Preface

The aim of this work is to present a tool for students interested in partial differential equations, both those working toward a Master's degree in pure or applied mathematics and those with PhD research in this field. It gathers results from functional analysis that make it easier to understand the nature and properties of the functions occurring in these equations, as well as the constraints they must obey to qualify as solutions. We present modern resolution methods for a class of such problems and interpret the solutions we obtain by studying their regularity.

Let us recall that the domain in which we study a partial differential equation is an open subset Ω of $\mathbb{R}^N$. The equation is a relation that an unknown function u and its partial derivatives (cf. the preliminary chapter) must satisfy. Moreover, we impose certain conditions on the function u and possibly on some of its derivatives (see the Dirichlet and Neumann problems in the preliminary chapter), namely that they equal given functions on the boundary $\partial\Omega$ of the open set under consideration. These relations are called *boundary conditions*.

Looking for such a function is the aim of a so-called *boundary problem*. We find many examples of these in physics.

If we consider the derivatives in the usual sense in the interior of the open set, classical analysis proves to be ineffective for solving such problems, as can be illustrated with examples. Indeed, the *solutions* obtained in these examples sometimes do not belong to the spaces of differentiable functions in the classical sense because of their *irregularity*. Moreover, we can find examples in physics where the right-hand side f of the given equation has discontinuities.

Let us consider the simple example in $\mathbb{R}$ of the differential equation

$$y'' + y' + y = f,$$

where f is discontinuous at the point $t = 0$. Any solution cannot be $\mathcal{C}^2$ on $\mathbb{R}$. We can, however, look for a solution of class $\mathcal{C}^1$ with derivative y'' almost everywhere, or such that y'' is a derivative of y' in the sense of distributions. Assuming that f is even more irregular, but can be considered as a distribution that we denote by $[f]$, we are led to look for solutions that are distributions $[u]$. In this case, for every infinitely differentiable function φ with compact support in $\mathbb{R}$, we have $\langle [u], \varphi'' - \varphi' + \varphi \rangle = \langle [f], \varphi \rangle$. These solutions, which we can also consider when f is regular, are also called weak solutions of the equation.

All of this leads, by replacing the usual differentiability with that in the sense of distributions, to the concept of weak solutions for general PDEs and leads us to study certain spaces of functions whose distributional derivatives can be identified with summable pth power functions. We therefore study Sobolev spaces $W^{m,p}(\Omega)$, which are normed and complete, so that the classical theorems from functional analysis apply to them.

When there are boundary conditions, the functions in these spaces need to be extended to the boundary of Ω, since they are only defined in its interior. The existence of such extensions depends *a priori* on the regularity of the boundary. We therefore in particular study the space $W^{m,p}(\Omega)$ when the boundary of the open set Ω is a manifold that is either differentiable or piecewise differentiable. This allows us to give, for the functions in these spaces, an interpretation of the boundary conditions that is in accordance with physics.

Consequently, in many situations, the great flexibility of differentiation in the sense of distributions leads us to state limit problems under equivalent forms that are better suited to establishing existence and uniqueness theorems.

Of course, the results we obtain necessitate preliminaries. These concern the functional spaces that we can use, in particular, normed spaces, completeness, density, and the generalization of the notion of function and integration. The aim of Chapter 1 is to describe these.

Contents of this Book

Chapter 1 is titled *Notions from Topology and Functional Analysis.* In it, we first recall the definition of topological vector spaces, including the important example of normed spaces, and in particular Banach spaces. We state the Baire theorem, the open image theorem, the Banach–Steinhaus theorem and the Hahn–Banach theorem. After defining continuous linear maps, we introduce dual topology on a normed space. To illustrate the different types of convergence of sequences of functions that are most common, which are less strict than (for example) uniform convergence, we introduce weak topologies on a space and on its dual. We also define reflexive spaces, in particular Hilbert spaces, and uniform convex spaces, whose properties we use in many examples in this book. We study the space of continuous functions on an open subset of $\mathbb{R}^N$ before recalling the definitions of distribution spaces, their topologies and the operators that we define on them, as well as convergence properties of sequences. The chapter concludes with the spaces $L^p(\Omega)$, their completion and reflexivity, and the density of the regular functions.

This last part of the chapter thus forms an introduction to the Sobolev spaces that we study in later chapters.

Chapter 2 concerns these *Sobolev spaces*, which give a suitable functional setting for most of the elliptic limit problems (cf. the preliminary chapter) from physics. An important part of this chapter deals with Sobolev embedding theorems. We first present the notion of the differentiation of functions in the weak, or generalized sense, that is, differentiation in the sense of distributions. After introducing the spaces L^p, this allows us to define the Sobolev spaces $W^{m,p}(\Omega)$. The properties of $L^p(\Omega)$ lead to density results for the regular functions in the spaces $W^{m,p}(\Omega)$. The most important result of the chapter is the Sobolev embedding theorem, which gives the inclusion of the elements of $W^{m,p}(\Omega)$ in $L^q(\Omega)$ for $q > p$, or in spaces of continuous Lipschitz or Hölder functions. Some of these embeddings are compact. These compactness results, which hold for bounded open sets, form a key argument for showing the existence of solutions of coercive minimization problems (cf. Chapter 5). In the second part of the chapter, we study possible extensions of functions in $W^{m,p}(\Omega)$ to elements of $W^{m,p}(\mathbb{R}^N)$, for which we need regularity conditions on the boundary $\partial\Omega$. At this point, we define the Lipschitz open sets and the open sets of class $\mathcal{C}^m$. The chapter concludes with a trace theorem that allows us, on such open sets, to extend $u \in W^{1,p}(\Omega)$ to the boundary, giving a function in $L^p(\partial\Omega)$. This generalizes the restriction to $\partial\Omega$ for functions that are in principle only defined in the open set Ω. This theorem is very useful when stating boundary conditions for a limit problem.

Chapter 3 deals with the image of the trace map on $W^{1,p}(\Omega)$ when the open set is regular. This is our first example of a fractional Sobolev space, namely $W^{1-1/p,p}(\partial\Omega)$. The chapter also contains Green's formulas and embedding theorems. These can be deduced from the embedding results on Sobolev spaces with integer exponents that they generalize.

Chapter 4 deals with more general fractional spaces $W^{s,p}(\Omega)$ (for s a noninteger real number). It also contains embedding and compact embedding results.

In Chapter 5, we use all the theory presented up to now to prove the existence of solutions of elliptic PDEs. There are, however, two exceptions, namely minimal surfaces and linear elasticity in the case of small deformations. For the first, the theoretical justifications from functions of a measure are given in the following chapter. The second necessitates the use of Korn's inequalities, which form the main subject of Chapter 7. In many situations, the existence theorems concerning these elliptic PDEs result from rewriting these limit problems in a variational form. The solutions then appear as functions minimizing a convex and coercive functional. Next, we study the regularity of the solutions of some of these problems, using for example approximations of the derivative by finite difference or *a priori* estimation methods. We conclude the chapter with properties characterizing these PDEs, namely the maximum principle in its weak form followed by its strong form.

In Chapter 6, we study spaces related to the Sobolev spaces, in particular the space of distributions whose derivative tensor, which is symmetric and is also called the *deformation tensor*, is in $L^p(\Omega)$ for $p \in [1, \infty[$. We also study the case $p = 1$ and the spaces where the deformation is a bounded measure. In particular, we give embedding theorems analogous to those for the classical Sobolev spaces, as well as existence results for a trace on the boundary when the open set is sufficiently regular. We conclude with a section devoted to functions of a measure.

In the setting of harmonic analysis, the results of Chapter 7 lead up to a proof of Korn's inequalities in $W^{1,p}$.

We conclude the book with an appendix concerning the regularity of the solutions of the p-Laplacian problems. As a complement to Chapter 5, we establish more technical results that we obtain using *a priori* estimation methods.

Organization of the Book

Each chapter is followed by a number of exercises. In most cases we give hints for the solution. The level of the exercises varies. Some of them, indicated with a [$*$], offer additional details to a result given in the chapter, an application of the results with explicit computations to illustrate them, or a different proof for such a result. Other exercises, indicated with a [$**$], offer complements to a given subject. In some cases, the results are presented in dimension $N = 1$ or $N = 2$, where the nature of the problems and the specifics of the proposed methods can be highlighted. In these small dimensions, the methods may lead to explicit computations that can help the reader better understand the notions that are being studied.

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