Chapter 2
Hopf Bifurcation and Normal Form Computation

In this chapter, we discuss the computation of normal forms. First we present a general approach which combines center manifold theory with computation of the normal form. Then, we focus on a perturbation method which has proved efficient in computation. Efficient computation is also discussed, and a sufficient and necessary condition for determining Hopf critical points of high-dimensional systems is given.

Note that for computing the normal form of systems with bifurcation (perturbation) parameters, one usually takes two steps. First, at a critical point (at which the dynamic system has a singularity) one sets the parameters to zero to obtain a so-called "reduced" (or "simplified") system and then normal form theory is applied to this system to obtain the normal form. Having found the normal form of the reduced system, one adds "unfolding" terms to get a parametric normal form for bifurcation analysis. However, this way one usually does not know the relationship between the original system parameters and the unfolding.

2.1 Hopf Bifurcation

Now suppose that system (1.5), which is rewritten below for convenience,

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu), \quad \boldsymbol{x} \in \boldsymbol{R}^n, \mu \in \boldsymbol{R}, \qquad \boldsymbol{f} : \boldsymbol{R}^{n+1} \to \boldsymbol{R}^n, \tag{2.1}$$

has an equilibrium, given by $\boldsymbol{x} = \boldsymbol{p}(\mu)$. Suppose the Jacobian, $D\boldsymbol{f}(\mu_0)$, of the system evaluated on the equilibrium at a critical point μ_0 has a simple pair of purely imaginary eigenvalues, $\pm \mathrm{i}\omega$ ($\omega > 0$), and no other eigenvalues with zero real part. The implicit function theorem guarantees (since $D\boldsymbol{f}(\mu_0)$ is invertible) that for each μ near μ_0 there will be an equilibrium $\boldsymbol{p}(\mu)$ near $\boldsymbol{p}(\mu_0)$ which varies smoothly with μ. Nonetheless, the dimensions of stable and unstable manifolds of $\boldsymbol{p}(\mu)$ do change if the eigenvalues of $D\boldsymbol{f}(\boldsymbol{p}(\mu))$ cross the imaginary axis at μ_0. This qualitative change in the local flow near $\boldsymbol{p}(\mu)$ must be marked by some other local changes in the phase portraits not involving fixed points.

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A clue to what happens in the generic bifurcation problem involving an equilibrium with purely imaginary eigenvalues can be gained from examining linear systems in which there is a change of this type. For example, consider the system

$$\begin{aligned} \dot{x} &= \mu x - \omega y, \\ \dot{y} &= \omega x + \mu y, \end{aligned} \tag{2.2}$$

whose solutions have the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\mu t} \begin{bmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{bmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{2.3}$$

When $\mu < 0$, solutions spiral into the origin, and when $\mu > 0$, solutions spiral away from the origin. When $\mu = 0$, all solutions are periodic. Even in a one-parameter family of equations, it is highly special to find a parameter value at which there is a whole family of periodic orbits, but there is still a surface of periodic orbits which appears in the general problem.

The normal form theorem gives us the required information about how the generic problem differs from system (2.2). By smooth changes of coordinates, the Taylor series of degree 3 for the general problem can be brought to the following form (see next section):

$$\begin{aligned} \dot{x} &= \left[d\mu + a\left(x^2 + y^2\right)\right]x - \left[\omega + c\mu + b\left(x^2 + y^2\right)\right]y, \\ \dot{y} &= \left[\omega + c\mu + b\left(x^2 + y^2\right)\right]x + \left[d\mu + a\left(x^2 + y^2\right)\right]y, \end{aligned} \tag{2.4}$$

which is expressed in polar coordinates as

$$\begin{aligned} \dot{r} &= \left(d\mu + ar^2\right)r, \\ \dot{\theta} &= \omega + c\mu + br^2. \end{aligned} \tag{2.5}$$

Since the $\dot{r}$ equation in (2.5) separates from θ, we see that there are periodic orbits of (2.4) which are circles $r = \text{const.}$, obtained from the nonzero solutions of $\dot{r} = 0$ in (2.5). If $a \neq 0$ and $d \neq 0$ these solutions lie along the parabola $\mu = -(\frac{a}{d})r^2$. This implies that the surface of periodic orbits has a quadratic tangency with its tangent plane $\mu = 0$ in $\boldsymbol{R}^2 \times \boldsymbol{R}$.

In the following, we first introduce the Hopf bifurcation theorem and then discuss in detail the computation of normal forms associated with various singularities (including Hopf bifurcation). It should be mentioned that in the original paper [100] (with English translation included in [162]) that Hopf did not know the terminology of the Poincaré normal form.

Theorem 2.1 ([100]) *Suppose that the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu)$, $\boldsymbol{x} \in \boldsymbol{R}^n$, $\mu \in \boldsymbol{R}$, has an equilibrium $(\boldsymbol{x}_0, \mu_0)$ at which the following properties are satisfied.*

(H1) *$D_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}_0, \mu_0)$ has a simple pair of purely imaginary eigenvalues and no other eigenvalues with zero real parts.*

Then (H1) *implies that there is a smooth curve of equilibria* $(\boldsymbol{x}(\mu), \mu)$ *with* $\boldsymbol{x}(\mu_0) = \boldsymbol{x}_0$. *The eigenvalues* $\lambda(\mu)$, $\bar{\lambda}(\mu)$ *of* $D_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}(\mu), \mu_0)$, *which are imaginary at* $\mu = \mu_0$, *vary smoothly with* μ.

If, moreover,

$$\text{(H2)} \qquad \frac{d}{d\mu}\big(\operatorname{Re}\lambda(\mu)\big)\big|_{\mu=\mu_0} = d \neq 0,$$

then there is a unique three-dimensional center manifold passing through $(\boldsymbol{x}_0, \mu_0)$ *in* $\boldsymbol{R}^n \times \boldsymbol{R}$ *and a smooth system of coordinates* (*preserving the planes* $\mu = const.$) *for which the Taylor expansion of degree* 3 *on the center manifold is given by* (2.4). *If* $a \neq 0$, *there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of* $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$ *agreeing to second order with the paraboloid*

$$\mu = -\left(\frac{a}{d}\right)\left(x^2 + y^2\right). \tag{2.6}$$

If $a < 0$, *then these periodic solutions are stable limit cycles*, *while if* $a > 0$, *the periodic solutions are repelling*.

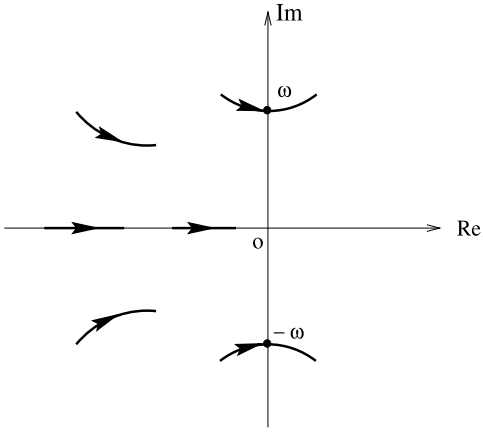

Fig. 2.1 Transversality of Hopf bifurcation

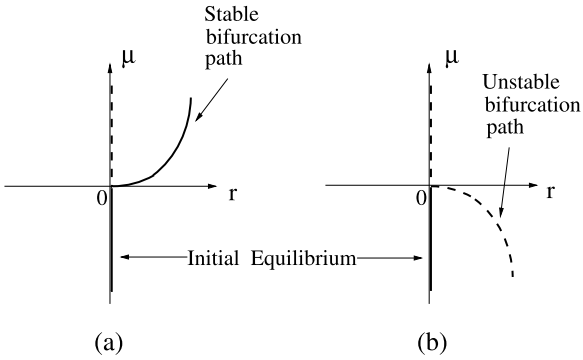

Fig. 2.2 Post-critical bifurcation path for a Hopf bifurcation: (**a**) stable; (**b**) unstable

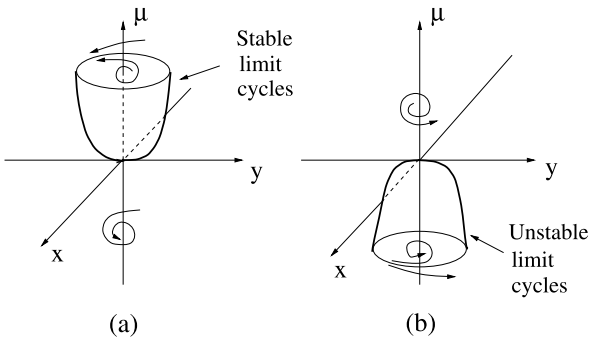

Fig. 2.3 Bifurcating periodic solutions (limit cycles): (**a**) stable; (**b**) unstable

This theorem can be proved by a direct application of the center manifold and normal form theorems. The transversality conditions given in (H1) and (H2) are illustrated in Fig. 2.1. The parameter–amplitude relation (2.6) is shown in Fig. 2.2, where $r = \sqrt{x^2 + y^2}$, and the bifurcating periodic solutions depicted in three-dimensional space are given in Fig. 2.3.

Computation of center manifolds and normal forms will be discussed in the following sections.

2.2 Computation of Normal Forms

Many computational methods such as the Poincaré method (also called the Takens method), briefly discussed in the last section have been developed (e.g., see [47, 72, 166]). Usually, given a dynamical system described in differential equations, the center manifold theory (e.g., see [27]) is first applied to obtain a locally invariant small-dimensional manifold—a center manifold. Then additional nonlinear transformations are introduced to further simplify the center manifold to a normal form. To find the "form" of a normal form, first a homogeneous polynomial vector field of degree k is found in a space complementary to the range of the so-called "homological operator". Then the original vector field is decomposed into two parts: one of them, called the *nonresonant* terms, is eliminated and the other, called the *resonant* terms, is kept in the normal form. This simple form can be used conveniently for analyzing the local dynamic behavior of the original system.

For a practical system, not only the possible qualitative dynamical behavior of the system of concern, but also the quantitative relationship between the normal forms and the equations of the original system needs to be established. Normal forms are, in general, not uniquely defined and finding a normal form for a given system of differential equations is not a simple task. In particular, finding the explicit formulas for normal forms in terms of the coefficients of the original nonlinear system is not easy. Therefore, the crucial part in computing a normal form is the computational efficiency in finding the coefficients of the normal forms and the corresponding nonlinear transformations. Furthermore, the algebraic manipulation becomes very

involved as the order of approximation increases. Thus, symbolic computations using symbolic computer languages such as Maple, Mathematica and Macsyma have been introduced for computing normal forms. However, it seems that even with a symbolic manipulator the computation of normal forms is still limited to lower-order approximations. This is because a program which computes a normal form usually quickly runs out of computer memory as the order of approximation increases. Therefore, computationally efficient methodology and software packages need to be developed for computing higher-order normal forms.

The idea of normal form theory is to use successive nonlinear transformations to derive a new set of differential equations by removing as many nonlinear terms from the system as possible. The terms remaining in the normal form are called the *resonant* terms. If the Jacobian matrix of the linearized system evaluated at an equilibrium can be transformed into diagonal form, then the bases of the nonlinear transformations are decoupled from each other. However, for a general singular vector field such as a system with non-semisimple double-zero or triple-zero eigenvalues, these bases are coupled. Such a coupling makes computation of the normal forms complicated.

First, we introduce a general approach for computing normal forms based on the work of [20, 21, 207, 226]. Consider a system described by the general nonlinear ordinary differential equation,

$$\dot{\boldsymbol{y}} = A\boldsymbol{y} + \boldsymbol{F}(\boldsymbol{y}), \quad \boldsymbol{y} \in \boldsymbol{R}^n, \qquad \boldsymbol{F}(\boldsymbol{y}) : \boldsymbol{R}^n \to \boldsymbol{R}^n, \tag{2.7}$$

where a dot indicates differentiation with respect to time t, $A\boldsymbol{y}$ represents the linear part, and $\boldsymbol{F}$ is a nonlinear vector function and assumed to be analytic, satisfying $\boldsymbol{F}(\boldsymbol{0}) = \boldsymbol{0}$ and $D_{\boldsymbol{y}}\boldsymbol{F}(\boldsymbol{0}) = \boldsymbol{0}$, $i = 1, 2, \ldots, n$. By introducing the linear transformation,

$$\boldsymbol{y} = T\boldsymbol{x}, \tag{2.8}$$

(2.7) can be transformed into

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \tag{2.9}$$

where J is in Jordan canonical form. If the eigenvalues of A with zero real parts are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_{n_0}$, and those with nonzero real parts are given by λ_{n_0+1}, $\lambda_{n_0+2}, \ldots, \lambda_n$, then $J = T^{-1}AT = \mathrm{diag}(J_0, J_1)$, where $J_0 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n_0})$ and $J_1 = \mathrm{diag}(\lambda_{n_0+1}, \lambda_{n_0+2}, \ldots, \lambda_n)$. Next, let $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)^T$, where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are variables associated with the eigenvalues with zero real parts and nonzero real parts, respectively, then (2.9) can be written as

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= J_0\boldsymbol{x}_1 + \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2), \\ \dot{\boldsymbol{x}}_2 &= J_1\boldsymbol{x}_2 + \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2). \end{aligned} \tag{2.10}$$

Now the center manifold of the system may be defined in the form

$$\boldsymbol{x}_2 = \boldsymbol{N}(\boldsymbol{x}_1), \qquad \boldsymbol{N}(\boldsymbol{0}) = \boldsymbol{0}, \qquad D\boldsymbol{N}_{\boldsymbol{x}_1}(\boldsymbol{0}) = \boldsymbol{0}, \tag{2.11}$$

which satisfies

$$D_{\boldsymbol{x}_1}\boldsymbol{N}(\boldsymbol{x}_1)\big[J_0\boldsymbol{x}_1+\boldsymbol{f}_1\big(\boldsymbol{x}_1,\boldsymbol{N}(\boldsymbol{x}_1)\big)\big]=J_1\boldsymbol{N}(\boldsymbol{x}_1)+\boldsymbol{f}_2\big(\boldsymbol{x}_1,\boldsymbol{N}(\boldsymbol{x}_1)\big). \tag{2.12}$$

Once $\boldsymbol{N}(\boldsymbol{x}_1)$ is determined from (2.12), the differential equation describing the dynamics on the center manifold is given by

$$\dot{\boldsymbol{x}_1}=J_0\boldsymbol{x}_1+\boldsymbol{f}_1\big(\boldsymbol{x}_1,\boldsymbol{N}(\boldsymbol{x}_1)\big). \tag{2.13}$$

Further, (2.13) may be transformed into a set of simpler differential equations—the so-called *normal form*,

$$\dot{\boldsymbol{u}}=J_0\boldsymbol{u}+\boldsymbol{C}(\boldsymbol{u}), \tag{2.14}$$

under a series of nonlinear transformations given by

$$\boldsymbol{x}_1=\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\equiv\boldsymbol{u}+\sum_{k\geq 2}\boldsymbol{H}_k(\boldsymbol{u}), \tag{2.15}$$

where $\boldsymbol{H}_k(\boldsymbol{u})$ is a kth-order vector of homogeneous polynomials in $\boldsymbol{u}$. Substituting (2.15) into (2.13) results in

$$\big(I+D_{\boldsymbol{u}}\boldsymbol{H}(\boldsymbol{u})\big)\dot{\boldsymbol{u}}=J_0\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)+\boldsymbol{f}_1\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)\big), \tag{2.16}$$

and then substituting (2.14) into (2.16) yields

$$\begin{aligned}&\big(I+D_{\boldsymbol{u}}\boldsymbol{H}(\boldsymbol{u})\big)\big(J_0\boldsymbol{u}+\boldsymbol{C}(\boldsymbol{u})\big)\\&\quad=J_0\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)+\boldsymbol{f}_1\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)\big),\end{aligned} \tag{2.17}$$

which can be rearranged as

$$\begin{aligned}&D_{\boldsymbol{u}}\boldsymbol{H}(\boldsymbol{u})J_0\boldsymbol{u}-J_0\boldsymbol{H}(\boldsymbol{u})\\&\quad=\boldsymbol{f}_1\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)\big)-D_{\boldsymbol{u}}\boldsymbol{H}(\boldsymbol{u})\boldsymbol{C}(\boldsymbol{u})-\boldsymbol{C}(\boldsymbol{u}).\end{aligned} \tag{2.18}$$

To find the expression of the center manifold, one may substitute (2.15) into (2.12) to obtain

$$\begin{aligned}&D_{\boldsymbol{x}_1}\boldsymbol{N}(\boldsymbol{x}_1)\big[J_0\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)+\boldsymbol{f}_1\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)\big)\big]\\&\quad=J_1\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)+\boldsymbol{f}_2\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)\big).\end{aligned} \tag{2.19}$$

Now rewrite (2.18) as

$$\begin{aligned}&J_0\boldsymbol{H}(\boldsymbol{u})+\boldsymbol{f}_1\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)\big)\\&\quad=D_{\boldsymbol{u}}\boldsymbol{H}(\boldsymbol{u})J_0\boldsymbol{u}+D_{\boldsymbol{u}}\boldsymbol{H}(\boldsymbol{u})\boldsymbol{C}(\boldsymbol{u})+\boldsymbol{C}(\boldsymbol{u}),\end{aligned} \tag{2.20}$$

and substitute this into (2.19) to obtain

$$\begin{aligned}&D_{\boldsymbol{x}_1}\boldsymbol{N}(\boldsymbol{x}_1)\big(I+D_{\boldsymbol{u}}\boldsymbol{H}(\boldsymbol{u})\big)\big(J_0\boldsymbol{u}+\boldsymbol{C}(\boldsymbol{u})\big)\\&\quad=J_1\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)+\boldsymbol{f}_2\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)\big)\end{aligned} \tag{2.21}$$

which can be written as

$$\begin{aligned} &D_{\boldsymbol{u}}\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)J_0\boldsymbol{u}-J_1\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)\\ &\quad=\boldsymbol{f}_2\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)\big)-D_{\boldsymbol{u}}\boldsymbol{N}\big(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u})\big)\boldsymbol{C}(\boldsymbol{u}). \end{aligned} \tag{2.22}$$

Combining (2.18) and (2.22) yields

$$\begin{aligned} &D_{\boldsymbol{u}}\begin{pmatrix}\boldsymbol{H}(\boldsymbol{u})\\ \boldsymbol{h}(\boldsymbol{u})\end{pmatrix}J_0\boldsymbol{u}-\begin{bmatrix}J_0 & 0\\ 0 & J_1\end{bmatrix}\begin{pmatrix}\boldsymbol{H}(\boldsymbol{u})\\ \boldsymbol{h}(\boldsymbol{u})\end{pmatrix}\\ &\quad=\begin{pmatrix}\boldsymbol{f}_1(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{h}(\boldsymbol{u}))\\ \boldsymbol{f}_2(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{h}(\boldsymbol{u}))\end{pmatrix}-D_{\boldsymbol{u}}\begin{pmatrix}\boldsymbol{H}(\boldsymbol{u})\\ \boldsymbol{h}(\boldsymbol{u})\end{pmatrix}\boldsymbol{C}(\boldsymbol{u})-\begin{pmatrix}\boldsymbol{C}(\boldsymbol{u})\\ \boldsymbol{0}\end{pmatrix}, \end{aligned} \tag{2.23}$$

where $\boldsymbol{h}(\boldsymbol{u})=\boldsymbol{N}(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}))$. Furthermore, let

$$\overline{\boldsymbol{H}}(\boldsymbol{u})=\begin{pmatrix}\boldsymbol{H}(\boldsymbol{u})\\ \boldsymbol{h}(\boldsymbol{u})\end{pmatrix},\qquad \boldsymbol{f}(\boldsymbol{u})=\begin{pmatrix}\boldsymbol{f}_1(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{h}(\boldsymbol{u}))\\ \boldsymbol{f}_2(\boldsymbol{u}+\boldsymbol{H}(\boldsymbol{u}),\boldsymbol{h}(\boldsymbol{u}))\end{pmatrix},\qquad \overline{\boldsymbol{C}}(\boldsymbol{u})=\begin{pmatrix}\boldsymbol{C}(\boldsymbol{u})\\ \boldsymbol{0}\end{pmatrix},$$

then one can write (2.23) in a more compact form:

$$D_{\boldsymbol{u}}\overline{\boldsymbol{H}}(\boldsymbol{u})J_0\boldsymbol{u}-J\overline{\boldsymbol{H}}(\boldsymbol{u})=\boldsymbol{f}(\boldsymbol{u})-D_{\boldsymbol{u}}\overline{\boldsymbol{H}}(\boldsymbol{u})\boldsymbol{C}(\boldsymbol{u})-\overline{\boldsymbol{C}}(\boldsymbol{u}). \tag{2.24}$$

If (2.24) can be solved for $\overline{\boldsymbol{H}}(\boldsymbol{u})$ and $\overline{\boldsymbol{C}}(\boldsymbol{u})$ explicitly, one then will have the normal form as well as the nonlinear transformation. In general, however, the exact solution of (2.24) cannot be obtained. Thus, approximate solutions may be sought and assumed in the form

$$\begin{aligned} \overline{\boldsymbol{H}}(\boldsymbol{u})&=\sum_{m\geq 2}\overline{\boldsymbol{H}}^m(\boldsymbol{u})=\sum_{m\geq 2}\Big(\sum\overline{\boldsymbol{H}}_{\tilde{m}}u_1^{m_1}u_2^{m_2}\cdots u_{n_0}^{m_{n_0}}\Big),\\ \boldsymbol{C}(\boldsymbol{u})&=\sum_{m\geq 2}\boldsymbol{C}^m(\boldsymbol{u})=\sum_{m\geq 2}\Big(\sum\boldsymbol{C}_{\tilde{m}}u_1^{m_1}u_2^{m_2}\cdots u_{n_0}^{m_{n_0}}\Big), \end{aligned} \tag{2.25}$$

where $\overline{\boldsymbol{H}}_{\tilde{m}}=(\boldsymbol{H}_{\tilde{m}},\boldsymbol{h}_{\tilde{m}})^T$, $\tilde{m}\triangleq m_1m_2\cdots m_{n_0}$, denoting a choice of the values of $m_1,m_2,\ldots,m_{n_0}$ which satisfies $m_1+m_2+\cdots+m_{n_0}=m$ and $m_i\geq 0, i=1,2,\ldots,n_0$. The summation given in the parentheses of (2.25) represents the sum of all possible choices for $\tilde{m}$. Similarly, suppose that $\boldsymbol{f}(\boldsymbol{u})-D_{\boldsymbol{u}}\overline{\boldsymbol{H}}(\boldsymbol{u})\boldsymbol{C}(\boldsymbol{u})$ can be expressed as

$$\begin{aligned} \tilde{\boldsymbol{f}}(\boldsymbol{u})&\triangleq\boldsymbol{f}(\boldsymbol{u})-D_{\boldsymbol{u}}\overline{\boldsymbol{H}}(\boldsymbol{u})\boldsymbol{C}(\boldsymbol{u})=\sum_{m\geq 2}\tilde{\boldsymbol{f}}^m(\boldsymbol{u})\\ &=\sum_{m\geq 2}\Big(\sum\tilde{\boldsymbol{f}}_{\tilde{m}}u_1^{m_1}u_2^{m_2}\cdots u_{n_0}^{m_{n_0}}\Big). \end{aligned} \tag{2.26}$$

Then, it can be shown that $\tilde{\boldsymbol{f}}^m(\boldsymbol{u})$ only depends upon $\overline{\boldsymbol{H}}^{m'}(\boldsymbol{u})$ and $\boldsymbol{C}^{m'}(\boldsymbol{u})$ for $m'<m$. Therefore, once $\overline{\boldsymbol{H}}^{m'}(\boldsymbol{u})$ and $\boldsymbol{C}^{m'}(\boldsymbol{u})$ $(m'<m)$ are determined, the ex-

pression of $\tilde{\boldsymbol{f}}^{m}(\boldsymbol{u})$ is definitely defined. To explicitly calculate the coefficients $\overline{\boldsymbol{H}}_{\tilde{m}}$ and $\boldsymbol{C}_{\tilde{m}}$, we have the following result.

Theorem 2.2 *If some of the eigenvalues of A with zero real parts are non-semisimple, then (2.24) can be written as*

$$(\lambda_0 - J)\overline{\boldsymbol{H}}_{\tilde{m}} + \sum_{j=1}^{n_0-1} (J_0)_{j(j+1)}(m_j+1)\overline{\boldsymbol{H}}_{m_1 m_2 \cdots (m_j+1)(m_{j+1}-1)\cdots m_{n_0}} = \tilde{\boldsymbol{f}}_{\tilde{m}} - \overline{\boldsymbol{C}}_{\tilde{m}}, \tag{2.27}$$

where $\lambda_0 = m_1\lambda_1 + m_2\lambda_2 + \cdots + m_{n_0}\lambda_{n_0}$, n_0 is the number of eigenvalues with zero real parts and $(J_0)_{j(j+1)}$ denotes the element of J_0 located on the jth row and $(j+1)$th column. In addition, $m_j \le n_0 - 1$ and $m_{j+1} \ge 1$.

Proof We can prove this result as follows. First note that if some of the eigenvalues of A with zero real parts are non-semisimple, then J_0 can be decomposed into

$$J_0 = S + N, \tag{2.28}$$

where $S = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n_0})$ and N is given in the form

$$N_{ij} = \begin{cases} 1 \text{ or } 0 & \text{if } i+1 = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.29}$$

It is clear that some of the nondiagonal elements of J_0 are nonzero, which causes some of the equations to be coupled. Therefore,

$$\begin{aligned}
&D_{\boldsymbol{u}}\overline{\boldsymbol{H}}_{m_1\cdots m_{j-1}(m_j+1)(m_{j+1}-1)m_{j+2}\cdots m_{n_0}} \\
&\quad \times u_1^{m_1}\cdots u_{j-1}^{m_{j-1}} u_j^{m_j+1} u_{j+1}^{m_{j+1}-1} u_{j+2}^{m_{j+2}}\cdots u_{n_0}^{m_{n_0}} N\boldsymbol{u} \\
&= \sum_{j=1}^{n_0-1} \frac{\partial}{\partial u_j}\big(\overline{\boldsymbol{H}}_{m_1\cdots m_{j-1}(m_j+1)(m_{j+1}-1)m_{j+2}\cdots m_{n_0}} \\
&\quad \times u_1^{m_1}\cdots u_j^{m_j+1} u_{j+1}^{m_{j+1}-1}\cdots u_{n_0}^{m_{n_0}}\big) N_{j(j+1)} u_{j+1} \\
&= \sum_{j=1}^{n_0-1} N_{j(j+1)}(m_j+1)\overline{\boldsymbol{H}}_{m_1\cdots m_{j-1}(m_j+1)(m_{j+1}-1)m_{j+2}\cdots m_{n_0}} \\
&\quad \times u_1^{m_1}\cdots u_{j-1}^{m_{j-1}} u_j^{m_j} u_{j+1}^{m_{j+1}} u_{j+2}^{m_{j+2}}\cdots u_{m_{n_0}}^{m_{n_0}},
\end{aligned}$$

where $m_j \le n_0 - 1$ and $m_{j+1} \ge 1$. In fact, the two elements m_j and m_{j+1} in the original subscripts of $\overline{\boldsymbol{H}}$, $m_1 m_2 \cdots m_{j-1} m_j m_{j+1} m_{j+2}\cdots m_{n_0}$, have been replaced by $m_j + 1$ and $m_{j+1} - 1$, respectively. Thus, for a fixed m, the expression of $D_{\boldsymbol{u}}\overline{\boldsymbol{H}}^{m}(\boldsymbol{u})J_0\boldsymbol{u}$ becomes

$$
\begin{aligned}
&D_{\boldsymbol{u}}\overline{\boldsymbol{H}}^{m}(\boldsymbol{u})J_0\boldsymbol{u}\\
&\quad = D_{\boldsymbol{u}}\overline{\boldsymbol{H}}^{m}(\boldsymbol{u})S\boldsymbol{u} + D_{\boldsymbol{u}}\overline{\boldsymbol{H}}^{m}(\boldsymbol{u})N\boldsymbol{u}\\
&\quad = \sum\Bigg[\lambda_0\overline{\boldsymbol{H}}_{\tilde{m}}u_1^{m_1}u_2^{m_2}\cdots u_{n_0}^{m_{n_0}}\\
&\qquad + \sum_{j=1}^{n_0-1} N_{j(j+1)}(m_j+1)\overline{\boldsymbol{H}}_{m_1m_2\cdots(m_j+1)(m_{j+1}-1)\cdots m_{n_0}}\\
&\qquad \times u_1^{m_1}\cdots u_j^{m_j}u_{j+1}^{m_{j+1}}\cdots u_{n_0}^{m_{n_0}}\Bigg]\\
&\quad = \sum\Bigg[\lambda_0\overline{\boldsymbol{H}}_{\tilde{m}} + \sum_{j=1}^{n_0-1} N_{j(j+1)}(m_j+1)\overline{\boldsymbol{H}}_{m_1m_2\cdots(m_j+1)(m_{j+1}-1)\cdots m_{n_0}}\Bigg]\\
&\qquad \times u_1^{m_1}\cdots u_{n_0}^{m_{n_0}}.
\end{aligned}
$$

Now substituting the above equation into (2.24) and balancing the corresponding coefficients of $u_1^{m_1}u_2^{m_2}\cdots u_{n_0}^{m_{n_0}}$ in the resulting equation gives

$$
(\lambda_0 - J)\overline{\boldsymbol{H}}_{\tilde{m}} + \sum_{j=1}^{n_0-1}(J_0)_{j(j+1)}(m_j+1)\overline{\boldsymbol{H}}_{m_1m_2\cdots(m_j+1)(m_{j+1}-1)\cdots m_{n_0}} = \tilde{\boldsymbol{f}}_{\tilde{m}} - \overline{\boldsymbol{C}}_{\tilde{m}},
$$

which is (2.27). □

For the semisimple case, it can be seen that the summation term on the left-hand side of (2.27) disappears. Thus, if the eigenvalues of A with zero real parts are all semisimple, then (2.24) can be expressed in a simpler form

$$
(\lambda_0 I - J)\overline{\boldsymbol{H}}_{\tilde{m}} = \tilde{\boldsymbol{f}}_{\tilde{m}} - \overline{\boldsymbol{C}}_{\tilde{m}}, \tag{2.30}
$$

or

$$
(\lambda_0 I - J_0)\boldsymbol{H}_{\tilde{m}} = \tilde{\boldsymbol{f}}_{1\tilde{m}} - \boldsymbol{C}_{\tilde{m}}, \tag{2.31}
$$

$$
(\lambda_0 I - J_1)\boldsymbol{h}_{\tilde{m}} = \tilde{\boldsymbol{f}}_{2\tilde{m}}. \tag{2.32}
$$

For non-semisimple cases, since the summation term also contains the $\overline{\boldsymbol{H}}$ coefficients, the computation becomes more involved.

In the following, we show how to compute the normal form for semisimple cases [21], and present the computation for two non-semisimple cases—a double-zero singularity [20] and a 1:1 resonant double-Hopf singularity [226].

2.2.1 Semisimple Case

If we can find all the coefficients $\overline{\boldsymbol{H}}_{\tilde{m}}$, $\boldsymbol{C}_{\tilde{m}}$ and $\boldsymbol{C}_{m_1 m_2 \cdots m_{n_0}}$ satisfying (2.31) and (2.32), then we have solved the problem. To achieve this, first note that the diagonal matrix $\lambda_0 I - J_1$ is nonsingular since the real part of every component of $\lambda_0 I$ is zero while every component of J_1 is not zero. Therefore, (2.32) can be solved to uniquely determine $\boldsymbol{h}_{\tilde{m}}$:

$$\boldsymbol{h}_{\tilde{m}} = (\lambda_0 I - J_1)^{-1} \tilde{\boldsymbol{f}}_{2\tilde{m}}. \tag{2.33}$$

For (2.31), we need to treat the nonresonant and resonant terms separately.

(1) Nonresonant terms. In this case, $\lambda_0 I - J_0$ is nonsingular, i.e., $\det(\lambda_0 I - J_0) \neq 0$. Therefore, $\boldsymbol{H}_{\tilde{m}}$ can always be determined for a given $\boldsymbol{C}_{\tilde{m}}$. In order to obtain a normal form for this particular order as simply as possible, one may choose $\boldsymbol{C}_{\tilde{m}} = \boldsymbol{0}$, and thus,

$$\boldsymbol{C}_{\tilde{m}} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{H}_{\tilde{m}} = (\lambda_0 I - J_0)^{-1} \tilde{\boldsymbol{f}}_{1\tilde{m}}. \tag{2.34}$$

(2) Resonant terms. For this case, $\lambda_0 I - J_0$ is singular, i.e., $\det(\lambda_0 I - J_0) = 0$, implying that some components of $\lambda_0 I - J_0$ equal zero (remember that $\lambda_0 I - J_0$ is a diagonal matrix). Note that (2.31) has n_0 linear equations. This equation can be rewritten in component form:

$$(\lambda_0 I - J_0)_k \boldsymbol{H}_{\tilde{m}k} = \tilde{\boldsymbol{f}}_{1\tilde{m}k} - \boldsymbol{C}_{\tilde{m}k}, \tag{2.35}$$

where the subscript k denotes the kth component of the coefficient matrix and the kth components of the vectors $\boldsymbol{H}_{\tilde{m}}$, $\tilde{\boldsymbol{f}}_{1\tilde{m}}$ and $\boldsymbol{C}_{\tilde{m}}$. Thus, it is easy to find the following rules.

(i) When $(\lambda_0 I - J_0)_k \neq 0$,

$$\boldsymbol{C}_{\tilde{m}k} = 0, \qquad \boldsymbol{H}_{\tilde{m}k} = \frac{\tilde{\boldsymbol{f}}_{1\tilde{m}k}}{(\lambda_0 I - J_0)_k}. \tag{2.36}$$

(ii) When $(\lambda_0 I - J_0)_k = 0$,

$$\boldsymbol{H}_{\tilde{m}k} = 0, \qquad \boldsymbol{C}_{\tilde{m}k} = \tilde{\boldsymbol{f}}_{1\tilde{m}k}. \tag{2.37}$$

Summarizing the above results we have the following theorem.

Theorem 2.3 *For a fixed m, the coefficients, $\boldsymbol{C}_{\tilde{m}}$, of the normal form and the corresponding coefficients, $\boldsymbol{H}_{\tilde{m}}$ and $\boldsymbol{h}_{\tilde{m}}$, of the nonlinear transformation are determined from the following formulas.*

(i) $$\boldsymbol{h}_{\tilde{m}} = (\lambda_0 I - J_1)^{-1} \tilde{\boldsymbol{f}}_{2\tilde{m}}.$$

(ii) *If $\lambda_0 I - J_0$ is nonsingular, then*

$$\boldsymbol{C}_{\tilde{m}} = \boldsymbol{0}, \qquad \boldsymbol{H}_{\tilde{m}} = (\lambda_0 I - J_0)^{-1} \tilde{\boldsymbol{f}}_{1\tilde{m}}.$$

(iii) *If $\lambda_0 I - J_0$ is singular, then*

$$\boldsymbol{C}_{\tilde{m}k} = 0, \qquad \boldsymbol{H}_{\tilde{m}k} = \frac{\tilde{\boldsymbol{f}}_{1\tilde{m}k}}{(\lambda_0 I - J_0)_k} \quad \text{for } (\lambda_0 I - J_0)_k \neq 0,$$

and

$$\boldsymbol{H}_{\tilde{m}k} = 0, \qquad \boldsymbol{C}_{\tilde{m}k} = \tilde{\boldsymbol{f}}_{1\tilde{m}k} \quad \text{for } (\lambda_0 I - J_0)_k = 0,$$

where the subscript k indicates the kth linear equation of (2.31).

(iv) *The center manifold $\boldsymbol{N}(\boldsymbol{u} + \boldsymbol{H}(\boldsymbol{u}))$ is given by*

$$\boldsymbol{N}\big(\boldsymbol{u} + \boldsymbol{H}(\boldsymbol{u})\big) \equiv \boldsymbol{h}(\boldsymbol{u}) = \sum_{m \geq 2} \Big(\sum \boldsymbol{h}_{\tilde{m}} u_1^{m_1} u_2^{m_2} \cdots u_{n_0}^{m_{n_0}} \Big).$$

To end this section, we show several numerical examples and the corresponding normal forms, obtained using the Maple programs, which can be found on "Springer Extras" (by visiting extras.springer.com and searching for the book using its ISBN).

Example 2.4 First, let us consider a simple nonlinear oscillator, given by

$$\begin{aligned} \dot{x} &= -y + x\big[1 - \big(x^2 + y^2\big)\big], \\ \dot{y} &= x + y\big[1 - \big(x^2 + y^2\big)\big]. \end{aligned} \tag{2.38}$$

By using the polar coordinate transformations $x = r\cos\theta$ and $y = r\sin\theta$, one can transform (2.38) into

$$\begin{aligned} \dot{r} &= r\big(1 - r^2\big), \\ \dot{\theta} &= 1. \end{aligned} \tag{2.39}$$

Note: multiplying the r equation by $2r$ results in

$$\frac{dr^2}{dt} = 2r\dot{r} = 2r^2\big(1 - r^2\big), \tag{2.40}$$

which becomes the logistic equation if we let $r^2 = R > 0$.

Solving the θ equation yields $\theta = t + \theta_0$. The phase portrait in $\boldsymbol{R}^2$ is shown in Fig. 2.4.

It is easy to show that the equilibrium point $r = 0$ (i.e., $(x, y) = (0, 0)$) is unstable and the solution $r = 1, \theta = t + \theta_0$ approaches the *unit circle*, which is called a *limit cycle*.

Note that (2.39) is the normal form of system (2.38), which is an exact result. In fact, if we apply the Maple programs (which can be found on "Springer Extras" by visiting extras.springer.com and searching for the book using its ISBN) to execute system (2.38), we would obtain

$$\begin{aligned} \dot{r} &= r + v_1 r^3 + v_2 r^5 + v_3 r^7 + \cdots, \\ \dot{\theta} &= 1 + \tau_1 r^2 + \tau_2 r^4 + \tau_3 r^6 + \cdots, \end{aligned}$$

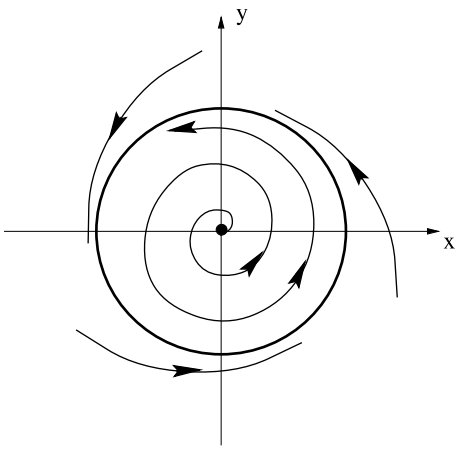

Fig. 2.4 The phase portrait of system (2.38)

where $v_1 = -1, v_i = 0$, $i = 2, 3, \ldots$, and $\tau_i = 0$, $i = 1, 2, \ldots$, agreeing with the form (2.39).

Example 2.5 The first example is a simple two-dimensional system with a Hopf-type singularity, described by

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2, \\ \dot{x}_2 &= -x_1 + x_2^2 + x_2^3, \end{aligned} \tag{2.41}$$

whose Jacobian evaluated at the origin is in Jordan canonical form, with a purely imaginary pair of eigenvalues: $\pm$i. Executing the Maple program yields the normal form given in polar coordinates as

$$\begin{aligned} \dot{r} &= \frac{3}{8} r^3 + \frac{5}{16} r^4 + \cdots, \\ \dot{\theta} &= 1 - \frac{1}{3} r^2 + \frac{2023}{6912} r^4 + \cdots, \end{aligned} \tag{2.42}$$

up to fifth order.

Example 2.6 The second example is an expansion of the above example, leading to a five-dimensional system described by

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2 - x_1 x_3, \\ \dot{x}_2 &= -x_1 + x_2^2 + x_1 x_4 + x_2^3, \\ \dot{x}_3 &= -x_3 + x_1^2, \\ \dot{x}_4 &= -x_4 + x_5 + x_1^2, \\ \dot{x}_5 &= -x_4 - x_5 + x_2^2. \end{aligned} \tag{2.43}$$

The Jacobian of the system is in Jordan canonical form, with eigenvalues $\pm$i, -1, $-1 \pm$ i, indicating that the system has a Hopf singularity at the equilibrium point $x_i = 0$. Executing the Maple program yields the normal form of the reduced system in polar coordinates as

$$\begin{aligned} \dot{r} &= \frac{3}{40} r^3 - \frac{14867}{68000} r^5 + \cdots, \\ \dot{\theta} &= 1 - \frac{1}{72} r^2 + \frac{5691403}{14688000} r^4 + \cdots, \end{aligned} \tag{2.44}$$

up to fifth order. This example, as well as the one above, will be discussed again when we study the perturbation method in Sect. 2.3.

Example 2.7 Consider the following six-dimensional system:

$$\begin{aligned} \dot{x}_1 &= -(x_1 - x_2 - x_4)^2, \\ \dot{x}_2 &= x_3 - (x_1 - x_2 + x_5)^2, \\ \dot{x}_3 &= -x_2 - (x_2 - x_3 + x_4)^2, \\ \dot{x}_4 &= -x_4 + (x_1 - x_5)^2, \\ \dot{x}_5 &= -x_5 + x_6 + (x_1 - x_4)^2, \\ \dot{x}_6 &= -x_5 - x_6 + (x_2 + x_5)^2. \end{aligned} \tag{2.45}$$

The system has an equilibrium at the origin, and the eigenvalues of the Jacobian of the system evaluated at the origin are $0, \pm$i, -1, $-1 \pm$ i, implying that the dimension of the center manifold is 3. Executing our Maple programs gives the following normal form up to fifth order:

$$\begin{aligned} \dot{y} &= -y^2 - \frac{1}{2} r^2 + 2y^3 - 2yr^2 + 5y^4 - \frac{411}{160} r^4 + \frac{33}{8} y^2 r^2 - \frac{11}{2} y^5 \\ &\quad + \frac{1013}{40} y^3 r^2 - \frac{503}{800} yr^4 + \cdots, \\ \dot{r} &= yr - \frac{1}{40} r^3 + \frac{1}{2} y^2 r - \frac{53}{20} y^3 r + \frac{83}{50} yr^3 - \frac{75597}{136000} r^5 \\ &\quad + \frac{759}{100} y^4 r + \frac{7423}{1000} y^2 r^3 + \cdots, \\ \dot{\theta} &= 1 - \frac{67}{40} r^2 + \frac{3}{2} y^2 + \frac{33}{10} y^3 + \frac{7}{100} yr^2 - \frac{219}{50} y^4 - \frac{9022679}{816000} r^4 \\ &\quad + \frac{192391}{6000} y^2 r^2 + \cdots. \end{aligned} \tag{2.46}$$

The nonlinear transformation is not listed, but can be obtained from the computer output.

Example 2.8 The equations are described by

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_1^3 - x_1^2 x_5 + x_1^2 x_7,\\
\dot{x}_2 &= -x_1 - 2x_1 x_3^2,\\
\dot{x}_3 &= \sqrt{2}x_4 + x_1^2 x_3 - 4x_5^3,\\
\dot{x}_4 &= -\sqrt{2}x_3,\\
\dot{x}_5 &= -x_5 + (x_1 - x_5)^2,\\
\dot{x}_6 &= -x_6 + x_7 + (x_1 - x_4)^2,\\
\dot{x}_7 &= -x_6 - x_7 + (x_2 - x_6)^2.
\end{aligned} \tag{2.47}$$

The Jacobian of this system evaluated at the origin has two pairs of purely imaginary eigenvalues: $\pm\mathrm{i}$ and $\pm\sqrt{2}\mathrm{i}$, and three noncritical eigenvalues: $-1, -1 \pm \mathrm{i}$. This is an *internal nonresonant* case. The normal form up to fifth order given in polar coordinates is

$$\begin{aligned}
\dot{r}_1 &= \frac{3}{8}r_1^3 + \frac{157}{1360}r_1^5 - \frac{9}{40}r_1^3 r_2^2 + \cdots,\\
\dot{\theta}_1 &= 1 + \frac{1}{2}r_2^2 - \frac{5543}{21760}r_1^4 - \frac{1}{16}r_2^4 - \frac{3}{80}r_1^2 r_2^2 + \cdots,\\
\dot{r}_2 &= \frac{1}{4}r_1^2 r_2 - \frac{1}{16}r_1^2 r_2^3 + \cdots,\\
\dot{\theta}_2 &= \sqrt{2} - \frac{\sqrt{2}}{32}r_1^4 + \cdots.
\end{aligned} \tag{2.48}$$

It should be noted from the normal form (2.48) that the two equations describing the amplitudes r_1 and r_2 are decoupled from the two equations describing the phases θ_1 and θ_2, as expected from the characteristics of nonresonance.

More examples can be found in [21].

2.2.2 A Double-Zero Eigenvalue

For a system with non-semisimple double-zero eigenvalues, Takens [194] and Bogdanov [22] obtained the "form" of the normal forms and gave a very detailed bifurcation analysis. (That's why this case is also called the Takens–Bogdanov zero singularity.) Their results have been extended to the study of global bifurcations and chaos. Baider and Sanders [13] gave a complete formal classification of such a system, while Broer [23] presented a universal local bifurcation analysis for such a

system. Yu and Huseyin [244], on the other hand, studied the static and dynamic bifurcations for this case, and obtained explicit formulas for the bifurcation solutions and stability conditions. But the results are limited to first-order approximations.

It is well known that the two standard normal forms for a double-zero singularity are given by (e.g., see [72])

$$\dot{x} = y + \sum_{j=2}^{k} a_j x^j, \qquad \dot{y} = \sum_{j=2}^{k} b_j x^j, \tag{2.49}$$

and

$$\dot{x} = y, \qquad \dot{y} = \sum_{j=2}^{k} \left(a_j x^j + b_j x^{j-1} y\right), \tag{2.50}$$

where the a_j's and b_j's are constant coefficients.

In the following, we apply Theorem 2.2 to derive the explicit formulas for the normal forms and associated nonlinear transformations of a system characterized by a non-semisimple double-zero eigenvalue.

In this case, $n_0 = 2$ and the Jacobian, J, of (2.9) is now in the form

$$J = \begin{bmatrix} J_0 & 0 \\ 0 & J_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J_1 \end{bmatrix}, \tag{2.51}$$

where J_1 has eigenvalues with nonzero real parts, given in Jordan canonical form. By using (2.25) and (2.26), one may assume the following forms:

$$\begin{aligned} \boldsymbol{C}(\boldsymbol{u}) &= \sum_{m\geq 2} \boldsymbol{C}^m(\boldsymbol{u}) = \sum_{m\geq 2}\sum_{j=0}^{m} \boldsymbol{C}_{j(m-j)} u_1^j u_2^{m-j}, \\ \overline{\boldsymbol{H}}(\boldsymbol{u}) &= \sum_{m\geq 2} \overline{\boldsymbol{H}}^m(\boldsymbol{u}) = \sum_{m\geq 2}\sum_{j=0}^{m} \overline{\boldsymbol{H}}_{j(m-j)} u_1^j u_2^{m-j}, \\ \tilde{\boldsymbol{f}}(\boldsymbol{u}) &= \sum_{m\geq 2} \tilde{\boldsymbol{f}}^m(\boldsymbol{u}) = \sum_{m\geq 2}\sum_{j=0}^{m} \tilde{\boldsymbol{f}}_{j(m-j)} u_1^j u_2^{m-j}, \end{aligned} \tag{2.52}$$

and thus, (2.24) can be written as

$$\begin{aligned} &\sum_{m\geq 2}\left\{\sum_{j=0}^{m-1}\left[(j+1)\overline{\boldsymbol{H}}_{(j+1)(m-j-1)} - J\overline{\boldsymbol{H}}_{j(m-j)}\right]u_1^j u_2^{m-j} - J\overline{\boldsymbol{H}}_{m0} u_1^m\right\} \\ &\quad = \sum_{m\geq 2}\sum_{j=0}^{m}[\tilde{\boldsymbol{f}}_{j(m-j)} - \overline{\boldsymbol{C}}_{j(m-j)}]u_1^j u_2^{m-j}. \end{aligned} \tag{2.53}$$

Comparing the coefficients of the components $u_1^j u_2^{k-j}$ in (2.53) yields

$$(j+1)\overline{\boldsymbol{H}}_{(j+1)(m-j-1)} - J\overline{\boldsymbol{H}}_{j(m-j)} = \tilde{\boldsymbol{f}}_{j(m-j)} - \overline{\boldsymbol{C}}_{j(m-j)}, \quad j = 0, 1, \ldots, m-1, \tag{2.54}$$

and

$$-J\overline{\boldsymbol{H}}_{m0} = \tilde{\boldsymbol{f}}_{m0} - \overline{\boldsymbol{C}}_{m0}. \tag{2.55}$$

Let

$$\begin{aligned} \overline{\boldsymbol{H}}_{ij} &= (\overline{H}_{1,ij},\ \overline{H}_{2,ij}, \overline{H}_{3,ij}, \ldots, \overline{H}_{n,ij})^T, \\ \tilde{\boldsymbol{f}}_{ij} &= (\tilde{f}_{1,ij}, \tilde{f}_{2,ij}, \tilde{f}_{3,ij}, \ldots, \tilde{f}_{n,ij})^T, \\ \overline{\boldsymbol{C}}_{ij} &= (C_{1,ij}, C_{2,ij}, 0, \ldots, 0)^T, \end{aligned} \tag{2.56}$$

then (2.55) can be written as

$$\begin{aligned} -\overline{H}_{2,m0} &= \tilde{f}_{1,m0} - C_{1,m0}, \\ 0 &= \tilde{f}_{2,m0} - C_{2,m0}, \end{aligned} \tag{2.57}$$

and

$$-\lambda_s \overline{H}_{s,m0} = \tilde{f}_{s,m0}, \quad s = 3, 4, \ldots, n. \tag{2.58}$$

On the other hand, the following equations can be found from (2.54):

$$(j+1)\overline{H}_{1,(j+1)(m-j-1)} - \overline{H}_{2,j(m-j)} = \tilde{f}_{1,j(m-j)} - C_{1,j(m-j)}, \tag{2.59}$$

$$(j+1)\overline{H}_{2,(j+1)(m-j-1)} = \tilde{f}_{2,j(m-j)} - C_{2,j(m-j)}, \tag{2.60}$$

and

$$(j+1)\overline{H}_{s,(j+1)(m-j-1)} - \lambda_s \overline{H}_{2,j(m-j)} = \tilde{f}_{s,j(m-j)}, \quad s = 3, 4, \ldots, n, \tag{2.61}$$

where $j = 1, 2, \ldots, m-1$. Note that the coefficients associated with the Jacobian J_1, determined by (2.58) and (2.61) are decoupled (2.59) and (2.60), and thus can be solved independently. Now, let us first find the coefficients associated with the Jacobian J_0, which are determined by (2.59) and (2.60). To achieve this, solving (2.60) for $\overline{H}_{2,(j+1)(m-j-1)}$ and then replacing j by $j-1$ for $j \geq 2$ in the resulting expression yields

$$\overline{H}_{2,j(m-j)} = \frac{1}{j}[\tilde{f}_{2,(j-1)(m-j+1)} - C_{2,(j-1)(m-j+1)}]. \tag{2.62}$$

Then, substituting (2.62) into (2.59) gives

$$\overline{H}_{1,(j+1)(m-j-1)} = \frac{1}{j+1}\left\{\tilde{f}_{1,j(m-j)} - C_{1,j(m-j)} + \frac{1}{j}[\tilde{f}_{2,(j-1)(m-j+1)} - C_{2,(j-1)(m-j+1)}]\right\}, \tag{2.63}$$

where $j = 2, 3, \ldots, m$. Setting $j = m$ in (2.62) and $j = m-1$ in (2.63) results in

$$\overline{H}_{2,m0} = \frac{1}{m}[\tilde{f}_{2,(m-1)1} - C_{2,(m-1)1}], \tag{2.64}$$

$$\overline{H}_{1,m0} = \frac{1}{m}\left\{\tilde{f}_{1,(m-1)1} - C_{1,(m-1)1} + \frac{1}{m-1}[\tilde{f}_{2,(m-2)2} - C_{2,(m-2)2}]\right\}. \tag{2.65}$$

By using (2.64) to eliminate $\overline{H}_{2,m0}$ from (2.57), one can rewrite (2.57) as

$$\begin{aligned} \frac{1}{m}[\tilde{f}_{2,(m-1)1} - C_{2,(m-1)1}] + \tilde{f}_{1,m0} - C_{1,m0} = 0, \\ \tilde{f}_{2,m0} - C_{2,m0} = 0. \end{aligned} \tag{2.66}$$

Since no assumptions have been made in the above derivations, all coefficients $C_{1,ij}$ and $C_{2,ij}$, except those involved in (2.66), can be chosen arbitrarily. In order to find as simple a normal form as possible, these coefficients may be chosen to equal zero. On the other hand, the coefficients $C_{1,m0}$, $C_{2,(m-1)1}$ and $C_{2,m0}$, which must satisfy (2.66), cannot be chosen arbitrarily and may have to be kept in the resulting normal form. Thus, the best choice of the values of these coefficients is one such that the normal form is as simple as possible and (2.66) is also satisfied. Obviously, there are two possible choices:

$$\begin{aligned} &\text{if } C_{2,(m-1)1} = 0 \quad \text{then } C_{1,m0} = \tfrac{1}{m}\tilde{f}_{2,(m-1)1} + \tilde{f}_{1,m0}; \\ &\text{if } C_{1,m0} = 0 \quad \text{then } C_{2,(m-1)1} = \tilde{f}_{2,(m-1)1} + m\tilde{f}_{1,m0}. \end{aligned}$$

Combining these two choices for different order normal forms may result in infinitely many different normal forms. However, if these two choices are used consistently for all orders of normal form, then one may have the following two different normal forms.

Theorem 2.9 *The normal form of the double-zero singularity up to an arbitrary order k is given by*

$$\begin{aligned} \dot{u}_1 &= u_2 + \sum_{m=2}^{k} C_{1,m0} u_1^m, \\ \dot{u}_2 &= \sum_{m=2}^{k} C_{2,m0} u_1^m, \end{aligned} \tag{2.67}$$

or

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= \sum_{m=2}^{k} \left(C_{2,(m-1)1} u_1^{m-1} u_2 + C_{2,m0} u_1^m\right). \end{aligned} \tag{2.68}$$

The normal form coefficients $C_{2,(m-1)1}$ *and* $C_{2,m0}$ *for the form* (2.68) *are then given by*

$$\begin{aligned} C_{2,(m-1)1} &= \tilde{f}_{2,(m-1)1} + m\tilde{f}_{1,m0}, \\ C_{2,m0} &= \tilde{f}_{2,m0}. \end{aligned} \tag{2.69}$$

The above two forms, (2.67) and (2.68), are the same as those given by (2.49) and (2.50), respectively. Usually, the second form, (2.68), is chosen for bifurcation analysis. Moreover, the coefficients of the corresponding nonlinear transformation are determined as

$$\begin{aligned} &\overline{H}_{2,(j+1)(m-j-1)} = \frac{1}{j+1}[\tilde{f}_{2,j(m-j)} - C_{2,j(m-j)}], \\ &\quad j = 0, 1, \ldots, m-1, \\ &\overline{H}_{1,(j+1)(m-j-1)} = \frac{1}{j+1}\left[\tilde{f}_{1,j(m-j)} + \frac{1}{j}\tilde{f}_{2,(j-1)(m-j+1)}\right], \\ &\quad j = 1, 2, \ldots, m-1, \\ &\overline{H}_{2,m0} = -\tilde{f}_{1,m0}. \end{aligned} \tag{2.70}$$

Here, one may observe from (2.70) that the three coefficients $\overline{H}_{1,1(m-1)}$, $\overline{H}_{1,0m}$ and $\overline{H}_{2,0m}$ are not defined in these equations. In addition, since $\overline{H}_{1,0m}$ does not appear in any of these equations, it can be chosen arbitrarily. The other two coefficients, $\overline{H}_{1,1(m-1)}$ and $\overline{H}_{2,0m}$, must satisfy an equation obtained from (2.54) by setting $j = 0$, i.e.,

$$\overline{H}_{1,1(m-1)} - \overline{H}_{2,0m} = \tilde{f}_{1,0m}, \tag{2.71}$$

which indicates that these two coefficients are not independent. Thus, one of them can be chosen arbitrarily and the other is then determined. To make the normal form definite, we adopt the following choices:

$$\begin{aligned} &\overline{H}_{1,0m} = 0, \\ &\overline{H}_{2,0m} = 0, \\ &\overline{H}_{1,1(m-1)} = \tilde{f}_{1,0m}. \end{aligned} \tag{2.72}$$

Finally, the coefficients of the nonlinear transformation associated with the Jacobian J_1 can be uniquely determined as follows. First, from (2.58), one can find

$$\overline{H}_{s,m0} = -\frac{1}{\lambda_s}\tilde{f}_{s,m0}, \quad s = 3, 4, \ldots, n. \tag{2.73}$$

Then, an iteration equation can be derived from (2.61) as

$$\overline{H}_{s,j(m-j)} = -\frac{1}{\lambda_s}\left(\tilde{f}_{s,j(m-j)} - (j+1)\overline{H}_{s,(j+1)(m-j-1)}\right) \tag{2.74}$$

for $j = 1, 2, \ldots, (m-1)$ and $s = 3, 4, \ldots, n$.

Therefore, the normal form and the associated nonlinear transformation are uniquely determined. A symbolic computation software package has been developed for (non-semisimple) double-zero and triple-zero singularities [20]. An example of the application of a double-zero singularity is shown in Sect. 5.2.2.

2.2.3 1:1 Resonant Hopf Bifurcation

Hopf and generalized Hopf bifurcations have been extensively studied by many researchers (e.g., see [47, 54, 55, 63, 72, 102, 166, 218]), and are associated with a pair of purely imaginary eigenvalues at an equilibrium. If the Jacobian of a system evaluated at a critical point involves two pairs of purely imaginary eigenvalues, the so-called "double-Hopf bifurcation" may occur. Such bifurcations may exhibit more complicated and interesting dynamic behavior such as quasiperiodic motions on tori, and chaos (e.g., see [21, 72, 102, 166, 221, 223, 226, 231]). A bifurcation is called *nonresonant* if the ratio of the two eigenvalues is not a rational number, otherwise it is called *resonant*. The most important resonance is the 1:1 non-semisimple case, which is also usually called *1:1 double-Hopf bifurcation*. In this subsection, the method developed above is applied to consider the 1:1 non-semisimple case for general n-dimensional systems which are not necessarily described on a four-dimensional center manifold.

For this case, $J = \text{diag}[J_0\, J_1]$, and both J_0 and J_1 are assumed to be in Jordan canonical form. In particular, J_0 has eigenvalues $\pm \mathrm{i}\omega$. Without loss of generality, we may assume $\omega = 1$, and thus J_0 is given in the form

$$J_0 = \begin{bmatrix} \mathrm{i} & 1 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 1 \\ 0 & 0 & 0 & -\mathrm{i} \end{bmatrix}. \tag{2.75}$$

It should be noted that the third and fourth equations of (2.9) are the complex conjugates of the first and second equations, respectively, and thus $x_3 = \bar{x}_1$ and $x_4 = \bar{x}_2$, where the bar denotes complex conjugate. J_1, on the other hand, is assumed to have eigenvalues with nonzero real parts. One may assume that J_1 is given in the following general block form:

$$J_1 = \begin{bmatrix} J_{15} & 0 & \cdots & 0 \\ 0 & J_{16} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{1i} \end{bmatrix}, \tag{2.76}$$

where

$$J_{1j} = \begin{bmatrix} \lambda_j & s & \cdots & 0 & 0 \\ 0 & \lambda_j & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_j & s \\ 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix}, \tag{2.77}$$

for $j = 5, 6, \ldots, i$. Note that the dimension of J_1 is $(n-4) \times (n-4)$. Also note that the "s" given in (2.77), on the shoulder of the λ_j's, only takes the value of either 1 or 0. This indicates that both J_0 and J_1 are non-semisimple.

Now, (2.23) or (2.24) can be rewritten as

$$\begin{aligned} &\sum_m \begin{pmatrix} [\lambda_0 I - J_0]\boldsymbol{H}_{m_1m_2m_3m_4} \\ [\lambda_0 I - J_1]\boldsymbol{h}_{m_1m_2m_3m_4} \end{pmatrix} u_1^{m_1} u_2^{m_2} u_3^{m_3} u_4^{m_4} \\ &\quad + \sum_m m_1 \begin{pmatrix} \boldsymbol{H}_{m_1m_2m_3m_4} \\ \boldsymbol{h}_{m_1m_2m_3m_4} \end{pmatrix} u_1^{m_1-1} u_2^{m_2+1} u_3^{m_3} u_4^{m_4} \\ &\quad + \sum_m m_3 \begin{pmatrix} \boldsymbol{H}_{m_1m_2m_3m_4} \\ \boldsymbol{h}_{m_1m_2m_3m_4} \end{pmatrix} u_1^{m_1} u_2^{m_2} u_3^{m_3-1} u_4^{m_4+1} \\ &= \sum_m \begin{pmatrix} \tilde{\boldsymbol{f}}_{1m_1m_2m_3m_4} - \boldsymbol{C}_{m_1m_2m_3m_4} \\ \tilde{\boldsymbol{f}}_{2m_1m_2m_3m_4} \end{pmatrix} u_1^{m_1} u_2^{m_2} u_3^{m_3} u_4^{m_4} \end{aligned} \tag{2.78}$$

where

$$\lambda_0 = (m_1 + m_2 - m_3 - m_4)\mathrm{i} \quad \text{and} \quad m_1 + m_2 + m_3 + m_4 = m. \tag{2.79}$$

Equation (2.78) can be used for solving for the coefficients $\boldsymbol{C}_m$, $\boldsymbol{H}_m$ and $\boldsymbol{h}_m$, order by order, starting from $m = 2$, for all possible choices such that $m_1 + m_2 + m_3 + m_4 = m$. Here, $\tilde{\boldsymbol{f}}_{1m}$ and $\tilde{\boldsymbol{f}}_{2m}$ denote the mth-order coefficients of $\boldsymbol{f}_1(\boldsymbol{u} + \boldsymbol{H}(\boldsymbol{u}), \boldsymbol{h}(\boldsymbol{u})) - D\boldsymbol{H}(\boldsymbol{u})\boldsymbol{C}(\boldsymbol{u})$ and $\boldsymbol{f}_2(\boldsymbol{u} + \boldsymbol{H}(\boldsymbol{u}), \boldsymbol{h}(\boldsymbol{u})) - D\boldsymbol{h}(\boldsymbol{u})\boldsymbol{C}(\boldsymbol{u})$, respectively. In the following we shall present a procedure for solving (2.78).

Firstly, it is noted from (2.23) that the mth-order coefficients $\tilde{\boldsymbol{f}}_{1m}$ and $\tilde{\boldsymbol{f}}_{2m}$ contain $\boldsymbol{C}$, $\boldsymbol{H}$ and $\boldsymbol{h}$ coefficients whose orders are less than m. Therefore, for computing the mth-order coefficients, we only need to consider the terms on the left-hand side of (2.78) and the $\boldsymbol{C}_m$ term on the right-hand side. Secondly, since λ_0 only contains the eigenvalues of J_0 (with zero real parts) and all eigenvalues of J_1 have nonzero real parts, $[\lambda_0 I - J_1]$ cannot equal zero in any of its components. This suggests that $\boldsymbol{h}_m$ can be uniquely determined from (2.78). However, note that the second and third terms on the left-hand side of (2.78) do not have the same index as other terms. Thus, the solution procedure is not as simple as the semisimple case [226]. Making

the index consistent in (2.78) yields

$$\begin{aligned}&[\lambda_0 I - J_0]\boldsymbol{H}_{m_1m_2m_3m_4} + (m_1+1)\boldsymbol{H}_{(m_1+1)(m_2-1)m_3m_4}\\&\quad + (m_3+1)\boldsymbol{H}_{m_1m_2(m_3+1)(m_4-1)}\\&= \tilde{\boldsymbol{f}}_{1m_1m_2m_3m_4} - \boldsymbol{C}_{m_1m_2m_3m_4}\end{aligned} \tag{2.80}$$

and

$$\begin{aligned}&[\lambda_0 I - J_1]\boldsymbol{h}_{m_1m_2m_3m_4} + (m_1+1)\boldsymbol{h}_{(m_1+1)(m_2-1)m_3m_4}\\&\quad + (m_3+1)\boldsymbol{h}_{m_1m_2(m_3+1)(m_4-1)}\\&= \tilde{\boldsymbol{f}}_{2m_1m_2m_3m_4}.\end{aligned} \tag{2.81}$$

Note that (2.81) does not contain the normal form coefficients $\boldsymbol{C}_{m_1m_2m_3m_4}$. In addition, it is easy to show that the matrix $[\lambda_0 I - J_1]$ is nonsingular. Therefore, one can uniquely determine $\boldsymbol{h}_{m_1m_2m_3m_4}$ from (2.81) by using the following recursive formulas, where $m_1+m_2+m_3+m_4=m, m_i \geq 0, i=1,2,3,4$. For convenience, let $k_1 = m_1+m_2, k_2 = m_3+m_4$ and so $k_1+k_2=m$.

(1) For $m_2 = m_4 = 0$:

$$\boldsymbol{h}_{k_10k_20} = [\lambda_0 I - J_1]^{-1}\tilde{\boldsymbol{f}}_{2k_10k_20}. \tag{2.82}$$

(2) For $m_4 = 0, m_2 = 1, 2, \ldots, k_1$:

$$\begin{aligned}&\boldsymbol{h}_{(k_1-m_2)m_2k_20}\\&\quad = [\lambda_0 I - J_1]^{-1}\big(\tilde{\boldsymbol{f}}_{2(k_1-m_2)m_2k_20} - (k_1-m_2+1)\boldsymbol{h}_{(k_1-m_2+1)(m_2-1)k_20}\big).\end{aligned} \tag{2.83}$$

(3) For $1 \leq m_4 \leq k_2, m_2 = 0, 1, 2, \ldots, k_1$:

$$\begin{aligned}&\boldsymbol{h}_{k_10(k_2-m_4)m_4}\\&\quad = [\lambda_0 I - J_1]^{-1}\big(\tilde{\boldsymbol{f}}_{2k_10(k_2-m_4)m_4} - (k_2-m_4+1)\boldsymbol{h}_{k_10(k_2-m_4+1)(m_4-1)}\big)\end{aligned} \tag{2.84}$$

when $m_2 = 0$, and

$$\begin{aligned}\boldsymbol{h}_{(k_1-m_2)m_2(k_2-m_4)m_4} = [\lambda_0 I - J_1]^{-1}\big(&\tilde{\boldsymbol{f}}_{2(k_1-m_2)m_2(k_2-m_4)m_4}\\&- (k_1-m_2+1)\boldsymbol{h}_{(k_1-m_2+1)(m_2-1)(k_2-m_4)m_4}\\&- (k_2-m_4+1)\boldsymbol{h}_{(k_1-m_2)m_2(k_2-m_4+1)(m_4-1)}\big)\end{aligned} \tag{2.85}$$

when $m_2 = 1, 2, \ldots, k_1$.

Now, we solve (2.80) for $\boldsymbol{H}_{m_1m_2m_3m_4}$ and $\boldsymbol{C}_{m_1m_2m_3m_4}$. Since we want to choose the normal form as simply as possible, we may set all the coefficients $\boldsymbol{C}_{m_1m_2m_3m_4}$ to zero if the matrix $[\lambda_0 I - J_0]$ is nonsingular, and then the coefficients $\boldsymbol{H}_{m_1m_2m_3m_4}$ are uniquely determined from (2.80). However, the matrix $[\lambda_0 I - J_0]$ can be singular, giving rise to the so-called *resonant* terms which have to be retained in the normal form. Because the third and fourth equations of (2.7) are the complex conjugates of the first and second equations, respectively, thus the third and fourth components of $\boldsymbol{H}_m$ and $\boldsymbol{C}_m$ are, respectively, the complex conjugates of the first and second components of $\boldsymbol{H}_m$ and $\boldsymbol{C}_m$. Therefore, we only need to consider the first two equations of (2.80) to solve $\boldsymbol{H}_m^* = (H_{1m_1m_2m_3m_4}, H_{2m_1m_2m_3m_4})^T$ and $\boldsymbol{C}_m^* = (C_{1m_1m_2m_3m_4}, C_{2m_1m_2m_3m_4})^T$.

Since

$$[\lambda_0 I - J_0] = \begin{bmatrix} \lambda_0 \mathrm{i} & 1 & 0 & 0 \\ 0 & \lambda_0 \mathrm{i} & 0 & 0 \\ 0 & 0 & -\lambda_0 \mathrm{i} & 1 \\ 0 & 0 & & -\lambda_0 \mathrm{i} \end{bmatrix}, \tag{2.86}$$

where $\lambda_0 = k_1 - k_2 - 1$, there exist two cases: (i) $k_1 - k_2 \neq 1$ (i.e., $\lambda_0 \neq 0$) which is called the *nonresonant* case; and (ii) $k_1 - k_2 = 1$ (i.e., $\lambda_0 = 0$) called the *resonant* case.

(A) Nonresonant Case: $m_1 + m_2 \neq m_3 + m_4 + 1$ For the nonresonant case, $k_1 - k_2 \neq 1$, i.e., $m_1 + m_2 \neq m_3 + m_4 + 1$, we may set all $\boldsymbol{C}^*_{m_1m_2m_3m_4} = 0$, and then similarly determine $\boldsymbol{H}^*_{m_1m_2m_3m_4}$ from (2.80). The detailed steps for solving for $\boldsymbol{H}^*_{m_1m_2m_3m_4}$ are given below.

(1) For $m_2 = m_4 = 0$:

$$\boldsymbol{H}^*_{k_10k_20} = [\lambda_0 I - J_0]^{-1} \tilde{\boldsymbol{f}}^*_{1k_10k_20}. \tag{2.87}$$

(2) For $m_4 = 0$, $m_2 = 1, 2, \ldots, k_1$:

$$\begin{aligned} \boldsymbol{H}^*_{(k_1-m_2)m_2k_20} = [\lambda_0 I - J_0]^{-1} \big(& \tilde{\boldsymbol{f}}^*_{1(k_1-m_2)m_2k_20} - (k_1 - m_2 + 1) \\ & \times \boldsymbol{H}^*_{(k_1-m_2+1)(m_2-1)k_20} \big). \end{aligned} \tag{2.88}$$

(3) For $1 \leq m_4 \leq k_2$, $m_2 = 0, 1, 2, \ldots, k_1$:

$$\begin{aligned} \boldsymbol{H}^*_{k_10(k_2-m_4)m_4} = [\lambda_0 I - J_0]^{-1} \big(& \tilde{\boldsymbol{f}}^*_{1k_10(k_2-m_4)m_4} - (k_2 - m_4 + 1) \\ & \times \boldsymbol{H}^*_{k_10(k_2-m_4+1)(m_4-1)} \big) \end{aligned} \tag{2.89}$$

when $m_2 = 0$, and

$$\begin{aligned} \boldsymbol{H}^*_{(k_1-m_2)m_2(k_2-m_4)m_4} = [\lambda_0 I - J_0]^{-1} \big(& \tilde{\boldsymbol{f}}^*_{1(k_1-m_2)m_2(k_2-m_4)m_4} \\ & - (k_1 - m_2 + 1) \boldsymbol{H}^*_{(k_1-m_2+1)(m_2-1)(k_2-m_4)m_4} \\ & - (k_2 - m_4 + 1) \boldsymbol{H}^*_{(k_1-m_2)m_2(k_2-m_4+1)(m_4-1)} \big) \end{aligned} \tag{2.90}$$

when $m_2 = 1, 2, \ldots, k_1$.

(B) Resonant Case: $m_1 + m_2 = m_3 + m_4 + 1$ When $m_1 + m_2 = m_3 + m_4 + 1$ or $k_1 = k_2 + 1$, the matrix $[\lambda_0 I - J_0]$ is singular and we may not be able to set all $\boldsymbol{C}^*_{m_1 m_2 m_3 m_4}$ to zero. These nonzero $\boldsymbol{C}^*_m$ terms consist of the normal form. From $k_1 + k_2 = m$ and $k_1 - k_2 = 1$, we obtain $k_1 = \frac{1}{2}(m+1)$ and $k_2 = \frac{1}{2}(m-1)$. Since both k_1 and k_2 are non-negative integers, m must be a positive odd integer. Let $m = 2k+1$, then $k_1 = k+1$ and $k_2 = k$ for $k = 1, 2, \ldots$. Thus, (2.80) becomes

$$\begin{aligned}(k+2-m_2)\boldsymbol{H}_{1(k+2-m_2)(m_2-1)(k-m_4)m_4} &- \boldsymbol{H}_{2(k+1-m_2)m_2(k-m_4)m_4} \\ = \tilde{\boldsymbol{f}}_{11(k+1-m_2)m_2(k-m_4)m_4} &- \boldsymbol{C}_{1(k+1-m_2)m_2(k-m_4)m_4} \\ - (k+1-m_4)\boldsymbol{H}_{1(k+1-m_2)m_2(k+1-m_4)(m_4-1)}&, \end{aligned} \tag{2.91}$$

and

$$\begin{aligned}(k+2-m_2)\boldsymbol{H}_{2(k+2-m_2)(m_2-1)(k-m_4)m_4} & \\ = \tilde{\boldsymbol{f}}_{12(k+1-m_2)m_2(k-m_4)m_4} &- \boldsymbol{C}_{2(k+1-m_2)m_2(k-m_4)m_4} \\ - (k+1-m_4)\boldsymbol{H}_{2(k+1-m_2)m_2(k+1-m_4)(m_4-1)}&. \end{aligned} \tag{2.92}$$

It seems that we may similarly find recursive formulas from (2.91) and (2.92). However, the procedure is actually not as straightforward as the nonresonant case since one has to determine which components of $\boldsymbol{C}^*_m$ should be retained.

Nevertheless, we can still follow a procedure similar to that used for the nonresonant case to obtain the following results.

(1) $m_4 = 0$, $m_2 = 0, 1, \ldots, k+1$:

$$\begin{aligned}(k+2-m_2)\boldsymbol{H}_{1(k+2-m_2)(m_2-1)k0} &- \boldsymbol{H}_{2(k+1-m_2)m_2k0} \\ = \tilde{\boldsymbol{f}}_{11(k+1-m_2)m_2k0} &- \boldsymbol{C}_{1(k+1-m_2)m_2k0}, \end{aligned} \tag{2.93}$$

$$\begin{aligned}(k+2-m_2)\boldsymbol{H}_{2(k+2-m_2)(m_2-1)k0} & \\ = \tilde{\boldsymbol{f}}_{12(k+1-m_2)m_2k0} &- \boldsymbol{C}_{2(k+1-m_2)m_2k0}, \end{aligned} \tag{2.94}$$

when $m_2 = 1, 2, \ldots, k+1$; and

$$-\boldsymbol{H}_{2(k+1)0k0} = \tilde{\boldsymbol{f}}_{11(k+1)0k0} - \boldsymbol{C}_{1(k+1)0k0}, \tag{2.95}$$

$$0 = \tilde{\boldsymbol{f}}_{12(k+1)0k0} - \boldsymbol{C}_{2(k+1)0k0}, \tag{2.96}$$

when $m_2 = 0$.

It follows from (2.96) that

$$\boldsymbol{C}_{2(k+1)0k0} = \tilde{\boldsymbol{f}}_{12(k+1)0k0}. \tag{2.97}$$

Then, a careful consideration of (2.93)–(2.96) shows that the following three equations:

$$(k+1)\boldsymbol{H}_{2(k+1)0k0} = \tilde{\boldsymbol{f}}_{12k1k0} - \boldsymbol{C}_{2k1k0}, \tag{2.98}$$

$$-\boldsymbol{H}_{2(k+1)0k0} = \tilde{\boldsymbol{f}}_{11(k+1)0k0} - \boldsymbol{C}_{1(k+1)0k0}, \tag{2.99}$$

$$\boldsymbol{H}_{11kk0} - \boldsymbol{H}_{20(k+1)k0} = \tilde{\boldsymbol{f}}_{110(k+1)k0} - \boldsymbol{C}_{10(k+1)k0}, \tag{2.100}$$

are unsolved, while the other equations are used to obtain

$$\begin{aligned} \boldsymbol{C}_{1(k+1-m_2)m_2k0} &= 0 \quad \text{for } m_2 = 1, 2, \ldots, k, \\ \boldsymbol{C}_{2(k+1-m_2)m_2k0} &= 0 \quad \text{for } m_2 = 2, 3, \ldots, k+1, \end{aligned} \tag{2.101}$$

and

$$\boldsymbol{H}_{2(k+2-m_2)(m_2-1)k0} = \frac{1}{k+2-m_2} \tilde{\boldsymbol{f}}_{12(k+1-m_2)m_2k0} \tag{2.102}$$

for $m_2 = 2, 3, \ldots, k+1$; and then

$$\begin{aligned} &\boldsymbol{H}_{1(k+2-m_2)(m_2-1)k0} \\ &= \frac{1}{k+2-m_2} [\tilde{\boldsymbol{f}}_{11(k+1-m_2)m_2k0} + \boldsymbol{H}_{2(k+1-m_2)m_2k0}], \end{aligned} \tag{2.103}$$

for $m_2 = 1, 2, \ldots, k$.

Next, we shall consider (2.98)–(2.100) and determine how to choose the $\boldsymbol{C}_m$ coefficients which must be used to solve these equations. First note that (2.100) is decoupled from (2.98) and (2.99), so we may set $\boldsymbol{C}_{10(k+1)k0}$ to zero and then use either $\boldsymbol{H}_{11kk0}$ or $\boldsymbol{H}_{20(k+1)k0}$ to solve the equation. Hence, we obtain

$$\begin{aligned} \boldsymbol{H}_{11kk0} &= \tilde{\boldsymbol{f}}_{110(k+1)k0}, \\ \boldsymbol{C}_{10(k+1)k0} &= \boldsymbol{H}_{20(k+1)k0} = 0. \end{aligned} \tag{2.104}$$

Equations (2.98) and (2.99) can eliminate only one of $\boldsymbol{C}_{1(k+1)0k0}$ and $\boldsymbol{C}_{2k1k0}$, however, since there is only one $\boldsymbol{H}_m$ coefficient involved in these two equations (i.e., $\boldsymbol{H}_{2(k+1)0k0}$). Further, note that $\boldsymbol{H}_{10(k+1)k0}$ does not appear in (2.93)–(2.96) and can thus be set to zero. In summary, for $m_4 = 0$, we have two choices:

(i) $\boldsymbol{C}_{1(k+1)0k0} \neq 0, \boldsymbol{C}_{2k1k0} = 0$;
(ii) $\boldsymbol{C}_{1(k+1)0k0} = 0, \boldsymbol{C}_{2k1k0} \neq 0$.

The other components of $\boldsymbol{C}_m^*$ and $\boldsymbol{H}_m^*$ are

$$\begin{aligned} \boldsymbol{C}_{1(k+1-m_2)m_2k0} &= 0 \quad \text{for } m_2 = 1, 2, \ldots, k, \\ \boldsymbol{C}_{2(k+1-m_2)m_2k0} &= 0 \quad \text{for } m_2 = 2, 3, \ldots, k+1, \\ \boldsymbol{C}_{2(k+1)0k0} &\neq 0 \quad \text{(see (2.97))}, \\ \boldsymbol{H}_{1(k+2-m_2)(m_2-1)k0} &\neq 0 \quad \text{for } m_2 = 1, 2, \ldots, k+1, \\ \boldsymbol{H}_{2(k+2-m_2)(m_2-1)k0} &\neq 0 \quad \text{for } m_2 = 1, 2, \ldots, k+1 \\ &\quad \text{(see (2.98), (2.99), (2.103) and (2.104))}, \\ \boldsymbol{H}_{10(k+1)k0} &= \boldsymbol{H}_{20(k+1)k0} = 0. \end{aligned}$$

(2) When $1 \le m_4 \le k$, we have similar formulas to $0 \le m_2 \le k+1$ (like those for case $m_4 = 0$). The detailed formulations are omitted here but the results are listed below. There are two choices:
(i) $\boldsymbol{C}_{1(k+1)0(k-m_4)m_4} \neq 0, \boldsymbol{C}_{2k1(k-m_4)m_4} = 0$;
(ii) $\boldsymbol{C}_{1(k+1)0(k-m_4)m_4} = 0, \boldsymbol{C}_{2k1(k-m_4)m_4} \neq 0$.
The other components of $\boldsymbol{C}_m^*$ and $\boldsymbol{H}_m^*$ are

$$\begin{aligned}
\boldsymbol{C}_{1(k+1-m_2)m_2(k-m_4)m_4} &= 0 \quad \text{for } m_2 = 1, 2, \ldots, k,\\
\boldsymbol{C}_{2(k+1-m_2)m_2(k-m_4)m_4} &= 0 \quad \text{for } m_2 = 2, 3, \ldots, k+1,\\
\boldsymbol{C}_{2(k+1)0(k-m_4)m_4} &\neq 0,\\
\boldsymbol{H}_{1(k+2-m_2)(m_2-1)(k-m_4)m_4} &\neq 0 \quad \text{for } m_2 = 1, 2, \ldots, k+1,\\
\boldsymbol{H}_{2(k+2-m_2)(m_2-1)(k-m_4)m_4} &\neq 0 \quad \text{for } m_2 = 1, 2, \ldots, k+1,\\
\boldsymbol{H}_{10(k+1)(k-m_4)m_4} &= \boldsymbol{H}_{20(k+1)(k-m_4)m_4} = 0.
\end{aligned}$$

Summarizing the above results yields the following theorem.

Theorem 2.10 *The two normal forms associated with non-semisimple* 1:1 *Hopf bifurcation are*

$$\begin{aligned}
\dot{u}_1 &= \mathrm{i}u_1 + u_2 + \sum_{k\ge 1}\sum_{j=0}^{k} \boldsymbol{C}_{1(k+1)0(k-j)j} u_1^{k+1} \bar{u}_1^{k-j} \bar{u}_2^{j},\\
\dot{u}_2 &= \mathrm{i}u_2 + \sum_{k\ge 1}\sum_{j=0}^{k} \boldsymbol{C}_{2(k+1)0(k-j)j} u_1^{k+1} \bar{u}_1^{k-j} \bar{u}_2^{j},
\end{aligned} \tag{2.105}$$

and

$$\begin{aligned}
\dot{u}_1 &= \mathrm{i}u_1 + u_2,\\
\dot{u}_2 &= \mathrm{i}u_2 + \sum_{k\ge 1}\sum_{j=0}^{k} \boldsymbol{C}_{2(k+1)0(k-j)j} u_1^{k+1} \bar{u}_1^{k-j} \bar{u}_2^{j}\\
&\quad + \sum_{k\ge 1}\sum_{j=0}^{k} \boldsymbol{C}_{2k1(k-j)j} u_1^{k} u_2 \bar{u}_1^{k-j} \bar{u}_2^{j}.
\end{aligned} \tag{2.106}$$

The form (2.106) is usually used in applications since its first equation has only two terms, which greatly simplifies finding equilibrium solutions.

To end this section, we give an example to illustrate the application of Theorem 2.10. The results are obtained directly by executing a Maple program, developed on the basis of Theorem 2.10.

Example 2.11 A nonlinear electrical circuit, shown in Fig. 2.5, consists of a DC source E, two capacitors C_1, C_2, two inductors L_1, L_2, a resistor R, and a conductance. L_1 and C_1 are connected in parallel, while L_2, C_2 and R are in series. All five

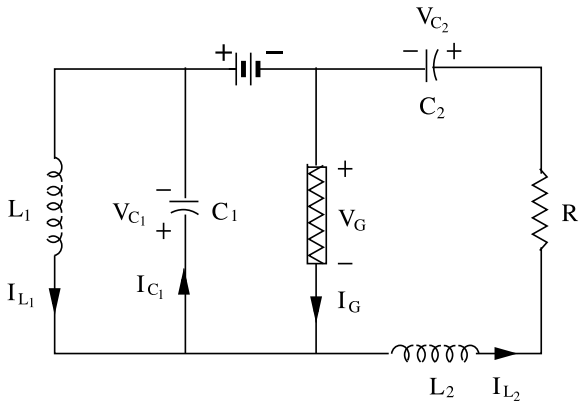

Fig. 2.5 A nonlinear electrical circuit

elements, L_1, L_2, C_1, C_2 and R are assumed to be linear time-invariant elements, but C_1 and R may be varied as control parameters. The conductance, however, is a nonlinear element with the characteristics

$$I_G = -\alpha V_G + \beta V_G^3, \quad \alpha > 0, \beta > 0, \tag{2.107}$$

where I_G and V_G represent the current and voltage of the conductance, respectively. The two parameters α and β may also be considered as control parameters.

The voltages across the capacitors and the currents in the inductors are chosen as the state variables (as shown in Fig. 2.5), leading to the equations of the circuit, given by

$$\begin{aligned}
\dot{z}_1 &= \frac{1}{C_1}\left(\alpha z_1 + z_2 - z_4 - \beta z_1^3\right),\\
\dot{z}_2 &= -\frac{1}{L_1} z_1,\\
\dot{z}_3 &= \frac{1}{C_2} z_4,\\
\dot{z}_4 &= \frac{1}{L_2}(E + z_1 - z_3 - R z_4),
\end{aligned} \tag{2.108}$$

where z_1, z_2, z_3 and z_4 denote the state variables V_{C_1}, I_{L_1}, V_{C_2} and I_{L_2}, respectively. It is clear that $(z_1, z_2, z_3, z_4) = (0, 0, E, 0)$ is the unique equilibrium of the system. Let all the parameters $C_1, C_2, L_1, L_2, R, E, \alpha$ and β equal one unit, then the Jacobian of system (2.108) evaluated at the equilibrium has eigenvalues $\lambda_{1,2} = \lambda_{3,4} = \pm \mathrm{i}$, indicating a double-Hopf bifurcation. Applying the linear transformation, given by

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{bmatrix} 2 & 0 & 4 & 1 \\ 0 & -2 & -1 & -4 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{2.109}$$

to (2.108) results in

$$\begin{aligned}
\dot{x}_1 &= -x_2 + x_3,\\
\dot{x}_2 &= x_1 + x_4 + \frac{1}{8}(2x_1 + 4x_3 + x_4)^3,\\
\dot{x}_3 &= -x_4 - \frac{1}{4}(2x_1 + 4x_3 + x_4)^3,\\
\dot{x}_4 &= x_3.
\end{aligned} \tag{2.110}$$

Executing the Maple program gives the following complex normal form up to third-order terms:

$$\begin{aligned}
\dot{u}_1 &= \mathrm{i}u_1 + u_2,\\
\dot{u}_2 &= \mathrm{i}u_2 - \frac{3}{4}u_1^2\bar{u}_1 - \frac{3}{2}u_1^2\bar{u}_2 - \left(3 - \frac{3}{2}\mathrm{i}\right)u_1u_2\bar{u}_1\\
&\quad - \left(\frac{3}{2} + 3\mathrm{i}\right)u_1u_2\bar{u}_2 + \cdots,
\end{aligned} \tag{2.111}$$

which can be transformed to real form by using the polar coordinates, $u_1 = R_1e^{\mathrm{i}\theta_1}$, $u_2 = R_2e^{\mathrm{i}\theta_2}$, as follows:

$$\dot{R}_1 = R_2\cos\phi, \tag{2.112}$$

$$\begin{aligned}
\dot{R}_2 &= -\frac{3}{4}R_1^3\cos\phi - 3R_1^2R_2\\
&\quad - \frac{3}{2}R_1^2R_2\cos 2\phi - \frac{3}{2}R_1R_2^2\cos\phi + 3R_1R_2^2\sin\phi,
\end{aligned} \tag{2.113}$$

$$\begin{aligned}
\dot{\phi} &= -\frac{R_2}{R_1}\sin\phi - \frac{3}{2}R_1^2 + 3R_2R_1\cos\phi + \frac{3}{2}R_2R_1\sin\phi\\
&\quad + \frac{3}{2}R_1^2\sin 2\phi + \frac{3R_1^3}{4R_2}\sin\phi,
\end{aligned} \tag{2.114}$$

where $\phi = \theta_1 - \theta_2$ is the phase difference of motion.

Usually, there exist periodic solutions bifurcating from a 1:1 double-Hopf bifurcation, and they can be determined by the steady-state solution of (2.114).

Letting $\dot{R}_1 = \dot{R}_2 = \dot{\phi} = 0$ yields nonzero steady-state solutions (periodic solutions). It follows from (2.112) that

$$\phi = \phi_+ = \frac{1}{2}\pi \quad \text{or} \quad \phi = \phi_- = \frac{3}{2}\pi, \tag{2.115}$$

where the subscripts $\pm$ indicate that $\sin\phi_\pm = \pm 1$. Substituting $\phi = \phi_\pm$ into (2.113) and (2.115) yields two sets of polynomial equations:

$$-\frac{3}{2}R_1^2R_2 \pm 3R_1R_2^2 = 0, \tag{2.116}$$

$$\frac{1}{4R_1R_2}\left(\mp 4R_2^2 - 6R_1^3R_2 \pm 6R_1^2R_2^2 \pm 3R_1^4\right) = 0. \tag{2.117}$$

It is easy to see from (2.116) that there is no nonzero solution for (R_1, R_2) when $\phi = \phi_+ = \frac{1}{2}\pi$. When $\phi = \phi_- = \frac{3}{2}\pi$, the only nonzero solution (phase-locked solution) is

$$(R_1, R_2, \phi) = \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{3}{2}\pi\right). \tag{2.118}$$

To find the stability of this periodic solution, evaluating the Jacobian of (2.112)–(2.114), given by

$$J = \begin{bmatrix} 0 & 0 & -R_2 \\ -3R_1R_2 + 3R_2^2 & -\frac{3}{2}R_1^2 + 6R_1R_2 & \frac{3}{2}R_1R_2^2 + \frac{3}{4}R_1^3 \\ -3R_1 + \frac{3}{2}R_2 + \frac{R_2}{R_1^2} + \frac{9R_1^2}{4R_2} & -\frac{1}{R_1} + \frac{3}{2}R_1 - \frac{3R_1^3}{4R_2^2} & -3R_1R_2 - 3R_1^2 \end{bmatrix} \tag{2.119}$$

at the solution (2.118) yields

$$J = \begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{6}} \\ -\frac{1}{2} & 1 & \frac{3}{2\sqrt{6}} \\ \sqrt{6} & -\sqrt{6} & -3 \end{bmatrix}.$$

The three eigenvalues of the above Jacobian matrix are

$$-2.1245702691, \qquad -0.4268172555, \qquad 0.5513875246,$$

implying that this periodic solution is unstable. So we expect that a trajectory starting from a nonzero initial point would converge to the equilibrium point, with a very slow convergence speed, particularly in the neighborhood of the equilibrium point, since at the critical value $\alpha = 1$ ($\mu = 0$), the equilibrium point is an elementary center. Numerical simulation results for this case are depicted in Fig. 2.6.

Now we consider adding unfolding to (2.111). For example, let

$$\alpha = \alpha_c + \mu = 1 + \mu, \tag{2.120}$$

where μ is a bifurcation parameter, then (2.110) becomes

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_3, \\ \dot{x}_2 &= x_1 + x_4 - \frac{1}{8}(2x_1 + 4x_3 + x_4)\mu + \frac{1}{8}(2x_1 + 4x_3 + x_4)^3, \\ \dot{x}_3 &= -x_4 + \frac{1}{4}(2x_1 + 4x_3 + x_4)\mu - \frac{1}{4}(2x_1 + 4x_3 + x_4)^3, \\ \dot{x}_4 &= x_3. \end{aligned} \tag{2.121}$$

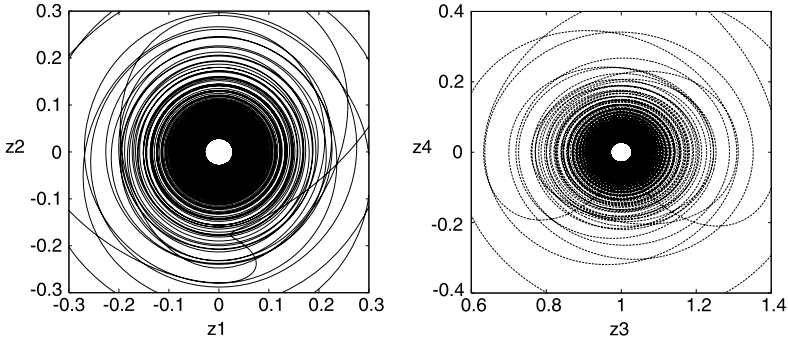

Fig. 2.6 Simulation result of system (2.108) with initial condition $(z_1, z_2, z_3, z_4) = (2.0, -1.0, 0.5, 1.0)$ for $\mu = 0$ ($\alpha = 1.0$), converging to the equilibrium point $(z_1, z_2, z_3, z_4) = (0, 0, 1, 0)$

The corresponding real normal form is given by (2.112)–(2.114) but now the second equation $\dot{R}_2$ needs to add an unfolding term $\frac{1}{2}\mu$. Hence the Jacobian matrix given by (2.119) is not changed.

Similarly, substituting $\phi = \phi_\pm$ into (2.113) and (2.115) yields two sets of polynomial equations:

$$\begin{aligned} \frac{1}{2}\mu - \frac{3}{2}R_1^2 R_2 \pm 3R_1 R_2^2 &= 0, \\ \frac{1}{4R_1 R_2}\left(\mp 4R_2^2 - 6R_1^3 R_2 \pm 6R_1^2 R_2^2 \pm 3R_1^4\right) &= 0, \end{aligned}$$

from which eliminating R_2 results in

$$R_2 = \frac{1}{3R_1^2(3R_1^2+2)}\left(\pm 9R_1^5 - 3\mu R_1^2 + 2\mu\right), \tag{2.122}$$

and a resultant,

$$F_1 = 81R_1^{10} - 54R_1^8 \mp 54\mu R_1^7 \pm 108\mu R_1^5 + 18\mu^2 R_1^4 - 24\mu^2 R_1^2 + 8\mu^2. \tag{2.123}$$

It is easy to observe from (2.123) that if for $\phi = \phi_+ = \frac{1}{2}\pi$, F_1 has a solution R_1^*, then $-R_1^*$ must be a solution for $\phi = \phi_- = \frac{3}{2}\pi$. But it is clear from (2.122) that R_2 does not change sign for these two cases. It has been shown that for $\mu < 0.063364025$, both sets ($\phi = \phi_+ = \frac{1}{2}\pi$ and $\phi = \phi_- = \frac{3}{2}\pi$) have two positive solutions, and one of them is stable; while for $\mu > 0.063364025$, only the second set (i.e., for $\phi = \phi_- = \frac{3}{2}\pi$) has one positive solution which is stable.

In the following, we take three values of $\mu = -0.1, 0.05, 0.2$ to demonstrate the above predictions. For $\mu = -0.1$, it is easy to show that the equilibrium solution $R_1 = R_2 = 0$ (i.e., $(z_1, z_2, z_3, z_4) = (0, 0, 1, 0)$) is stable. There exists one positive

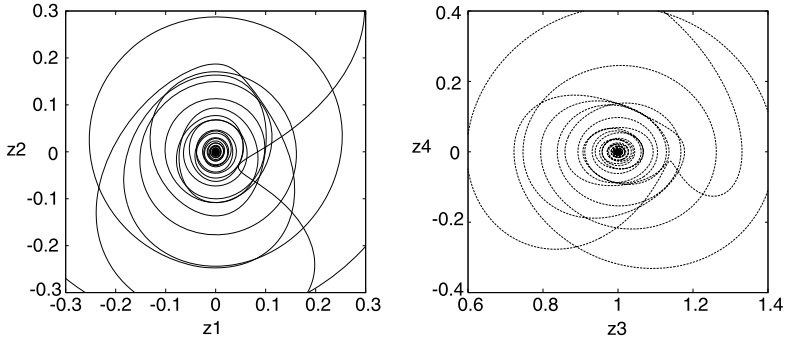

Fig. 2.7 Simulation result of system (2.108) with initial condition $(z_1, z_2, z_3, z_4) = (2.0, -1.0, 0.5, 1.0)$ for $\mu = -0.1$ ($\alpha = 0.9$), converging to the equilibrium point $(z_1, z_2, z_3, z_4) = (0, 0, 1, 0)$

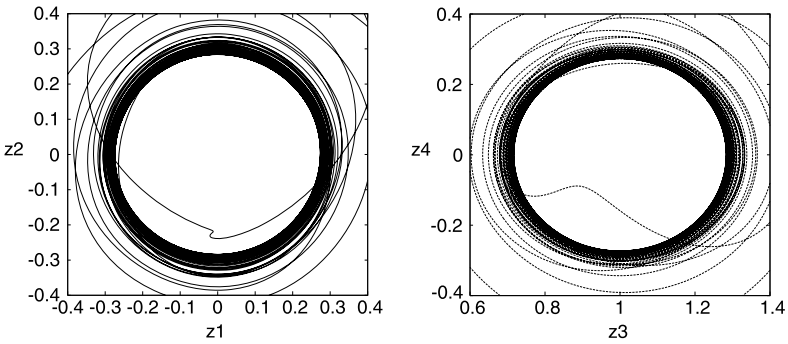

Fig. 2.8 Simulation result of system (2.108) with initial condition $(z_1, z_2, z_3, z_4) = (2.0, -1.0, 0.5, 1.0)$ for $\mu = 0.05$ ($\alpha = 1.05$), converging to a larger limit cycle from outside

solution:

$$(R_1, R_2, \phi) = \left(0.8851664606, 0.4816736981, \frac{1}{2}\pi\right),$$

for which the three eigenvalues of the Jacobian are

$$-0.3452769287, \qquad -2.8895657071, \qquad 0.9880883600,$$

indicating that the periodic solution is unstable, as expected, and so all trajectories around the equilibrium point converge to the equilibrium; this is depicted in Fig. 2.7. Note that due to the unfolding parameter $\mu < 0$, the convergence is very fast compared with that shown in Fig. 2.6.

For $\mu = 0.05$ and $\mu = 0.2$, we list the results in Table 2.1, where the solutions and their stability based on the eigenvalues of the Jacobian are given. The corresponding simulation results are shown in Figs. 2.8–2.10.

Table 2.1 Bifurcation of periodic solutions (PS) for 1:1 resonant case

μ	PS (R_1, R_2, ϕ)	Eigenvalues of the Jacobian	Stability
0.05	$(0.538140, 0.185664, \frac{\pi}{2})$	$-0.459453 \pm 0.826837\mathrm{i}, -0.084528$	Stable
	$(0.739432, 0.336194, \frac{\pi}{2})$	$-0.808474, -1.105175, 0.199009$	Unstable
	$(0.297963, 0.108584, \frac{3\pi}{2})$	$-0.062462 \pm 0.753452\mathrm{i}, -0.371659$	Stable
0.20	$(0.386313, 0.212635, \frac{3\pi}{2})$	$-0.071537 \pm 1.089584\mathrm{i} - 0.774928$	Stable

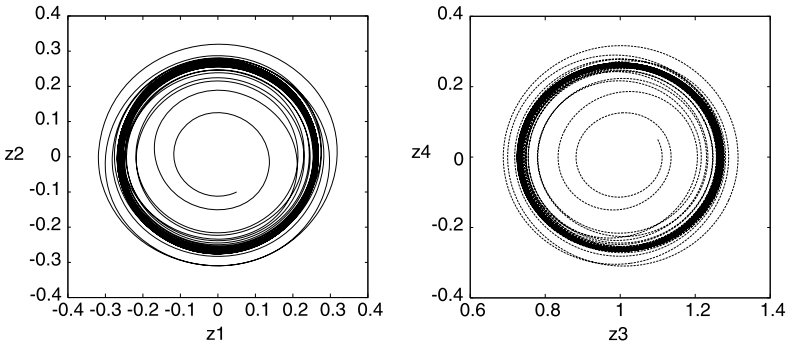

Fig. 2.9 Simulation result of system (2.108) with initial condition $(z_1, z_2, z_3, z_4) = (0.05, -0.1, 1.1, 0.05)$ for $\mu = 0.05$ $(\alpha = 1.05)$, converging to a smaller limit cycle from inside

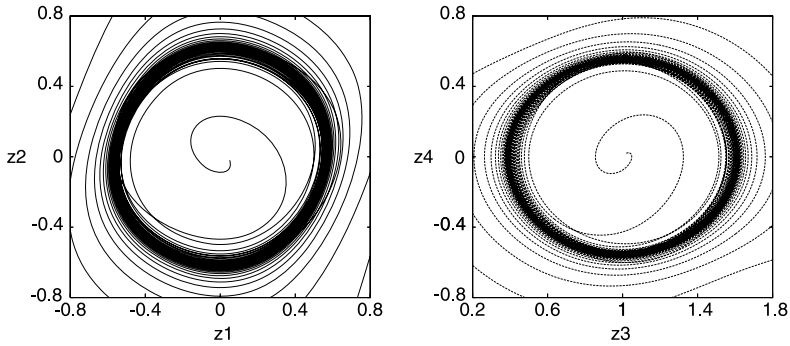

Fig. 2.10 Simulation result of system (2.108) with initial conditions $(z_1, z_2, z_3, z_4) = (2.0, 1.0, 1.5, 1.0)$ and $(0.05, -0.02, 1.02, 0.02)$ for $\mu = 0.2$ $(\alpha = 1.2)$, converging to a limit cycle

Remark 2.12 It can be seen from Figs. 2.8 and 2.9 that when $\mu = 0.05$, there exist two different stable limit cycles. Therefore, depending upon the initial conditions, the trajectories may converge to different limit cycles (see the convergence in Fig. 2.8 from outside while in Fig. 2.9 it is from inside). The result for $\mu = 0.2$ is given in Fig. 2.10 showing that trajectories starting from different initial conditions converge to a unique limit cycle.

2.3 A Perturbation Method Based on Multiple Timescales

In this section, we present a perturbation technique which combines the method of multiple timescales (MTS) or simply multiple scales (MS) [166, 168] and harmonic balancing [102] to study nonlinear vibration and bifurcation problems. Huseyin and Lin [103] used this approach to obtain the explicit formulas for simplified differential equations (which are actually normal forms) up to first-order approximation. Later, this method was extended to compute the normal forms of Hopf and generalized Hopf bifurcations up to an arbitrary order [218]. This method does not need the application of center manifold theory and can be directly applied to general n-dimensional systems. Moreover, user-friendly symbolic programs written in Maple were developed [218], which can be executed "automatically" on a computer system. The crucial part in the computation of normal forms using a computer algebra system is the memory problem. A computer may quickly run out of memory if an inefficient computational approach is used. For example, it is difficult to obtain a fifth-order normal form for a three-dimensional system using a matrix approach, even with a fast computer. Therefore, developing efficient methodologies for computing normal forms is necessary.

Another difficulty in the application of normal forms is that many end users may not be familiar with normal form theory and may not be good at coding symbolic programs. However they do want to apply a method or a program to study their own specific problems which usually have large dimension. Therefore, not only the computational efficiency of a method, but also the ease of use of the method needs to be considered. A perturbation method [218] was developed to provide "automatic" symbolic programs for computing the normal forms of Hopf and generalized Hopf bifurcations. Later, this method was extended to consider double-Hopf bifurcations [221, 223] and Hopf-zero singularities [226]. The Maple programs developed in these papers only require a user to prepare a very straightforward input file.

By comparison with existing results, the MTS method is believed to yield correct normal forms (e.g., see [218]); however, no rigorous mathematical proof was given for a long time. The first proof, to the best of our knowledge, was given in [254], showing that the normal form obtained using the MTS method is indeed equivalent to that derived by Poincaré normal form theory—both are based on the concept of *resonant* terms.

2.3.1 Basic Idea of the MTS

In order to show the basic idea of using the perturbation technique based on the MTS method to find the normal form of differential equations, we first consider a simple, well-known example—van der Pol's equation (1.1)—before dealing with general n-dimensional systems. We rewrite van der Pol's equation here as

$$\ddot{x} + x + \varepsilon\left(x^2 - 1\right)\dot{x} = 0, \tag{2.124}$$

where ε is a small, non-negative, real number (i.e., $0 \leq \varepsilon \ll 1$). These kinds of system are called *weakly nonlinear* systems and perturbation methods can be applied to find approximate periodic solutions.

Van der Pol's equation (2.124) has been studied by many researchers. Recently, this equation was re-investigated by using, in addition to the regular (direct) perturbation method, four frequently-used perturbation approaches: the Lindstedt–Poincaré procedure, time averaging, multiple timescales and intrinsic harmonic balancing [260]. It has been shown that the regular perturbation method yields an unbounded solution which contains secular terms. The Lindstedt–Poincaré procedure cannot be used for stability analysis though it produces the same accurate approximation as that obtained using the MTS and intrinsic harmonic balancing approaches. The first-order time-averaging method has the simplest solution procedure and can be used for stability analysis, but its solution is less accurate. The intrinsic harmonic balancing technique, unlike the other three perturbation methods, does not require the solution of differential equations. However, this approach needs the "normal form" (governing equations) to be constructed for stability analysis, which is usually not straightforward, in particular, for highly codimensional systems. Moreover, the "normal form" obtained using the intrinsic harmonic balancing is only valid up to the leading-order term. Therefore, this approach is not suitable for finding higher-order normal forms.

The MTS method can be used to find not only the approximate solutions but also the normal forms. More importantly, its procedure for finding higher-order normal forms is systematic, and its formulas can be easily implemented using computer algebra systems. It has been shown [260] that the MTS method is the best approach, among the four perturbation methods mentioned above, for the study of nonlinear oscillating systems, in particular, for computing the normal forms.

To apply the MTS method, one begins by introducing the new independent variables,

$$T_k = \varepsilon^k t, \quad k = 0, 1, 2, \ldots. \tag{2.125}$$

It follows that the derivatives with respect to t become expansions in terms of the partial derivatives with respect to T_n according to

$$\begin{aligned} \frac{d}{dt} &= \frac{dT_0}{dt}\frac{\partial}{\partial T_0} + \frac{dT_1}{dt}\frac{\partial}{\partial T_1} + \frac{dT_2}{dt}\frac{\partial}{\partial T_2} + \cdots \\ &\equiv D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \\ \frac{d^2}{dt^2} &= D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2\left(D_1^2 + 2D_0 D_2\right) + \cdots, \\ &\text{etc.}, \end{aligned} \tag{2.126}$$

where $D_i, i = 1, 2, \ldots$, denotes the differentiation operator $\frac{\partial}{\partial T_i}$.

Next, assume that the solution of van der Pol's equation (2.124) is represented by an expansion in the form of

$$x(t;\varepsilon) = x_0(T_0, T_1, T_2, \ldots) + \varepsilon x_1(T_0, T_1, T_2, \ldots) + \varepsilon^2 x_2(T_0, T_1, T_2, \ldots) + \cdots . \tag{2.127}$$

Note that the number of independent timescales used in the solution depends upon the order to which the expansion is carried out. For example, if the expansion is expanded to $O(\varepsilon^2)$, then T_0, T_1 and T_2 are needed. In general, if we want to find the approximate solution up to order $O(\varepsilon^n)$, then the scaled times $T_0, T_1, \ldots, T_n$ should be used. It should be pointed out that the same ε is used in both time and space scales. In other words, we treat the scaling of the motion of the system uniformly for time and space.

Applying formulas (2.126) and (2.127) to system (2.124) and balancing like powers of ε results in the following ordered perturbation equations:

$$\varepsilon^0 : D_0^2 x_0 + x_0 = 0, \tag{2.128}$$

$$\varepsilon^1 : D_0^2 x_1 + x_1 = -2D_1 D_0 x_0 - \left(x_0^2 - 1\right) D_0 x_0, \tag{2.129}$$

etc.

The solution of the ε^0-order equation (2.128) can be expressed as

$$x_0 = a(T_1, T_2, \ldots)\cos\left[T_0 + \phi(T_1, T_2, \ldots)\right] \equiv a\cos(\theta). \tag{2.130}$$

Then substitute this solution into the ε^1-order equation (2.129) to obtain

$$D_0^2 x_1 + x_1 = \left[2D_1 a - a\left(1 - \frac{1}{4}a^2\right)\right]\sin(T_0 + \phi) + 2aD_1\phi\cos(T_0 + \phi) + \frac{1}{4}a^3\sin 3(T_0 + \phi). \tag{2.131}$$

Eliminating the secular terms, which may appear in solution x_1, requires that

$$\begin{aligned} D_1 a &= \frac{1}{2}a\left(1 - \frac{1}{4}a^2\right), \\ D_1\phi &= 0, \end{aligned} \tag{2.132}$$

and thus (2.131) becomes

$$D_0^2 x_1 + x_1 = \frac{1}{4}a^3\sin 3(T_0 + \phi), \tag{2.133}$$

which, in turn, yields the solution

$$x_1 = -\frac{1}{32}a^3\sin 3(T_0 + \phi). \tag{2.134}$$

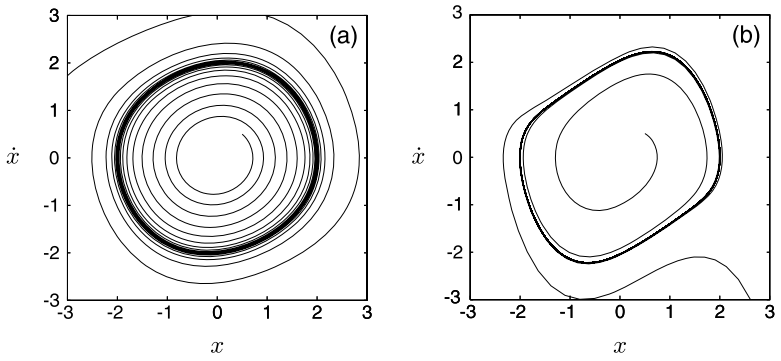

Fig. 2.11 Simulated phase portraits of van der Pol's equation (2.124) with initial conditions $(x, \dot{x}) = (0.5, 0.5)$ and $(3, -4)$ for (**a**) $\varepsilon = 0.1$; and (**b**) $\varepsilon = 0.5$

Hence, the approximate solution up to first order is obtained as

$$x(t, \varepsilon) = a\cos(t + \phi) - \frac{\varepsilon}{32}a^3 \sin 3(t + \phi). \tag{2.135}$$

It should be noted that we only find the *particular* solution from (2.133) since we can leave the *homogeneous* solution part to be included in a and ϕ. In fact, if we add the homogeneous solution, given by $a_1 \cos(t + \phi_1)$ where a_1 and ϕ_1 are determined from the initial conditions, to the particular solution (2.134), we can see that the homogeneous solution can indeed be combined with the first term of solution (2.135).

Finally, the governing equations for the amplitude a and the phase ϕ of the periodic solution above can be obtained, up to $O(\varepsilon^2)$, as follows:

$$\frac{da}{dt} = \frac{\partial a}{\partial T_1}\frac{\partial T_1}{\partial t} + O\left(\varepsilon^2\right) = \varepsilon D_1 a + O\left(\varepsilon^2\right) \approx \frac{\varepsilon}{2}a\left(1 - \frac{1}{4}a^2\right) \tag{2.136}$$

and

$$\frac{d\theta}{dt} = 1 + \frac{d\phi}{dt} = 1 + \frac{\partial \phi}{\partial T_1}\frac{\partial T_1}{\partial t} + O\left(\varepsilon^2\right) = 1 + \varepsilon D_1 \phi + O\left(\varepsilon^2\right) \approx 1. \tag{2.137}$$

These two equations (well-known results in the existing literature) are in fact the normal form of van der Pol's equation up to order ε. The approximate amplitude of the periodic solution is $\overline{a} = 2$, confirmed by the numerical simulation result, shown in Fig. 2.11, where ε takes two values: $\varepsilon = 0.1, 0.5$. It can be seen that for the small value of ε (0.1), the limit cycle is close to a circle, while for the large value of ε (0.5), the limit cycle is no longer close to a circle, but its amplitude is still close to 2, as expected.

Now we want to extend the above procedure for a second-order differential equation to general n-dimensional systems. We use Hopf bifurcation as an example to provide a proof for the perturbation technique based on the MTS method. Other cases having similar procedures can be similarly proved.

2.3.2 Hopf and Generalized Hopf Bifurcations

Consider the general n-dimensional autonomous system (2.9),

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \boldsymbol{R}^n, \qquad \boldsymbol{f} : \boldsymbol{R}^n \to \boldsymbol{R}^n, \tag{2.138}$$

where J is an $n \times n$ Jacobian matrix, and $J\boldsymbol{x}$ is the linear part of the system. The function $\boldsymbol{f}$ represents the nonlinear terms, and is assumed to be analytic. In addition, $\boldsymbol{f}$ and its first derivative vanish at the origin $\mathbf{0}$, indicating that $\mathbf{0}$ is an equilibrium (fixed point) of the system. J is given by

$$J = \begin{bmatrix} A_0 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}, \tag{2.139}$$

where A_0 is a 2×2 matrix with a pair of purely imaginary eigenvalues, given by

$$A_0 = \begin{bmatrix} 0 & \omega_c \\ -\omega_c & 0 \end{bmatrix}, \tag{2.140}$$

and the eigenvalues of A_1 and A_2 have negative real parts, which implies that the center manifold of the system has dimension 2. The eigenvalues of J are in the Siegel domain [115] and one encounters much greater computational complexity than for a Hopf critical point (having a pair of purely imaginary eigenvalues).

Matrix A_1 is an $n_1 \times n_1$ matrix having negative real eigenvalues:

$$A_1 = \begin{bmatrix} -\alpha_3 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -\alpha_{2+n_1} \end{bmatrix}, \tag{2.141}$$

and A_2 is an $n_2 \times n_2$ matrix whose eigenvalues are complex conjugates with negative real parts:

$$A_2 = \begin{bmatrix} -\alpha_{n_1+3} & \omega_{n_1+3} & 0 & 0 & \dots & 0 & 0 \\ -\omega_{n_1+3} & -\alpha_{n_1+3} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -\alpha_{n_1+5} & \omega_{n_1+5} & \dots & 0 & 0 \\ 0 & 0 & -\omega_{n_1+5} & -\alpha_{n_1+5} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\alpha_{n-1} & \omega_{n-1} \\ 0 & 0 & 0 & 0 & \dots & -\omega_{n-1} & -\alpha_{n-1} \end{bmatrix}, \tag{2.142}$$

where ω_c, α_p, $p = 3, 4, \dots, n_1 + 2$, and $\alpha_q, q = n_1 + 3, n_1 + 5, \dots, n - 1$, are positive, and $2 + n_1 + 2n_2 = n$.

For the convenience of the analysis using the MTS method, one may write (2.138) in component form:

$$\begin{aligned}\dot{x}_1 &= \omega_c x_2 + f_1(\boldsymbol{x}),\\ \dot{x}_2 &= -\omega_c x_1 + f_2(\boldsymbol{x}),\end{aligned} \tag{2.143}$$

$$\dot{x}_p = -\alpha_p x_p + f_p(\boldsymbol{x}), \quad p = 3, \ldots, n_1 + 2, \tag{2.144}$$

$$\begin{aligned}\dot{x}_q &= -\alpha_q x_q + \omega_q x_{q+1} + f_q(\boldsymbol{x}),\\ \dot{x}_{q+1} &= -\omega_q x_q - \alpha_q x_{q+1} + f_{q+1}(\boldsymbol{x}),\\ q &= n_1 + 3, n_1 + 5, \ldots, n - 1.\end{aligned} \tag{2.145}$$

Based on the above equations, the MTS method can be used to find the normal forms. First, assume that the solution of system (2.138) is given in the form

$$\begin{aligned}x_j(t;\varepsilon) &= \varepsilon x_{j,1}(T_0, T_1, \ldots) + \varepsilon^2 x_{j,2}(T_0, T_1, \ldots) + \cdots,\\ j &= 1, \ldots, n,\end{aligned} \tag{2.146}$$

and then substitute (2.146) into (2.143)–(2.145) with the aid of (2.125) and (2.126) and balance like powers of ε to obtain the following ordered perturbation equations:

$$\begin{aligned}\varepsilon^1 &: D_0 x_{1,1} = \omega_c x_{2,1},\\ & D_0 x_{2,1} = -\omega_c x_{1,1},\end{aligned} \tag{2.147}$$

$$D_0 x_{p,1} = -\alpha_p x_{p,1}, \quad p = 3, \ldots, n_1 + 2, \tag{2.148}$$

$$\begin{aligned}& D_0 x_{q,1} = -\alpha_q x_{q,1} - \omega_q x_{(q+1),1},\\ & D_0 x_{(q+1),1} = \omega_q x_{q,1} - \alpha_q x_{(q+1),1},\\ & q = n_1 + 3, n_1 + 5, \ldots, n - 1;\end{aligned} \tag{2.149}$$

$$\begin{aligned}\varepsilon^2 &: D_0 x_{1,2} = \omega_c x_{2,2} - D_1 x_{1,1} + f_{1,2}(\boldsymbol{x}_1),\\ & D_0 x_{2,2} = -\omega_c x_{1,2} - D_1 x_{2,1} + f_{2,2}(\boldsymbol{x}_1),\end{aligned} \tag{2.150}$$

$$D_0 x_{p,2} = -\alpha_p x_{p,2} + f_{p,2}(\boldsymbol{x}_1), \quad p = 3, \ldots, n_1 + 2, \tag{2.151}$$

$$\begin{aligned}& D_0 x_{q,2} = -\alpha_q x_{q,2} - \omega_q x_{(q+1),2} + f_{q,2}(\boldsymbol{x}_1),\\ & D_0 x_{(q+1),2} = \omega_q x_{q,2} - \alpha_q x_{(q+1),2} + f_{(q+1),2}(\boldsymbol{x}_1),\\ & q = n_1 + 3, n_1 + 5, \ldots, n - 1;\end{aligned} \tag{2.152}$$

etc.,

where $\boldsymbol{x}_1$ represents the first-order approximation of $\boldsymbol{x}$, $\boldsymbol{x}_{i,j}$ represents the jth-order approximation of $\boldsymbol{x}_i$, and

$$f_{i,2} = \left.\frac{d^2(f_i(x_1, x_2, \ldots)/\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0}, \tag{2.153}$$

which are functions of $\boldsymbol{x}_{i,1}$ only. In general, the function $f_{i,k}$ only involves the ordered approximations $\boldsymbol{x}_{i,1}, \boldsymbol{x}_{i,2}, \ldots, \boldsymbol{x}_{i,k-1}$, which have been found from the previous $(k-1)$ perturbation equations. It should be noted that unlike (2.127) which starts with a zero-order term, the solution form (2.146) starts with a first-order term. This is because van der Pol's equation has ε in the nonlinear term, while for the general nonlinear system (2.138), usually the first step is to use scaling $\boldsymbol{x} \to \varepsilon \boldsymbol{x}$ to separate different order terms, and then use the solution form starting with a zero-order term. Here the first step has been included in solution form (2.146).

To find the normal form of the system, we start with (2.146). Differentiating the first equation of (2.147) and then substituting the second equation into the resulting equation yields

$$D_0^2 x_{1,1} + \omega_c^2 x_{1,1} = 0 \tag{2.154}$$

which is a free-vibrating system with the solution

$$x_{1,1} = r(T_1, T_2, \ldots)\cos\big(\omega_c T_0 + \phi_1(T_1, T_2, \ldots)\big) \equiv r\cos\theta, \tag{2.155}$$

where r and θ are the amplitude and phase of motion, respectively. Having found $x_{1,1}$, one can easily find the solution of $x_{2,1}$ as

$$x_{2,1} = r\sin\theta \tag{2.156}$$

from the first equation of (2.147). Thus the first-order solutions for $x_{1,1}$ and $x_{2,1}$ are found.

Since we are interested in the steady-state (asymptotic) solutions of the system, the ε^1-order solutions for $x_{j,1}$, $j = 3, \ldots, n$, associated with the eigenvalues having negative real parts, are equal to 0, i.e.,

$$x_{j,1} = 0, \quad j = 3, \ldots, n. \tag{2.157}$$

It follows from solution (2.155) that

$$D_0 r = D_0 \phi = 0. \tag{2.158}$$

Next, to solve the ε^2-order perturbation equation (2.150), one may apply the above procedure and substitute the ε^1-order solution into (2.150) to obtain the following second-order nonhomogeneous ODE (Ordinary Differential Equation):

$$D_0^2 x_{1,2} + \omega_c^2 x_{1,2} = -D_1 D_0 x_{1,1} - D_1 x_{2,1} + D_0 f_{1,2} + f_{2,2}. \tag{2.159}$$

Note that the right-hand side of (2.159) is a polynomial in the first-order solutions $x_{1,1}$ and $x_{2,1}$, so the solution of $x_{1,2}$ can be expressed by a finite Fourier series

$$x_{1,2} = \sum_{j=0}^{2} C_j \cos(j\theta) + S_j \sin(j\theta). \tag{2.160}$$

In order to determine the coefficients C_j and S_j and thus solve for $x_{1,2}$, one may substitute (2.160) into (2.159) and then balance the harmonics. However, as usual, the resulting equation may involve terms which will generate secular terms in the solution of $x_{1,2}$. To eliminate the secular terms we must set their coefficient to zero. This yields two algebraic equations to determine $D_1 r$ and $D_1 \phi$ which are called *resonant* terms and will be retained in the normal form.

The solutions for $x_{j,2}$, $j = 3, \ldots, n$, can also be found in the same form of Fourier series as that of $x_{1,2}$, except that no secular terms would appear. Hence, they can be uniquely determined by a straightforward harmonic balancing approach.

The general solution of the n_rth-order perturbation equations of the system is given in the following theorem.

Theorem 2.13 *The solutions of the n_rth-order perturbation equations of an autonomous system can be expressed by a finite Fourier series*:

$$x_{i,n_r} = \sum_{j=0}^{n_r} C_j \cos(j\theta) + S_j \sin(j\theta). \tag{2.161}$$

Proof First, we may rewrite the first-order solution of (2.147)–(2.149) as

$$x_{i,1} = \sum_{j=0}^{1} C_j \cos(j\theta) + S_j \sin(j\theta), \tag{2.162}$$

where C_j and S_j are constants.

Next, for the second-order perturbation equations (2.150)–(2.152), it is easy to observe that the nonlinear terms are second-degree polynomials in the first-order solutions. So the highest-order terms are $x_{i,1}x_{j,1}$, $i, j = 1, 2, \ldots, n$. Therefore, the highest-order terms involved in the second-order solutions are $\sin^2\theta$, $\cos^2\theta$ and $\sin\theta\cos\theta$, which can be rewritten as $\frac{1}{2}[1 - \cos 2\theta]$, $\frac{1}{2}[1 + \cos 2\theta]$, and $\frac{1}{2}\sin 2\theta$, respectively.

Similarly, the highest-order terms in the third-order perturbation equations come from the multiplication of first- and second-order solutions, i.e., $x_{i,1}x_{j,2}$, $i, j = 1, 2, \ldots, n$. So the highest-order terms in third-order solutions are

$$\sin 2\theta \sin\theta, \qquad \sin 2\theta \cos\theta, \qquad \cos 2\theta \sin\theta, \qquad \cos 2\theta \cos\theta.$$

Therefore, the highest-order terms in the third-order solutions can be written as $\sin 3\theta$ or $\cos 3\theta$.

The above procedure can be easily extended to discuss higher-order perturbation equations. More rigorously, one may apply the method of mathematical induction to show that the highest-order terms in the n_rth-order solutions can be written in the form $\sin(n_r\theta)$ or $\cos(n_r\theta)$. This completes the proof. □

As discussed above, the normal form terms $D_1 r$ and $D_1 \phi$ are obtained from the second-order perturbation equations by removing the secular terms from the

solutions. In general, for the n_rth-order equations, the normal form terms $D_{n_r-1}r$ and $D_{n_r-1}\phi$ are obtained by eliminating the secular terms.

Finally, the normal form of system (2.138) associated with a Hopf-type singularity, up to any order, can be written in polar coordinates as

$$\begin{aligned}\frac{dr}{dt} &= \frac{\partial r}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial r}{\partial T_2}\frac{\partial T_2}{\partial t} + \frac{\partial r}{\partial T_3}\frac{\partial T_3}{\partial t} + \cdots \\ &= \varepsilon D_1 r + \varepsilon^2 D_2 r + \varepsilon^3 D_3 r + \cdots, \qquad (2.163)\\ r\frac{d\theta}{dt} &= R\left(\omega_c + \frac{\partial \phi}{\partial T_1}\frac{\partial T_1}{\partial t} + \frac{\partial \phi}{\partial T_2}\frac{\partial T_2}{\partial t} + \frac{\partial \phi}{\partial T_3}\frac{\partial T_3}{\partial t} + \cdots\right) \\ &= r\left(\omega_c + \varepsilon D_1\phi + \varepsilon^2 D_2\phi + \varepsilon^3 D_3\phi + \cdots\right). \qquad (2.164)\end{aligned}$$

It should be noted that only odd order terms are retained in the normal form, implying that

$$D_{2k-1}r = D_{2k-1}\phi = 0, \quad \text{where } k \text{ is a positive integer.} \qquad (2.165)$$

Note that the subscript $2k-1$ is not the order of the term. More specifically, we have the following theorem.

Theorem 2.14 *The "form" (in polar coordinates) of the normal form of an autonomous system with a Hopf-type singularity is given by*

$$\dot{r} = rP\left(r^2\right), \qquad r\dot{\phi} = rQ\left(r^2\right), \qquad (2.166)$$

where P and Q are polynomials in r^2.

Remark 2.15 The equations given in (2.166) are actually the Poincaré normal form for Hopf bifurcation.

Proof It is easy to prove the theorem using complex formulas. Thus, introduce the following transformations:

$$x_1 = \frac{1}{2}(z_1 + z_2), \qquad x_2 = \frac{1}{2\mathrm{i}}(z_1 - z_2), \qquad (2.167)$$

$$x_p = z_p, \quad p = 3, \ldots, n_1 + 2, \qquad (2.168)$$

$$x_q = \frac{1}{2}(z_q + z_{q+1}), \qquad x_{q+1} = \frac{1}{2\mathrm{i}}(z_q - z_{q+1}),$$
$$q = n_1 + 3, \ldots, n-1, \qquad (2.169)$$

where i is the imaginary unit satisfying $\mathrm{i}^2 = -1$. It should be noted in the above expressions that z_2 and z_{q+1} are the complex conjugates of z_1 and z_q, respectively.

Then (2.143)–(2.145) can be transformed into complex form as follows:

$$\begin{aligned}\dot{z}_1 &= \mathrm{i}\omega_i z_1 + f_1 + \mathrm{i} f_2,\\ \dot{z}_2 &= -\mathrm{i}\omega_i z_2 + f_1 - \mathrm{i} f_2,\end{aligned} \tag{2.170}$$

$$\dot{z}_p = -\alpha_p z_p + f_p, \quad p = 3, 4, \ldots, n_1 + 2, \tag{2.171}$$

$$\begin{aligned}\dot{z}_q &= (-\alpha_q + \mathrm{i}\omega_q) z_q + f_q + \mathrm{i} f_{q+1},\\ \dot{z}_{q+1} &= -(\alpha_q + \mathrm{i}\omega_q) z_{q+1} + f_q - \mathrm{i} f_{q+1},\\ q &= n_1 + 3, n_1 + 5, \ldots, n - 1,\end{aligned} \tag{2.172}$$

where

$$f_i = f_i\big(x_1(\mathbf{z}), x_2(\mathbf{z}), \ldots, x_n(\mathbf{z})\big), \tag{2.173}$$

and the $x_i(\mathbf{z})$ are given by transformations (2.167)–(2.169). Similarly, applying the MTS method to the above equations results in complex ordered perturbation equations, which are similar to the real forms (2.147)–(2.149).

The solutions to the first-order complex equations (which can be readily obtained from (2.170)–(2.172) by removing f terms) are

$$z_{1,1} = re^{\mathrm{i}\theta}, \qquad z_{2,1} = \overline{z}_{1,1}, \tag{2.174}$$

$$z_{p,1} = 0, \quad p = 3, \ldots, n, \tag{2.175}$$

where $\overline{z}$ represents the complex conjugate of z, $r = r(T_1, T_2, \ldots)$ is real and positive, and $\theta = \omega T_0 + \phi(T_1, T_2, \ldots)$.

Similarly to the real analysis, it can been shown that the nonlinear terms on the right-hand side of the complex perturbation equations can be written as polynomials in the first-order solutions in the form of

$$F = \sum C_j z_1^{a_j} \overline{z}_1^{b_j} = \sum C_j r^{(a_j + b_j)} e^{\mathrm{i}(a_j - b_j)\theta}, \tag{2.176}$$

where the C_j are complex constants and a_j, b_j, are non-negative integers.

For the first equation, the secular term should be in the form $e^{\mathrm{i}\theta}$, and thus it follows from (2.176) that the terms producing the secular terms for the first equation satisfy

$$a_j - b_j = 1. \tag{2.177}$$

Therefore, the powers of the normal form terms for the first equation are the solution of (2.177). Now substituting (2.177) into (2.176) results in the secular term

$$S = \sum C_j r^{2b_j + 1} e^{\mathrm{i}\theta}, \tag{2.178}$$

which is balanced by the term $D_{2b_j}(re^{\mathrm{i}\theta})$. Thus the normal form terms $D_{2b_j} r$ and $D_{2b_j}\phi$ can be found from the equation

$$D_{2b_j} r + \mathrm{i} r D_{2b_j}\theta = S = \sum C_j r^{2b_j + 1} e^{\mathrm{i}\theta} \tag{2.179}$$

which, in turn, yields

$$\begin{aligned} D_{2b_j} r &= r \sum \mathrm{Re}(C_j) r^{2b_j} = r P_{b_j}\left(r^2\right), \\ r D_{2b_j} \theta &= R \sum \mathrm{Im}(C_j) r^{2b_j} = r Q_{b_j}\left(r^2\right), \end{aligned} \tag{2.180}$$

where Re and Im represent the real and imaginary parts, respectively, and P_{b_j} and Q_{b_j} are the b_jth-degree polynomials in r^2. Note that the above formulas indicate that only odd order terms are retained in the normal form.

The proof of Theorem 2.14 is complete. □

Similar theorems and proofs for the equivalence of other cases such as double-Hopf and Hopf-zero singularities can be established, and are not repeated here.

Therefore, for the general system (2.138), by a back-scaling, the normal form for Hopf and generalized Hopf bifurcations can be rewritten as

$$\dot{r} = r\left(v_0\mu + v_1 r^2 + \cdots + v_k r^{2k} + \cdots\right), \tag{2.181}$$

$$\dot{\theta} = \omega_c + \tau_0\mu + \tau_1 r^2 + \tau_2 r^4 + \cdots + \tau_k r^{2k} + \cdots, \tag{2.182}$$

where μ is an unfolding (bifurcation parameter) when the original system involves parameters, v_k is the kth-order *focus value*. v_0 and τ_0 can be determined from linear analysis.

The Maple source program for computing the coefficients v_k, τ_k, and nonlinear transformation of the normal form, as well as sample input files, can be found on "Springer Extras" (by visiting extras.springer.com and searching for the book using its ISBN). In the following, we give two examples to illustrate the application of the method. In order to apply the Maple program to analyze a problem, the system must first be transformed such that its Jacobian evaluated at an equilibrium point should be in real Jordan canonical form, as shown in the sample file.

Example 2.16 The first example is the same as Example 2.6, which is a two-dimensional system, with a Hopf singularity. Executing the Maple program yields the following normal form given in polar coordinates:

$$\dot{r} = \frac{3}{8} r^3 + \frac{5}{16} r^5 + \cdots, \tag{2.183}$$

$$\dot{\theta} = 1 - \frac{1}{3} r^2 - \frac{29}{6912} r^4 + \cdots. \tag{2.184}$$

Note that the normal form given in the above equations is not exactly the same as that obtained by using the method developed in Sect. 2.2. The coefficient of r^4 in (2.184) is different from that given in the second equation of (2.42). This is not surprising since the normal form is not unique. The verification scheme described in [218] can be used to prove that both normal forms are correct! In fact, by the simplest normal form theory, one can further remove the r^4 term from the phase equation (e.g., see [219]).

Based on the normal form up to fifth order, it is easy to see that limit cycles do not exist since both the coefficients of r^3 and r^5 in the amplitude equation are positive.

Example 2.17 The second example is the same as Example 2.7, which is a five-dimensional system, with a Hopf singularity. Executing the Maple program yields the following normal form in polar coordinates up to fifth order:

$$\dot{r} = \frac{3}{40}r^3 - \frac{14867}{68000}r^5 + \cdots, \tag{2.185}$$

$$\dot{\theta} = 1 - \frac{7}{12}r^2 + \frac{8093503}{14688000}r^4 + \cdots, \tag{2.186}$$

which are again different from those given in (2.44) (see Sect. 2.2) by one term in the phase equation. Now, based on (2.185), we can easily find the steady-state solutions by setting $\dot{r} = 0$, yielding $\overline{r} = 0$, corresponding to the initial equilibrium point $\boldsymbol{x} = \mathbf{0}$, and a nontrivial solution

$$\overline{r} = \sqrt{5100/14867}. \tag{2.187}$$

This nontrivial solution represents a periodic motion and its asymptotic solution is given by (obtained from the computer output):

$$\begin{aligned}
x_1 &= r\cos\theta + \frac{1}{6}r^2(3 + \cos 2\theta + 2\sin 2\theta) - \frac{1}{480}r^3(28\cos 3\theta - 13\sin 3\theta) + \cdots,\\
x_2 &= -r\cos\theta - \frac{1}{6}r^2(3 - \cos 2\theta + 2\sin 2\theta)\\
&\quad - \frac{1}{480}r^3(260\cos\theta - 168\sin\theta + 17\cos 3\theta + 28\sin 3\theta) + \cdots,\\
x_3 &= \frac{1}{10}r^2(5 + \cos 2\theta + 2\sin 2\theta)\\
&\quad - \frac{1}{12}r^3(5\cos\theta + 3\sin\theta - \cos 3\theta + \sin 3\theta) + \cdots,\\
x_4 &= \frac{1}{10}r^2(5 + 2\cos 2\theta + \sin 2\theta)\\
&\quad + \frac{1}{510}r^3(119\cos\theta + 408\sin\theta - 23\cos 3\theta + 44\sin 3\theta) + \cdots,\\
x_5 &= -\frac{1}{10}r^2(\cos 2\theta + 3\sin 2\theta)\\
&\quad - \frac{1}{510}r^3(68\cos\theta - 119\sin\theta - 24\cos 3\theta + 57\sin 3\theta) + \cdots.
\end{aligned} \tag{2.188}$$

The stability of the periodic solution is determined by the Jacobian of (2.185) evaluated at $r = \overline{r}$, giving

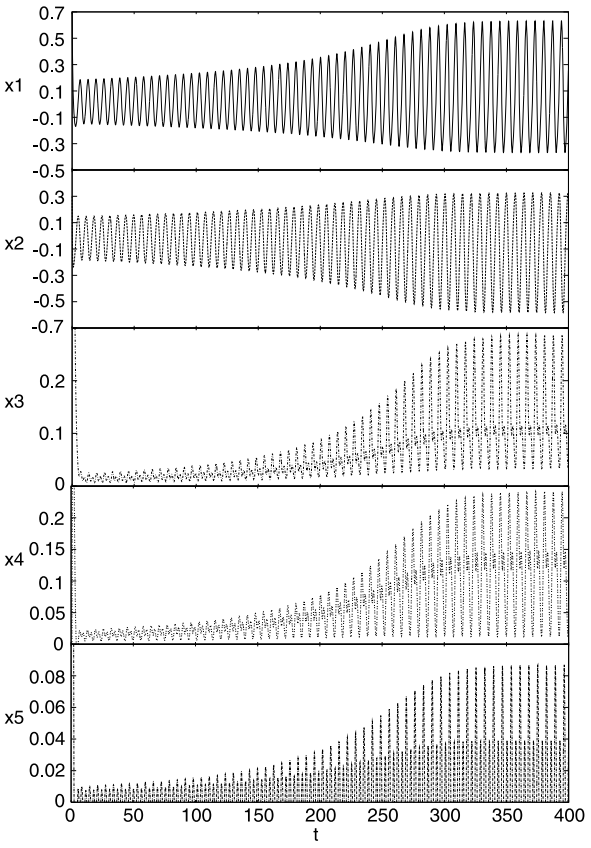

Fig. 2.12 Simulated time history for Example 2.17 with initial condition $(x_1, x_2, x_3, x_4, x_5) = (0.7, 0.3, 3.0, -2.0, 2.0)$, converging to a stable limit cycle

$$J(r = \overline{r}) = \left.\frac{d\dot{r}}{dr}\right|_{r=\overline{r}} = -\frac{765}{14867} < 0.$$

Therefore, the periodic solution is stable, and the frequency of the periodic motion is approximated by

$$\omega = \dot{\theta}\big|_{r=\overline{r}} = 1 - \frac{1}{12}\left(\frac{5100}{14867}\right)\left(7 - \frac{8093503}{1224000} \times \frac{5100}{14867}\right) = \frac{9174269227}{10609329072}$$
$$\approx 0.864736,$$

which indicates that the period of the motion is

$$T = \frac{2\pi}{\omega} \approx 7.266246 \text{ seconds}.$$

The numerical simulation results are shown in Figs. 2.12 and 2.13. It is observed that the convergence of the bifurcating limit cycle is very slow. In fact, the limit cycle is quite weak, in the sense that when one of the first two components of the

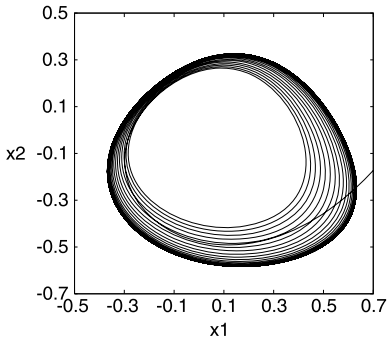

Fig. 2.13 Simulated phase portrait for Example 2.17 with initial condition $(x_1, x_2, x_3, x_4, x_5) = (0.7, 0.3, 3.0, -2.0, 2.0)$, projected on the x_1–x_2 plane, showing a stable limit cycle

initial condition (x_1, x_2) is increased a little bit from the simulating value $(0.7, 0.3)$ to, say, $(0.7, 0.8)$, the trajectory diverges to infinity.

2.4 Efficient Computation

Since normal form computation usually requires heavy algebraic manipulation, computer algebra systems such as Maple, Mathematica, Reduce, etc., have been used extensively. A basic procedure in the symbolic computation of normal forms is to substitute an obtained lower-order ($<k$) normal form and nonlinear transformation to the original differential equation to yield an expression for the kth-order computation, which contains not only the kth-order terms, but also lower-order ($<k$) and higher-order ($>k$) terms. One must extract the kth-order terms from the expression to obtain the kth-order algebraic equation. This unnecessarily increases the computational burden and takes too much computer memory, especially when computing higher-order normal forms. Therefore, removing the unnecessary lower- and higher-order terms from the kth-order computation becomes essential in order to reduce the computation time and memory requirement. In particular, efficient algebraic methods for computing focal values have been discussed in the literature, for example, in [67] and the recent books [60, 182].

In this section, we present an efficient approach to compute the kth-order ($k \geq 2$, an arbitrary integer) algebraic equation which only contains the terms belonging to the kth-order equation. Based on the Lie bracket operator (e.g., see [108]), a recursive formula is derived which can be applied to consider any singularities. Moreover, the new method does not require solving large matrix equations; instead it solves linear algebraic equations, one by one, and is therefore computationally efficient. In addition, unlike most normal form methods which use separate nonlinear transformations at each order, the new approach uses a consistent nonlinear transformation through all order computations. This provides a convenient, one-step transformation between the original system and the normal form, which is particularly useful in real applications.

In the following, we first derive the general formula for an efficient computational method, which is summarized in a theorem, and then discuss symbolic computation.

2.4.1 Theoretical Analysis

Consider the general system described by

$$\dot{\boldsymbol{x}} = J\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}) \equiv \boldsymbol{v}_1 + \boldsymbol{f}_2(\boldsymbol{x}) + \boldsymbol{f}_3(\boldsymbol{x}) + \cdots + \boldsymbol{f}_k(\boldsymbol{x}) + \cdots, \quad (2.189)$$

where $\boldsymbol{x} \in \boldsymbol{R}^n$, $\boldsymbol{v}_1 = J\boldsymbol{x}$ represents the linear part, and the Jacobian matrix J is, without loss of generality, in a standard Jordan canonical form. It is assumed that all eigenvalues of J have zero real parts, implying that the dynamics of system (2.189) are described on an n-dimensional center manifold. $\boldsymbol{f}_k(\boldsymbol{x})$ denotes a kth-order vector of homogeneous polynomials in $\boldsymbol{x}$. It is further assumed that system (2.189) has an equilibrium at the origin, $\boldsymbol{x} = \boldsymbol{0}$.

The basic idea of normal form theory is to find a near-identity nonlinear transformation

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}) \equiv \boldsymbol{y} + \boldsymbol{h}_2(\boldsymbol{y}) + \boldsymbol{h}_3(\boldsymbol{y}) + \cdots + \boldsymbol{h}_k(\boldsymbol{y}) + \cdots, \quad (2.190)$$

such that the resulting system

$$\dot{\boldsymbol{y}} = J\boldsymbol{y} + \boldsymbol{g}(\boldsymbol{y}) \equiv J\boldsymbol{y} + \boldsymbol{g}_2(\boldsymbol{y}) + \boldsymbol{g}_3(\boldsymbol{y}) + \cdots + \boldsymbol{g}_k(\boldsymbol{y}) + \cdots, \quad (2.191)$$

becomes as simple as possible. Here, $\boldsymbol{h}_k(\boldsymbol{y})$ and $\boldsymbol{g}_k(\boldsymbol{y})$ denote a kth-order vector of homogeneous polynomials in $\boldsymbol{y}$.

Define the Lie bracket operator [72] as

$$[U_k, \boldsymbol{v}_1] = D\boldsymbol{v}_1 \cdot U_k - DU_k \cdot \boldsymbol{v}_1. \quad (2.192)$$

The following theorem summarizes the results for the recursive and computationally efficient approach, which can be used to compute the kth-order normal form and the associated nonlinear transformation [225, 226, 233, 253].

Theorem 2.18 *The recursive formula for computing the coefficients of the normal form and the nonlinear transformation is given by*

$$\begin{aligned}\boldsymbol{g}_k = \boldsymbol{f}_k + [\boldsymbol{h}_k, \boldsymbol{v}_1] + \sum_{i=2}^{k-1}\left\{[\,\boldsymbol{h}_{k-i+1}, \boldsymbol{f}_i] + D\boldsymbol{h}_i(\boldsymbol{f}_{k-i+1} - \boldsymbol{g}_{k-i+1})\right\} \\ + \sum_{m=2}^{[\frac{k}{2}]} \frac{1}{m!} \sum_{i=m}^{k-m} D^m \boldsymbol{f}_i \\ \times \sum_{\substack{l_1+l_2+\cdots+l_m=k-(i-m) \\ 2\le l_1,l_2,\ldots,l_m\le k-(i-m)-2(m-1)}} \boldsymbol{h}_{l_1}\boldsymbol{h}_{l_2}\cdots\boldsymbol{h}_{l_m}, \end{aligned} \quad (2.193)$$

for $k = 2, 3, \ldots$, where $\boldsymbol{f}_k$, $\boldsymbol{h}_k$, and $\boldsymbol{g}_k$ are kth-order vectors of homogeneous polynomials in $\boldsymbol{y}$ (where $\boldsymbol{y}$ has been dropped for simplicity). $\boldsymbol{f}_k$ represents the kth-order terms of the original system, $\boldsymbol{h}_k$ is the kth-order nonlinear transformation, and $\boldsymbol{g}_k$ denotes the kth-order normal form.

Remark 2.19 The notation $D^m \boldsymbol{f}_i \boldsymbol{h}_{l_1} \boldsymbol{h}_{l_2} \cdots \boldsymbol{h}_{l_m}$ denotes the mth-order terms of the Taylor expansion of $\boldsymbol{f}_i(\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y}))$ about $\boldsymbol{y}$. More precisely,

$$D^m \boldsymbol{f}_i(\boldsymbol{y} + \boldsymbol{h}) = D\big(D\big(\cdots D\big((D\boldsymbol{f}_i)\boldsymbol{h}_{l_1}\big)\boldsymbol{h}_{l_2}\big)\cdots \boldsymbol{h}_{l_{m-1}}\big)\boldsymbol{h}_{l_m}, \tag{2.194}$$

where each differential operator D affects only function $\boldsymbol{f}_i$, not $\boldsymbol{h}_{l_j}$ (i.e., $\boldsymbol{h}_{l_j}$ is treated as a constant vector in the process of differentiation), and thus $m \le i$. Note that at each level of the differentiation, the D operator is actually a Frechét derivative, giving rise to a matrix, which is multiplied by a vector to generate another vector, and then to another level of Frechét derivative, and so on.

Proof First differentiating (2.177) results in

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{y}} + D\boldsymbol{h}(\boldsymbol{y})\dot{\boldsymbol{y}} = \big(I + D\boldsymbol{h}(\boldsymbol{y})\big)\dot{\boldsymbol{y}}. \tag{2.195}$$

Then substituting (2.189) and (2.191) into (2.195) yields

$$\begin{aligned} &J\boldsymbol{x} + \boldsymbol{f}_2(\boldsymbol{x}) + \boldsymbol{f}_3(\boldsymbol{x}) + \cdots + \boldsymbol{f}_k(\boldsymbol{x}) + \cdots \\ &\quad = \big(I + D\boldsymbol{h}(\boldsymbol{y})\big)\big(J\boldsymbol{y} + \boldsymbol{g}_2(\boldsymbol{y}) + \boldsymbol{g}_3(\boldsymbol{y}) + \cdots + \boldsymbol{g}_k(\boldsymbol{y}) + \cdots\big). \end{aligned} \tag{2.196}$$

Next substituting (2.177) into (2.196) and rearranging the resulting equation gives

$$\begin{aligned} &\boldsymbol{g}_2(\boldsymbol{y}) + \boldsymbol{g}_3(\boldsymbol{y}) + \cdots + \boldsymbol{g}_k(\boldsymbol{y}) + \cdots \\ &\quad = J\boldsymbol{h}(\boldsymbol{y}) - D\boldsymbol{h}(\boldsymbol{y})J\boldsymbol{y} \\ &\qquad - D\boldsymbol{h}(\boldsymbol{y})\boldsymbol{g}_2(\boldsymbol{y}) - D\boldsymbol{h}(\boldsymbol{y})\boldsymbol{g}_3(\boldsymbol{y}) - \cdots - D\boldsymbol{h}(\boldsymbol{y})\boldsymbol{g}_k(\boldsymbol{y}) - \cdots \\ &\qquad + \boldsymbol{f}_2\big(\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y})\big) + \cdots + \boldsymbol{f}_k\big(\boldsymbol{y} + \boldsymbol{h}(\boldsymbol{y})\big) + \cdots, \end{aligned} \tag{2.197}$$

which can be rewritten, using the Taylor expansion about $\boldsymbol{y}$, as

$$\begin{aligned} &\boldsymbol{g}_2(\boldsymbol{y}) + \boldsymbol{g}_3(\boldsymbol{y}) + \cdots + \boldsymbol{g}_k(\boldsymbol{y}) + \cdots \\ &\quad = \boldsymbol{f}_2(\boldsymbol{y}) + \boldsymbol{f}_3(\boldsymbol{y}) + \cdots + \boldsymbol{f}_k(\boldsymbol{y}) + \cdots + \sum_{i=2}^{\infty}\big\{J\boldsymbol{h}_i(\boldsymbol{y}) - D\boldsymbol{h}_i(\boldsymbol{y})J\boldsymbol{y}\big\} \\ &\qquad + \sum_{i=2}^{\infty} D\,\boldsymbol{h}(\boldsymbol{y})\big\{\boldsymbol{f}_i(\boldsymbol{y}) - \boldsymbol{g}_i(\boldsymbol{y})\big\} + \sum_{i=2}^{\infty}\big\{D\boldsymbol{f}_i(\boldsymbol{y})\boldsymbol{h}(\boldsymbol{y}) - D\boldsymbol{h}(\boldsymbol{y})\boldsymbol{f}_i(\boldsymbol{y})\big\} \\ &\qquad + \frac{1}{2!}\big\{D^2\boldsymbol{f}_2(\boldsymbol{y})\boldsymbol{h}^2(\boldsymbol{y}) + D^2\boldsymbol{f}_3(\boldsymbol{y})\boldsymbol{h}^2(\boldsymbol{y}) + \cdots\big\} \end{aligned}$$

$$+\frac{1}{3!}\left\{D^3\boldsymbol{f}_3(\boldsymbol{y})\boldsymbol{h}^3(\boldsymbol{y})+D^3\boldsymbol{f}_4(\boldsymbol{y})\boldsymbol{h}^3(\boldsymbol{y})+\cdots\right\}+\cdots$$
$$+\frac{1}{k!}\left\{D^k\boldsymbol{f}_k(\boldsymbol{y})\boldsymbol{h}^k(\boldsymbol{y})+D^k\boldsymbol{f}_{k+1}(\boldsymbol{y})\boldsymbol{h}^k(\boldsymbol{y})+\cdots\right\}+\cdots. \tag{2.198}$$

Further, one can use Lie bracket notation to rewrite the Taylor expansion in component form according to the order of the terms:

$$\begin{aligned}\sum_{i=2}^{\infty}\boldsymbol{g}_i(\boldsymbol{y})=&\sum_{i=2}^{\infty}\boldsymbol{f}_i(\boldsymbol{y})+\sum_{i=2}^{\infty}\big[\boldsymbol{h}_i(\boldsymbol{y}),\boldsymbol{v}_1(\boldsymbol{y})\big]\\&+\sum_{p=4}^{\infty}\sum_{\substack{i+j=p\\i,j\geq 2}}^{\infty}D\boldsymbol{h}_j(\boldsymbol{y})\big\{\boldsymbol{f}_i(\boldsymbol{y})-\boldsymbol{g}_i(\boldsymbol{y})\big\}\\&+\sum_{p=4}^{\infty}\sum_{\substack{i+j=p\\i,j\geq 2}}^{\infty}\big\{D\boldsymbol{f}_i(\boldsymbol{y})\boldsymbol{h}_j(\boldsymbol{y})-D\boldsymbol{h}_j(\boldsymbol{y})\boldsymbol{f}_i(\boldsymbol{y})\big\}\\&+\sum_{m=2}^{\infty}\frac{1}{m!}\sum_{i=m}^{\infty}D^m\boldsymbol{f}_i(\boldsymbol{y})\big\{\boldsymbol{h}_2(\boldsymbol{y})+\boldsymbol{h}_3(\boldsymbol{y})+\cdots\big\}^m.\end{aligned} \tag{2.199}$$

Finally, we may round off (2.199) up to kth order, which is enough for the proof, and put it in ascending order:

$$\begin{aligned}\sum_{i=2}^{k}\boldsymbol{g}_i(\boldsymbol{y})=&\sum_{i=2}^{k}\boldsymbol{f}_i(\boldsymbol{y})+\sum_{i=2}^{k}\big[\boldsymbol{h}_i(\boldsymbol{y}),\boldsymbol{v}_1(\boldsymbol{y})\big]\\&+\sum_{j=3}^{k}\sum_{i=2}^{j-1}\big[\boldsymbol{h}_{j-i+1}(\boldsymbol{y}),\boldsymbol{f}_i(\boldsymbol{y})\big]\\&+\sum_{j=3}^{k}\sum_{i=2}^{j-1}D\boldsymbol{h}_i(\boldsymbol{y})\big\{\boldsymbol{f}_{k-i+1}(\boldsymbol{y})-\boldsymbol{g}_{k-i+1}(\boldsymbol{y})\big\}\\&+\sum_{j=4}^{k}\sum_{m=2}^{[\frac{j}{2}]}\frac{1}{m!}\sum_{i=m}^{j-m}D^m\boldsymbol{f}_i(\boldsymbol{y})\\&\times\sum_{\substack{l_1+l_2+\cdots+l_m=j-(i-m)\\2\leq l_1,l_2,\ldots,l_m\leq j-(i-m)-2(m-1)}}\boldsymbol{h}_{l_1}(\boldsymbol{y})\cdots\boldsymbol{h}_{l_m}(\boldsymbol{y}),\end{aligned} \tag{2.200}$$

where the property of the Lie bracket,

$$[\boldsymbol{X}_i,\boldsymbol{Y}_j]\in\mathcal{H}_{i+j-1}\quad\text{for }\boldsymbol{X}_i\in\mathcal{H}_i\text{ and }\boldsymbol{Y}_j\in\mathcal{H}_j, \tag{2.201}$$

has been used, where $\mathcal{H}_k$ denotes linear vector space consisting of kth-degree homogeneous polynomials.

Now by taking the terms in (2.200) according to their order one obtains

$$\begin{aligned}
\boldsymbol{g}_2 &= \boldsymbol{f}_2 + [\boldsymbol{h}_2, \boldsymbol{v}_1], \\
\boldsymbol{g}_3 &= \boldsymbol{f}_3 + [\boldsymbol{h}_3, \boldsymbol{v}_1] + [\boldsymbol{h}_2, \boldsymbol{f}_2] + D\boldsymbol{h}_2(\boldsymbol{f}_2 - \boldsymbol{g}_2), \\
\boldsymbol{g}_4 &= \boldsymbol{f}_4 + [\boldsymbol{h}_4, \boldsymbol{v}_1] + [\boldsymbol{h}_3, \boldsymbol{f}_2] + [\boldsymbol{h}_2, \boldsymbol{f}_3] \\
&\quad + D\boldsymbol{h}_2(\boldsymbol{f}_3 - \boldsymbol{g}_3) + D\boldsymbol{h}_3(\boldsymbol{f}_2 - \boldsymbol{g}_2) + \frac{1}{2}D^2\boldsymbol{f}_2\boldsymbol{h}_2^2, \\
&\text{etc.},
\end{aligned} \tag{2.202}$$

where the variable $\boldsymbol{y}$ has been dropped for simplicity. For general k, we have

$$\begin{aligned}
\boldsymbol{g}_k = \boldsymbol{f}_k + [\boldsymbol{h}_k, \boldsymbol{v}_1] + \sum_{i=2}^{k-1}\left\{[\boldsymbol{h}_i, \boldsymbol{f}_{k-i+1}] + D\boldsymbol{h}_i(\boldsymbol{f}_{k-i+1} - \boldsymbol{g}_{k-i+1})\right\} \\
+ \sum_{m=2}^{[\frac{k}{2}]} \frac{1}{m!} \sum_{i=m}^{k-m} D^m \boldsymbol{f}_i \sum_{\substack{l_1+l_2+\cdots+l_m=k-(i-m) \\ 2\le l_1,l_2,\ldots,l_m\le k-(i-m)-2(m-1)}} \boldsymbol{h}_{l_1}\boldsymbol{h}_{l_2}\cdots\boldsymbol{h}_{l_m},
\end{aligned}$$

which is (2.193) and the proof is thus completed. □

Remark 2.20 The following observations are made from (2.193).

(1) The only operation appearing in the formula is the Frechét derivative involved in $D\boldsymbol{h}_i$, $D^m\boldsymbol{f}_i$ and the Lie bracket $[\bullet, \bullet]$. This operation can be easily implemented on computers using a computer algebra system.
(2) The kth-order equation contains all the kth-order and only the kth-order terms. The equation is given in a recursive form.
(3) The kth-order equation depends upon the known vector homogeneous polynomials $\boldsymbol{v}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, \ldots, \boldsymbol{f}_{k-1}$, and upon $\boldsymbol{h}_2, \boldsymbol{h}_3, \ldots, \boldsymbol{h}_{k-1}$, as well as $\boldsymbol{g}_2, \boldsymbol{g}_3, \ldots, \boldsymbol{g}_{k-1}$, which have been explicitly determined from the lower-order equations.
(4) The equation involves the coefficients of the nonlinear transformation $\boldsymbol{h}$ and the coefficients of the kth-order normal form $\boldsymbol{g}_k$. If the jth-order ($j < k$) coefficients of $\boldsymbol{h}_j$ are completely determined from the jth-order equation, then the only unknown coefficients in the kth-order equation are $\boldsymbol{h}_k$ and $\boldsymbol{g}_k$, which yields the normal form.
(5) If the kth-order equation contains lower-order coefficients of $\boldsymbol{h}_j (j < k)$ which are undetermined in the lower-order ($<k$) equations, they may be used to eliminate some coefficients of $\boldsymbol{g}_k$, and thus the normal form can be further simplified.
(6) For most of the approaches in computing the simplest normal forms (e.g., see [2, 3, 54, 55, 202, 219, 255]), the nonlinear vector field, $\boldsymbol{f}(\boldsymbol{x})$, given in (2.189) is assumed to be a conventional normal form in order to simplify symbolic computations. All the approaches described in the above-mentioned references

generate the kth-order algebraic equation which contains lower-order ($<k$) as well as higher-order ($>k$) terms. This is extremely time consuming in symbolic computations and it also takes too much computer memory. With the above efficient recursive formulas, the kth-order equation exactly contains the kth-order terms, which greatly saves computer memory and computational time. Therefore, for our approach, the vector field $\boldsymbol{f}(\boldsymbol{x})$ can be assumed to be a general analytic function, not necessarily a conventional normal form.

2.4.2 Symbolic Computation

The recursive formula given in (2.193) has been directly used to develop a symbolic computational program based on Maple. The main operation involved in the computation is the multiplication of a matrix by a vector (a Lie bracket operator consists of two such multiplications). The computations for the second term and the third term in (2.193) (i.e., the Lie bracket and the first summation) are straightforward, while the last summation in (2.193) needs careful consideration in order to achieve the minimum number of operations. First it should be noted that the variable used in functions $\boldsymbol{h}_{l_1}, \boldsymbol{h}_{l_2}, \ldots, \boldsymbol{h}_{l_m}$, must be different (i.e., not the $\boldsymbol{y}$ variable) from that of $\boldsymbol{f}_i$, so that the operator D can treat them as "constant" in the processing of differentiation (see (2.194)). When the differentiation is complete, the variables in these $\boldsymbol{h}$ functions should be changed back to the original variable $\boldsymbol{y}$. Secondly, in order to achieve the minimum number of operations for the last summation, note that many terms in the summation are actually the same due to the fact that the indices $l_1, l_2, \ldots, l_m$ can be equal, and due to the fact that

$$D\big((D\boldsymbol{f}_i)\boldsymbol{h}_1\big)\boldsymbol{h}_2 = D\big((D\boldsymbol{f}_i)\boldsymbol{h}_2\big)\boldsymbol{h}_1, \tag{2.203}$$

which can be proved by direct calculation as follows.

$$\begin{aligned} &D\big((D\boldsymbol{f}_i)\boldsymbol{h}_1\big)\boldsymbol{h}_2 \\ &\quad = D\left(\begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{pmatrix} h_{11} \\ h_{12} \\ \vdots \\ h_{1n} \end{pmatrix}\right)\boldsymbol{h}_2 \\ &\quad = D\begin{pmatrix} \sum_{j=1}^{n} f_{1j}h_{1j} \\ \sum_{j=1}^{n} f_{2j}h_{2j} \\ \vdots \\ \sum_{j=1}^{n} f_{nj}h_{nj} \end{pmatrix}\begin{pmatrix} h_{21} \\ h_{22} \\ \vdots \\ h_{2n} \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} \sum_{j=1}^n f_{1j1}h_{1j} & \sum_{j=1}^n f_{1j2}h_{1j} & \cdots & \sum_{j=1}^n f_{1jn}h_{1j} \\ \sum_{j=1}^n f_{2j1}h_{1j} & \sum_{j=1}^n f_{2j2}h_{1j} & \cdots & \sum_{j=1}^n f_{2jn}h_{1j} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^n f_{nj1}h_{1j} & \sum_{j=1}^n f_{nj2}h_{1j} & \cdots & \sum_{j=1}^n f_{njn}h_{1j} \end{bmatrix} \begin{pmatrix} h_{21} \\ h_{22} \\ \vdots \\ h_{2n} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{l=1}^n (\sum_{j=1}^n f_{1jl}h_{1j})h_{2l} \\ \sum_{l=1}^n (\sum_{j=1}^n f_{2jl}h_{1j})h_{2l} \\ \vdots \\ \sum_{l=1}^n (\sum_{j=1}^n f_{njl}h_{1j})h_{2l} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n (\sum_{l=1}^n f_{1jl}h_{2l})h_{1j} \\ \sum_{j=1}^n (\sum_{l=1}^n f_{2jl}h_{2l})h_{1j} \\ \vdots \\ \sum_{j=1}^n (\sum_{l=1}^n f_{njl}h_{2l})h_{1j} \end{pmatrix} \\
&= D\big((D\boldsymbol{f}_i)\boldsymbol{h}_2\big)\boldsymbol{h}_1, \qquad (2.204)
\end{aligned}
$$

where the fact that h_{l_j} is not affected by the operator D has been used. Thus the order of differentiation in (2.194) with respect to h_{l_j} has no influence. Then the last summation in (2.193) can be rewritten as

$$
\begin{aligned}
&\sum_{m=2}^{[\frac{k}{2}]} \frac{1}{m!} \sum_{i=m}^{k-m} D^m \boldsymbol{f}_i \sum_{\substack{l_1+l_2+\cdots+l_m=k-(i-m) \\ 2\le l_1,l_2,\ldots,l_m \le k-(i-m)-2(m-1)}} \boldsymbol{h}_{l_1}\boldsymbol{h}_{l_2}\cdots\boldsymbol{h}_{l_m} \\
&= \sum_{m=2}^{[\frac{k}{2}]} \frac{1}{m!} \sum_{i=m}^{k-m} D^m \boldsymbol{f}_i \sum_{\substack{q_1l_1+\cdots+q_pl_p=k-(i-m) \\ 2\le l_p<\cdots<l_1\le (k-(i-m))/m}} \frac{m!}{q_1!\cdots q_p!}\boldsymbol{h}_{l_1}^{q_1}\cdots\boldsymbol{h}_{l_p}^{q_p} \\
&= \sum_{m=2}^{[\frac{k}{2}]} \sum_{i=m}^{k-m} D^m \boldsymbol{f}_i \sum_{\substack{q_1l_1+q_2l_2+\cdots+q_pl_p=k-(i-m) \\ 2\le l_p<\cdots<l_1\le (k-(i-m))/m}} \frac{\boldsymbol{h}_{l_1}^{q_1}\boldsymbol{h}_{l_2}^{q_2}\cdots\boldsymbol{h}_{l_p}^{q_p}}{q_1!q_2!\cdots q_p!}, \qquad (2.205)
\end{aligned}
$$

where q_j, $j = 1, 2, \ldots, p$, are nonzero positive integers.

Based on (2.193) and (2.205), Maple programs have been developed which only require simple preparation of an input file by a user. The algorithm is outlined below.

(1) Read a prepared input file. The input file lists the order of the normal form to be computed, `ord`, and the case of singularity to be considered.
(2) Separate the different order terms from the given input differential equations.
(3) A procedure for computing the Lie bracket operator.

(4) A procedure for computing the Jacobian matrix and the multiplication of a matrix by a vector.
(5) A procedure for solving a linear algebraic equation.
(6) Obtain the ordered terms for functions $F[k]$, $G[k]$ and $H[k]$.
(7) For order k, recursively compute the algebraic equations which will be used for finding the coefficients of the normal form and the corresponding nonlinear transformation.
   (i) Compute the Lie bracket term $[\boldsymbol{h}_k, \boldsymbol{v}_1]$.
   (ii) Compute the summation $\sum_{i=2}^{k-1}\{[\boldsymbol{h}_{k-i+1}, \boldsymbol{f}_i] + D\boldsymbol{h}_i(\boldsymbol{f}_{k-i+1} - \boldsymbol{g}_{k-i+1})\}$.
   (iii) Find the indices $q_1, q_2, \ldots, q_p$ for the term $h_{l_1}^{q_1} h_{l_2}^{q_2} \cdots h_{l_p}^{q_p}$, where $2 \le l_p \le \cdots \le l_1 \le (k-(i-m))/m$, satisfying $q_1 l_1 + q_2 l_2 + \cdots + q_p l_p = k - (i-m)$, which results in the coefficient $\frac{m!}{i_1! i_2! \ldots i_p!}$ for this term.
(8) Call the subroutine for a case study of the given singularity.
   (i) Obtain the kth-order algebraic equation from the main program, which is stored in the variable `cof`$[i, j, k]$.
   (ii) For complex analysis, get the real and imaginary part from the coefficient `cof`$[i, j, k]$.
   (iii) For a suborder k, determine the b coefficients and then solve for the relative c coefficients.
   (iv) Return to the main program.
(9) Write the normal form into the output file `Nform`.

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