Chapter 2
Distributed-Order Linear Time-Invariant System (DOLTIS) and Its Stability Analysis

2.1 Introduction

By using distributed-order concept, we can describe the dynamical properties of real world system more accurately, so distributed-order system identification problem was studied in Hartley and Lorenzo (1999, 2003, 2004). In the following sections, the stability analysis of distributed-order linear time-invariant systems in four cases are first studied, then the frequency-domain responses are presented, and time-domain responses on the basis of numerical inverse Laplace transform technique are shown in details.

2.2 Stability Analysis of DOLTIS in Four Cases

Consider a distributed-order system described by the following linear time-invariant (LTI) distributed-order differential equation (DODE) and algebraic output equation:

$$
\begin{aligned}
{}_0\mathrm{D}_t^{w(\alpha)} x(t) = \int_0^1 w(\alpha)\, {}_0\mathrm{D}_t^{\alpha} x(t)\mathrm{d}\alpha &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t)
\end{aligned} \tag{2.1}
$$

where $w(\alpha)$ is the function of distribution of order $\alpha \in [0, 1]$, ${}_0\mathrm{D}_t^{\alpha}$ denotes the Caputo fractional-order derivative operator, A, B, C, D are matrices with appropriate dimensions.

Remark 2.1 Since any interval (γ_1, γ_2) can be converted to $(0, 1)$ through variable substitution, without loss of generality, the integral interval in (2.1) is considered to be $(0, 1)$.

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For the distributed-order derivative operator $\mathrm{D}^{w(\alpha)}x(t)$, the Laplace transform is

$$L\left\{\mathrm{D}^{w(\alpha)}x(t)\right\}(s)=\tilde{x}(s)\int_0^1 w(\alpha)s^{\alpha}\mathrm{d}\alpha-x(0)\frac{1}{s}\int_0^1 w(\alpha)s^{\alpha}\mathrm{d}\alpha,\ \ s\in \mathrm{C}\backslash\left(-\infty,0\right]$$

where $\tilde{x}(s)=L\left\{x(t)\right\}(s):=\int_0^{\infty}x(t)e^{-st}\mathrm{d}t$. By applying the Laplace transform to (2.1) with the assumptions that $x(0)=0$, $u(t)=\delta(t)$ ($\delta(t)$ is the Dirac delta distribution), one obtains

$$\tilde{x}(s)\int_0^1 w(\alpha)s^{\alpha}\mathrm{d}\alpha=A\tilde{x}(s)+B$$

i.e.,

$$\tilde{x}(s)=\left(\left(\int_0^1 w(\alpha)s^{\alpha}\mathrm{d}\alpha\right)I-A\right)^{-1}B$$

where I is the identity matrix. Application of the inverse Laplace transform to the previous expression yields

$$x(t)=L^{-1}\left[\left(\left(\int_0^1 w(\alpha)s^{\alpha}\mathrm{d}\alpha\right)I-A\right)^{-1}B\right](t),\ \ t>0. \tag{2.2}$$

In the following, four different cases of the weighting function of order are discussed respectively.

Case 1 $w(\alpha)=1$

In this case, it can be followed by (2.2) that

$$\begin{aligned}x(t)&=L^{-1}\left[\left(\left(\int_0^1 s^{\alpha}\mathrm{d}\alpha\right)I-A\right)^{-1}B\right](t)\\&=L^{-1}\left[\left(\frac{s-1}{\ln s}I-A\right)^{-1}B\right](t)\\&=L^{-1}\left[\ln s(sI-(I+\ln sA))^{-1}B\right](t).\end{aligned} \tag{2.3}$$

Remark 2.2 It is well known from complex analysis (Asmar and Jones 2002) that complex logarithm $\ln z=\ln|z|+i\arg z$ ($z\neq 0$) defines a multiple-valued function, because $\arg z$ is multiple-valued. For term $\ln s$ in (2.3), we know that it is a multi-valued function of the complex variable s whose domain can be seen as a Riemann surface (Cuadrado and Cabanes 1989; Westerlund and Ekstam 1994) of a number of sheets which is infinite. Note that in multiple-valued functions only the first Riemann sheet has its physical significance (Gross and Braga 1961), so we can make $\ln s$ a single-valued function by specifying a single-valued $-\pi<\arg s<\pi$. Because

$s = 0$ and s on the negative real axis are nonremovable discontinuities, the branch cut of $\ln s$ is $(-\infty, 0]$.

Definition 2.1 A distributed-order system $H(s)$ defined by its impulse response $h(t) = L^{-1}\{H(s)\}$ is BIBO stable if and only if $\forall u \in L^{\infty}(R^{+})$, $h * u \in L^{\infty}(R^{+})$. $*$ stands for the convolution product and $L^{\infty}(R^{+})$ stands for the Lebesgue space of measurable function h such that $ess \sup\limits_{t\in R^{+}} |h(t)| < \infty$.

Based on Definition 2.1 and the above analysis, the following theorem can be established.

Theorem 2.1 *The distributed-order linear time-invariant system (2.1) with transfer function $G_1(s) = C \ln s(sI - (I + A\ln s))^{-1}B + D$ is BIBO stable, if and only if all the eigenvalues of A lie on the left of curve $l_1 := l_a \bigcup l_b$ in the complex plane, where l_a and l_b are symmetrical with respect to the real axis, and*

$$l_a := \left\{ x - iy \,\middle|\, x = \frac{2\pi\omega - 4\ln\omega}{4(\ln\omega)^2 + \pi^2},\, y = \frac{4\omega\ln\omega + 2\pi}{4(\ln\omega)^2 + \pi^2} \right\}$$

with $\omega \in [0, \infty)$.

Proof (if part) Note that the final value theorem implies that $\lim\limits_{t\to\infty} g(t) = sG_1(s) \to 0$, if all poles of $sG_1(s)$ are in the left half-plane when $s \to 0$. It can be easily known that all the poles of $sG_1(s)$ satisfy the transcendental characteristic equation of the form

$$|(s-1)I - A\ln s| = 0. \tag{2.4}$$

From (2.4) we know that $\frac{s-1}{\ln s} = \sigma_i(A)$ $(i = 1, \cdots, n)$, where $\sigma(A)$ denotes the set of eigenvalues of A. As all the zeros of (2.4) should lie in the left half-plane to ensure the BIBO stability of distributed-order system $G_1(s)$, it is necessary to derive the range of $\lambda = \frac{s-1}{\ln s}$ when s belongs to the left half-plane.

It is natural to determine the range of $\lambda = \frac{s-1}{\ln s}$ when s lies on the imaginary axis. Then, for $s = j\omega$, $(-\infty < \omega < 0)$, we have

$$\lambda = \frac{(2\pi(-\omega) - 4\ln(-\omega)) + j\,(4(-\omega)\ln(-\omega) + 2\pi)}{4(\ln(-\omega))^2 + \pi^2}$$

while for $s = j\omega$, $(0 \le \omega < \infty)$, we have

$$\lambda = \frac{(2\pi\omega - 4\ln\omega) - j\,(4\omega\ln\omega + 2\pi)}{4(\ln\omega)^2 + \pi^2}$$

which means that the imaginary axis is mapped to a curve denoted by l_1, which is symmetrical with respect to the real axis. By choosing a point s randomly which lies on the left of the imaginary axis, the range of $\lambda = \frac{s-1}{\ln s}$ lies on the left of curve l_1, which means that the stable region of distributed-order system (2.1) is the left region

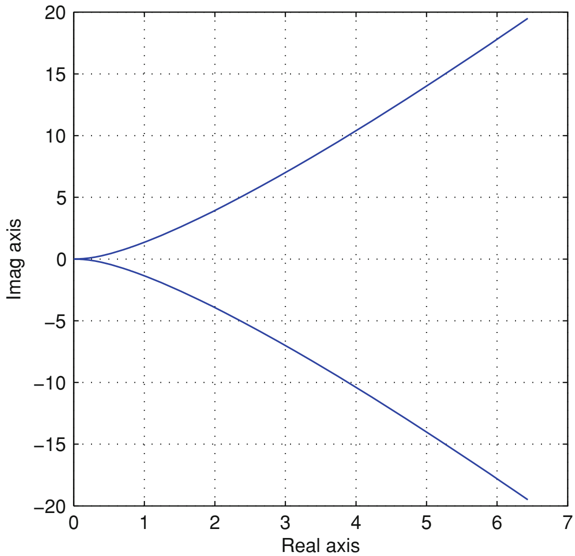


Fig. 2.1 The stable boundary of the distributed-order system (2.1) $G_1(s)$

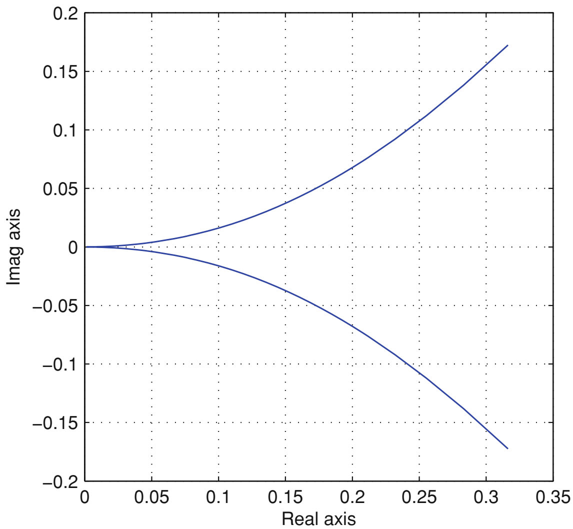


Fig. 2.2 The stable boundary of the distributed-order system (2.1) $G_1(s)$ (*Zoomed*)

of curve l_1. In the following, l_1 is plotted in Fig. 2.1, with the local property around 0 zoomed in Fig. 2.2.

It can be easily known from the above analysis that if all the eigenvalues of A lie on the left of curve l_1, all the poles of $sG_1(s)$ lie on the left half-plane. From the final value theorem, we further know that $\lim_{t\to\infty} g(t) = 0$, which means the distributed-order system with transfer function $G_1(s) = C \ln s(sI - (I + A \ln s))^{-1} B + D$ is BIBO stable.

(only if part) It is obviously known from Definition 2.1 that $G_1(s)$ lies in H_∞, the space of bounded analytic functions on the right half plane of the complex plane, which means that all the poles of $G_1(s)$ lie in the left half plane of the complex plane. From the proof of (if part), it is known that $\{s_k\}_{k=1,2,\cdots,n}$ lie in the open left half plane, which is equivalents to that all the eigenvalues of A lie in the left region with respect to l_1.

Remark 2.3 It is easy to conclude that the slope of the curve l_1 at the original point is 0, and is infinity at the infinite point, which means that any ray in the first quadrant starts at point 0 will have point of intersection with the curve l_1. This means any constant fractional-order approximation of DODS is problematic, since the stability domains are different.

Case 2 $w(\alpha) = \alpha$

In this case, the following can be obtained under the similar analysis procedure in Case 1,

$$\begin{aligned} x(t) &= L^{-1}\left[\left(\left(\int_0^1 \alpha s^\alpha \mathrm{d}\alpha\right) I - A\right)^{-1} B\right] \\ &= L^{-1}\left[\left(\frac{1 - s + s\ln s}{\ln^2 s} I - A\right)^{-1} B\right] \\ &= L^{-1}\left[\ln^2 s\Big((1 - s + s\ln s)\, I - \ln^2 s A\Big)^{-1} B\right]. \end{aligned}$$

Theorem 2.2 *The distributed-order linear time-invariant system (2.1) with transfer function* $G_2(s) = C\ln^2 s\big((1 - s + s\ln s)I - A\ln^2 s\big)^{-1} B + D$ *is BIBO stable, if and only if all the eigenvalues of* A *lie on the left of curve* $l_2 := l_c \bigcup l_d$*, where* l_c *and* l_d *are symmetrical with respect to the real axis, and* $l_c := \{x + iy \,|x = x_\omega, y = y_\omega, \omega \in (0, \infty)\}$*, with notations*

$$x_\omega = \frac{\left(\ln\omega - \frac{\pi^2}{4}\right)\left(1 - \frac{\pi}{2}\omega\right) - \pi\ln\omega\,(\omega - \omega\ln\omega)}{\left(\ln^2\omega + \frac{\pi^2}{4}\right)^2}$$

and

$$y_\omega = \frac{\left(\ln\omega - \frac{\pi^2}{4}\right)(\omega - \omega\ln\omega) + \pi\ln\omega\left(1 - \frac{\pi}{2}\omega\right)}{\left(\ln^2\omega + \frac{\pi^2}{4}\right)^2}.$$

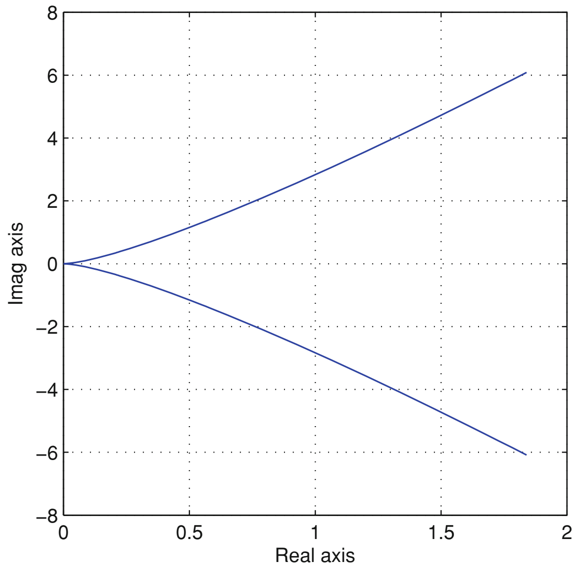


Fig. 2.3 The stable boundary of distributed-order system (2.1) $G_2(s)$

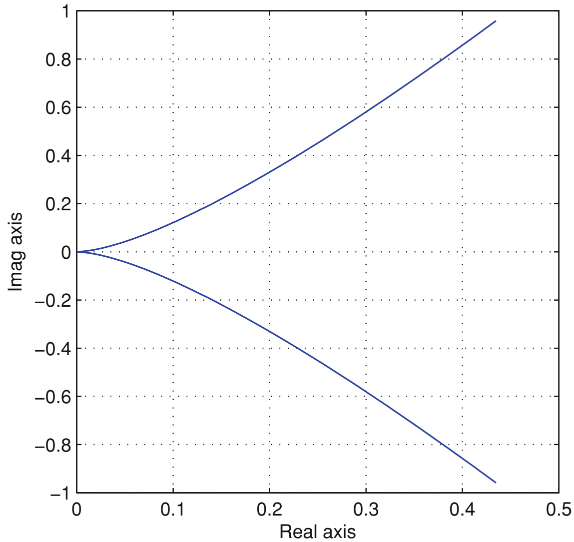


Fig. 2.4 The stable boundary of distributed-order system (2.1) $G_2(s)$ (*Zoomed*)

The proof of Theorem 2.2 can be given by the similar procedures in Theorem 2.1, the stable boundary for distributed-order system $G_2(s)$ is shown in Fig. 2.3, with the local property around 0 shown in Fig. 2.4.

Case 3 $w(\alpha) = \delta(\alpha - \beta),\ (0 < \beta < 1)$

In this case, the DODE (2.1) converts to a constant-order fractional-order system described by

$$\begin{aligned} {}_0\mathrm{D}_t^{\beta} x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{2.5}$$

Using Laplace transform, the irrational transfer function of fractional-order system (2.5) with null initial conditions is

$$G_3(s) = C\left(s^{\beta} I - A\right)^{-1} B + D. \tag{2.6}$$

Remark 2.4 Note that term s^{β} in (2.6) defines a multi-valued function of the complex variable s whose domain can be seen as a Riemann surface (Cuadrado and Cabanes 1989; Westerlund and Ekstam 1994) of a number of sheets which is finite in the case of $\beta \in \mathrm{Q}^+$, and infinite in the case of $\beta \in \mathrm{R}^+ \backslash \mathrm{Q}^+$. It is well known that in multiple-valued functions only the principal sheet defined by $-\pi < \arg s < \pi$ has its physical significance (Gross and Braga 1961).

The following can be obtained under the similar analysis procedure in the previous cases,

$$\begin{aligned} x(t) &= L^{-1}\left[\left(\left(\int_0^1 \delta(\alpha - \beta) s^{\alpha} \mathrm{d}\alpha\right) I - A\right)^{-1} B\right](t) \\ &= L^{-1}\left[\left(s^{\beta} I - A\right)^{-1} B\right](t). \end{aligned}$$

The following theorem which corresponds to the stability condition of fractional-order system obtained in Matignon (1996) can be given.

Theorem 2.3 *The fractional-order linear time-invariant system with transfer function* $G_3(s) = C\left(s^{\beta} I - A\right)^{-1} B + D$ *is BIBO stable, if and only if all the eigenvalues of* A *lie on the left of curve* $l_3 := l_e \bigcup l_f$*, where* l_e *and* l_f *are symmetrical with respect to the real axis, and* $l_e := \left\{re^{i\theta} \,\middle|\, r = \omega^{\beta}, \theta = \pi\beta/2, \omega \in (0, \infty)\right\}$.

The proof of Theorem 2.3 can be given by the similar procedures in Theorem 2.1, the stable region for fractional-order system $G_3(s)$ with $\beta = 0.5$ is shown in Fig. 2.5.

Case 4 $w(\alpha) = \sum_{k=1}^{n} b_k \delta(\alpha - k\beta),\ (0 < n\beta < 1)$.

In this case, the DODE (2.1) converts to the so-called LTI commensurate fractional-order system

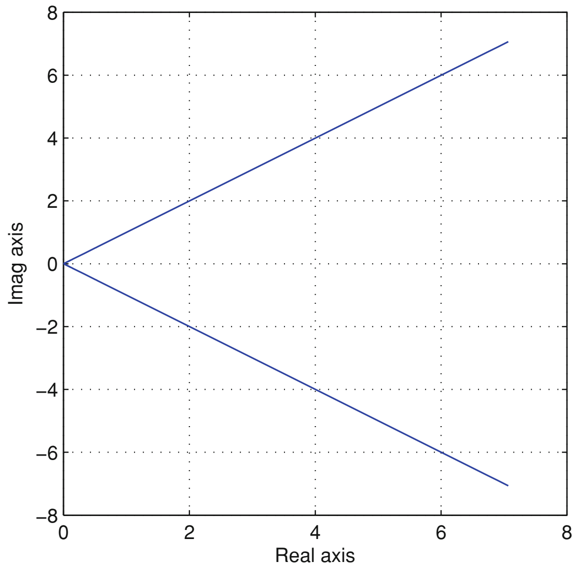


Fig. 2.5 The stable boundary of fractional-order system (2.5) $G_3(s)$

$$\sum_{k=1}^{n} b_{k0}\mathrm{D}_t^{k\beta}x(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t). \tag{2.7}$$

Let $\hat{x}(t) = \begin{bmatrix} x(t) & \mathrm{D}^{\beta}x(t) & \mathrm{D}^{2\beta}x(t)\cdots\mathrm{D}^{(n-1)\beta}x(t)\end{bmatrix}^T$, (2.7) can be converted to the following equivalent form

$$_0\mathrm{D}t^{\beta}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \tag{2.8}$$

where $\hat{A} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ \frac{A}{b_n} & -\frac{b_1}{b_n}I & -\frac{b_2}{b_n}I & \cdots & -\frac{b_{n-1}}{b_n}I \end{bmatrix}$, $\hat{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{B}{b_n} \end{bmatrix}$.

Now we have changed Case 4 to Case 3, which can be similarly analyzed.

The following can be obtained under the similar analysis procedure in the previous cases,

$$x(t) = L^{-1}\left[\left(\left(\int_0^1 \sum_{n=1}^{N} b_n \delta(\alpha - \beta_n) s^{\alpha} \mathrm{d}\alpha\right) I - A\right)^{-1} B\right]$$
$$= L^{-1}\left[\left(\sum_{n=1}^{N} b_n s^{\beta_n} I - A\right)^{-1} B\right].$$

In the following, Case 4 will not be considered.

2.3 Time-Domain Analysis: Impulse Responses

Case 1 $w(\alpha) = 1$

As the transfer function of distributed-order system for Case 1 with the assumption that $D = 0$ is $G_1(s) = C \ln s((s-1)I - A \ln s)^{-1} B$, using the similar method of impulse response for distributed-order integrator/differentiator in Li et al. (2010), the inverse Laplace transform of $G_1(s)$ can be derived as follows.

$$\begin{aligned} y_1(t) &= L^{-1}\{G_1(s)\} \\ &= C\left(\frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{-st} \ln s (sI - (I + \ln s A))^{-1} \mathrm{d}s\right) B \\ &= C\left(\int_0^{\infty} e^{-xt} (x+1) A_1^{-1} \mathrm{d}x\right) B \end{aligned} \tag{2.9}$$

where $A_1 := ((x+1)I + A \ln x)^2 + (A\pi)^2$.

Case 2 $w(\alpha) = \alpha$

Following the same procedures, the transfer function of distributed-order system for Case 2 with $D = 0$ is $G_2(s) = C\ln^2 s\big((s - 1 - \ln s)I - A\ln^2 s\big)^{-1} B$, using the similar method of impulse response for distributed-order integrator/differentiator in Li et al. (2010), the inverse Laplace transform of $G_2(s)$ can be derived as follows.

$$\begin{aligned} y_2(t) &= L^{-1}\{G_2(s)\} \\ &= C\left(\frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} \ln^2 s \Big((1 - s + s\ln s) I - \ln^2 s A\Big)^{-1} \mathrm{d}s\right) B \\ &= C\left(\int_0^{\infty} e^{-xt} \begin{pmatrix} \big((1 + x - x\ln x)\, I + \big(\ln^2 x - \pi^2\big)\, A\big)^2 \\ +\pi^2 (xI + 2\ln x A)^2 \end{pmatrix} A_2^{-1} \mathrm{d}x\right) B \end{aligned} \tag{2.10}$$

where $A_2 := \big((1 + x - x\ln x)\, I + \big(\ln^2 x - \pi^2\big)\, A\big)^2 + \pi^2 (xI + 2\ln x A)^2$.

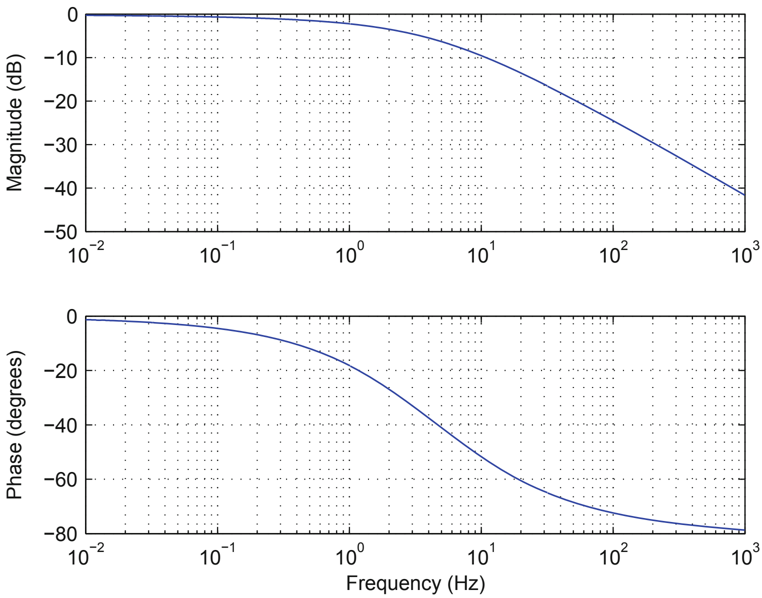


Fig. 2.6 The Bode plot of distributed-order system $g_2(s) = \frac{\ln^2 s}{1-s+s\ln s+\ln^2 s}$

Case 3 $w(\alpha) = \delta(\alpha - \beta),\ (0 < \beta < 1)$

The transfer function of fractional-order system for Case 3 with the assumption that $D = 0$ is $G_3(s) = C\left(s^{\beta} I - A\right)^{-1} B$, the inverse Laplace transform of $G_3(s)$ with null initial condition is

$$y_3(t) = C\left(t^{\beta-1} E_{\beta,\beta}(At^{\beta})\right) B \tag{2.11}$$

where $E_{\alpha,\beta}(\cdot)$ is the Mittag-Leffler function in two parameters defined as in Podlubny (1999)

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ (\Re(\alpha,\beta) > 0).$$

Remark 2.5 Computing (2.9), (2.10) and (2.11) can be easily realized in MATLAB numerically.

2.4 Frequency-Domain Response: Bode Plots

Generally, the frequency domain response has to be obtained by the direct evaluation of the irrational transfer function of distributed-order system along the imaginary axis for $s = j\omega,\ \omega \in (0, \infty)$. For simplicity, Bode plots of some scalar transfer functions for Case 1 to Case 3 are shown as follows.

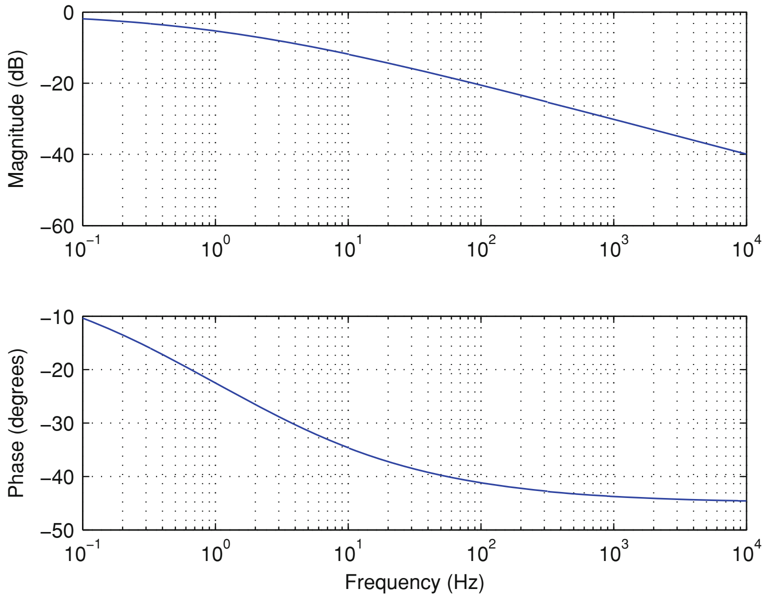


Fig. 2.7 The Bode plot of fractional-order system $g_3(s) = \frac{1}{s^{0.5}+1}$

For $w(\alpha) = 1$, the frequency-domain response of $g_1(s) = \frac{\ln s}{s-1+\ln s}$ is shown in Fig. 2.6.

For $w(\alpha) = \alpha$, the frequency-domain response of $g_2(s) = \frac{\ln^2 s}{1-s+s\ln s+\ln^2 s}$ is shown in Fig. 2.7.

For $w(\alpha) = \delta(\alpha - \beta),\ (0 < \beta < 1)$, the frequency-domain response of $g_3(s) = \frac{1}{s^{0.5}+1}$ is shown in Fig. 2.8.

2.5 Numerical Examples

In this section, numerical examples are shown to demonstrate the effectiveness of the proposed results.

Example 1 Consider a distributed-order system with Case 1 described with parameters given as $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \end{bmatrix}$ and $D = 0$.

The eigenvalues of A are $\lambda_1 = 1 + 2j$ and $\lambda_2 = 1 - 2j$, so it can be known from Theorem 2.1 that this distributed-order system is bounded-input bounded-output stable. Using MATLAB to derive numerically, the states of impulse response with null initiations are shown in Figs. 2.9 and 2.10, respectively.

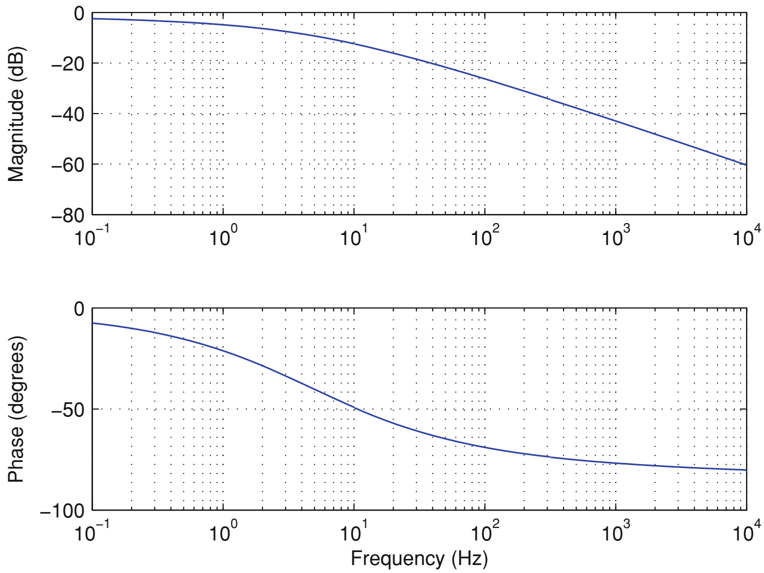
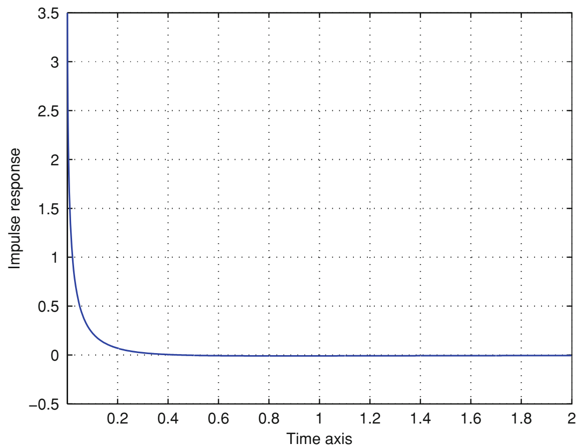


Fig. 2.8 The Bode plot of distributed-order system $g_1(s) = \frac{\ln s}{s-1+\ln s}$

Fig. 2.9 The state x_1 of stable distributed-order system (2.1) for Case 1



Example 2 Consider a distributed-order system with Case 1 described with parameters given as $A = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \end{bmatrix}$ and $D = 0$.

The eigenvalues of A are $\lambda_1 = 2+2j$ and $\lambda_2 = 2-2j$, and it can be known from Theorem 2.1 that this distributed-order system is not bounded-input bounded-output stable. Using MATLAB to derive numerically, the states of impulse response with null initiations are shown in Figs. 2.11 and 2.12, respectively.

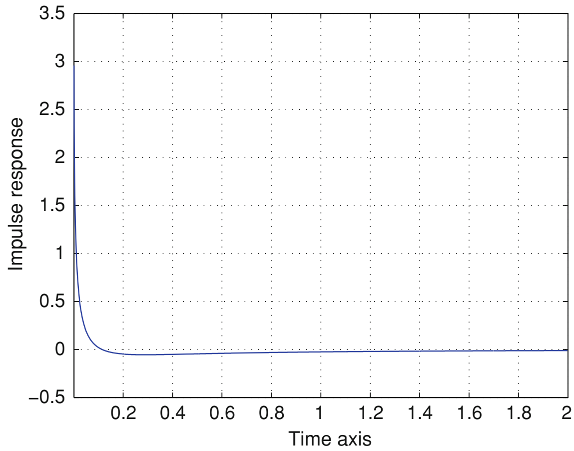


Fig. 2.10 The state x_2 of stable distributed-order system (2.1) for Case 1

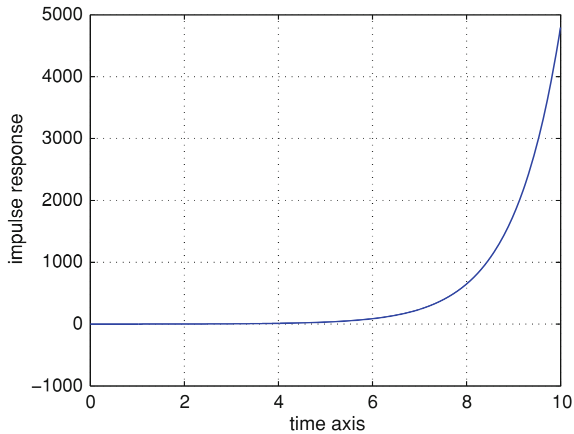


Fig. 2.11 The state x_1 of unstable distributed-order system (2.1) for Case 1

Example 3 Consider a distributed-order system with Case 2 described with parameters given as $A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \end{bmatrix}$ and $D = 0$.

The eigenvalues of A are $\lambda_1 = 1 + 3j$ and $\lambda_2 = 1 - 3j$, so it can be known from Theorem 2.2 that this distributed-order system is bounded-input bounded-output stable, and the states of impulse response with null initiations are shown in Figs. 2.13 and 2.14, respectively.

Example 4 Consider a distributed-order system with Case 2 described with parameters given as $A = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \end{bmatrix}$ and $D = 0$.

The eigenvalues of A are $\lambda_1 = 2 + 2j$ and $\lambda_2 = 2 - 2j$, it can be known from Theorem 2.2 that this distributed-order system is not bounded-input bounded-output

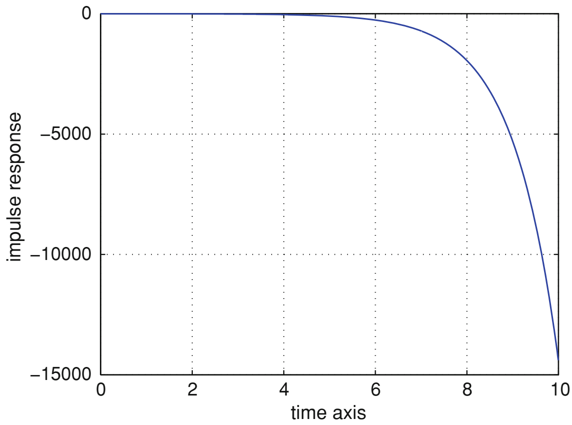


Fig. 2.12 The state x_2 of unstable distributed-order system (2.1) for Case 1

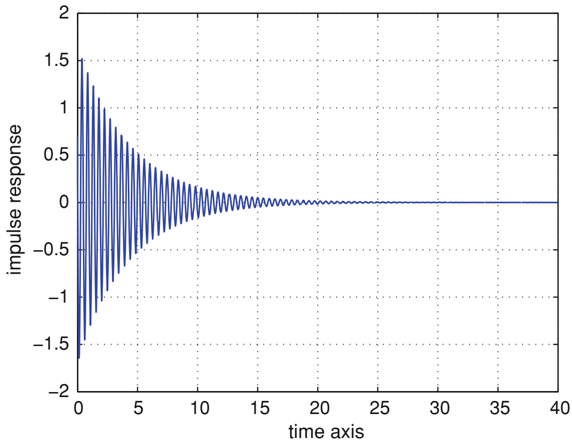


Fig. 2.13 The state x_1 of stable distributed-order system (2.1) for Case 2

stable, and by using MATLAB to derive numerically, the states of impulse response with null initiations are shown in Figs. 2.15 and 2.16, respectively.

Example 5 Consider a fractional-order system for Case 3 described with parameters given as $\alpha = 0.5$, $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \end{bmatrix}$ and $D = 0$.

The eigenvalues of A are $\lambda_1 = 2j$ and $\lambda_2 = -2j$, it can be known from Theorem 2.3 that this fractional-order system is bounded-input bounded-output stable. Using MATLAB to derive numerically, the states of impulse response with null initiations are shown in Figs. 2.17 and 2.18, respectively.

Example 6 Consider a fractional-order system for Case 3 described with parameters given as $\alpha = 2/3$, $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \end{bmatrix}$ and $D = 0$.

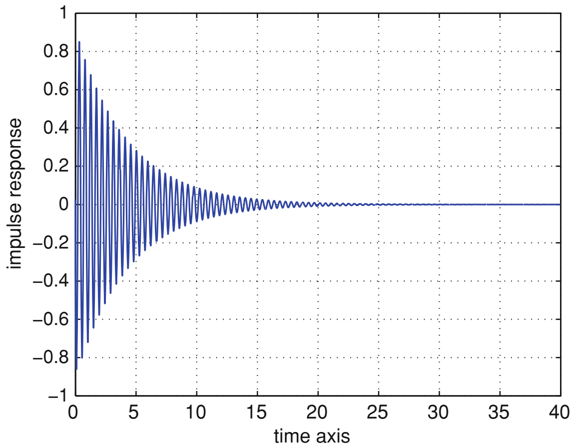


Fig. 2.14 The state x_2 of stable distributed-order system (2.1) for Case 2

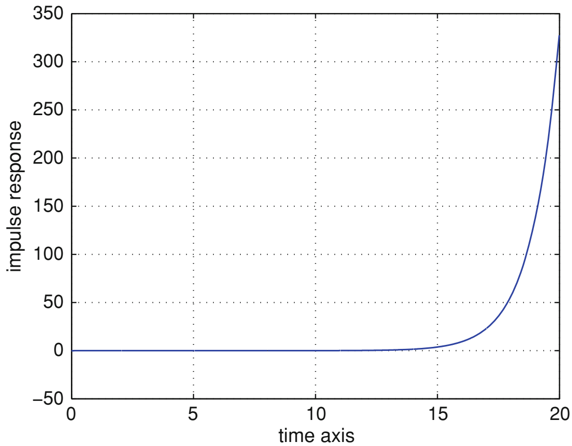


Fig. 2.15 The state x_1 of unstable distributed-order system (2.1) for Case 2

Since the eigenvalues of A are $\lambda_1 = 1 + j$ and $\lambda_2 = 1 - j$, it can be known from Theorem 2.3 that this fractional-order system is bounded-input bounded-output stable. Using MATLAB to derive numerically, the states of impulse response with null initiations are shown in Figs. 2.19 and 2.20, respectively.

2.6 Chapter Summary

In this chapter, the bounded-input bounded-output stability conditions for four kinds of linear time-invariant distributed-order system whose integral interval being (0, 1) have been derived for the first time. Based on the final value property of Laplace transform, sufficient and necessary conditions of stability for distributed-order sys-

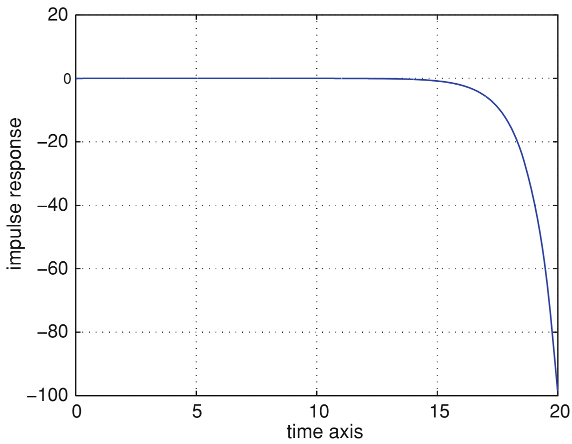


Fig. 2.16 The state x_2 of unstable distributed-order system (2.1) for Case 2

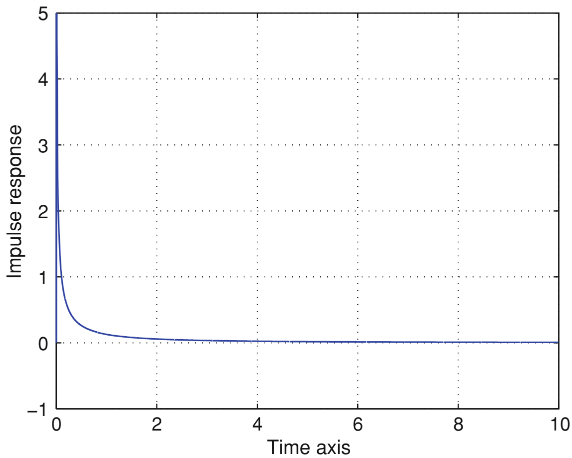


Fig. 2.17 The state x_1 of stable fractional-order system (2.5) for Case 3

tems are presented. In addition, time-domain and frequency-domain responses are presented with six illustrative numerical examples. Detailed MATLAB codes are shown in Appendix A.

Fig. 2.18 The state x_2 of stable fractional-order system (2.5) for Case 3

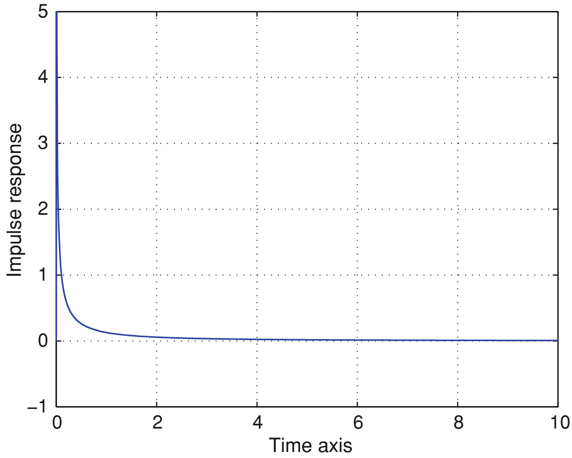
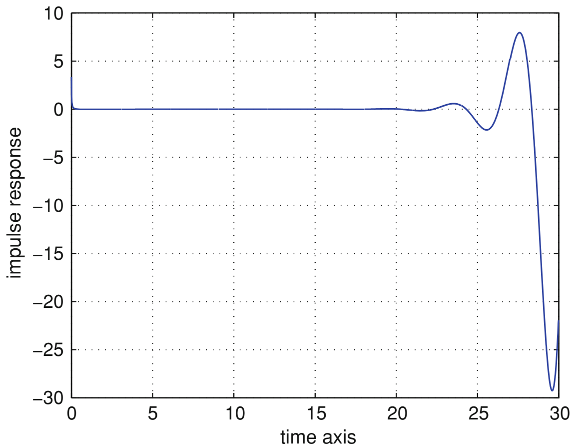


Fig. 2.19 The state x_1 of unstable fractional-order system (2.5) for Case 3



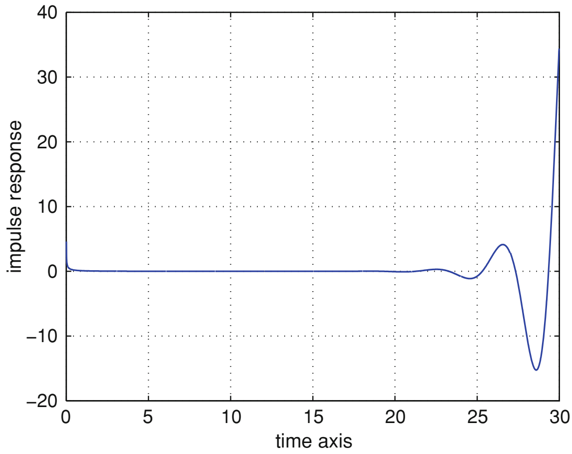


Fig. 2.20 The state x_2 of unstable fractional-order system (2.5) for Case 3

References

Asmar NH, Jones GC (2002) Applied complex analysis with partial differential equations. Prentice Hall, Upper Saddle River

Cuadrado M, Cabanes R (1989) T. de Variable Compleja. Servicio de Publicaciones de la ETSIT UPM, Madrid

Gross B, Braga EP (1961) Singularities of linear system functions. Elsevier, New York

Hartley TT, Lorenzo CF (1999) Fractional system identification: an approach using continuous order-distributions. NASA Tech Memo 209640:20

Hartley TT, Lorenzo CF (2003) Fractional-order system identification based on continuous order-distributions. Signal Process 83(11):2287–2300

Hartley TT, Lorenzo CF (2004) A frequency-domain approach to optimal fractional-order damping. Nonlinear Dyn 38(1–2):69–84

Li Y, Sheng H, Chen YQ (2010) On distributed order integrator/differentiator. Signal Process 91(5):1079–1084

Matignon D (1996) Stability results on fractional differential equations with applications to control processing. In: Multiconference on computational engineering in systems and application, pp 963–968

Podlubny I (1999) Fractional differential equations. Academic Press, San Diego

Westerlund S, Ekstam L (1994) Capacitor theory. IEEE Trans Dielectr Electr Insul 1(5):826–839

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