 0, we can write u as

$$(*) \qquad u = u \star \Delta(\zeta E) - u \star \nabla\zeta \cdot \nabla E - u \star (\Delta\zeta)E.$$

Note that when $p \geqslant 1$, the last two terms of $(*)$, namely $u \star \nabla\zeta \cdot \nabla E$ and $u \star (\Delta\zeta)E$, can each be expressed as the convolution of $u \in L^p$ with a function in $\mathcal{D}(\mathbb{R}^N)$. It follows that this convolution is in L^k for every $k \geqslant p$. We are therefore reduced to considering the first term of $(*)$, which can be written as $u \star \Delta(\zeta E) = \nabla u \star \nabla(\zeta E)$.

Let, for example, $p = 1$. Noting that $\nabla(\zeta E) \in L^q$ with $q < N/(N-1)$, and then using the properties of a convolution with an L^1 function, we obtain, thanks to $(*)$, that $u \in L^q$ whenever $q < N/(N-1)$.

The same computation shows that if $1 < p < N$, we still have $u \in L^q$ for every $q < pN/(N-p)$.

To proceed up to the critical exponent in the case $1 < p < N$ with $N \geqslant 2$, we use the *Sobolev lemma* (cf. [60]), where one of the factors of the convolution is the radial function $x \mapsto r^{-s}$. The lemma can be applied to the present situation when $p > 1$ by choosing the exponent $s = N - 1$, in accordance with the definition of $\nabla(\zeta E)$, regardless whether $N = 2$ or not. The statement of the lemma is as follows.

Lemma 2.36 (Sobolev). *Let f be an element of $L^p(\mathbb{R}^N)$ with compact support, where $p \geqslant 1$. Consider the convolution $g = r^{-s} \star f$. The following holds:*

(1) *If $p > 1$, then the function g belongs to L^q on every compact subset of $\mathbb{R}^N$, provided that q satisfies*

$$\frac{1}{q} \geqslant \sup\left\{\frac{1}{q_1}, 0\right\}, \quad \textit{where} \quad \frac{1}{q_1} = \frac{1}{p} + \frac{s}{N} - 1.$$

(2) *If $p = 1$, then the function g belongs to L^q on every compact subset, provided that $1/q > 1/q_1 = s/N$.*

(3) *If $1/p + s/N = 1$, the function g belongs to L^q on every compact subset for every $q < \infty$.*

In all cases, we have upper bounds of the following type on every compact subset:

$$\|g\|_q \leqslant C\|f\|_p,$$

where the constant C depends on q, on the compact on which we bound g, and on the compact support of f.

The proof of this lemma is difficult for the cases not covered by the Riesz–Thorin theorem and will not be given in this book.

Remark 2.37. The critical exponent $N/(N-1)$ for $p = 1$ is not covered by Sobolev's lemma. In what follows, we use more elementary arguments than those in Sobolev's proof.

2.2.3 The Structure of the Proof of Sobolev's Theorem

Step A. We establish the following inequality for the functions φ in $\mathcal{D}(\mathbb{R}^N)$:

$$\|\varphi\|_{L^{N/(N-1)}(\mathbb{R}^N)} \leqslant C\|\varphi\|_{W^{1,1}(\mathbb{R}^N)}.$$

Statement (1) of the theorem for the case $p = m = 1$ follows, using Remark 2.32.

Step B. We establish the following inequality for the functions φ in $\mathcal{D}(\mathbb{R}^N)$ in the case $p < N$:

$$\|\varphi\|_{L^{Np/(N-p)}(\mathbb{R}^N)} \leqslant C\|\varphi\|_{W^{1,p}(\mathbb{R}^N)}.$$

Step C. We use induction to establish the following inequality for the functions φ in $\mathcal{D}(\mathbb{R}^N)$ in the case $m \geqslant 2$ and $mp < N$:

$$\|\varphi\|_{L^{Np/(N-mp)}(\mathbb{R}^N)} \leqslant C\|\varphi\|_{W^{m,p}(\mathbb{R}^N)}.$$

Combining these three steps and Remarks 2.32 and 2.33 gives us statement (1).

Step D. We establish the following inequality for the functions φ in $\mathcal{D}(\mathbb{R}^N)$:

$$\|\varphi\|_\infty \leqslant C\|\varphi\|_{W^{N,1}(\mathbb{R}^N)}.$$

Using the density of the regular functions, we deduce from this statement (2) of the theorem.

Step E. We prove statement (3) of the theorem, beginning with the case $m = 1$ and $p = N$, followed by the case $m \geqslant 2$ and $Np = m$.

Step F. We show that last two statements, (4) and (5), of the theorem.

2.2.4 Proof of Sobolev's Theorem

Proof of Step A. We must prove that

$$\exists\, C, \quad \forall\, \varphi \in \mathcal{D}(\mathbb{R}^N), \quad \|\varphi\|_{L^{N/(N-1)}} \leqslant C\|\varphi\|_{W^{1,1}}. \tag{2.38}$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$; then for every index $i \in [1, N]$, we have

$$\forall\, x \in \mathbb{R}^N, \quad \varphi(x) = \int_{-\infty}^{x_i} \partial_i\varphi(x + (s - x_i)e_i)ds.$$

Consequently,

$$|\varphi(x)| \leqslant \int_{\mathbb{R}} \big|\partial_i\varphi(x + (s - x_i)e_i)\big|ds. \tag{2.39}$$

Note that the integral on the right-hand side of (2.39) does not depend on the component x_i of x. We denote the $(N-1)$-tuple $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$ by $\breve{x}_i^{(N)}$. On $\mathbb{R}^{N-1}$, we define the function φ_i with compact support by setting

$$\varphi_i(\breve{x}_i^{(N)}) = \int_{\mathbb{R}} \big|\partial_i\varphi(x + (s - x_i)e_i)\big|ds.$$

The inequalities (2.39) can now be written as

$$\forall\, i \in [1, N],\ \forall\, x \in \mathbb{R}^N, \quad |\varphi(x)| \leqslant \varphi_i(\breve{x}_i^{(N)}).$$

Since our goal is to study $\|\varphi\|_{L^{N/(N-1)}}$, we note that

$$\forall\, x \in \mathbb{R}^N, \quad |\varphi(x)|^{N/(N-1)} \leqslant \prod_1^N \big[\varphi_i(\breve{x}_i^{(N)})\big]^{1/(N-1)}.$$

Next, we use the following lemma.

Lemma 2.40. *Let $N \geqslant 2$. Consider N functions F_i, each belonging to $L^{N-1}(\mathbb{R}^{N-1})$. We have*

$$\prod_{1\leqslant i\leqslant N} F_i(\breve{x}_i^{(N)}) \in L^1(\mathbb{R}^N)$$

and the inequality

$$\int_{\mathbb{R}^N} \prod_i \bigl|F_i(\breve{x}_i^{(N)})\bigr| dx \leqslant \prod_i \Bigl(\int_{\mathbb{R}^{N-1}} \bigl|F_i(\breve{x}_i^{(N)})\bigr|^{N-1} d\breve{x}_i^{(N)} \Bigr)^{1/(N-1)}. \tag{2.41}$$

Proof of Lemma 2.40. The proof uses induction on N. For $N = 2$, it is the following known property:

$$\int_{\mathbb{R}^2} F_1(x_2)F_2(x_1)dx_1dx_2 = \int_{\mathbb{R}} F_1(x_2)dx_2 \int_{\mathbb{R}} F_2(x_1)dx_1. \tag{2.42}$$

Let us assume that the property has been proved up to order N. For $1 \leqslant j \leqslant N+1$, consider elements F_j of $L^N(\mathbb{R}^N)$, each a function of the variable $\breve{x}_j^{(N+1)}$.

Fixing x_{N+1}, consider the following integration over $x = (x_1, x_2, \dots, x_N)$:

$$I_N = \int_{\mathbb{R}^N} \Bigl[\prod_{1\leqslant i\leqslant N} \bigl|F_i(\breve{x}_i^{(N)}, x_{N+1})\bigr| \Bigr] \bigl|F_{N+1}(x)\bigr| dx \leqslant +\infty.$$

In this integral, where x_{N+1} is fixed, we apply Hölder's inequality with exponents N and $N/(N-1)$. This consists in the inequality

$$(*) \quad I_N \leqslant \Bigl(\int_{\mathbb{R}^N} \Bigl(\prod_{1\leqslant i\leqslant N} \bigl|F_i(\breve{x}_i^{(N)}, x_{N+1})\bigr| \Bigr)^{N/(N-1)} dx \Bigr)^{(N-1)/N} \cdot \Bigl(\int_{\mathbb{R}^N} \bigl|F_{N+1}\bigr|^N (x) dx \Bigr)^{1/N}.$$

Next, consider the N functions h_i, which for x_{N+1} fixed and $i \leqslant N$, are defined by

$$h_i(\breve{x}_i^{(N)}, x_{N+1}) = \bigl|F_i(\breve{x}_i^{(N)}, x_{N+1})\bigr|^{N/(N-1)}. \tag{2.43}$$

By the induction hypothesis at order N, as the function $(h_i)^{N-1}$ is summable on $\mathbb{R}^{N-1}$, the product of these functions is in $L^1(\mathbb{R}^N)$. The inequality $(*)$ above then gives $I_N < +\infty$. Let

$$[g_i(x_{N+1})]^N = \int_{\mathbb{R}^{N-1}} \bigl|F_i(\breve{x}_i^{(N)}, x_{N+1})\bigr|^N d\breve{x}_i^{(N)}.$$

By the induction hypothesis, the functions h_i satisfy (2.41), namely

$$(**)\quad \Big(\int_{\mathbb{R}^N} \prod_{1\leqslant i\leqslant N} h_i(\breve{x}_i^{(N)}, x_{N+1})dx\Big)^{(N-1)/N} \leqslant \prod_{1\leqslant i\leqslant N}\Big(\int_{\mathbb{R}^{N-1}} \big|F_i(\breve{x}_i^{(N)}, x_{N+1})\big|^N d\breve{x}_i^{(N)}\Big)^{1/N}.$$

The right-hand side of this inequality is $\prod_{1\leqslant i\leqslant N}[g_i(x_{N+1})]$.

The integral

$$I_{N+1} = \int_{\mathbb{R}^{N+1}} \prod_{1\leqslant j\leqslant N+1} \big|F_j(\breve{x}_j^{(N+1)})\big| dx\, dx_{N+1}$$

is the integral of

$$I_N = \int_{\mathbb{R}^N} \prod_{1\leqslant i\leqslant N} \big|F_i(\breve{x}_i^{(N)}, x_{N+1})\big|\,\big|F_{N+1}(x)\big|\, dx$$

over $\mathbb{R}$. We apply Hölder's inequality to I_N and note that

$$K_N = \Big[\int_{\mathbb{R}^N} \big|F_{N+1}(x)\big|^N dx\Big]^{1/N}$$

is independent of x_{N+1}. By the definitions of h_i and g_i and the inequalities $(*)$ and $(**)$, this leads to

$$\begin{aligned} I_N &\leqslant K_N\Big(\int_{\mathbb{R}^N} \prod_{1\leqslant i\leqslant N} h_i(\breve{x}_i^{(N)}, x_{N+1})dx\Big)^{(N-1)/N} \\ &\leqslant K_N \prod_{1\leqslant i\leqslant N} g_i(x_{N+1}). \end{aligned}$$

Finally, integrating over $\mathbb{R}$, applying the generalized Hölder inequality (cf. Subsection 1.5.1) with N exponents that are all equal to $1/N$, and using Fubini's formula for the integrals of g_i, we obtain

$$\begin{aligned} I_{N+1} &\leqslant K_N \prod_{1\leqslant i\leqslant N}\Big[\int_{\mathbb{R}} (g_i(x_{N+1}))^N dx_{N+1}\Big]^{1/N} \\ &= \Big[\prod_{1\leqslant j\leqslant N+1} \int_{\mathbb{R}^N} \big|F_j(\breve{x}_j^{(N+1)})\big|^N\, d\breve{x}_j^{(N+1)}\Big]^{1/N}. \end{aligned}$$

We thus obtain relation 2.41 for the rank $N+1$ case, concluding the proof of Lemma 2.40. $\square$

Let us conclude Step A. We apply Lemma 2.40 to the functions $F_i = |\varphi_i|^{1/(N-1)}$. The inequality $|\varphi(x)| \leqslant \prod_{1\leqslant i\leqslant N} |\varphi_i(\breve{x}_i)|^{1/(N-1)}$ then gives the following results for the norm $\Phi = \|\varphi\|_{L^{N/(N-1)}}$:

$$\begin{aligned}
\Phi &\leqslant \Big[\int_{\mathbb{R}^N} \prod_{1\leqslant i\leqslant N} F_i(\breve{x}_i)dx\Big]^{(N-1)/N} \\
&\leqslant \prod_{1\leqslant i\leqslant N} \Big[\int_{\mathbb{R}^{N-1}} |\varphi_i(\breve{x}_i)| d\breve{x}_i\Big]^{1/N} \\
&= \Big[\prod_{1\leqslant i\leqslant N} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} |\partial_i \varphi(x+se_i)| ds\, d\breve{x}_i\Big]^{1/N} \\
&= \Big[\prod_{1\leqslant i\leqslant N} \|\partial_i\varphi\|_{L^1(\mathbb{R}^N)}\Big]^{1/N} \\
&\leqslant \frac{1}{N} \sum_{1\leqslant i\leqslant N} \|\partial_i\varphi\|_{L^1(\mathbb{R}^N)} \leqslant \frac{1}{N}\|\varphi\|_{W^{1,1}(\mathbb{R}^N)}.
\end{aligned}$$

We therefore have an embedding $W^{1,1}(\mathbb{R}^N) \hookrightarrow L^{N/(N-1)}(\mathbb{R}^N)$. Moreover, by Remark 2.32, statement (1) of the theorem has now been proved in the case $p = m = 1$.

Remark 2.44. The last inequality, which states the continuity of the injection, can be written more precisely as follows:

$$\|\varphi\|_{N/(N-1)} \leqslant C\|\nabla\varphi\|_1. \tag{2.45}$$

Proof of Step B.

Let us now assume that $m = 1$ and $p < N$. Consider, for $u \in \mathcal{D}(\mathbb{R}^N)$, the function $v = |u|^{p(N-1)/(N-p)-1}u$, where the exponent is positive since $p \geqslant 1$. By the definition $|u|^\alpha = \exp(\alpha \ln(|u|))$, the partial derivative $\partial_i v$ can be written as

$$\partial_i v = \frac{p(N-1)}{N-p}|u|^{p(N-1)/(N-p)-1}\partial_i u.$$

Moreover, the previous remark and Hölder's inequality give

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^N} |v(x)|^{N/(N-1)}dx\Big)^{(N-1)/N} \\
&\qquad \leqslant C\int_{\mathbb{R}^N} \frac{p(N-1)}{N-p}|u(x)|^{p(N-1)/(N-p)-1}|\nabla u(x)| \\
&\qquad \leqslant C\Big(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx\Big)^{1/p}\Big(\int_{\mathbb{R}^N} |u(x)|^{Np/(N-p)}dx\Big)^{1-1/p}.
\end{aligned}$$

The left-hand side is none other than $\|u\|_{Np/(N-p)}^{(N-1)p/(N-p)}$. Hence, dividing by $\|u\|_{Np/(N-p)}^{N(p-1)/(N-p)}$, we obtain the inequality

$$\|u\|_{Np/(N-p)} \leqslant C\|\nabla u\|_p. \tag{2.46}$$

We have thus proved statement (1) of the theorem for $m = 1$ and $1 < p < N$.

Proof of Step C. Let us give a proof by induction on m.

Assume that $m \geqslant 2$ and $m\,p < N$. We therefore have $(m-1)p < N$ and $p < N$. Let D denote the differential operator of order 1. By the existence of an embedding $W^{m-1,p} \hookrightarrow L^{Np/(N-(m-1)p)}$ which we assume proved, we have $Du \in W^{m-1,p}$, and therefore $Du \in L^{Np/(N-(m-1)p)}$. Since $u \in W^{m,p}$, we have $u \in W^{m-1,p}$, hence also $u \in L^{Np/(N-(m-1)p)}$.

Finally, setting $q = Np/(N-(m-1)p)$, we have $u \in W^{1,q}$. By the embedding theorem for $m = 1$ and because $q < N$, we have

$$u \in L^{Nq/(N-q)} = L^{Np/(N-mp)},$$

where the equality of the spaces follows from $q/(N-q) = p/(N-mp)$. This completes the proof of step C.

We have now proved statement (1) of the theorem.

Proof of Step D.

We move on to the proof of statement (2) by showing that $W^{N,1} \hookrightarrow L^\infty$. The density of the regular functions will then imply the existence of an embedding $W^{N,1} \hookrightarrow \mathcal{C}_b(\mathbb{R}^N)$.

In the proof of result (1) (cf. (2.39)), we have already shown that if $u \in W^{1,1}(\mathbb{R}^N)$, then

$$\forall\, x' \in \mathbb{R}^{N-1}, \quad \|u\|_\infty(x', \cdot) \leqslant \int_{\mathbb{R}} |\partial_N u(x', t)| dt.$$

Let us make the following induction hypothesis. If $v \in W^{N-1,1}(\mathbb{R}^{N-1})$, then $v \in L^\infty(\mathbb{R}^{N-1})$ and

$$\|v\|_\infty \leqslant \sum_{\substack{\alpha\in\mathbb{N}^{N-1}\\ |\alpha|\leqslant N-1}} \int_{\mathbb{R}^{N-1}} |D^\alpha v(x')| dx'.$$

Applying this inequality to the function $\partial_N u(x', x_N)$ for fixed x_N gives

$$\sup_{x'\in\mathbb{R}^{N-1}} |\partial_N u(x', x_N)| \leqslant \sum_{\substack{\alpha\in\mathbb{N}^{N-1}\\ |\alpha|\leqslant N-1}} \int_{\mathbb{R}^{N-1}} |D^\alpha(\partial_N u)|(x', x_N) dx'.$$

We then integrate with respect to x_N:

$$\begin{aligned}\sup_{\substack{x'\in\mathbb{R}^{N-1}\\ x_N\in\mathbb{R}}} |u(x',x_N)| &\leqslant \int_{\mathbb{R}} \sup_{x'} |\partial_N u(x',x_N)| dx_N \\ &\leqslant \sum_{\substack{\alpha\in\mathbb{N}^{N-1}\\ |\alpha|\leqslant N-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} |D^\alpha(\partial_N u)(x',x_N)| dx' dx_N \\ &\leqslant \sum_{\substack{\alpha\in\mathbb{N}^{N}\\ |\alpha|\leqslant N}} \int_{\mathbb{R}^N} |D^\alpha u|(x) dx.\end{aligned}$$

We have thus obtained the embedding $W^{N,1} \hookrightarrow L^\infty$.

Let us return to statement (2). Let $u \in W^{N,1}(\mathbb{R}^N)$ and let $\{u_n\}$ be a sequence in $\mathcal{D}(\mathbb{R}^N)$ such that $\|u_n - u\|_{W^{N,1}(\mathbb{R}^N)} \to 0$. By the above, we can deduce that $\|u_n - u\|_{L^\infty(\mathbb{R}^N)} \to 0$, which means that $\{u_n\} \to u$ uniformly on $\mathbb{R}^N$. Consequently, u is continuous on $\mathbb{R}^N$. Since $u \in L^\infty$, it follows that $u \in \mathcal{C}_b(\mathbb{R}^N)$. Moreover, the inequality $\|u\|_{L^\infty} \leqslant C\|u\|_{W^{N,1}}$ gives

$$\forall\, u \in W^{N,1}(\mathbb{R}^N), \quad \|u\|_{\mathcal{C}_b(\mathbb{R}^N)} \leqslant C\|u\|_{W^{N,1}}.$$

This concludes step D and the proof of statement (2).

Proof of Step E. Let us now assume that $mp = N$.

We begin with the case $m = 1$, $p = N > 1$.

Let $u \in W^{1,N}(\mathbb{R}^N)$. We will show that u belongs to L^q for every $q \geqslant N$. We begin by showing that $W^{1,N}(\mathbb{R}^N)$ has an embedding into L^q for every $q \in [N, N^2/(N-1)]$. For this, we note that if $u \in W^{1,N}$, then $u^N \in W^{1,1}$. This follows from $\nabla(u^N) = N u^{N-1}\nabla u$ and Hölder's inequality:

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u^N| &\leqslant N \int_{\mathbb{R}^N} |\nabla u||u^{N-1}| dx \\ &\leqslant N\Big(\int_{\mathbb{R}^N} |\nabla u|^N dx\Big)^{1/N} \Big(\int_{\mathbb{R}^N} |u|^N dx\Big)^{(N-1)/N}.\end{aligned}$$

Using the Sobolev embedding of $W^{1,1}$ into $L^{N/(N-1)}$, we deduce that u belongs to $L^{N^2/(N-1)}$.

Let us now show that u belongs to all L^q with $q > N^2/(N-1)$. For this, we note that q can be written as $q = q'N/(N-1)$ with $q' > N$. Suppose that φ is a regular function tending to u in $W^{1,N}(\mathbb{R}^N)$. We consider

$$A = \Big(\int_{\mathbb{R}^N} \big|\varphi^{q'N/(N-1)}\big| dx\Big)^{(N-1)/N} = \|\varphi^{q'}\|_{L^{N/(N-1)}}.$$

Using $\nabla(|\varphi|^{q'}) = q'|\varphi|^{q'-2}\varphi\nabla\varphi$, Remark 2.44, that is, the upper bound (2.45), and then Hölder's inequality, we obtain the following upper bounds for A:

$$
\begin{aligned}
A &\leqslant q'C\int_{\mathbb{R}^N}|\varphi|^{q'-1}|\nabla\varphi|dx \\
&\leqslant q'C\Big(\int_{\mathbb{R}^N}|\varphi|^{(q'-1)N/(N-1)}dx\Big)^{(N-1)/N}\Big(\int_{\mathbb{R}^N}|\nabla\varphi|^N dx\Big)^{1/N}.
\end{aligned}
\tag{2.47}
$$

We see that $(q'-1)N/(N-1) \in [N, q'N/(N-1))$. Therefore, there exists a $\theta \in [0,1]$, namely $\theta = 1/(q'+1-N)$, such that

$$\frac{(q'-1)N}{(N-1)} = \theta N + (1-\theta)\frac{q'N}{(N-1)}.$$

Consequently, once more using Hölder's inequality, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N}|\varphi(x)|^{(q'-1)N/(N-1)}dx \\
&\qquad\leqslant \Big(\int_{\mathbb{R}^N}|\varphi(x)|^{q'N/(N-1)}dx\Big)^{1-\theta}\Big(\int_{\mathbb{R}^N}|\varphi(x)|^N dx\Big)^{\theta}.
\end{aligned}
$$

Substituting this in inequality (2.47) above, we find

$$
\begin{aligned}
&\Big(\int_{\mathbb{R}^N}|\varphi(x)|^{q'N/(N-1)}\Big)^{(N-1)/(Nq')} \\
&\leqslant Cq'^{(q'-N+1)/q'}\Big(\int_{\mathbb{R}^N}|\varphi(x)|^N\Big)^{(N-1)/(Nq')}\Big(\int_{\mathbb{R}^N}|\nabla\varphi(x)|^N\Big)^{(q'-N+1)/(q'N)}.
\end{aligned}
$$

We have thus established (cf. Remark 2.32) that $u \in L^{q'N/(N-1)}$.

Note that we cannot conclude that $u \in L^\infty$, as the scalar sequence $q'^{(q'-N+1)/q'}$ is not bounded. Moreover, there exist examples of unbounded $W^{1,N}$ functions with $N \geqslant 2$.

Let us assume that $m \geqslant 2$ and $mp = N$.

We then have $(m-1)p < N$. From $u \in W^{m,p}$, we deduce that $u \in W^{m-1,p}$ and that for every j, $\partial_j u \in W^{m-1,p}$. Hence, by statement (1) of the theorem, we know that u and $\partial_j u$ are elements of L^r with $r = Np/(N-(m-1)p)$.

From $mp = N$, we deduce that $r = N$. Hence $u \in W^{1,r}$, which by the above implies that $u \in L^q$ for every q, concluding the proof of step E.

Proof of Step F. Let us now assume that $mp > N$.

We begin with the case $p > N$, $m = 1$.

Let $u \in W^{1,p}(\mathbb{R}^N)$ and let $p > N$. We will give two proofs that we then have $u \in L^\infty(\mathbb{R}^N)$.

First proof that $u \in L^\infty(\mathbb{R}^N)$ in step F. This proof is based on the integration of the function over a cone $C_{h,\theta}$ with vertex 0, opening angle θ, and bounded by a sphere of radius h in $\mathbb{R}^N$. This proof can therefore also be used for an open subset Ω with the uniform cone property, that is, an open subset for which there exist h, θ such that for every $x \in \Omega$, there exists a rotation R of $\mathbb{R}^N$ with $x + R(C_{h,\theta}) \subset \Omega$. This is of course the case for Lipschitz open sets, whose definition we will give further on. It does not hold for the open set in Example 2.9, in which $\partial\Omega$ has a cusp.

We will show that

$$(2.48) \qquad \forall\,\varphi \in \mathcal{D}(\mathbb{R}^N), \quad \|\varphi\|_\infty \leqslant C_1 h^{-N/p}\|\varphi\|_p + C_2 h^{1-N/p}\|\nabla\varphi\|_p.$$

After applying a translation, if necessary, we reduce to finding an upper bound for $|\varphi(0)|$. We will use the polar coordinates (ρ,σ), where $\rho \in [0,h]$ and $\sigma \in A(\rho)$, with $A(\rho)$ the surface of intersection of $C_{h,\theta}$ and the sphere of radius ρ (cf. Figure 2.2 below). Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and let $\widetilde{\varphi}(\rho,\sigma)$ be its expression in polar coordinates.

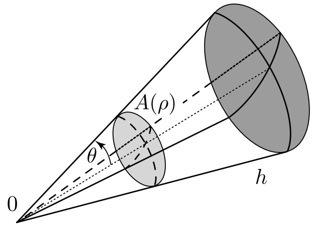

Fig. 2.2. The cone $C_{h,\theta}$.

We have

$$\varphi(0) = \widetilde{\varphi}(\rho,\sigma) + \int_\rho^0 \partial_\rho(\widetilde{\varphi})(\lambda,\sigma)d\lambda.$$

For the remainder of the proof, we set

$$I(\rho,\sigma) = \int_0^\rho |\partial_\rho(\widetilde{\varphi})(\lambda,\sigma)|d\lambda.$$

The volume element is defined by $dx = \rho^{N-1}s(\sigma)d\sigma d\rho$, where $s(\sigma)d\sigma$ is the $(N-1)$-dimensional surface element on the unit sphere S_N. Since the volume of the cone is proportional to h^N, by Fubini, integrating the inequality above over $C_{h,\theta}$ gives the following inequality, where $c_1 > 0$ is a constant bounded from below independently of h:

$$(*)\quad |\varphi(0)|h^N c_1 \leqslant \int_0^h \int_{A(\rho)} s(\sigma)|\widetilde{\varphi}(\rho,\sigma)|\rho^{N-1} d\sigma d\rho + \int_0^h \int_{A(\rho)} \rho^{N-1} s(\sigma) I(\rho,\sigma) d\sigma d\rho.$$

The first integral of $(*)$ is the same as $A = \int_{C_{h,\theta}} |\varphi(x)|dx$. Using Hölder's inequality with conjugate exponents p and p', we find

$$A \leqslant \Big(\int_{C_{h,\theta}} dx\Big)^{1/p'} \Big(\int_{C_{h,\theta}} |\varphi(x)|^p dx\Big)^{1/p} \leqslant c_1' h^{N/p'} \|\varphi\|_{L^p(C_{h,\theta})}.$$

We will now study the second integral B of $(*)$. First consider the integral $I(\rho,\sigma)$, which we write as

$$I(\rho,\sigma) = \int_0^\rho |\partial_\rho \widetilde{\varphi}|(\lambda,\sigma) \lambda^{(N-1)/p} \lambda^{(N-1)/p'} \lambda^{-(N-1)} d\lambda,$$

giving

$$I(\rho,\sigma) \leqslant \Big(\int_0^\rho |\partial_\rho(\widetilde{\varphi})(\lambda,\sigma)|^p \lambda^{N-1} d\lambda\Big)^{1/p} \Big(\int_0^\rho \lambda^{(N-1)(1-p')} d\lambda\Big)^{1/p'}$$

when we apply Hölder's inequality to it. We note that the exponent of the last integrand satisfies the relation $(N-1)(1-p') > -1$ as $p > N$, which implies the finiteness of this integral. The second integral B in $(*)$ therefore leads to the inequality

$$B \leqslant K \int_0^h \rho^{N-1} \int_{A(\rho)} s(\sigma) \cdot \Big(\int_0^\rho |\partial_\rho(\widetilde{\varphi})(\lambda,\sigma)|^p \lambda^{N-1} d\lambda\Big)^{1/p} \rho^{[(N-1)(1-p')+1]/p'} d\sigma d\rho.$$

Bounding the inner integral by the corresponding integral over $[0,h]$, we have

$$B \leqslant K \int_0^h \rho^{N/p'} \int_{A(\rho)} s(\sigma) \Big(\int_0^h |\partial_\rho(\widetilde{\varphi})(\lambda,\sigma)|^p \lambda^{N-1} d\lambda\Big)^{1/p} d\sigma d\rho.$$

Again applying Hölder's inequality, this time to the integral over $A(\rho)$, we have

$$B \leqslant K \int_0^h \rho^{N/p'} (\operatorname{mes} A(\rho))^{1/p'} \cdot \Big(\int_{A(\rho)} s(\sigma) \int_0^h |\partial_\rho(\widetilde{\varphi})(\lambda,\sigma)|^p \lambda^{N-1} d\lambda d\sigma\Big)^{1/p} d\rho.$$

Since the measure of $A(\rho)$ is bounded by the area of S_N, hence independently of h, the right-hand side of the inequality above can be interpreted as an integral over $C_{h,\theta}$. Since $|\partial_\rho(\widetilde{\varphi}(\lambda,\sigma)|$ is bounded from above by $|\nabla\varphi(x)|$, we can therefore write

$$B \leqslant K' h^{1+N/p'} \|\nabla\varphi\|_{L^p(C_{h,\theta})}.$$

Dividing by h^N and applying $(*)$, we obtain the desired inequality (2.48). We then extend to $W^{1,p}$ functions by density. Moreover, we will see later on that in the case of $\mathbb{R}^N$, as h can be any element of $\mathbb{R}$, the right-hand side of (2.48) is bounded from above, giving an optimal upper bound for the norm $\|\cdot\|_\infty$. □

Second proof that $u \in L^\infty(\mathbb{R}^N)$ in step F. Consider the fundamental solution E of the Laplacian. We can easily verify (cf. Exercise 2.19) that $E = k_N r^{2-N}$ for $N \geqslant 3$ and $E = k_2 \ln r$ for $N = 2$, with $k_2 = 1/(2\pi)$ and $k_N = 1/((2-N)\omega_{N-1})$, where ω_{N-1} is the $(N-1)$-dimensional surface area of the unit sphere in $\mathbb{R}^N$. Let θ be a function in $\mathcal{D}(\mathbb{R}^N)$ with value 1 on a ball with center 0. Let $F = \theta E$. We then have

$$\Delta F = \theta\delta_0 + 2\nabla\theta\cdot\nabla E + (\Delta\theta)E = \delta_0 + \psi,$$

where $\psi \in \mathcal{D}(\mathbb{R}^N)$. We can write

$$u = \delta_0 \star u = \Delta F \star u - \psi \star u$$

and

$$\Delta F \star u = \sum_{1\leqslant i\leqslant N} \partial_i F \star \partial_i u.$$

Moreover, the derivatives of F are of the form r^{1-N} in the neighborhood of 0 and have compact support on $\mathbb{R}^N$. Therefore they all belong to L^q for $q < N/(N-1)$. In particular, they belong to $L^{p'}$ because $p > N$. The convolution $\sum_i \partial_i F \star \partial_i u$ therefore belongs to L^∞. Since $\psi \in \mathcal{D}(\mathbb{R}^N)$ and, for example, $u \in L^1$, the convolution $u \star \psi$ is a bounded $\mathcal{C}^\infty$ function.

We have thus obtained the existence of a constant C such that

$$\|u\|_\infty \leqslant C\Big(\|\nabla F\|_{p'}\|\nabla u\|_p + \|\psi\|_{p'}\|u\|_p\Big),$$

completing the proof that $u \in L^\infty(\mathbb{R}^N)$. □

Note that we obtain an optimal estimate by using functions of the form $u_\lambda(x) = u(x/\lambda)$, where $\lambda > 0$. Indeed, the continuity inequality $\|u\|_\infty \leqslant C_1\|u\|_p + C_2\|\nabla u\|_p$ applied to u_λ gives

$$\|u\|_\infty \leqslant C_1\lambda^{N/p}\|u\|_p + C_2\lambda^{-1+N/p}\|\nabla u\|_p.$$

In particular, the minimum of the function of λ on the right-hand side is reached for $\lambda = M\|\nabla u\|_p\big(\|u\|_p\big)^{-1}$, where $M = C_2(p-N)/(NC_1)$. We thus obtain the following inequality, where C is a constant that depends only on N, p, and universal data:

$$\|u\|_\infty \leqslant C\Big(\|u\|_p^{1-N/p}\|\nabla u\|_p^{N/p}\Big).$$

We conclude the proof of step F by studying the Hölder continuity of u.

Let $h \in \mathbb{R}^N$. In Proposition 2.23, we have already noted that

$$\|\tau_h u - u\|_p \leqslant Ch\|\nabla u\|_p$$

and

$$\|\nabla(\tau_h u - u)\|_p \leqslant 2\|\nabla u\|_p,$$

so that applying the previous inequality gives

$$\|\tau_h u - u\|_\infty \leqslant Ch^{1-N/p}\|\nabla u\|_{L^p}.$$

This implies that u is a Hölder continuous function with exponent $1 - N/p$. We have thus proved that u is a Hölder continuous function for $m = 1$.

Let us now consider the case $m \geqslant 2$. If $mp > N$, $N/p \notin \mathbb{N}$, and $j = [N/p] + 1$, then

$$W^{m,p}(\mathbb{R}^N) \hookrightarrow \mathcal{C}_b^{m-j,j-N/p}(\mathbb{R}^N).$$

Indeed, let j be such that $jp > N > (j-1)p$; then

$$u \in W^{j,p}(\mathbb{R}^N) \Longrightarrow (u, Du) \in (W^{j-1,p}(\mathbb{R}^N))^2.$$

Hence $(u, Du) \in (L^{Np/(N-(j-1)p)}(\mathbb{R}^N))^2$ by the first Sobolev embedding, since $(j-1)p < N$. Consequently,

$$u \in W^{1,Np/(N-(j-1)p)}(\mathbb{R}^N).$$

By the above and the inequality $Np/(N-(j-1)p) > N$, we find that $u \in \mathcal{C}_b(\mathbb{R}^N)$ or, more precisely,

$$u \in \mathcal{C}_b^{0,1-N(N-(j-1)p)/(Np)} = \mathcal{C}_b^{0,j-N/p}(\mathbb{R}^N).$$

Next, let $u \in W^{m,p}(\mathbb{R}^N)$ with $pm > N$. Let j satisfy $(j-1)p \leqslant N < jp$. By the above, $D^{(m-j)}u \in W^{j,p}(\mathbb{R}^N)$, so that $u \in \mathcal{C}_b^{(m-j)}(\mathbb{R}^N)$ with $j = \big[N/p\big]+1$. Since $D^{m-j}u \in \mathcal{C}_b^{0,j-N/p}(\mathbb{R}^N)$, we have $u \in \mathcal{C}_b^{m-j,j-N/p}(\mathbb{R}^N)$.

If $u \in W^{j,p}(\mathbb{R}^N)$ with $j = (N/p) + 1 \in \mathbb{N}$, then $Du \in W^{j-1,p}(\mathbb{R}^N)$. Moreover, as $(j-1)p = N$, step E implies that $Du \in L^q$ for every $q < \infty$. By the above, $u \in \mathcal{C}_b^{0,\lambda}(\mathbb{R}^N)$ for every $\lambda < 1 - N/q$, that is, $u \in \mathcal{C}_b^{0,\lambda}(\mathbb{R}^N)$ for every $\lambda < 1$.

If $j = (N/p) + 1 \in \mathbb{N}$, then the above shows that $D^{m-j}u \in \mathcal{C}_b^{0,\lambda}(\mathbb{R}^N)$ for every $\lambda < 1$, whence $u \in \mathcal{C}_b^{m-N/p-1,\lambda}(\mathbb{R}^N)$ for every $\lambda < 1$.

This concludes step F and the proof of Theorem (2.31).

2.3 Generalization to Other Open Sets

In this section, we study certain classes of open subsets for which the statements of the Sobolev embedding theorem of Section 1.2 still hold.

2.3.1 Methods, Examples and Counterexamples

One method for obtaining the embeddings is as follows. If possible, we extend every function $u \in W^{m,p}(\Omega)$ outside of Ω to a function $\widetilde{u} \in W^{m,p}(\mathbb{R}^N)$. We then use the properties of Theorem 2.31 for $\widetilde{u}$. Returning to u, which is the restriction of $\widetilde{u}$ to Ω, we obtain the corresponding property for the space $W^{m,p}(\Omega)$.

We will see that the existence of such extensions are closely linked to the geometric structure of the open set Ω. Let us first give a counterexample.

Example 2.49. Consider the open set Ω defined by

$$\Omega = \{(x, y) \mid 0 < x < 1,\ 0 < y < x^2\}.$$

The Sobolev embeddings do not all hold for this open set (cf. [68]).

Indeed, the function $(x, y) \mapsto x^\alpha$ belongs to $H^1(\Omega)$ provided that $\alpha > -1/2$. On the other hand, it belongs to L^p if and only if $\alpha p + 2 > -1$. This implies that $u \in L^p$ for $p < 6$ but not for $p = 6$, while the classical Sobolev embedding would give the inclusion for arbitrary p.

Let us present a relatively large class of open sets for which the embedding theorems hold. The reader can consult [1] for counterexamples and more general open sets.

2.3.2 (m, p)-Extension Operators

Definition 2.50. We say that an open subset Ω of $\mathbb{R}^N$ has an (m, p)-extension if there exists a continuous linear operator E from $W^{m,p}(\Omega)$ to $W^{m,p}(\mathbb{R}^N)$ such that for every $x \in \Omega$, the operator satisfies $Eu(x) = u(x)$.

We have the following theorem.

Theorem 2.51. *Let Ω be an open subset of $\mathbb{R}^N$ that has an (m, p)-extension; then the results concerning $W^{m,p}$ in Theorem 2.31 extend to the case of Ω.*

Proof of Theorem 2.51.

Let us assume that $mp < N$. Let E be a continuous extension operator from $W^{m,p}(\Omega)$ to $W^{m,p}(\mathbb{R}^N)$. Let $q \leqslant Np/(N - mp)$. Since $Eu(x) = u(x)$ for x in Ω, we have

$$\|u\|_{L^q(\Omega)} \leqslant \|E(u)\|_{L^q(\mathbb{R}^N)} \leqslant C\|E(u)\|_{W^{m,p}(\mathbb{R}^N)} \leqslant C\|E\|\ \|u\|_{W^{m,p}(\Omega)}.$$

We use a similar method for the other cases (2) and (3) of the Sobolev embedding theorem. □

We will now give sufficient *geometric* conditions on the open set Ω for the existence of an (m,p)-extension.

2.3.3 The Case of the Half-Space $(\mathbb{R}^N)^+$

Let $(\mathbb{R}^N)^+ = \mathbb{R}^{N-1} \times]0,+\infty[$. We will show the existence of an (m,p)-extension in $W^{m,p}((\mathbb{R}^N)^+)$. We begin with a lemma stating the existence of a "trace" on the boundary. This result is a first encounter with the trace theorem that we will see in the next chapter.

Proposition 2.52. *There exists a continuous linear map*

$$\gamma_0 : W^{1,p}((\mathbb{R}^N)^+) \longrightarrow L^p(\mathbb{R}^{N-1})$$

such that if $u \in \mathcal{C}((\mathbb{R}^{N-1}) \times [0,+\infty[) \cap W^{1,p}((\mathbb{R}^N)^+)$, *then* $\gamma_0 u(x') = u(x',0)$. *Moreover, if* u *has compact support in* $\mathbb{R}^{N-1} \times [0,\infty[$, *then* $\gamma_0 u$ *has compact support in* $\mathbb{R}^{N-1}$ *and we have*

$$\int_{\mathbb{R}^{N-1}\times]0,\infty[} \partial_N u(x)dx = -\int_{\mathbb{R}^{N-1}} \gamma_0\, u(x')dx'. \tag{2.53}$$

Proof of Proposition 2.52.

Let us show that the sequence $x' \mapsto u(x',1/n)$ of functions in $L^p(\mathbb{R}^{N-1})$ is a Cauchy sequence. By Corollary 2.19 of Proposition 2.12, we have for almost all $x' \in \mathbb{R}^{N-1}$ that

$$\big|u(x',1/n) - u(x',1/m)\big| = \Big|\int_{1/m}^{1/n} \partial_N u(x',t)dt\Big|. \tag{$*$}$$

Applying Hölder's inequality with fixed x', taking the pth power, and integrating gives

$$\int_{\mathbb{R}^{N-1}} |u(x',1/n) - u(x',1/m)|^p dx' \leqslant \Big|\frac{1}{n} - \frac{1}{m}\Big|^{p-1} \int_{\mathbb{R}^{N-1}} \int_{1/m}^{1/n} \big|\partial_N u(x',t)\big|^p dt\, dx'.$$

Since the last integral is bounded by $\|\partial_N u\|_p^p$, we conclude that the sequence we are studying is a Cauchy sequence. Let $\gamma_0\, u$ be the function defined by $\gamma_0\, u(x') = \lim_{n\to+\infty} u(x',1/n)$. The above shows that $\gamma_0 u \in L^p(\mathbb{R}^{N-1})$. Moreover, the linearity of γ_0 is clear, and when $u \in \mathcal{C}^1((\mathbb{R}^N)^+)$, the limit is none other than $u(x',0)$, whence $\gamma_0(u)(x',0) = u(x',0)$.

Let us show the continuity of γ_0 on $W^{1,p}(\mathbb{R}^{N-1} \times]0,\infty[)$.

By applying Corollary 2.19 of Proposition 2.12 with $1/m$ and y and taking the limit in $(*)$ for m tending to $+\infty$, we find

$$\text{for almost all } \; y \in \mathbb{R}^+, \quad \gamma_0 u(x') = u(x',y) - \int_0^y \partial_N u(x',t)dt. \tag{$**$}$$

Integrating the pth power of $(**)$ with respect to $y \in [0,1]$ and $x' \in \mathbb{R}^{N-1}$ and applying Minkowski's inequality, we obtain

$$\|\gamma_0 u\|_{L^p(\mathbb{R}^{N-1})} \leqslant \Big(\int_0^1\int_{\mathbb{R}^{N-1}} |u(x',y)|^p dx'dy\Big)^{1/p} + \Big(\int_0^1\int_{\mathbb{R}^{N-1}} |\partial_N u|^p dx'dy\Big)^{1/p}.$$

The continuity of the map γ_0 follows from this.

Consider u in $W^{1,p}\big(\mathbb{R}^{N-1} \times [0,\infty[\big)$ with compact support. The formula $(**)$ tells us that

$$\forall\, x' \in \mathbb{R}^{N-1}, \quad \gamma_0 u(x') = -\int_0^\infty \partial_N u(x',t)dt.$$

We can now obtain (2.53) by integrating with respect to x' $\square$

This proposition is used in the proof of the following theorem.

Theorem 2.54. *For every $m \in \mathbb{N}^*$ and $1 \leqslant p < \infty$, the half-space $\mathbb{R}^{N-1} \times \mathbb{R}^+$ has an (m,p)-extension operator.*

Proof of Theorem 2.54.

For $u \in W^{m,p}(\mathbb{R}^{N+})$, we define the extension Eu of u for $x_N < 0$ by

$$Eu(x) = \sum_{1 \leqslant j \leqslant m} \lambda_j u(x', -jx_N), \tag{2.55}$$

where the m-tuple (λ_j) consists of the unique solution of the following system:

$$\forall\, k \in \{0,1,\dots,m-1\}, \quad \sum_{1 \leqslant j \leqslant m} (-j)^k \lambda_j = 1. \tag{2.56}$$

We can first remark that under these conditions, if $u \in \mathcal{C}^m((\mathbb{R}^N)^+)$, then for every $k \leqslant m-1$, the function u and the partial derivatives $\partial^k Eu/\partial x_N^k$ are continuous at the intersection with $\{x_N = 0\}$. Consequently, $Eu \in \mathcal{C}^{m-1}(\mathbb{R}^N)$, which we can show using the definition of the derivatives ∂_N^k along $\{x_N = 0\}$.

In Theorem 2.54, we can in fact use the given formula for Eu with m' numbers λ_j for any $m' > m$, provided that the m conditions in (2.56) are satisfied, this time with $1 \leqslant j \leqslant m'$. We apply this in the case $m = 1$ in Proposition 2.57 below, which provides a good beginning for the proof of Theorem 2.54.

Proposition 2.57. *Consider v in $W^{1,p}(\mathbb{R}^{N+})$ and $k \geqslant 1$ real numbers μ_j such that*

$$\sum_{1 \leqslant j \leqslant k} \mu_j = 1.$$

Let $\widetilde{v}$ be defined on $\mathbb{R}^N$ by

$$\widetilde{v}(x', x_N) = \begin{cases} v(x', x_N) & \text{if } x_N > 0, \\ \sum_{1 \leqslant j \leqslant k} \mu_j v(x', -jx_N) & \text{if } x_N < 0; \end{cases}$$

then $\widetilde{v} \in W^{1,p}(\mathbb{R}^N)$.

We will give the proof of Proposition 2.57 later. For the moment, we will admit the results of the proposition, in order to continue the proof of Theorem 2.54.

We must first show that $u \in W^{m,p}((\mathbb{R}^N)^+)$ implies $Eu \in W^{m,p}(\mathbb{R}^N)$. Let $u \in W^{m,p}((\mathbb{R}^N)^+)$ and let Eu be defined by (2.56). Assuming that we have proved that $Eu \in W^{m-1,p}((\mathbb{R}^N)^+)$, it suffices to verify that for every α with $|\alpha| = m - 1$, the derivative $D^\alpha\big(E(u)\big)$ satisfies the conditions of the proposition. In order to do this, let $D^\alpha = D^{\alpha'}\partial_N^k$ with $\alpha = (\alpha', k)$ and $k \leqslant m - 1$; then

$$D^\alpha(Eu)(x', x_N) = \sum_1^m \lambda_j(-j)^k D^{\alpha'}\partial_N^k u(x', -jx_N).$$

Since the $m \geqslant 1$ numbers $\mu_j = \lambda_j(-j)^k$ satisfy the relation $\sum_1^m \mu_j = 1$, the conditions of Proposition 2.57 are fulfilled. Consequently, $D^\alpha(Eu) \in W^{1,p}(\mathbb{R}^N)$.

We still need to prove the continuity of E. We will give its proof after that of Proposition 2.57.

Proof of Proposition 2.57. Let us show that $\widetilde{v}$ indeed belongs to $W^{1,p}(\mathbb{R}^N)$. For this we need the following lemma.

Lemma 2.58. *Let $v \in W^{1,p}((\mathbb{R}^N)^+)$ and let $\varphi \in \mathcal{D}(\mathbb{R}^N)$; then for every $i \in [1, N-1]$,*

$$\int_{(\mathbb{R}^N)^+} \partial_i v(x)\varphi(x)dx + \int_{(\mathbb{R}^N)^+} v(x)\partial_i\varphi(x)dx = 0. \tag{2.59}$$

If φ satisfies $\varphi(x', 0) = 0$, then

$$\int_{(\mathbb{R}^N)^+} \partial_N v(x)\varphi(x)dx + \int_{(\mathbb{R}^N)^+} v(x)\partial_N\varphi(x)dx = 0. \tag{2.60}$$

Proof of Lemma 2.58.

Let us show equality (2.59).

Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and let $\{v_n\}$ be a sequence in $\mathcal{C}^\infty((\mathbb{R}^N)^+) \cap W^{1,p}((\mathbb{R}^N)^+)$ that converges to v in $W^{1,p}((\mathbb{R}^N)^+)$. By the definition of the derivative $\partial_i v_n$ in the sense of distributions on $\mathbb{R}^{N-1}$, we have for almost all x_N,

$$\int_{\mathbb{R}^{N-1}} \partial_i v_n(x)(x', x_N)\varphi(x', x_N)dx' + \int_{\mathbb{R}^{N-1}} \partial_i\varphi(x', x_N)v_n(x', x_N)dx' = 0.$$

Integrating this equality with respect to x_N and taking the limit gives the desired result.

Let us now show equality (2.60).

If φ satisfies $\varphi(x', 0) = 0$, then the function $u\varphi$ is an element of $W^{1,p}((\mathbb{R}^N)^+)$ and has value 0 on the boundary $\{x_N = 0\}$. By Proposition 2.52, we have

$$\int_{(\mathbb{R}^N)^+} \partial_N(u\varphi)(x)dx = 0,$$

that is,

$$\int_{(\mathbb{R}^N)^+} \partial_N u(x)\varphi(x)dx = -\int_{(\mathbb{R}^N)^+} u(x)\partial_N\varphi(x)dx. \qquad \square$$

We conclude the proof of Proposition 2.57 by using derivation in the sense of distributions and Lemma 2.58.

Let $\varphi \in \mathcal{D}(\mathbb{R})$. The function $v(x', jx_N)$ is still an element of $W^{1,p}((\mathbb{R}^N)^+)$, and $\varphi(x', -x_N)$ is still an element of $\mathcal{D}(\mathbb{R}^N)$, so that by substituting $x_N \mapsto -x_N$ twice and using the first equality of Lemma 2.58, we have

$$\begin{aligned}\int_{(\mathbb{R}^N)^-} v(x', -jx_N)\partial_i\varphi(x)dx &= \int_{(\mathbb{R}^N)^+} v(x', jx_N)\partial_i\varphi(x', -x_N)dx \\ &= -\int_{(\mathbb{R}^N)^+} \partial_i v(x', jx_N)\varphi(x', -x_N)dx \\ &= -\int_{(\mathbb{R}^N)^-} \partial_i v(x', -jx_N)\varphi(x', x_N)dx\end{aligned}$$

for $i \leqslant N-1$.

Again by the first part of the lemma,

$$\begin{aligned}\langle \partial_i\widetilde{v}, \varphi\rangle &= -\langle \widetilde{v}, \partial_i\varphi\rangle \\ &= -\int_{(\mathbb{R}^N)^+} v(x)\partial_i\varphi(x)dx - \int_{(\mathbb{R}^N)^-} \sum_1^k \mu_j v(x', -jx_N)\partial_i\varphi(x)dx \\ &= \int_{(\mathbb{R}^N)^+} \partial_i v(x)\varphi(x)dx + \int_{(\mathbb{R}^N)^-} \sum_1^k \mu_j \partial_i v(x', -jx_N)\varphi(x)dx,\end{aligned}$$

where the right-hand side can also be written as

$$(*) \qquad \int_{\mathbb{R}^N} \Big[\partial_i\widetilde{v}\,\chi_{((\mathbb{R}^N)^+)} + \Big(\sum_1^k \mu_j\partial_i v(x', -jx_N)\Big)\,\chi_{((\mathbb{R}^N)^-)}\Big]\varphi(x)dx.$$

We have thus obtained

$$(2.61) \qquad \partial_i\widetilde{v} = \partial_i\widetilde{v}\,\chi_{((\mathbb{R}^N)^+)} + \Big(\sum_1^k \mu_j\partial_i v(x', -jx_N)\Big)\,\chi_{((\mathbb{R}^N)^-)}.$$

For the derivation in x_N, we substitute the variable $-jx_N$ for x_N:

$$\begin{aligned}
\langle \partial_N \widetilde{v}, \varphi \rangle &= -\langle \widetilde{v}, \partial_N \varphi \rangle \\
&= -\int_{(\mathbb{R}^N)^+} v(x)\partial_N \varphi(x)dx - \int_{(\mathbb{R}^N)^-} \sum_1^k \mu_j v(x', -jx_N)\partial_N \varphi(x)dx \\
&= -\int_{(\mathbb{R}^N)^+} v(x)\partial_N \varphi(x)dx - \int_{(\mathbb{R}^N)^+} \sum_1^k \frac{\mu_j}{j} v(x)(\partial_N \varphi)(x', -\frac{x_N}{j})dx \\
&= -\int_{(\mathbb{R}^N)^+} v(x', x_N)\partial_N \Big(\varphi(x', x_N) - \sum_1^k \mu_j \varphi(x', -\frac{x_N}{j}))\Big)dx \\
&= \int_{(\mathbb{R}^N)^+} \partial_N v \Big(\varphi(x', x_N) - \sum_1^k \mu_j \varphi(x', -\frac{x_N}{j})\Big)dx.
\end{aligned}$$

The last equality follows from the second part of Lemma 2.58 applied to the function $\varphi(x', x_N) - \sum_1^k \mu_j \varphi(x', -x_N/j)$, which is zero on $\{x_N = 0\}$ by the hypothesis $\sum_1^k \mu_j = 1$. After another change of variables, we have

$$\langle \partial_N \widetilde{v}, \varphi \rangle = \int_{((\mathbb{R}^N)^+)} \partial_N v(x)\varphi(x)dx - \int_{((\mathbb{R}^N)^-)} \sum_1^k \mu_j j \partial_N v(x', -jx_N)\varphi(x)dx.$$

It follows that

$$\partial_N \widetilde{v} = \partial_N v(x', x_N)\chi_{((\mathbb{R}^N)^+)} - \sum_1^m j\mu_j \partial_N v(x', -jx_N)\chi_{((\mathbb{R}^N)^-)}. \tag{2.62}$$

The two relations (2.61) and (2.62) show that all $\partial_i \widetilde{v}$ for $i \leqslant N$ belong to $L^p(\mathbb{R}^N)$.

We have thus completed the proof of Proposition 2.57. □

Let us finish the proof of Theorem 2.54 by proving the continuity of E. The previous equalities show that for all $i \leqslant N$,

$$|\partial_i \widetilde{v}|_{L^p(\mathbb{R}^N)} \leqslant 2\|\partial_i v\|_{L^p((\mathbb{R}^N)^+)}.$$

It follows that there exists a constant C such that

$$\|Eu\|_{m,p} \leqslant C\|u\|_{W^{m,p}((\mathbb{R}^N)^+)}.$$

The continuity of the operator E follows from this. □

Corollary 2.63. *The space $W_0^{1,p}((\mathbb{R}^N)^+)$ is the subspace of $W^{1,p}((\mathbb{R}^N)^+)$ consisting of the functions u such that $\gamma_0 u = 0$, that is, the functions u whose extension by 0 outside of $(\mathbb{R}^N)^+$ is an element of $W^{1,p}(\mathbb{R}^N)$.*

Proof of Corollary 2.63.

It is clear, using the continuity of the trace map γ_0, that for every sequence of functions with compact support that converges in $W^{1,p}((\mathbb{R}^N)^+)$, the trace of the limit is zero. It follows that if $u \in W_0^{1,p}((\mathbb{R}^N)^+)$, then $\gamma_0 u = 0$.

Conversely, let u satisfy $\gamma_0 u = 0$. Let $\widetilde{u}$ denote the extension by 0 for $x_N < 0$. Then for $i \leqslant N-1$, by the first equality (2.59) of Lemma 2.58, computing the derivative of this extension in the direction e_i gives

$$\forall\, \varphi \in \mathcal{D}(\mathbb{R}^N), \quad \langle \partial_i \widetilde{u}, \varphi \rangle = -\langle \widetilde{u}, \partial_i \varphi \rangle = -\int_{x_N>0} u \partial_i \varphi = \int_{x_N>0} \partial_i u \varphi.$$

For $i = N$, by the second equality (2.60) of Lemma 2.58 and since the trace of $u\varphi$ is zero, we have

$$\langle \partial_N \widetilde{u}, \varphi \rangle = -\langle \widetilde{u}, \partial_N \varphi \rangle = -\int_{x_N>0} u \partial_N \varphi = \int_{x_N>0} \partial_N u \varphi.$$

Let $v_n(x') = \widetilde{u}(x', x_N - 1/n)$; then the sequence $\{v_n\}$ with compact support in $(\mathbb{R}^N)^+$ converges to $\widetilde{u}$ in $W^{1,p}(\mathbb{R}^N)$. To see this, note that

$$\forall\, w \in L^p(\mathbb{R}^N), \quad \lim_{h \to 0} \|\tau_h w - w\|_p = 0. \tag{2.64}$$

Indeed, let $\varepsilon > 0$ and let ψ be an element of $\mathcal{C}_c(\mathbb{R}^N)$ such that $\|w - \psi\|_p \leqslant \varepsilon/3$. By the continuity of ψ, there exists an h_0 such that

$$\forall\, h, \quad |h| \leqslant h_0 \Longrightarrow \|\tau_h \psi - \psi\|_\infty \leqslant \frac{\varepsilon}{3|\operatorname{supp}(\psi)|^{1/p'}}.$$

Hence, for $|h| \leqslant h_0$, we have

$$\|w - \tau_h w\|_p \leqslant \|w - \psi\|_p + \|\psi - \tau_h \psi\|_p + \|\tau_h \psi - \tau_h w\|_p \leqslant \varepsilon.$$

It follows that

$$\lim_{n \to +\infty} \|v_n - \widetilde{u}\|_p = 0 \quad \text{and} \quad \forall\, j \in [1, N], \quad \lim_{n \to +\infty} \|\partial_j v_n - \partial_j \widetilde{u}\|_p = 0.$$

Next, let ρ be a function in $\mathcal{D}(\mathbb{R}^N)$. We set $\rho_{2n} = (2n)^N \rho(2nx)$ and $u_n = \rho_{2n} \star v_n$; then $\{u_n\}$ is a sequence of regular functions with compact support in $(\mathbb{R}^N)^+$ that converges to u in $W^{1,p}(\mathbb{R}^N)$, completing the proof. $\square$

2.3.4 Lipschitz Open Sets, $\mathcal{C}^m$ Open Sets

Let us begin with the definition of a uniformly Lipschitz open set, followed by that of a uniformly $\mathcal{C}^1$ open set.

Definition 2.65. We call Ω a uniformly Lipschitz open set if:

(1) There exists an open cover $(\Omega_i)_{i\geqslant 0}$ of Ω such that $d(\Omega_0, \partial\Omega) > 0$, for every $i \geqslant 1$, Ω_i is bounded and $\Omega_i \cap \partial\Omega \neq \varnothing$, and either the family $\{\Omega_i\}$ is finite or

$$\exists\, k \geqslant 2, \quad |i-j| \geqslant k \implies \Omega_i \cap \Omega_j = \varnothing.$$

(2) There exists an open subset $\mathcal{O}'_i$ of $\mathbb{R}^{N-1}$, a function a_i that is Lipschitz on $\mathcal{O}'_i$, and a system of coordinates such that, after permuting the coordinates if necessary,

$$\begin{aligned} \Omega_i \cap \Omega &\subset \{(x', x_N) \mid x' \in \mathcal{O}'_i,\ x_N > a_i(x')\}, \\ \Omega_i \cap \partial\Omega &= \{(x', a_i(x')) \mid x' \in \mathcal{O}'_i\}. \end{aligned}$$

(3) There exist a partition of unity $(\varphi_i)_i$ subordinate to the cover of Ω by the Ω_i (cf. Definition 2.11) and constants C_1 and C_2 such that

$$\forall\, i, \quad \|\varphi_i\|_{W^{1,\infty}(\mathbb{R}^N)} \leqslant C_1 \quad \text{and} \quad \|a_i\|_{W^{1,\infty}(\mathcal{O}'_i)} \leqslant C_2.$$

Definition 2.66. We say that an open set is uniformly of class $\mathcal{C}^1$ if it is uniformly Lipschitz with functions a_i of class $\mathcal{C}^1$.

Remark 2.67. To simplify the terminology, we will from now on often omit the adjective *regular* or *uniformly* and simply use the terms $\mathcal{C}^1$, $\mathcal{C}^k$, or *Lipschitz.*

Lipschitz open sets have the $(1,p)$-extension property. Proposition 2.70 below states this result. Further on, we will define a class of open sets that have the (m,p)-extension property. Note that the latter is not necessary for the embedding theorems, as we will see that being "Lipschitz" is sufficient. However, when an open set is of class $\mathcal{C}^m$ with $m > 1$, it is possible to define higher order traces (cf. next chapter) and, consequently, to obtain results concerning the regularity up to the boundary. We will use these results when studying the solutions of elliptic equations (cf. Chapter 5).

When using the definition above, it helps to know the relation between the inclusion of restrictions of $u \in W^{1,p}(\Omega)$ in each of the spaces $W^{1,p}(\Omega \cap \Omega_i)$, as well as the relation between the corresponding norms. These are as follows.

Proposition 2.68. *Let Ω be a Lipschitz open set. If for every i, $u \in L^p(\Omega)$ satisfies $u \in W^{1,p}(\Omega \cap \Omega_i)$, then $u \in W^{1,p}(\Omega)$. Moreover, there exist constants C and C' that do not depend on u such that*

$$\begin{cases} \sum_i \|\varphi_i u\|_{W^{1,p}(\Omega_i \cap \Omega)} \leqslant C \|u\|_{W^{1,p}(\Omega)}, \\ \|u\|_{W^{1,p}(\Omega)} \leqslant C' \sum_i \|u\|_{W^{1,p}(\Omega \cap \Omega_i)}. \end{cases} \tag{2.69}$$

Proof of Proposition 2.68. The first part of the proposition is obvious. Let us show the inequalities concerning the norms.

Let $u \in L^p(\Omega)$. By condition (1) of Definition 2.65, we can divide up the sequence $\{\Omega_i\}$ into the union of k sequences of open sets $\{\Omega_{i_n}\}$ such that the intersections $\Omega \cap \Omega_{i_n}$ are two-by-two disjoint. For such a sequence, the sum $\sum_n \|u\|^p_{L^p(\Omega_{i_n})}$ is bounded from above by $\|u\|^p_{L^p(\Omega)}$.

From this, we can deduce the inequality $\sum_i \|u\|^p_{L^p(\Omega\cap\Omega_i)} \leqslant k\|u\|^p_{L^p(\Omega)}$. Next, let $u \in W^{1,p}(\Omega)$. Using the chain rule for $\varphi_i u$ and the uniform upper bounds, we find that in condition (3) of Definition 2.65, the norm $\|\varphi_i u\|_{W^{1,p}(\Omega\cap\Omega_i)}$ is uniformly bounded from above by $K\|u\|_{W^{1,p}(\Omega\cap\Omega_i)}$. The previous upper bound therefore leads to

$$\sum_i \|\varphi_i u\|^p_{W^{1,p}(\Omega\cap\Omega_i)} \leqslant kK\|u\|^p_{W^{1,p}(\Omega)}.$$

The second inequality follows from $u = \sum_i \varphi_i u$. □

We will now give a first important extension result for Lipschitz open sets.

Proposition 2.70. *If Ω is Lipschitz, then for every $p \geqslant 1$, there exists a $(1,p)$-extension operator from Ω to $\mathbb{R}^N$.*

Proof of Proposition 2.70.

Let $u \in W^{1,p}(\Omega)$ and let $i \in \mathbb{N}$; then by the definition of the partition of unity $\{\varphi_i\}$, the function $\varphi_i u$ has compact support contained in $\Omega_i \cap \overline{\Omega}$. Moreover, $\varphi_i u \in W^{1,p}(\Omega_i \cap \Omega)$. We use the composition of $\varphi_i u$ and a *symmetry* on $\mathcal{O}'_i \times \mathbb{R}$ with respect to the hypersurface $\{x_N = a_i(x')\}$ (see Figure 2.3).

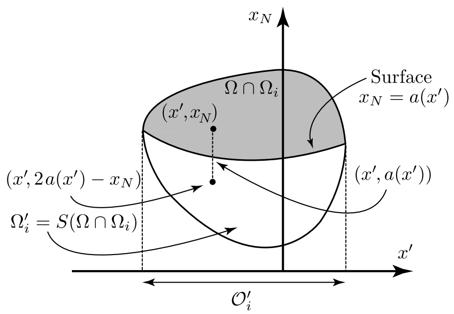

Fig. 2.3. Construction of the $(1,p)$-extension.

This *symmetry* S is defined on $\mathcal{O}'_i \times \mathbb{R}$ by $S(x', x_N) = (x', 2a_i(x') - x_N)$. The image of the bounded open set $\Omega_i \cap \Omega$ under S is a bounded open set Ω'_i. We let $U_i = (\Omega_i \cap \Omega) \cup (\partial\Omega \cap \Omega_i) \cup \Omega'_i$.

Let us begin by extending $\varphi_i u$. We will use local coordinates to define the extension $P_i(\varphi_i u)$ from $\Omega_i \cap \Omega$ to U_i. For every $(x', x_N) \in U_i$, we set

$$P_i(\varphi_i u)(x', x_N) = \begin{cases} (\varphi_i u)(x', x_N) & \text{if } x_N > a_i(x'), \\ (\varphi_i u)(x', 2a_i(x') - x_N) & \text{if } x_N < a_i(x'). \end{cases}$$

For $(x', x_N) \notin U_i$, we set $P_i(\varphi_i u) = 0$.

Let us verify that this extended function is an element of $W^{1,p}(\mathbb{R}^N)$ with norm in $W^{1,p}(\mathbb{R}^N)$ bounded from above by the norm $\|u\|_{W^{1,p}(\Omega)}$ multiplied by a constant depending only on the constants C_1 and C_2 of Definition 2.65.

We note that the symmetry S, which is its own inverse, is continuous because a_i is. Moreover, it satisfies

$$|S(x_1) - S(x_2)| \leqslant \bigl(1 + 4\|\nabla a_i\|_\infty\bigr)^{1/2} |x_1 - x_2|.$$

It follows that we can apply Lemma 2.22 to the function $P_i(\varphi_i u)$ on the open set $\Omega_i \cap \Omega$ and on its image under S. Let v be defined on $\mathbb{R}^{N-1} \times \,]0, +\infty[$ by

$$v(x', t) = \varphi_i u(x', a_i(x') + t).$$

The extension of v by reflection, that is, $\widetilde{v}(x', t) = \varphi_i u(x', a_i(x') - t)$ for $t < 0$, is the same as the previous reflection after the change of variable $t = x_N - a_i(x')$. It follows from Lemma 2.22 that $v \in W^{1,p}(\mathbb{R}^{N-1} \times \,]0, +\infty[)$. Since $\widetilde{v}$ results from a $(1,p)$-extension on $\mathbb{R}^N$, we find that $\widetilde{\varphi_i u} \in W^{1,p}(\mathbb{R}^N)$. Moreover, the constant c in Lemma 2.22 depends only on the Lipschitz constants of S and S^{-1}, and hence depends only on $\|\nabla a_i\|_\infty$, by the upper bound given earlier. We therefore have

$$\|\widetilde{v}\|_{W^{1,p}(\mathbb{R}^N)} \leqslant C(1 + \|\nabla a_i\|_\infty) \|\varphi_i u\|_{W^{1,p}(\Omega \cap \Omega_i)}.$$

Moreover, as the norms $\|\nabla a_i\|_\infty$ are bounded from above by C_2 (cf. Definition 2.65), setting $C_3 = C(1 + C_2)$, we have

$$\begin{aligned} \|\widetilde{\varphi_i u}\|_{W^{1,p}(\mathbb{R}^N)} &\leqslant (1 + C_2) \|\widetilde{v}\|_{W^{1,p}(\mathbb{R}^N)} \\ &\leqslant C_3 \|\varphi_i u\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Let us return to the open set Ω. Let

$$E(u) = \sum_i P_i(\varphi_i u).$$

By Proposition 2.68, we have $E(u) \in W^{1,p}(\mathbb{R}^N)$. The same proposition also gives

$$\|E(u)\|_{W^{1,p}(\mathbb{R}^N)} \leqslant \sum_i \|P_i(\varphi_i u)\|_{W^{1,p}(\mathbb{R}^N)} \leqslant C_3 \|u\|_{W^{1,p}(\Omega)}$$

This inequality implies the continuity of the extension operator E, completing the proof of Proposition 2.70. $\square$

Corollary 2.71. *If Ω is Lipschitz, then $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$.*

Proof of the Corollary.

Let $u \in W^{m,p}(\Omega)$ and let $v_n \in \mathcal{D}(\mathbb{R}^N)$ converge to $E(u)$ in $W^{m,p}(\mathbb{R}^N)$. The restrictions of the v_n to Ω then converge to the restriction of u to Ω, which is u itself. □

In compliance with the principles announced earlier, Proposition 2.70 allows us to prove the Sobolev embedding theorem.

Theorem 2.72. *Given a Lipschitz open set Ω, we have:*

(1) *If $N > mp$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \leqslant Np/(N-mp)$.*
(2) *If $N = mp$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q < \infty$. If $p = 1$, then $W^{N,1} \hookrightarrow \mathcal{C}_b(\Omega)$.*
(3) *If $mp > N$ with $N/p \notin \mathbb{N}$ and if j satisfies $(j-1)p < N < jp$, then we have*

$$W^{m,p}(\Omega) \longrightarrow \mathcal{C}_b^{m-j,\lambda}(\Omega), \quad \forall\, \lambda \leqslant j - N/p.$$

If $N/p \in \mathbb{N}$ and $m \geqslant j = N/p + 1$, then $W^{m,p}(\Omega) \hookrightarrow \mathcal{C}_b^{m-(N/p)-1,\lambda}(\Omega)$ for every $\lambda < 1$.

For the proof, which is left to the reader, it suffices to first understand that we can use the techniques of the proof of Theorem 2.31 to reduce to the case $m = 1$. After that, use the extension operator given in Proposition 2.70.

Let us continue with the (m,p)-extension operators, where $m > 1$.

Definition 2.73. An open set is called uniformly $\mathcal{C}^m$ if it is Lipschitz with functions a_i of class $\mathcal{C}^m$ and with the following uniform upper bounds in condition (3) of Definition 2.65:

$$\|a_i\|_{C^m(\mathcal{O}_i)} + \|\varphi_i\|_{C^m} \leqslant C_3. \tag{2.74}$$

Theorem 2.75. *A $\mathcal{C}^m$ open set has the (m,p)-extension property for every $p \in [1, \infty[$.*

Proof of Theorem 2.75.

Using local coordinate systems, we reduce the problem to the extension of a function of type $\varphi_i u$. Leaving out the indexes i in the function a_i and in the local coordinates for the sake of simplicity, we define

$$v(x', t) = u(x', a(x') + t),$$

which gives an element of $W^{m,p}((\mathbb{R}^N)^+)$ thanks to the properties of a. We then use the extension provided by Theorem 2.54. The continuity of the extension is an immediate consequence of the properties of $\mathcal{C}^m$-regularity, and the property of an extension on $\mathbb{R}^N$. □

Note that we can also define $\widetilde{u}$ directly using the formula

$$\widetilde{u}(x', x_N) = \sum_{j=1}^{m} \lambda_j u(x', -jx_N + (1+j)a(x')),$$

where the λ_j satisfy

$$\forall\, k \in [0, m-1], \quad \sum_j (-j)^k \lambda_j = 1.$$

However, in this case the computations are longer as we need to use the conservation of the tangential derivatives along $\partial\Omega$, for example, at order 1, $\partial_i u + \partial_i(a)\partial_N u$ for every $i \in [1, N-1]$.

2.4 Compact Embeddings in the Case of a Bounded Open Set

Let us now give compactness results for the Sobolev embeddings in bounded Lipschitz open sets. We begin by giving counterexamples in the case of the critical exponent for a bounded set, and for all embeddings in the unbounded case.

2.4.1 Two Preliminary Counterexamples

Example 2.76. Let us show that if $\Omega = B(0,1)$, $N > p$, and $m = 1$, then the embedding $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where q is the critical exponent $Np/(N-p)$, is not compact.

Let F be a $\mathcal{C}^1$ function on $\mathbb{R}^N$ with compact support in $B(0,1)$ that is not identically equal to zero. Let $\{F_n\}$ be the sequence of functions on $B(0,1)$ defined by $F_n(x) = n^{(N/p)-1}F(nx)$. We can easily see that $\{F_n\}$ tends to 0 almost everywhere and in $L^p(B(0,1))$. Moreover, its gradient is bounded in $L^p(B(0,1))$. Indeed,

$$\int_{B(0,1)} n^{(N/p-1+1)p} |\nabla F|^p(nx)dx = \|\nabla F\|_{L^p}^p. \tag{2.77}$$

In particular, $\{F_n\}$ is bounded in $W^{1,p}(\Omega)$. Moreover, we have

$$\|F_n\|_{L^{Np/(N-p)}(\Omega)} = \|F\|_{L^{Np/(N-p)}(\Omega)}. \tag{2.78}$$

It easily follows (cf. Section 6.1) that $|F_n|^{Np/(N-p)}$ converges vaguely to $|F|_{L^{Np/(N-p)}(\Omega)}^{Np/(N-p)}\delta_0$, where δ_0 denotes the Dirac measure at zero. Nevertheless, $\{F_n\}$ does not tend to 0 in $L^{Np/(N-p)}$.

Let us now give a counterexample to the existence of the compact embeddings when Ω is unbounded.

Example 2.79. Let us show that the embedding of $W^{1,1}(\mathbb{R}^N)$ in $L^1(\mathbb{R}^N)$ is not compact.

Consider $F \in \mathcal{D}(\mathbb{R}^N)$, non-identically zero, and a sequence $\{x_n\}$ that tends to infinity; then the sequence $\{F_n\}$ defined by $F_n(x) = F(x - x_n)$ is bounded in $W^{1,p}(\mathbb{R}^N)$ and converges almost everywhere to 0. Therefore, if it were to converge strongly in L^1, we would have $\|F_n\|_1 = \|F\|_1 = 0$, giving a contradiction.

2.4.2 Compactness Results

Theorem 2.80. *Let Ω be a bounded Lipschitz open subset of $\mathbb{R}^N$, where $N > 1$. If $N > mp$, then the embedding*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact for $q < Np/(N - mp)$.

Proof of Theorem 2.80. Let us first prove two lemmas.

Lemma 2.81. *For any bounded Lipschitz open subset Ω of $\mathbb{R}^N$, we have*

$$W^{1,1}(\Omega) \hookrightarrow_c L^1(\Omega).$$

Proof of Lemma 2.81. Let B be a bounded subset of $W^{1,1}(\Omega)$. We use the criteria for the compactness of bounded subsets of $L^p(\Omega)$ given in Theorem 1.95 of Chapter 1. Let us verify the two conditions of that theorem.

Let $\varepsilon > 0$ be given. We first show that there exists a compact subset K of Ω such that

$$\forall\, u \in B, \quad \int_{\Omega \smallsetminus K} |u(x)|dx \leqslant \varepsilon.$$

Indeed, using Hölder's inequality with exponents N and $N/(N-1)$, we have

$$\int_{\Omega \smallsetminus K} |u(x)|dx \leqslant \Big[\int_{\Omega \smallsetminus K} dx\Big]^{1/N} \Big[\int_{\Omega \smallsetminus K} |u(x)|^{N/(N-1)} dx\Big]^{(N-1)/N}.$$

Since the open set Ω is bounded, we can choose mes K sufficiently large that the measure of $(\Omega \smallsetminus K)$ is arbitrarily small, giving the desired result.

Next, we prove that there exists a δ such that if $\widetilde{u}$ denotes the extension of $u \in B$ by 0 outside of Ω, we have

$$\forall\, h, \quad |h| \leqslant \delta \implies \int_{\Omega} \Big|\widetilde{u}(x+h) - u(x)\Big| dx \leqslant \varepsilon.$$

Let $h_0 > 0$ be given. Let B_0 denote the closure of the union of the family $\mathcal{B}_{h_0}$ of all open balls with center in $\partial\Omega$ and radius h_0. Let $\omega = \Omega \setminus B_0$. This is an open set contained in Ω for which we can easily see that if $|h| < h_0$, then $x \in \omega \Rightarrow x + h \in \Omega$. Consequently, for every $x \in \omega$, $\widetilde{u}(x+h) = u(x+h)$. Consider the composed function $t \mapsto u(x+th)$. For $u \in B$, we have

$$\int_\omega |u(x+h) - u(x)| = \int_\omega \Big| \int_0^1 \frac{d}{dt}(u(x+th)dt \Big| dx.$$

Differentiating the absolutely continuous function $t \mapsto u(x+th)$ (cf. Exercise 2.3), we obtain

$$\frac{d}{dt}u(x+th) = \sum_1^N h_j \partial_j(u)(x+th) = h \cdot \nabla u(x+th),$$

whence

$$\int_\omega |u(x+h) - u(x)| \leqslant \int_\omega |h|\, |\nabla u(x+th)| dx.$$

Consequently, the last integral is bounded from above by $|h|\, \|\nabla u\|_{L^1(\Omega)}$, as $x + th \in \Omega$, hence by $C|h|$, as $u \in B$. Therefore, there exists an $h_1 < h_0$ such that

$$|h| \leqslant h_1 \Longrightarrow \int_\omega |u(x+h) - u(x)| \leqslant C|h| \leqslant \frac{\varepsilon}{2}.$$

We still need to bound the integral over $\Omega \setminus \omega$. For this, we use the inequality

$$\int_{\Omega\setminus\omega} |\widetilde{u}(x+h) - u(x)| \leqslant \int_{\Omega\setminus\omega} \big(|u(x+h)| + |u(x)|\big).$$

The argument given in the first part of the proof then implies the existence of a $\delta < h_1$ such that $|h| \leqslant \delta \Rightarrow 2\int_{d(x,\partial\Omega)\leqslant 2\delta} |u(x)|dx < \varepsilon$. Finally,

$$\forall\, u \in B, \quad |h| \leqslant \delta \Longrightarrow \int_\Omega |\widetilde{u}(x+h) - u(x)| dx \leqslant \varepsilon.$$

Theorem 1.95 now implies that B is relatively compact in $L^1(\Omega)$. □

Lemma 2.82. *Let Ω be an open subset of $\mathbb{R}^N$. Let $\{u_n\}$ be a sequence that is convergent in $L^k(\Omega)$ and bounded in $L^q(\Omega)$ for some $q > k$; then it converges in every $L^p(\Omega)$ with $k \leqslant p < q$.*

Proof of Lemma 2.82. We use Hölder's inequality to write $p = \theta k + (1-\theta)q$, where $\theta \in \,]0,1[$. We have

$$\|u_n - u_m\|_{L^p(\Omega)} \leqslant \|u_n - u_m\|^{\theta}_{L^k(\Omega)} \|u_n - u_m\|^{1-\theta}_{L^q(\Omega)}. \tag{2.83}$$

The right-hand side tends to zero when n and m tend to infinity, as it is the product of a bounded sequence and a sequence that tends to zero. We conclude that $\{u_n\}$ is a Cauchy sequence in $L^p(\Omega)$, and therefore converges in $L^p(\Omega)$. □

Let us return to the proof of Theorem 2.80.

Let $\{u_n\}$ be a bounded sequence in $W^{m,p}(\Omega)$. As Ω is bounded, $L^p(\Omega) \hookrightarrow L^1(\Omega)$ and $\{u_n\}$ is also bounded in $W^{1,1}(\Omega)$. By Lemma 2.81, the latter is relatively compact in $L^1(\Omega)$. Moreover, by Theorem 2.72, the sequence $\{u_n\}$ is bounded in $L^q(\Omega)$ with $q \leqslant Np/(N - mp)$. By Lemma 2.82, $\{u_n\}$ is relatively compact in all $L^q(\Omega)$ with $p \leqslant q < Np/(N - mp)$. □

Let us now consider, when $mp > N$, the compact embeddings into the spaces of Hölder continuous functions.

Theorem 2.84. *Let Ω be a Lipschitz open set. Let $mp > N$ and let $j = [N/p] + 1$; then for all $\lambda < j - N/p$, the embeddings*

$$W^{m,p}(\Omega) \hookrightarrow C^{m-j,\lambda}(\overline{\Omega})$$

are compact.

Proof of Theorem 2.84.

Let us begin with the case $m = 1$ and $p > N$. We will use the following result, whose proof we will give later.

Lemma 2.85. *Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $\{u_n\}$ be a sequence in $\mathcal{C}^{0,\lambda}(\Omega)$ that is relatively compact in $\mathcal{C}(\overline{\Omega})$; then for every μ satisfying $0 < \mu < \lambda$, the sequence $\{u_n\}$ is relatively compact in $\mathcal{C}_b^{0,\mu}(\Omega)$.*

Let us now show that the embedding of $W^{1,p}(\Omega)$ in $\mathcal{C}(\overline{\Omega})$ is compact. We will use the Ascoli–Arzelà theorem. Let K be a bounded set in $W^{1,p}(\Omega)$. The set $\{u(x) \mid u \in K\}$ is then uniformly bounded for every $x \in \Omega$. Indeed, as we already know that the injection is continuous (cf. Theorem 2.72), we have

$$\|u(x)\|_\infty \leqslant \|u\|_{W^{1,p}(\Omega)} \leqslant C$$

for all $u \in K$. Let us show that K is equicontinuous. Indeed, by the continuity of the embedding of $W^{1,p}(\Omega)$ in $\mathcal{C}^{0,1-N/p}(\Omega)$ (Theorem 2.72, again), we have

$$\forall\, (x, x+h) \in \overline{\Omega}^2, \quad |u(x+h) - u(x)| \leqslant C h^{1-N/p} \Big(\int_\Omega |\nabla u|^p dx \Big)^{1/p}.$$

This implies that K is uniformly Hölder, hence in particular equicontinuous. Lemma 2.85 allows us to conclude the proof in the case $m = 1$ and $p > N$.

Next, let K be a bounded subset of $W^{j,p}(\Omega)$ with $(j-1)p \leqslant N < jp$. We can easily see as above that K is relatively compact in $\mathcal{C}(\overline{\Omega})$. We again use Lemma 2.85 to conclude that K is compact in $\mathcal{C}^{0,\lambda}(\Omega)$ for every $\lambda < j - (N/p)$.

For the general case, let K be a bounded subset of $W^{m,p}(\Omega)$ and let $j = [N/p] + 1$. Let $\{u_n\}$ be a sequence of points of K. Since $\{u_n\}$ is bounded in $W^{m,p}(\Omega)$, both this sequence and the sequences consisting of its derivatives $\{D^{m-j}u_n\}$ are bounded in $W^{j,p}(\Omega)$. By the above, we can extract subsequences that converge in $\mathcal{C}_b^{0,\lambda}(\Omega)$ to u and $v_{m,j}$, respectively. For the sake of simplicity, we keep the same notation for the subsequences. They satisfy

$$\|u_n - u\|_\infty \longrightarrow 0 \quad \text{and} \quad \|D^{m-j}u_n - v_{m,j}\|_\infty \longrightarrow 0.$$

Since the convergence in L^∞ implies the convergence in the sense of distributions, we have $v_{m,j} = D^{m-j}u$. Moreover, by the above, $\{D^{m-j}u_n\}$ converges to $D^{m-j}u$ in $\mathcal{C}^{0,\lambda}(\Omega)$ for every $\lambda < j - N/p$.

It follows that for every $\lambda < j - N/p$, $\{u_n\}$ tends to u in $\mathcal{C}_b^{m-j,\lambda}(\Omega)$. This implies the compactness of the embedding of $W^{m,p}(\Omega)$ in $\mathcal{C}_b^{0,\mu}(\Omega)$, for every $\mu < j - N/p$. □

Proof of Lemma 2.85. Let $\theta \in \,]0,1[$ satisfy $\mu = \theta\lambda$. Let $\{u_{\sigma(n)}\}$ be a subsequence of $\{u_n\}$ that converges in $\mathcal{C}(\overline{\Omega})$. For any pair of indexes (n,m), set

$$d_{n,m} = \bigl|(u_{\sigma(n)} - u_{\sigma(m)})(x+h) - (u_{\sigma(n)} - u_{\sigma(m)})(x)\bigr|.$$

We have $d_{n,m} = d_{n,m}^{\theta} d_{n,m}^{1-\theta}$. Thanks to the convergence of $\{u_{\sigma(n)}\}$ in $\mathcal{C}(\overline{\Omega})$), we can choose n_0 sufficiently large that if $n, m \geqslant n_0$ and x and $x+h$ are elements of Ω with $|h| < h_0$, we have the following inequality:

$$d_{n,m}^{1-\theta} = \bigl|(u_{\sigma(n)} - u_{\sigma(m)})(x+h) - (u_{\sigma(n)} - u_{\sigma(m)})(x)\bigr|^{(1-\theta)} \leqslant \varepsilon.$$

Hence, under these conditions,

$$d_{n,m} \leqslant 2h^{\theta\lambda}\varepsilon.$$

Consequently,

$$\|u_{\sigma(n)} - u_{\sigma(m)}\|_{\mathcal{C}^{0,\mu}(\Omega)} \leqslant 2\varepsilon.$$ □

2.5 The Trace on the Boundary of a $\mathcal{C}^1$ Open Set

Recall that we defined a uniformly $\mathcal{C}^1$ open set to be an open subset of $\mathbb{R}^N$ that is Lipschitz with functions a_i of class $\mathcal{C}^1$. In this situation, we can define the integration on the subsets $U_i = \partial\Omega \cap \Omega_i$ of the boundary, each of which is a dimension $N-1$ submanifold of class $\mathcal{C}^1$ in $\mathbb{R}^N$. Such a submanifold is defined by a Cartesian equation $x' \mapsto x_N = a_i(x')$, where a_i is $\mathcal{C}^1$ on the open subset $\mathcal{O}_i'$ of $\mathbb{R}^{N-1}$, so that the $(N-1)$-dimensional surface element on U_i is given by $d\sigma(m) = \sqrt{1 + |\nabla(a_i)|^2}(m)\,dm$. Recall that in this case, the integral of a function f that is summable in U_i is defined by

$$\int_{U_i} f(m)dm = \int_{\mathcal{O}_i'} f(x', a_i(x'))\sqrt{1 + |\nabla(a_i)(x')|^2}dx'.$$

In this section, we define the trace of a function u in $W^{1,p}(\Omega)$ on the boundary of Ω in the same manner as in the case of $(\mathbb{R}^N)^+$, or more generally, in the case of a straight boundary. More precisely, we have the following theorem.

Theorem 2.86. *Let Ω be a uniformly $\mathcal{C}^1$ open subset of $\mathbb{R}^N$; then there exists a continuous linear map γ_0, called the* trace map, *from $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$ such that if $u \in \mathcal{C}(\overline{\Omega}) \cap W^{1,p}(\Omega)$, then its image $\gamma_0(u)$ is the function $x \mapsto u(x)$, which is well defined on $\partial\Omega$.*

To see the importance of the *class* $\mathcal{C}^1$ hypothesis on Ω, let us give an example of a non $\mathcal{C}^1$ open set on which the functions of $W^{1,p}(\Omega)$ do not have a restriction to $\partial\Omega$ in L^p.

Example 2.87. Consider the open sets defined in Example 2.9. We take the function $u(x,y) = 1/y^2$ that belongs to $H^1(\Omega)$, where the open set Ω is defined using $k = 1/6$. This function is the restriction of a function v defined everywhere on $\overline{\Omega}$ except at the point $x = 0$. Let us study whether $v|_{\partial\Omega}$ is an element of $L^2(\partial\Omega)$.

We have already proved in Example 2.9 that this is the case. Let us restrict ourselves to the part of $\partial\Omega$ that can be identified with either the arc Γ defined by $\{x \in [0,1] \mid y = x^{1/6}\}$ or the arc $\{x = y^6 \mid y \in [0,1]\}$. The infinitesimal element of arc is $ds(y) = \sqrt{1 + 36t^{25}}\, dt$, hence $\int_0^1 v(y)^2 ds(y)$ diverges at 0. It follows that this restriction, or trace, does not belong to $L^2(\partial\Omega)$.

Proof of Theorem 2.86. Even though the existence of the trace in the case of a Lipschitz open set can be shown in a manner similar to the one used in the case of $W^{1,p}((\mathbb{R}^N)^+)$, we will give a proof in which the importance of the notions of Definitions 2.65 and 2.66 is more evident.

Let us assume that $u \in \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$. We begin by defining the trace of $v_i = \varphi_i u$ using the partition of unity and local coordinates. This function, which is an element of $W^{1,p}(\Omega_i)$, can be extended by 0 outside of its support in the open set $\mathcal{O}'_i \times \{x_N > a_i(x')\}$. By Corollary 2.19, we have the following equality for every integer $n > 0$ and every $y > 0$:

$$(*) \quad v_i(x', a_i(x') + 1/n) - v_i(x', a(x') + y) = -\int_{1/n}^{y} \partial_N(v_i)(x', a_i(x') + t)dt.$$

Let $u_n(x') = v_i(x', a_i(x') + 1/n)$. From $(*)$, we deduce that for every pair (n, m) of nonzero integers, we have

$$\left|u_n(x') - u_m(x')\right| \leqslant \left|\int_{1/m}^{1/n} \left|\partial_N(v_i)(x', a_i(x') + t)\right| dt\right|.$$

We then apply Hölder's inequality and take the pth power. Next, we multiply on the left by the element of surface $d\sigma_i$ and integrate with respect to $x' \in \mathcal{O}'_i$. This allows us to prove that $A_{n,m} = \|u_n - u_m\|_{L^p(\mathcal{O}'_i, d\sigma_i)} \to 0$:

$$A_{n,m} \leqslant \left|\frac{1}{n} - \frac{1}{m}\right|^{1-1/p} \cdot \Big[\int_{\mathcal{O}'_i} \sqrt{1 + |\nabla a_i(x')|^2}\Big(\int_{\{a_i(x')-1/m \leqslant x_N \leqslant a_i(x')-1/n\}} \big|\partial_N v_i(x)\big|^p\Big)\Big]^{1/p},$$

whence

$$A_{n,m} \leqslant \left|\frac{1}{n} - \frac{1}{m}\right|^{1-1/p} \Big(\sqrt{1 + \|\nabla a_i\|_\infty^2}\Big)^{1/p} \|\partial_N(v_i)\|_{L^p(\Omega_i)}. \tag{2.88}$$

By Definition 2.66, this expresses the fact that $|\nabla a_i(x')|$ is bounded from above. When $p > 1$ and n and m tend to infinity, the right-hand side tends to zero, and therefore so does the left-hand side. When $p = 1$, the right-hand side still tends to zero, by the definition of L^1 functions. In all cases, $\{u_n\}$ is a Cauchy sequence in $L^p(\mathcal{O}'_i, d\sigma_i)$, the Lebesgue space for the bounded measure $d\sigma_i$, which is therefore complete. This sequence therefore converges in $L^p(\mathcal{O}'_i, d\sigma_i)$ to a function $w_i \in L^p(\mathcal{O}'_i, d\sigma_i)$. Moreover, there exists a subsequence $\{u_{\eta(n)}\}$ of $\{u_n\}$ that converges almost everywhere in $\mathcal{O}'_i$ to $w_i(x')$. Now, saying that $\lim(\varphi_i u)(x', a(x') + 1/(\eta(n)))$ exists almost everywhere is equivalent to saying that the function $x' \mapsto \varphi_i u(x', a(x')) = w_i(x')$ is well defined.

This extension w_i of $\varphi_i u$ on $\partial\Omega \cap \Omega_i$ is the desired trace. We therefore set $\gamma_0(\varphi_i u) = w_i$. By the above, this function belongs to $L^p(\mathcal{O}'_i, d\sigma_i)$, and therefore to $L^p(\partial\Omega \cap \Omega_i)$. Moreover, by taking the limit in $(*)$ and taking y sufficiently large that $v_i(x', a_i(x') + y) = 0$, we find that

(2.89)

$$\text{for almost all } x' \in \mathcal{O}'_i, \quad \gamma_0(\varphi_i u)(x') = -\lim\Big[\int_{1/\eta(n)}^{+\infty} \partial_N(v_i)(x', a_i(x') + t)dt\Big] = -\int_0^{+\infty} \partial_N(\varphi_i u)(x', a_i(x') + t)dt.$$

We must now define the trace of u by *gluing*.

Let $\gamma_0 u = \sum_i \gamma_0(\varphi_i u)$. This sum is locally finite, and by condition (1) of Definition (2.65), we conclude that $\gamma_0(u) \in L^p(\partial\Omega)$. We can show that the resulting trace does not depend on the choice of the elements in Definition (2.65).

If we assume that $u \in \mathcal{C}^1(\overline{\Omega})$, we can use the previous arguments. In particular, equality (2.89) gives us $\gamma_0(\varphi_i u)(x', a_i(x')) = \varphi_i \widetilde{u}(x', a_i(x'))$. It follows that $\gamma_0 u$ is the extension by continuity of u to the boundary $\partial\Omega$ (cf. the definition of $\mathcal{C}(\overline{\Omega})$).

To conclude we need only prove that the map γ_0 is continuous. For this, we start out with equality (2.89) and carry out the same computations we used to obtain (2.88). This gives

$$\|\gamma_0(\varphi_i u)\|_{L^p(\mathcal{O}'_i, d\sigma_i)} \leqslant C\Big(\sqrt{1+\|\nabla a_i\|_\infty^2}\Big)^{1/p} \|\partial_N(\varphi_i u)\|_{L^p(\Omega_i)}.$$

By condition (3) of Definition 2.65, this leads to the inequalities

$$\begin{aligned}\|\gamma_0 u\|_{L^p(\partial\Omega)} &\leqslant C\sup_i\Big(\sqrt{1+\|\nabla a_i\|_\infty^2}\Big)^{1/p}\sum_i \|\nabla(\varphi_i u)\|_{L^p(\Omega_i)}\\ &\leqslant C'\sum_i \|u\nabla\varphi_i + \varphi_i\nabla u\|_{L^p(\Omega_i)}\\ &\leqslant C'\sup_i\{\|\varphi_i\|_\infty, \|\partial_N\varphi_i\|_\infty\}\sum_i \|u\|_{W^{1,p}(\Omega_i)}.\end{aligned}$$

Using condition (2.66), we deduce that there exists a constant C^* that does not depend on the elements of Definition (2.65), such that

$$\forall\, u \in \mathcal{C}^\infty(\Omega)\cap W^{1,p}(\Omega), \quad \|\gamma_0 u\|_{L^p(\partial\Omega)} \leqslant C^*\|u\|_{W^{1,p}(\Omega)}.$$

We have thus defined the trace of u when $u \in \mathcal{C}^\infty(\Omega)\cap W^{1,p}(\Omega)$. For $u \in W^{1,p}(\Omega)$, we use the density stated in Proposition 2.12 to approximate u with $u_n \in \mathcal{C}^\infty(\Omega)\cap W^{1,p}(\Omega)$. By taking the limit, formula (2.89) gives $\gamma_0 u = -\int_0^{+\infty}\partial_N(\varphi_i u)(x', a_i(x')+t)dt$, whence $\gamma_0 u_n \to \gamma_0 u$ in $L^p(\partial\Omega\cap\Omega_i)$. The continuity follows, namely

$$\forall\, u \in W^{1,p}, \quad \|\gamma_0 u\|_{L^p(\partial\Omega)} \leqslant \|u\|_{W^{1,p}(\Omega)}. \qquad \square$$

Remark 2.90. The induced norm provides a way to define a norm on the image space of the trace map without giving it explicitly. We will give an explicit and intrinsic form of the norm in Chapter 3.

Let u be the trace of a function $U \in W^{1,p}(\Omega)$ on the boundary $\partial\Omega$. Let

$$|||u||| = \inf_{\{U\in W^{1,p}(\Omega)\,|\,u=U|_{\partial\Omega}\}} \|U\|_{W^{1,p}(\Omega)}. \tag{2.91}$$

This defines a norm for which the image space $\gamma_0(W^{1,p}(\Omega))$ is a Banach space. Indeed, let u and v be elements of $\gamma_0(W^{1,p}(\Omega))$ and let U and V be elements of $W^{1,p}(\Omega)$ such that $U = u$ and $V = v$ on $\partial\Omega$, and

$$\|U\| \leqslant |||u||| + \varepsilon \quad \text{and} \quad \|V\| \leqslant |||v||| + \varepsilon.$$

We then have $U + V = u + v$ on $\partial\Omega$ and

$$|||u + v||| \leqslant \|U + V\| \leqslant \|U\| + \|V\| \leqslant |||u||| + |||v||| + 2\varepsilon,$$

concluding the proof of the subadditivity. The proof of the other properties and of the completeness of the image space are left to the reader.

We conclude this chapter by going back to the characterization of the space $W_0^{1,p}(\Omega)$ when Ω is $\mathcal{C}^1$.

Theorem 2.92. *Let Ω be an open set of class $\mathcal{C}^1$; then the following statements are equivalent.*

(1) $u \in W_0^{1,p}(\Omega)$.
(2) *(only if $p > 1$) There exists a constant C such that for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$,*

$$\Big|\int_\Omega (u\nabla\varphi)(x)\Big| dx \leqslant C\|\nabla u\|_{L^p(\Omega)}\|\varphi\|_{L^{p'}}.$$

(3) *The function $\widetilde{u}$ defined by*

$$\widetilde{u} = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise}, \end{cases}$$

is an element of $W^{1,p}(\mathbb{R}^N)$.
(4) *The trace of u on $\partial\Omega$ is zero, that is, $\gamma_0 u = 0$.*

Proof of Theorem 2.92.
The implication $1 \Rightarrow 2$ is always true, without any assumptions on either the open set or on p. Let $u \in W_0^{1,p}(\Omega)$ and let $\{u_n\} \in \mathcal{D}(\Omega)$ converge to u in $W^{1,p}(\Omega)$. We have

$$\Big|\int_\Omega u_n(x)\partial_i(\varphi)(x)dx\Big| = \Big|-\int_\Omega \partial_i u_n(x)\varphi(x)dx\Big| \leqslant \|\nabla u_n\|_{L^p}\|\varphi\|_{L^{p'}}.$$

The result follows by taking the limit.

It is clear that (2) $\Rightarrow$ (3), since if $\varphi \in \mathcal{D}(\mathbb{R}^N)$, then

$$\langle \widetilde{u}, \partial_i\varphi\rangle = \int_\Omega u\partial_i\varphi\, dx.$$

Moreover, using (2), we find that if $p > 1$, then $\widetilde{u} \in W^{1,p}(\mathbb{R}^N)$.

The implication (3) $\Rightarrow$ (4) follows from the uniqueness of the trace.

Let us show that (4) $\Rightarrow$ (1). We reduce it to showing that if $u = 0$ on $\partial\Omega$, then we can approximate $u\varphi_i$ with functions in $\mathcal{D}(\Omega)$. Indeed, let

$$u_{n,i} = \widetilde{u\varphi_i}\Big(x', -a_i(x') + x_N - \frac{1}{n}\Big).$$

The functions $u_{n,i}$ are elements of $W^{1,p}(\Omega)$ with compact support. The sequence $\{u_{n,i}\}$ converges to $\widetilde{u\varphi_i}$ in $W^{1,p}(\mathbb{R}^N)$, hence converges to $u\varphi_i$ in $W^{1,p}(\Omega)$. Regularizing by a suitable function, we find that $u \in W_0^{1,p}(\Omega)$. $\square$

Comments

There are many books on Sobolev spaces over open subsets of $\mathbb{R}^N$. The simplest and most complete, as far as we are concerned, is Adams's book [1], which has the advantage of also studying more general open sets than Lipschitz sets, for example open sets satisfying the uniform cone condition, or having the segment property. One can also consult the original papers by Sobolev and Nikolskii [53], Sobolev [62] and Uspenskii [73]. The book by Gilbarg and Trudinger [34] presents the essentials, emphasizing the main points of the results.

There also exists a vast literature on Sobolev spaces over Riemann varieties. Let us mention, for example, the book by E. Hebey [37], which gives complete results and is agreeable to read.

The case where the codomain has other topological properties than $\mathbb{R}^p$ is discussed by Bethuel [6] and Brezis, Bethuel and Coron [5].

2.6 Exercises for Chapter 2

Exercise [$*$] **2.1 (On the Completeness of the Sobolev Space $H^1(\Omega)$).** Let Ω be an open subset of $\mathbb{R}^N$. Recall the definition of $H^1(\Omega)$. Show that

$$(u,v) = \int_\Omega u(x)v(x)dx + \sum_1^N \Big(\int_\Omega \partial_j u(x) \partial_j v(x) dx \Big),$$

defines a scalar product on the space $H^1(\Omega)$. Show that $H^1(\Omega)$ is a Hilbert space.

Hints. Let $\{u_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $H^1(\Omega)$. Prove that the sequence of derivatives $\{\partial_j u_n\}$ converges in L^2 to u_j. Next, prove that these functions are distributional derivatives of $u = \lim u_n$. Conclude.

Exercise 2.2 (On the Construction of a Partition of Unity).
We call a cover $\{\Omega'_k\}$ of Ω finer than the cover $\{\Omega_j\}$ if for every k, there exists a j such that $\Omega'_k \subset \Omega_j$. We call the cover $\{\Omega_j\}$ locally finite if every element x of Ω admits a neighborhood that meets only a finite number of open subsets in the family $\{\Omega_j\}$.

(1) Let $\{\Omega_j\}$ be an open cover of the open subset Ω of $\mathbb{R}^N$. Show that we can find a locally finite cover $\{\Omega'_k\}$ of Ω that is finer than $\{\Omega_j\}$ and consists of relatively compact sets.
(2) Consider a cover $\{\Omega_j\}$ consisting of relatively compact open sets. Show that there exist $\gamma_j \in \mathcal{D}(\Omega_j)$ such that $\gamma_j \geqslant 0$ and $\gamma_j = 1$ on $\overline{\Omega'_j}$. Use these functions to construct a partition of unity associated with the given cover.

In the general case, we will use the open cover of Ω given by question (1), which is finer than $\{\Omega_j\}$ and consists of relatively compact sets.

Hints. For (1), use an increasing sequence $\{U_k\}$ of relatively compact open sets that covers Ω and satisfies

$$U_0 = \varnothing, \quad \overline{U_k} \subset U_{k+1}.$$

Next, use the compactness of $\overline{\Omega_j}$ to determine a cover of this compact set by a finite number of U_k. It is easy to deduce a cover of Ω with the desired properties from this.

For (2), the construction of the γ_j, set $K = \overline{\Omega'_j}$. Let V be a neighborhood of 0 and let U be a compact neighborhood such that $U+U \subset V$ (first prove the existence of U). Let ρ_ε be a regularizing function (cf. Section 1.4.2) with support contained in U and let χ be the characteristic function of $K+U+U$. Let $\gamma_j = \chi \star \rho_\varepsilon$.

Since the sum $\gamma = \sum \gamma_j$ is locally finite, we can define this sum at every point of Ω and, by division, obtain the functions of a partition. Check this.

Exercise [$*$] 2.3 (On the Absolute Continuity of the Functions on a Sobolev Space (cf. Remark 2.4)).

The definition of an absolutely continuous function is given in Exercise 1.29. For any two absolutely continuous functions on an interval I, the product UV is also absolutely continuous. Moreover, for every $[a,b] \subset I$, we have the following formula for integration by parts, where u and v are almost everywhere derivatives of U and V:

$$\int_a^b U(t)v(t)dt = U(b)V(b) - U(a)V(a) - \int_a^b V(t)u(t)dt. \tag{2.93}$$

Let u be defined almost everywhere in an open set $\Omega \subset \mathbb{R}^2$.

(1) Let $\Omega \subset \mathbb{R}^2$, and let $u \in W^{1,p}(\Omega)$, where $p \geqslant 2$. Let $[\partial_x u]$ denote the L^p function equal to the derivative of u with respect to x, seen as a distribution. We can cover Ω by squares C_j and set $v_j = \psi_j u$, where $\psi_j \in \mathcal{D}(C_j)$ and $\sum \psi_j = 1$ on Ω. We extend v_j by 0 outside of C_j. Let v be defined on Ω by $v = \sum v_j$. Below we also write v for v_j, for the sake of simplicity. Show that $v \in L^p(\Omega)$. Let v^* be defined by $v^*(x) = \int_{-\infty}^x [\partial_1 v](t,y)dt$ for every y satisfying $\int_{\mathbb{R}} \big|[\partial_1 v](t,y)\big| dt < +\infty$. Deduce from this that $v = v^*$ almost everywhere and that on almost all lines parallel to Ox, the function u is almost everywhere derivable with $[\partial_1 u] = \partial_1 u$ almost everywhere.

(2) Let $u \in L^1_{\mathrm{loc}}(\Omega)$ be absolutely continuous on almost all lines parallel to Ox and such that its derivative almost everywhere $\partial_{x_1} u$ is an element of $L^p(\Omega)$. Show that $[\partial_{x_1} u] = \partial_{x_1} u$ almost everywhere.

(3) Let $u \in W^{1,1}(\Omega)$. Suppose that $[x, x+h] \in \Omega$. Show that the derivative of $v : t \mapsto u(x+th)$ exists almost everywhere on $]0,1[$ and that $dv/dt\,(x+th) = h \cdot \nabla u(x+th)$.

Hints. For (2), it suffices to compute $\int_\Omega \varphi \partial_{x_1} u dx$ by integrating by parts.

For (3), use the decomposition of $v(t') - v(t)$ as a sum of differences of the type $u(x+t'h) - u(x_1+t'h_1, x_2+t'h_2, \dots, x_{N-1}+t'h_{N-1}, x_N+th_N)$. Write each of these differences as the integral of a partial derivative over some interval. Taking the limit uses the continuity of a Lebesgue integral with respect to its bounds.

Exercise [$*$] 2.4 (On the $(1,p)$-Extension in the Case of an Interval in $\mathbb{R}$).
Let $u \in W^{1,p}(]0,+\infty[)$. We extend u to $]-\infty,0[$ by setting $\widetilde{u}(x) = u(-x)$. Prove that this extension of u is an element of $W^{1,p}(\mathbb{R})$. Let $u \in W^{1,p}(I)$ where $I =]a,b[$. Prove that we can extend u to an element of $W^{1,p}(\mathbb{R})$.

Hints. First establish that $\widetilde{u} \in W^{1,p}(]-\infty,0[)$ by showing that $(\widetilde{u})' = -\widetilde{u'}$.

Exercise 2.5 (Product of Functions in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$).
Consider a Lipschitz open subset Ω of $\mathbb{R}^N$. Let $p < N$, let $q < N$ and let $1/s = 1/p + 1/q - 1/N$. Show that if $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$, then $uv \in W^{1,s}(\Omega)$.

Hints. Use the Sobolev theorem 2.31 with suitable exponents and Hölder's inequality.

Exercise 2.6 (Example of a Non-Lipschitz Open Set).
Let $\Omega = \{0 < x < 1,\ 0 < y < x^4\}$. Prove that the function $x \mapsto x^{-1}$ is an element of $H^1(\Omega)$ but not an element of $L^5(\Omega)$. Conclude.

Exercise [$*$] 2.7 (Injection into a Non-Compact Space of Hölder Functions).
Let $p > N$. Show that the injection of $W^{1,p}(B(0,1))$ into $\mathcal{C}_b^{0,1-N/p}(B(0,1))$ is not compact, as follows.

Let $F \in \mathcal{D}(B(0,1))$ satisfy $F \geqslant 0$ and $\sup_{|x|<1} F(x) = 1$. Show that the sequence $F_n(x) = n^{-1+N/p} F(nx)$ tends to 0 in all the spaces $\mathcal{C}_b^{0,\lambda}(B(0,1))$ and has a constant norm equal to 1 in $\mathcal{C}_b^{0,1-N/p}$. Conclude.

Exercise 2.8 (Gluing Two Functions over a Straight Edge).
Let γ^- be the trace operator defined in the same manner as over $(\mathbb{R}^N)^+$ but using the open set $\mathbb{R}^{N-1} \times \mathbb{R}^-$. Let $u^+ \in W^{1,p}((\mathbb{R}^N)^+)$ and let $u^- \in W^{1,p}((\mathbb{R}^N)^-)$. We set

$$\widetilde{u} = \begin{cases} u^+(x) & \text{if } x \in (\mathbb{R}^N)^+, \\ u^-(x) & \text{if } x \in (\mathbb{R}^N)^-. \end{cases}$$

Prove that $\widetilde{u} \in W^{1,p}(\mathbb{R}^N)$ if and only if $\gamma_0 u^+ = \gamma^- u^-$ on $\mathbb{R}^{N-1}$.

Exercise 2.9 (Generalized Poincaré Inequality).
Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$. Let $p \in [1,+\infty[$ and let $\mathcal{N}$ be a continuous seminorm on $W^{1,p}(\Omega)$; that is, a norm on the constant functions.

Show that there exists a constant $C > 0$ that depends only on Ω, N, p, such that

$$\|u\|_{W^{1,p}(\Omega)} \leqslant C\Big(\Big(\int_\Omega |\nabla u(x)|^p dx\Big)^{1/p} + \mathcal{N}(u)\Big).$$

Apply this result to $\mathcal{N}(u) = \int_{\Gamma_0} |u(x)| dx$, when Ω is a $\mathcal{C}^1$ open set and Γ_0 is a subset of $\partial\Omega$ with positive $(N-1)$-dimensional Lebesgue measure.

Hints. Prove the result by contradiction. Assume that there exists a sequence $\{u_n\}$ such that

$$\|u_n\|_{W^{1,p}(\Omega)} \geqslant n\Big(\Big(\int_\Omega |\nabla u_n|^p\Big)^{1/p} + \mathcal{N}(u_n)\Big).$$

Normalizing, that is, considering $w_n = u_n(\|u_n\|_{W^{1,p}(\Omega)})^{-1}$, gives

$$\|w_n\|_{W^{1,p}(\Omega)} = 1, \quad \mathcal{N}(w_n) \longrightarrow 0, \quad \|\nabla w_n\|_p \longrightarrow 0.$$

Use the boundedness of Ω and the relative compactness of $\{w_n\}$ in L^p to deduce a contradiction.

Exercise 2.10 (Function from Ω to $\mathbb{R}^N$ Whose Deformation Tensor is an Element of $L^p(\Omega)$).

Consider the space

$$X_p(\Omega) = \big\{u \in L^p(\Omega, \mathbb{R}^N) \mid \\ \forall\, (i,j) \in [1,N]^2,\ \varepsilon_{ij}(u) = \tfrac{1}{2}(\partial_j u_i + \partial_i u_j) \in L^p(\Omega)\big\}$$

where $p \in \,]1, +\infty[$ (cf. Chapter 6). For the moment, we admit that if Ω is a bounded Lipschitz open subset of $\mathbb{R}^N$, then $W^{1,p}(\Omega, \mathbb{R}^N)$ coincides with the space above when $p > 1$. More precisely, there exists a $C > 0$ such that for every $u \in W^{1,p}(\Omega, \mathbb{R}^N)$,

$$\|u\|_{W^{1,p}(\Omega)} \leqslant C\Big(\int_\Omega |u|^p dx + \int_\Omega \sum_{ij} |\varepsilon_{ij}(u)|^p dx\Big)^{1/p}.$$

We will show this in Chapter 7.

(1) Show that $X_p(\Omega)$ endowed with the norm

$$|u|_{X_p} = \Big[\int_\Omega |u(x)|^p dx + \int_\Omega \sum_{ij} |\varepsilon_{ij}(u)(x)|^p dx\Big]^{1/p},$$

is a Banach space.

(2) Taking the derivatives in the sense of distributions, note that $u_{i,jk} = \partial_k(\varepsilon_{ij})(u) + \partial_j(\varepsilon_{ik})(u) - \partial_i(\varepsilon_{jk})(u)$. Show that the set $\mathcal{R}$ of the $W^{1,p}$ functions satisfying $\varepsilon(u) = 0$ consists of the rigid displacements, that is, the functions of the form $u = A + B(x)$, where A is a constant vector and B is an antisymmetric matrix. Determine the dimension of $\mathcal{R}$.

(3) Consider a seminorm $\mathcal{N}$ on $W^{1,p}$ that is a norm on the rigid displacements. Show that there exists a constant $C > 0$ such that

$$\forall\, u \in W^{1,p}(\Omega), \quad \|u\|_{W^{1,p}(\Omega)} \leqslant C\Big[\mathcal{N}(u) + \Big(\int_\Omega |\varepsilon(u)(x)|^p dx\Big)^{1/p}\Big].$$

Exercise 2.11 (Best Constant for the Injection of $W^{1,p}(\mathbb{R}^N)$ in $L^k(\mathbb{R}^N)$).
Let $p < N$ and let $k \leqslant Np/(N-p)$. We know that there exist two constants C_1 and C_2 such that

$$\forall\, u \in W^{1,p}(\mathbb{R}^N), \quad \|u\|_k \leqslant C_1\|\nabla u\|_p + C_2\|u\|_p.$$

We say that C_1 is the best constant for the injection of $W^{1,p}$ in L^k if C_1 is the smallest constant for which there exists a C_2 satisfying the inequality above. Prove that if $k < Np/(N-p)$, then there does not exist any best constant.

Hints. Assume that C_1 exists and define, for $\lambda > 1$, the sequence $u_\lambda(x) = u(x/\lambda)$. Prove that

$$\|u_\lambda\|_k \leqslant \lambda^{-1+N/p-N/k} C_1\|\nabla u\|_p + C_2\lambda^{-N/k+N/p}\|u\|_p.$$

Use this to prove that there exists a constant that is better than C_1.

Exercise 2.12 (Function with One Derivative in L^1 and the Other in L^2).
Let $X_0^{1,2}$ be the closure of the $\mathcal{D}(\mathbb{R}^2)$ functions for the norm $|\partial_1 u|_1 + |\partial_2 u|_2$. Show that $X_0^{1,2} \hookrightarrow L^4(\mathbb{R}^2)$.

Hints. For a regular function u, write

$$u^4(x_1, x_2) = u^3(x_1, x_2)u(x_1, x_2).$$

Next, use that

$$\begin{aligned}|u^3(x_1, x_2)| &\leqslant 3\int_{\mathbb{R}} u^2(x_1, t)|\partial_2 u|(x_1, t)dt \\ &\leqslant 3\Big(\int_{\mathbb{R}} |u|^4(x_1, t)dt\Big)^{1/2}\Big(\int_{\mathbb{R}} |\partial_2 u|^2(x_1, t)dt\Big)^{1/2} \\ &= \varphi(x_1)\psi(x_1)\end{aligned}$$

and

$$|u(x_1, x_2)| \leqslant \int_{\mathbb{R}} |\partial_1 u(t, x_2)|dt = h(x_2).$$

Finally, use Fubini's formula and Hölder's inequality as follows:

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}} |u|^4 dx_1 dx_2 &\leqslant \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x_1)\psi(x_1)h(x_2)dx_1 dx_2 \\ &\leqslant \|\varphi\|_2\|\psi\|_2\|h\|_1 \\ &\leqslant 3\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} u^4 dx_1 dx_2\Big)^{1/2}\|\partial_2 u\|_2\|\partial_1 u\|_1.\end{aligned}$$

Conclude.

Exercise [∗] 2.13 (Upper Bound for an Element u of $W_0^{1,1}$ on an Interval).
Let $u \in W_0^{1,1}(]0,1[)$. Prove that $\|u\|_\infty \leqslant 1/2\|u'\|_1$ and that this inequality is the best possible.

Hints. Write

$$u(x) = \int_0^x u'(t)dt \quad \text{and} \quad u(x) = -\int_x^1 u'(t)dt.$$

Exercise [∗] 2.14 (Consequences of the Existence of $\gamma_0(u)$ for u Defined over an Interval in $\mathbb{R}$).
Show the following inequality, which specifies the continuity of the trace map on $W^{1,1}(]0,1[)$:

$$\forall\, u \in W^{1,1}(]0,1[), \quad |u(0)| + |u(1)| \leqslant \int_0^1 |u'|(t)dt + 2\int_0^1 |u(t)|dt. \tag{2.94}$$

Show that the only functions that satisfy the equality are the constant functions.

Hints. Since the function u is absolutely continuous, we have the equalities

$$\forall\, x \in [0,1], \quad u(x) = u(0) + \int_0^x u'(t)dt,$$
$$\forall\, x \in [0,1], \quad u(x) = u(1) + \int_1^x u'(t)dt.$$

Taking the absolute values and integrating the sum of the two resulting inequalities over $]0,1[$ gives (2.94).

Assuming equality in (2.94) and taking into account the inequalities

$$|u(0)| \leqslant |u(x)| + \int_0^x |u'(t)|dt \quad \text{and} \quad |u(1)| \leqslant |u(x)| + \int_x^1 |u'(t)|dt,$$

deduce that for every x, $|u(x)| \geqslant \int_0^1 |u(t)|dt$. Applying this inequality to a point x where the continuous function u reaches its minimum gives the desired result.

Exercise [∗∗] 2.15 (The Spaces $W^{1,p}(I)$ for an Interval I in $\mathbb{R}$).
Let $1 \leqslant p < \infty$.

(1) Using Exercise 1.29, show that $u \in W^{1,p}(I)$ if and only if $u \in L^p(I)$, u is absolutely continuous, and the derivative almost everywhere satisfies $u' \in L^p(I)$.
(2) Show that every function in $W^{1,p}(I)$ can be extended to a continuous function on $\overline{I}$.
(3) In this question, we will use that $W^{1,p}(\mathbb{R}) = W_0^{1,p}(\mathbb{R})$. Let u be a $\mathcal{C}^1$ function on $\mathbb{R}$ with compact support. Let $v = |u|^{p-1}u$. Show that v is $\mathcal{C}^1$

with compact support and that $v' = p|u|^{p-1}u'$. Use the equality $v(x) = \int_{-\infty}^{x} v'(t)dt$ to show that there exists a constant C such that

$$\forall\, x \in \mathbb{R}, \quad |u(x)| \leqslant C\|u\|_{W^{1,p}(\mathbb{R})}.$$

Deduce that $W^{1,p}(\mathbb{R})$ is embedded in $L^\infty(\mathbb{R})$. Show that the constant C can be chosen independently of p. Show that the result still holds true when the interval I is bounded.

Hints.

(1) If $u \in W^{1,p}(I)$, then Exercise 1.29 gives the desired properties. Conversely, use integration by parts to prove that

$$\forall\, \varphi, \quad \langle [u]', \varphi\rangle = \langle [u'], \varphi\rangle.$$

(2) Since u' is summable over I, u is absolutely continuous over $\overline{I}$, giving the continuity on $\overline{I}$.

(3) Starting with the given hint, use Hölder's inequality to determine the upper bound $p^{1/p}\|u\|_p^{1/p'}\|u'\|_p^{1/p}$ for $|u(x)|$, giving the result by using $p^{1/p} \leqslant e$ and Jensen's inequality

$$|u|_p^{1/p'}|u'|_p^{1/p} \leqslant \frac{1}{p}|u|_p + \frac{1}{p'}|u'|_p.$$

This leads to the density of the continuous functions with compact support. When I is bounded, use

$$u(x) = u(x_0) + \int_{x_0}^{t} u'(t)dt.$$

Exercise [$**$] **2.16 (Solving Limit Problems on an Interval).**
Let $I =]0,1[$. Given $f \in L^2(I)$, we wish to find a u that, in some sense, is a solution of

$$(*) \qquad \begin{cases} -u'' + u = f, \\ u(0) = u(1). \end{cases}$$

(1) Assume that $u \in \mathcal{C}^2(\overline{I}) \cap H_0^1(I)$ satisfies $(*)$. We multiply $(*)$ by a function $v \in H_0^1(I)$ and integrate by parts over I. Prove that if $(\cdot|\cdot)$ denotes the inner product on $H_0^1(I)$, then

$$\forall\, v \in H_0^1(\Omega), \quad (u|v)_{H^1(I)} = \langle f, v\rangle_{L^2(I)}.$$

Conversely, prove that if $u \in H_0^1(I)$ satisfies this relation, then u is a solution to the problem, where u'' is taken in the sense of distributions. Next, prove that $v \mapsto \int_I f(t)v(t)dt$ defines an element of the dual of $H_0^1(I)$ and deduce the existence and uniqueness of a solution of the given problem in $H_0^1(I)$ (use the Riesz representation theorem for a Hilbert space). Prove that this solution is in $H^2(I)$ and that if $f \in \mathcal{C}(I)$, then the solution is in $\mathcal{C}^2(I)$.

(2) Use, for example, the fundamental solution of $u'' - u = 0$ on $\mathbb{R}^+$ or variation of the constants to determine this solution explicitly using integrals pertaining to the function f.

Exercise 2.17 (Relation Between $\|\nabla u\|_{L^2}$ and $\|u/r\|_{L^2}$).

(1) Let $u \in \mathcal{C}_c(\mathbb{R}^N)$ with $N \geqslant 3$. By computing

$$\Big|\nabla u + \frac{(N-2)}{2}\frac{u(x)\,\overrightarrow{x}}{r^2}\Big|^2,$$

integrating over $\mathbb{R}^N$, and integrating

$$\int_{\mathbb{R}^N} 2u\,\partial_i u\,\frac{x_i}{r^2}\,dx = \int_{\mathbb{R}^N} \partial_i(u^2)\,\frac{x_i}{r^2}\,dx,$$

by parts, show that

$$\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \geqslant \frac{(N-2)^2}{4}\int_{\mathbb{R}^N} \frac{u^2}{r^2} dx.$$

(2) Deduce that if $N \geqslant 3$, we have $u \in H^1 \Rightarrow u/|x| \in L^2$. Show that this result does not hold for $N = 2$.

Exercise 2.18 (Generalization of the Previous Exercise).

(1) Show that if $u \in W^{1,p}(\mathbb{R}^N)$, $N > p$, and $1 < p < \infty$, then $u/|x| \in L^p$. In order to do this, show Jensen's inequality (where $1/p + 1/p' = 1$):

$$\forall\, X, Y \in \mathbb{R}^N, \quad X \cdot Y \leqslant \frac{1}{p}|Y|^p + \frac{1}{p'}|X|^{p'}.$$

(2) Apply this inequality to the vectors $Y = \nabla u$ and

$$X = \Big|\frac{(N-p)u\,\overrightarrow{x}}{pr^2}\Big|^{p-2}\frac{(p-N)u\,\overrightarrow{x}}{pr^2} = -\Big(\frac{N-p}{p}\Big)^{p-1}\frac{|u|^{p-2}u\,\overrightarrow{x}}{r^p}.$$

Integrating the term $\int_{\mathbb{R}^N} |u|^{p-2}u\,\overrightarrow{x}/r^p \cdot \nabla u dx$ by parts, deduce that

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geqslant \Big(\frac{N-p}{p}\Big)^p \int_{\mathbb{R}^N} \Big(\frac{u}{r}\Big)^p dx.$$

Hints. For Jensen's inequality, use $f(x) = |x|^p$, which has derivative $p|x|^{p-2}x$, giving the inequality $f(x+y) \geqslant f(x) + Df(x)\cdot y$.

Exercise $[**]$ **2.19 (Fundamental Solutions of the Laplacian).**
Show that there exists a constant k_2 such that $\Delta(\ln\sqrt{x^2+y^2}) = k_2\delta_0$ in $\mathbb{R}^2$ in the sense of distributions. Show that in $\mathbb{R}^N$ with $N > 2$, $\Delta(r^{2-N}) = k_N\delta_0$, where k_N can be expressed using the area ω_{N-1} of the unit sphere in $\mathbb{R}^N$. Use elementary computations of integrals in the cases $N = 2$ and $N = 3$. For the general case, use Green's second theorem.

Hints.

(1) First show that as functions and outside of the origin, we have $\partial_x[\ln r] = x/r^2$ and $\partial_y[\ln r] = y/r^2$. Next, show that these functions are locally summable. Finally, use the function $\widetilde{\varphi}(r,\theta) = \varphi(r\cos\theta, r\sin\theta)$, the formula

$$\partial_x\varphi = \cos\theta\,\partial_r\widetilde{\varphi} - \frac{\sin\theta}{r}\,\partial_\theta\widetilde{\varphi},$$

and the analogous formula for φ_y to deduce that

$$\Big\langle \partial_x \frac{x}{r^2} + \partial_y \frac{y}{r^2}, \varphi \Big\rangle = 2\pi\varphi(0).$$

(2) Assume that $N = 3$. Show that the three derivatives of $u = r^{-1}$ are locally summable and deduce from this that

$$\langle \Delta u, \varphi\rangle = -\int_{\mathbb{R}^3} r^{-3}\big[x\partial_x\varphi + y\partial_y\varphi + z\partial_z\varphi\big]\,dx\,dy\,dz.$$

The polar coordinates are defined by

$$x = r\cos\xi\cos\eta, \quad y = r\cos\xi\sin\eta, \quad z = r\sin\eta.$$

Compute the partial derivatives using those of $\widetilde{\varphi}$ with respect to r, ξ and η, and show that the previous integral is equal to $-4\pi\varphi(0)$.

(3) We admit Green's second theorem: given a bounded open set Ω of class $\mathcal{C}^1$, a $\mathcal{C}^2$ function f, and $\varphi \in \mathcal{D}(\mathbb{R}^N)$, we have

$$\int_\Omega \big[f(x)\Delta\varphi(x) - \varphi(x)\Delta f(x)\big]\,dx = \int_{\partial\Omega} \big[f(x)\partial_{\vec{n}}\varphi(x) - \varphi(x)\partial_{\vec{n}} f(x)\big]\,d\sigma, \tag{2.95}$$

where the normal derivative $\partial_{\vec{n}}$ on $\partial\Omega$ is oriented outward from Ω. Deduce from this that when $\varepsilon \to 0$, $\langle \Delta(r^{2-N}), \varphi\rangle$ is the limit of

$$\int_{r\geqslant\varepsilon} \varphi\Delta(r^{2-N})dx + \int_{r=\varepsilon} \big[\varphi(x)\partial_{\vec{n}}(r^{2-N}) - r^{2-N}\partial_{\vec{n}}(\varphi)\big]\varepsilon^{N-1}d\sigma.$$

Use this to prove that $\Delta(r^{2-N}) = (2-N)\omega_{N-1}\delta_{(0)}$.

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