

Chapter 8

Linear Operators in Hilbert Spaces

Motivated by the eigenvalue problems for ordinary and partial differential operators, we shall develop the spectral theory for linear operators in Hilbert spaces. Here we transform the unbounded differential operators into singular integral operators which are completely continuous. With his study of integral equations D. Hilbert, together with his students E. Schmidt, I. Schur, and H. Weyl, opened a new era for the analysis.

1 Various Eigenvalue Problems

At first, we consider the resolution of linear systems of equations: For the given matrix $A = (a_{ij})_{i,j=1,\dots,n} \in \mathbb{R}^{n \times n}$ we associate the mapping

$$x \mapsto Ax : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

and the system of equations

$$\sum_{k=1}^n a_{ik}x_k = y_i, \quad i = 1, \dots, n, \quad \text{or equivalently} \quad Ax = y$$

with the right-hand side $y = (y_1, \dots, y_n)^t$. The system $Ax = y$ has a solution for all $y \in \mathbb{R}^n$ if and only if the homogeneous equation $Ax = 0$ possesses only the trivial solution $x = 0$ and we have $x = A^{-1}y$. We remark that the concept of determinants is not necessary in this context.

In Theorem 4.6 from Chapter 7, Section 4 by F. Riesz, we have transferred this solvability theory to linear operators in Banach spaces: Let $\mathcal{B}$ be a real Banach space and $K : \mathcal{B} \rightarrow \mathcal{B}$ a linear completely continuous operator with the associate operator $Tx := x - Kx$, $x \in \mathcal{B}$. If the implication

$$Tx = 0 \quad \Rightarrow \quad x = 0$$

holds true, the equation

$$Tx = y, \quad x \in \mathcal{B}$$

possesses exactly one solution for all $y \in \mathcal{B}$.

We now consider the *principal axes transformation of Hermitian matrices*:

Let $A = (a_{ik})_{i,k=1,\dots,n} \in \mathbb{C}^{n \times n}$ denote a Hermitian matrix, which means $a_{ik} = \overline{a_{ki}}$ for all $i, k = 1, \dots, n$. Then A possesses a complete orthonormal system of eigenvectors $\varphi_1, \dots, \varphi_n \in \mathbb{C}^n$ with the real eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, more precisely

$$A\varphi_i = \lambda_i\varphi_i, \quad i = 1, \dots, n,$$

and we have

$$(\varphi_i, \varphi_k) = \delta_{ik}, \quad i, k = 1, \dots, n.$$

Here we have used the inner product $(x, y) := \overline{x^t} \cdot y$. By $\xi_i := (\varphi_i, x)$ we denote the i -th component of $x \in \mathbb{C}^n$ with respect to $(\varphi_1, \dots, \varphi_n)$ which implies

$$x = \sum_{i=1}^n \xi_i \varphi_i.$$

Then we note that

$$Ax = \sum_{i=1}^n \xi_i A\varphi_i = \sum_{i=1}^n \lambda_i \xi_i \varphi_i.$$

We define the diagonal matrix

$$\Lambda := \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} \in \mathbb{R}^{n \times n}$$

and the unitary matrix $U^{-1} := (\varphi_1, \dots, \varphi_n) \in \mathbb{C}^{n \times n}$. Now we obtain the representation $x = U^{-1}\xi$ with $\xi = (\xi_1, \dots, \xi_n)^t$ and consequently $\xi = Ux$. This implies the equation

$$U \circ A \circ U^{-1} \circ \xi = U \circ A \circ x = U \circ U^{-1} \circ \Lambda \circ \xi = \Lambda \circ \xi$$

and consequently the *unitary transformation*

$$\Lambda = U \circ A \circ U^{-1}.$$

Now we observe $U^{-1} = U^* = \overline{U}^t$, and we calculate the transformation of the associate Hermitian form

$$\begin{aligned}
\sum_{i,k=1}^n a_{ik} \bar{x}_i x_k &= (x, Ax) = \left(\sum_{i=1}^n (\varphi_i, x) \varphi_i, \sum_{j=1}^n (\varphi_j, x) \lambda_j \varphi_j \right) \\
&= \sum_{i,j=1}^n \overline{(\varphi_i, x)} (\varphi_j, x) \lambda_j \delta_{ij} \\
&= \sum_{i=1}^n |(\varphi_i, x)|^2 \lambda_i = \sum_{i=1}^n \lambda_i |\xi_i|^2.
\end{aligned}$$

In the present chapter we intend to deduce the corresponding theorems for operators in Hilbert spaces. Specializing them to integral operators we shall treat eigenvalue problems for ordinary and partial differential equations.

Example 1.1. Let the domain of definition

$$\mathcal{D} := \left\{ u = u(x) \in C^2[0, \pi] : u(0) = 0 = u(\pi) \right\}$$

and the differential operator

$$Lu(x) := -u''(x), \quad x \in [0, \pi], \quad \text{for } u \in \mathcal{D}$$

be given. Which numbers $\lambda \in \mathbb{R}$ admit a nontrivial solution of the eigenvalue problem

$$Lu(x) = \lambda u(x), \quad 0 \leq x \leq \pi, \quad u \in \mathcal{D}, \quad (1)$$

satisfying $u \in \mathcal{D}$ and $u \not\equiv 0$?

$\lambda = 0$: We have $u''(x) = 0$ for $x \in [0, \pi]$ and consequently $u(x) = ax + b$ with the constants $a, b \in \mathbb{R}$. The boundary conditions for u yield $0 = u(0) = b$ and $0 = u(\pi) = a\pi + b = a\pi$, and we obtain

$$u(x) \equiv 0, \quad x \in [0, \pi].$$

Therefore, $\lambda = 0$ is not an eigenvalue of (1).

$\lambda < 0$: Setting $\lambda = -k^2$ with $k \in (0, +\infty)$, we rewrite (1) into the form

$$u''(x) - k^2 u(x) = 0, \quad x \in [0, \pi].$$

Evidently, the functions $\{e^{kx}, e^{-kx}\}$ constitute a fundamental system of the differential equation. Taking $u(0) = 0$ into account, we infer

$$u(x) = Ae^{kx} + Be^{-kx} = A(e^{kx} - e^{-kx}) = 2A \sinh(kx).$$

Noting $u(\pi) = 0$, we finally obtain

$$u(x) \equiv 0, \quad x \in [0, \pi].$$

Therefore, negative eigenvalues $\lambda < 0$ of (1) do not exist.

$\lambda > 0$: We now consider $\lambda = k^2$ with a number $k \in (0, +\infty)$. Then (1) is written in the form

$$u''(x) + k^2 u(x) = 0, \quad x \in [0, \pi].$$

The fundamental system is given by $\{\cos(kx), \sin(kx)\}$ and the general solution by

$$u(x) = A \cos(kx) + B \sin(kx), \quad x \in [0, \pi].$$

From $0 = u(0) = A$ we infer $u(x) = B \sin(kx)$ for $x \in [0, \pi]$, and the boundary condition $0 = u(\pi) = B \sin(k\pi)$ implies $k \in \mathbb{N}$. In this way we obtain the eigenvalues $\lambda = k^2$ of (1) with the eigenfunctions

$$u_k(x) = \sin(kx), \quad x \in [0, \pi], \quad k = 1, 2, \dots$$

Example 1.2. Let the domain

$$G := \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \in (0, \pi), i = 1, \dots, n \right\} = (0, \pi)^n \subset \mathbb{R}^n$$

be given. On the domain of definition

$$\mathcal{D} := \left\{ u \in C^2(G) \cap C^0(\overline{G}) : u|_{\partial G} = 0 \right\}$$

we define the differential operator

$$Lu(x) := -\Delta u(x) = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u(x), \quad x \in G.$$

We consider the eigenvalue problem

$$Lu(x) = \lambda u(x), \quad x \in G, \quad (2)$$

for $u \in \mathcal{D}$ and $\lambda \in \mathbb{R}$. In order to solve this problem we propose the *ansatz of separation*

$$u(x) = u(x_1, \dots, x_n) := u_1(x_1) \cdot u_2(x_2) \cdot \dots \cdot u_n(x_n), \quad x \in G.$$

The differential equation (2) becomes

$$-\sum_{i=1}^n u_1(x_1) \cdot \dots \cdot u_{i-1}(x_{i-1}) u_i''(x_i) u_{i+1}(x_{i+1}) \cdot \dots \cdot u_n(x_n) = \lambda u_1(x_1) \cdot \dots \cdot u_n(x_n)$$

and consequently

$$-\sum_{i=1}^n \frac{u_i''(x_i)}{u_i(x_i)} = \lambda, \quad x \in G.$$

We now choose $u_i(x_i) := \sin(k_i x_i)$ with $k_i \in \mathbb{N}$ for $i = 1, \dots, n$, and we obtain

$$-\sum_{i=1}^n \frac{u_i''(x_i)}{u_i(x_i)} = \sum_{i=1}^n k_i^2 = \lambda \in (0, \infty).$$

The solutions of the eigenvalue problem (2) appear as follows:

$$u(x_1, \dots, x_n) := \sin(k_1 x_1) \cdot \dots \cdot \sin(k_n x_n) \quad \text{and} \quad \lambda = k_1^2 + \dots + k_n^2$$

for $k_1, \dots, k_n \in \mathbb{N}$. Normalizing

$$u_{k_1, \dots, k_n}(x_1, \dots, x_n) := \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \sin(k_1 x_1) \cdot \dots \cdot \sin(k_n x_n), \quad x \in G \quad (3)$$

and using the inner product

$$(u, v) := \int_G u(x_1, \dots, x_n) v(x_1, \dots, x_n) dx_1 \dots dx_n \quad u, v \in \mathcal{D},$$

we obtain the orthonormal system of functions

$$(u_{k_1 \dots k_n}, u_{l_1 \dots l_n}) = \delta_{k_1 l_1} \cdot \dots \cdot \delta_{k_n l_n} \quad \text{for} \quad k_1, \dots, k_n, l_1, \dots, l_n \in \mathbb{N}. \quad (4)$$

On account of

$$Lu_{k_1 \dots k_n} = (k_1^2 + \dots + k_n^2) u_{k_1 \dots k_n}$$

and

$$\|u_{k_1 \dots k_n}\|_{L^2(G)}^2 := (u_{k_1 \dots k_n}, u_{k_1 \dots k_n}) = 1$$

for all $k_1 \dots k_n \in \mathbb{N}$ we infer

$$\sup_{u \in \mathcal{D}, \|u\|=1} \|Lu\| \geq \sup_{k_1 \dots k_n \in \mathbb{N}} \|Lu_{k_1 \dots k_n}\| = \sup_{k_1 \dots k_n \in \mathbb{N}} (k_1^2 + \dots + k_n^2) = +\infty. \quad (5)$$

Consequently $L = -\Delta : L^2(G) \rightarrow L^2(G)$ represents an unbounded operator on the Hilbert space $L^2(G)$.

The following question is of central interest: Do the given functions

$$\{u_{k_1 \dots k_n}\}_{k_1 \dots k_n=1,2,\dots}$$

constitute a complete system? Can we expand an arbitrary function into such a series of functions?

We now consider the *Sturm-Liouville eigenvalue problem*:

Let the numbers $c_1, c_2, d_1, d_2 \in \mathbb{R}$ satisfying $c_1^2 + c_2^2 > 0$ and $d_1^2 + d_2^2 > 0$ be prescribed. We choose the linear space

$$\mathcal{D} := \left\{ f \in C^2([a, b], \mathbb{R}) : c_1 f(a) + c_2 f'(a) = 0 = d_1 f(b) + d_2 f'(b) \right\}$$

as our domain of definition, where the numbers $-\infty < a < b < +\infty$ are fixed. With the functions $p = p(x) \in C^1([a, b], (0, +\infty))$ and $q = q(x) \in C^0([a, b], \mathbb{R})$ we define the *Sturm-Liouville operator*

$$Lu(x) := -(p(x)u'(x))' + q(x)u(x), \quad x \in [a, b], \quad \text{for} \quad u \in \mathcal{D}.$$

Proposition 1.3. *The operator $L : \mathcal{D} \rightarrow \mathbb{C}^0([a, b], \mathbb{R})$ is linear and symmetric satisfying*

$$L(\alpha u + \beta v) = \alpha Lu + \beta Lv \quad \text{for all } u, v \in \mathcal{D}, \quad \alpha, \beta \in \mathbb{R}$$

and

$$\int_a^b u(x) (Lv(x)) dx = \int_a^b (Lu(x)) v(x) dx \quad \text{for all } u, v \in \mathcal{D}.$$

Proof: The linearity is evident, and we calculate for $u, v \in \mathcal{D}$ as follows:

$$\begin{aligned} vLu - uLv &= v(-(pu')' + qu) - u(-(pv')' + qv) \\ &= \frac{d}{dx} \left\{ p(x)(u(x)v'(x) - u'(x)v(x)) \right\}, \quad x \in [a, b]. \end{aligned} \quad (6)$$

This implies

$$\int_a^b (vLu - uLv) dx = \left[-p(x)(u(x), u'(x)) \cdot (-v'(x), v(x))^t \right]_a^b = 0,$$

since the vectors $(u(x), u'(x))$ and $(v(x), v'(x))$ are parallel for $x = a$ and $x = b$, respectively; here we take $u, v \in \mathcal{D}$ into account. q.e.d.

We now investigate the eigenvalue problem

$$Lu = \lambda u, \quad u \in \mathcal{D}. \quad (7)$$

Setting

$$(u, v) := \int_a^b u(x)v(x) dx \quad \text{for } u, v \in \mathcal{D},$$

we obtain an orthonormal system of eigenfunctions

$$u_k(x) \in \mathcal{D} \quad \text{with} \quad (u_k, u_l) = \delta_{kl}, \quad k, l \in \mathbb{N}$$

in Section 8 satisfying

$$Lu_k = \lambda_k u_k, \quad k = 1, 2, \dots$$

According to the Example 1.1 we expect the asymptotic behavior

$$-\infty < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \rightarrow +\infty \quad (8)$$

for the eigenvalues. Consequently, the operator L is unbounded. We shall derive the expansion into the series

$$f(x) = \sum_{k=1}^{\infty} c_k u_k(x) \quad \text{with} \quad c_k = (u_k, f)$$

for all functions $f \in \mathcal{D}$. At first, we require the

Assumption 0: The equation $Lu = 0$ with $u \in \mathcal{D}$ admits only the trivial solution $u \equiv 0$.

The domain $\mathcal{D}$ is not complete with respect to the norm $\|u\| := \sqrt{(u, u)}$ for $u \in \mathcal{D}$, and the operator L is unbounded in general. Therefore, we cannot prove the existence of the inverse L^{-1} by the Theorem of F. Riesz. With the *Assumption 0*, however, we shall construct the inverse with the aid of *Green's function for the Sturm-Liouville operator* $K = K(x, y)$. Having achieved this, we shall transform (7) equivalently into an eigenvalue problem for the bounded operator L^{-1} , namely

$$L^{-1}u = \frac{1}{\lambda}u, \quad u \in \mathcal{D}. \quad (9)$$

For the construction of the inverse we consider the ordinary differential equation

$$\begin{aligned} Lu(x) &= -(p(x)u'(x))' + q(x)u(x) \\ &= -p(x)u''(x) - p'(x)u'(x) + q(x)u(x) \\ &= f(x), \quad a \leq x \leq b. \end{aligned} \quad (10)$$

The homogeneous equation $Lu = 0$ possesses a fundamental system $\alpha = \alpha(x)$, $\beta = \beta(x)$ satisfying

$$L\alpha(x) \equiv 0 \equiv L\beta(x) \quad \text{in} \quad [a, b].$$

We construct a solution of (10) by the method *variation of the constants*

$$u(x) = A(x)\alpha(x) + B(x)\beta(x), \quad a \leq x \leq b, \quad (11)$$

under the subsidiary condition

$$A'(x)\alpha(x) + B'(x)\beta(x) = 0. \quad (12)$$

With the aid of (12) we calculate

$$u'(x) = A(x)\alpha'(x) + B(x)\beta'(x)$$

and

$$u''(x) = A(x)\alpha''(x) + B(x)\beta''(x) + A'(x)\alpha'(x) + B'(x)\beta'(x).$$

Together with the formula (11) we obtain

$$Lu(x) = A(x)L\alpha(x) + B(x)L\beta(x) - p(x)\{A'(x)\alpha'(x) + B'(x)\beta'(x)\} = f(x),$$

and therefore

$$-p(x)\left\{A'(x)\alpha'(x) + B'(x)\beta'(x)\right\} = f(x), \quad a \leq x \leq b. \quad (13)$$

By the ansatz

$$A'(x) = \beta(x)k(x), \quad B'(x) = -\alpha(x)k(x), \quad a \leq x \leq b, \quad (14)$$

with a continuous function $k = k(x)$, $x \in [a, b]$, the relation (12) is fulfilled and (13) becomes

$$-p(x)\{\beta(x)\alpha'(x) - \alpha(x)\beta'(x)\}k(x) = f(x), \quad a \leq x \leq b. \quad (15)$$

Proposition 1.4. *The relation $p(x)\{\alpha(x)\beta'(x) - \alpha'(x)\beta(x)\} = \text{const}$ in $[a, b]$ holds true.*

Proof: Applying (6) to $u = \alpha(x)$ and $v = \beta(x)$ we infer

$$0 = \frac{d}{dx} \left\{ p(x) (\alpha(x)\beta'(x) - \alpha'(x)\beta(x)) \right\} \quad \text{in } [a, b]. \quad \text{q.e.d.}$$

We now choose $\alpha = \alpha(x)$ and $\beta = \beta(x)$ to solve the homogeneous equation $Lu = 0$ satisfying

$$p(x)\left\{\alpha(x)\beta'(x) - \alpha'(x)\beta(x)\right\} \equiv 1 \quad \text{in } [a, b] \quad (16)$$

and

$$c_1\beta(a) + c_2\beta'(a) = 0 = d_1\alpha(b) + d_2\alpha'(b). \quad (17)$$

Here we solve the initial value problems

$$L\alpha = 0 \quad \text{in } [a, b], \quad \alpha(b) = d_2, \quad \alpha'(b) = -d_1$$

and

$$L\beta = 0 \quad \text{in } [a, b], \quad \beta(a) = \frac{1}{M}c_2, \quad \beta'(a) = -\frac{1}{M}c_1.$$

Thereby, we determine $M \neq 0$ such that

$$p(a)\left\{\alpha(a)\beta'(a) - \alpha'(a)\beta(a)\right\} = -\frac{1}{M}p(a)\left\{c_1\alpha(a) + c_2\alpha'(a)\right\} = 1$$

is fulfilled choosing

$$M = -p(a)\left\{c_1\alpha(a) + c_2\alpha'(a)\right\}.$$

The statement $M \neq 0$ is contained in the following

Proposition 1.5. *The functions $\{\alpha, \beta\}$ constitute a fundamental system.*

Proof: If the statement were violated, we have a number $\mu \neq 0$ with the property

$$\alpha(x) = \mu\beta(x), \quad a \leq x \leq b.$$

We deduce $\alpha \in \mathcal{D}$ from (17), and the *Assumption 0* yields a contradiction with $\alpha \equiv 0$. q.e.d.

The relations (15) and (16) imply

$$k(x) = f(x) \quad \text{in } [a, b], \quad (18)$$

and (14) yields

$$A(x) = \int_a^x \beta(y)f(y) dy + \text{const}, \quad B(x) = \int_x^b \alpha(y)f(y) dy + \text{const}. \quad (19)$$

We summarize our considerations to the following

Theorem 1.6. *The Sturm-Liouville equation $Lu = f$ for $u \in \mathcal{D}$ with the right-hand side $f \in C^0([a, b])$ is solved by the function*

$$u(x) = \alpha(x) \int_a^x \beta(y)f(y) dy + \beta(x) \int_x^b \alpha(y)f(y) dy = \int_a^b K(x, y)f(y) dy. \quad (20)$$

With the aid of the fundamental system $\{\alpha, \beta\}$ of $Lu = 0$ satisfying (16) and (17), we here define the Green's function of the Sturm-Liouville operator as follows:

$$K(x, y) = \begin{cases} \alpha(x)\beta(y), & a \leq y \leq x \\ \beta(x)\alpha(y), & x \leq y \leq b. \end{cases} \quad (21)$$

Proof: The derivation above implies that the function $u(x)$ from (20) satisfies the differential equation $Lu = f$. Furthermore, we see

$$u(a) = \beta(a) \int_a^b \alpha(y)f(y) dy, \quad u'(a) = \beta'(a) \int_a^b \alpha(y)f(y) dy,$$

and (17) gives us

$$c_1 u(a) + c_2 u'(a) = (c_1 \beta(a) + c_2 \beta'(a)) \int_a^b \alpha(y)f(y) dy = 0.$$

In the same way we determine

$$u(b) = \alpha(b) \int_a^b \beta(y) f(y) dy, \quad u'(b) = \alpha'(b) \int_a^b \beta(y) f(y) dy$$

and

$$d_1 u(b) + d_2 u'(b) = (d_1 \alpha(b) + d_2 \alpha'(b)) \int_a^b \beta(y) f(y) dy = 0.$$

q.e.d.

Theorem 1.6 directly implies the following

Theorem 1.7. *With the Assumption 0, the subsequent statements are equivalent:*

- I. *The function $u \in \mathcal{D}$ with $u \not\equiv 0$ satisfies $Lu = \lambda u$.*
- II. *The function $u \in \mathcal{D}$ with $u \not\equiv 0$ satisfies $\int_a^b K(x, y) u(y) dy = \frac{1}{\lambda} u(x)$ for $a \leq x \leq b$.*

We shall now address the *eigenvalue problem of the n -dimensional oscillation equation* considered by H. von Helmholtz : Let $G \subset \mathbb{R}^n$ be a bounded Dirichlet domain, which means that all continuous functions $g = g(x) : \partial G \rightarrow \mathbb{R}$ possess a solution of the Dirichlet problem

$$\begin{aligned} u &= u(x) \in C^2(G) \cap C^0(\overline{G}), \\ \Delta u(x) &= 0 \quad \text{in } G, \\ u(x) &= g(x) \quad \text{on } \partial G \end{aligned} \tag{22}$$

(compare Chapter 5, Section 3). The further assumption will be eliminated by Proposition 9.1 in Section 9, namely that G satisfies the conditions of the Gaussian integral theorem from Section 5 in Chapter 1. Then we can specify the *Green's function of the Laplace operator* for the domain G as follows:

$$H(x, y) = \begin{cases} -\frac{1}{2\pi} \log |y - x| + h(x, y), & n = 2 \\ \frac{1}{(n-2)\omega_n} \frac{1}{|y - x|^{n-2}} + h(x, y), & n \geq 3 \end{cases} \tag{23}$$

for $(x, y) \in G \otimes G := \{(\xi, \eta) \in G \times G : \xi \neq \eta\}$. Here we have $\Delta_y h(x, y) = 0$ in G and

$$h(x, y) = \begin{cases} \frac{1}{2\pi} \log |y - x|, & n = 2 \\ -\frac{1}{(n-2)\omega_n} \frac{1}{|y - x|^{n-2}}, & n \geq 3 \end{cases} \tag{24}$$

for $x \in G$ and $y \in \partial G$. Furthermore, ω_n denotes the area of the unit sphere in $\mathbb{R}^n$. Due to Chapter 5, Sections 1 and 2, we can represent a solution of the problem

$$\begin{aligned} u &= u(x) \in C^2(G) \cap C^0(\overline{G}), \\ -\Delta u(x) &= f(x) \quad \text{in } G, \\ u(x) &= 0 \quad \text{on } \partial G \end{aligned} \tag{25}$$

in the following form

$$u(x) = \int_G H(x, y) f(y) dy, \quad x \in G. \tag{26}$$

For the deduction of (26) we consider the domain $G_\varepsilon := \{y \in G : |y - x| > \varepsilon\}$ with a small $\varepsilon > 0$. The Gaussian integral theorem implies

$$\begin{aligned} & \int_{G_\varepsilon} \left(H(x, y) \Delta u(y) - u(y) \Delta_y H(x, y) \right) dy \\ &= \int_{\partial G} \left(H(x, y) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial H}{\partial \nu}(x, y) \right) d\sigma(y) \\ &\quad - \int_{y: |y-x|=\varepsilon} \left(H(x, y) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial H}{\partial \nu}(x, y) \right) d\sigma(y). \end{aligned}$$

Observing $\varepsilon \downarrow 0$ we obtain

$$\begin{aligned} - \int_G H(x, y) f(y) dy &= \lim_{\varepsilon \downarrow 0} \int_{r=|y-x|=\varepsilon} u(y) \left(\frac{1}{(n-2)\omega_n} (2-n)r^{1-n} \right) d\sigma(y) \\ &= - \lim_{\varepsilon \downarrow 0} \left(\frac{1}{\varepsilon^{n-1}\omega_n} \int_{r=|y-x|=\varepsilon} u(y) d\sigma(y) \right) \\ &= -u(x) \quad \text{for all } x \in G \end{aligned}$$

in the case $n \geq 3$, and similarly in the case $n = 2$.

We now derive the *symmetry of Green's function*, namely

$$H(x, y) = H(y, x) \quad \text{for all } (x, y) \in G \otimes G. \tag{27}$$

Here we choose the points $x, y \in G$ satisfying $x \neq y$, and on the domain

$$G_\varepsilon := \left\{ z \in G : |z - x| > \varepsilon \text{ and } |z - y| > \varepsilon \right\}$$

we consider the functions $p(z) := H(x, z)$ and $q(z) := H(y, z)$, $z \in G_\varepsilon$. For $\varepsilon \downarrow 0$ the Gaussian integral theorem implies

$$\begin{aligned}
 0 &= \lim_{\varepsilon \downarrow 0} \int_{G_\varepsilon} (q \Delta p - p \Delta q) dz = \lim_{\varepsilon \downarrow 0} \int_{\partial G_\varepsilon} \left(q \frac{\partial p}{\partial \nu} - p \frac{\partial q}{\partial \nu} \right) d\sigma(z) \\
 &= - \lim_{\varepsilon \downarrow 0} \int_{|z-x|=\varepsilon} \left(q \frac{\partial p}{\partial \nu} - p \frac{\partial q}{\partial \nu} \right) d\sigma(z) - \lim_{\varepsilon \downarrow 0} \int_{|z-y|=\varepsilon} \left(q \frac{\partial p}{\partial \nu} - p \frac{\partial q}{\partial \nu} \right) d\sigma(z) \\
 &= q(x) - p(y) = H(y, x) - H(x, y) \quad \text{for all } x, y \in G \text{ with } x \neq y.
 \end{aligned}$$

We now show a *growth condition for Green's function* $H(x, y)$ as follows: With $\varepsilon > 0$ given we define the harmonic function

$$W_\varepsilon(x, y) := \begin{cases} -\frac{1}{2\pi}(1+\varepsilon) \log \frac{|y-x|}{d}, & n=2 \\ \frac{1+\varepsilon}{(n-2)\omega_n} |y-x|^{2-n}, & n \geq 3 \end{cases}$$

setting $d := \text{diam } G$. We consider the function $\Phi_\varepsilon(x, y) := W_\varepsilon(x, y) - H(x, y)$ and choose $\delta > 0$ so small that

$$\Phi_\varepsilon(x, y) \geq 0 \quad \text{for all } y : |y-x| = \delta \text{ and all } y \in \partial G$$

is satisfied. Applying the maximum principle to the harmonic function $\Phi_\varepsilon(x, \cdot)$ on the domain $G_\delta := \{y \in G : |x-y| > \delta\}$ we infer $\Phi_\varepsilon(x, y) \geq 0$ in G_δ and consequently

$$H(x, y) \leq W_\varepsilon(x, y) \quad \text{for all } \varepsilon > 0.$$

Therefore, we obtain

$$0 \leq H(x, y) \leq \begin{cases} -\frac{1}{2\pi} \log \frac{|y-x|}{d}, & n=2 \\ \frac{1}{(n-2)\omega_n} |y-x|^{2-n}, & n \geq 3 \end{cases}$$

for all $(x, y) \in G \otimes G$ and finally the growth condition

$$|H(x, y)| \leq \frac{\text{const}}{|x-y|^\alpha} \quad \text{for all } (x, y) \in G \otimes G \quad (28)$$

with $\alpha := n-2 < n$.

Definition 1.8. Let $G \subset \mathbb{R}^n$ denote a bounded domain where $n \in \mathbb{N}$ holds true, and let the number $\alpha \in [0, n)$ be chosen arbitrarily. A function $K = K(x, y) \in C^0(G \otimes G, \mathbb{C})$ is called a *singular kernel of the order α* - briefly $K \in \mathcal{S}_\alpha(G, \mathbb{C})$ - if we have a constant $c \in [0, +\infty)$ satisfying

$$|K(x, y)| \leq \frac{c}{|x-y|^\alpha} \quad \text{for all } (x, y) \in G \otimes G. \quad (29)$$

We name the kernel $K \in \mathcal{S}_\alpha(G, \mathbb{C})$ *Hermitian*, if

$$K(x, y) = \overline{K(y, x)} \quad \text{for all } (x, y) \in G \otimes G \quad (30)$$

is valid. The real kernels belong to the class $\mathcal{S}_\alpha(G) := \mathcal{S}_\alpha(G, \mathbb{R})$, and these kernels $K \in \mathcal{S}_\alpha(G)$ are Hermitian if and only if they are symmetric in the following sense:

$$K(x, y) = K(y, x) \quad \text{for all } (x, y) \in G \otimes G. \quad (31)$$

We summarize our considerations about the n -dimensional oscillation equation to the following

Theorem 1.9. *Let $G \subset \mathbb{R}^n$, $n = 2, 3, \dots$ denote a Dirichlet domain satisfying the assumptions for the Gaussian integral theorem. Furthermore, we fix the domain of definition*

$$\mathcal{D} := \left\{ u = u(x) \in C^2(G) \cap C^0(\overline{G}) : u(x) = 0 \text{ for all } x \in \partial G \right\}.$$

Then the following two statements are equivalent:

I. The function $u \in \mathcal{D}$ with $u \not\equiv 0$ solves the differential equation

$$-\Delta u(x) = \lambda u(x) \quad \text{in } G$$

for a number $\lambda \in \mathbb{R}$.

II. The function $u \in \mathcal{D}$ with $u \not\equiv 0$ solves the integral equation

$$\int_G H(x, y) u(y) dy = \frac{1}{\lambda} u(x) \quad \text{in } G$$

for a number $\lambda \in \mathbb{R} \setminus \{0\}$.

Here Green's function $H(x, y)$ of the Laplace operator for the domain G represents a symmetric real singular kernel of the regularity class $\mathcal{S}_{n-2}(G)$.

We finally consider *singular integral operators*: On the bounded domain $G \subset \mathbb{R}^n$ with $n \in \mathbb{N}$, the singular kernel $K = K(x, y) \in \mathcal{S}_\alpha(G, \mathbb{C})$ of the order $\alpha \in [0, n)$ is defined. On the domain of definition

$$\begin{aligned} \mathcal{D} &:= \left\{ u(x) : G \rightarrow \mathbb{C} \in C^0(G, \mathbb{C}) : \begin{array}{l} \text{There exists a number } c \in [0, +\infty) \\ \text{satisfying } |u(x)| \leq c \text{ for all } x \in G \end{array} \right\} \\ &=: C_b^0(G, \mathbb{C}) = C^0(G, \mathbb{C}) \cap L^\infty(G, \mathbb{C}) \end{aligned}$$

we consider the integral operator $\mathbb{K} : \mathcal{D} \rightarrow C^0(G, \mathbb{C})$ given by

$$\mathbb{K}u(x) := \int_G K(x, y) u(y) dy, \quad x \in G, \quad \text{with } u \in \mathcal{D}.$$

Evidently, we obtain with $\mathbb{K} : \mathcal{D} \rightarrow C^0(G, \mathbb{C})$ a linear operator.

Theorem 1.10. *Let the kernel $K = K(x, y) \in \mathcal{S}_\alpha(G, \mathbb{C})$ with $\alpha \in [0, n)$ be Hermitian. Then we have the following statements:*

- a) *If $u \in \mathcal{D}$ is an eigenfunction of the associate integral operator, more precisely $u \neq 0$ and $\mathbb{K}u = \lambda u$ with $\lambda \in \mathbb{C}$, we then infer $\lambda \in \mathbb{R}$.*
- b) *Given the two eigenfunctions $u_i \in \mathcal{D}$ with $\mathbb{K}u_i = \lambda_i u_i$ and $i = 1, 2$ for the eigenvalues $\lambda_1 \neq \lambda_2$, we infer $(u_1, u_2) = 0$. Here we used the inner product*

$$(u, v) := \int_G \overline{u(x)} v(x) dx \quad \text{for } u, v \in \mathcal{D}.$$

Proof:

- a) Let $u \in \mathcal{D} \setminus \{0\}$ be a solution of the problem $\mathbb{K}u = \lambda u$ with a number $\lambda \in \mathbb{C}$. This implies

$$\lambda u(x) = \int_G K(x, y) u(y) dy, \quad x \in G.$$

We multiply the equation by $\overline{u(x)}$ and afterwards integrate over the domain G with respect to x , and we obtain

$$\lambda(u, u) = \int_G \int_G K(x, y) \overline{u(x)} u(y) dx dy \in \mathbb{R}.$$

Since the inner product (u, u) is a real expression, the number λ has to be real.

- b) Let the eigenfunctions $u_i \in \mathcal{D}$ satisfying $\mathbb{K}u_i = \lambda_i u_i$ with $i = 1, 2$ and the eigenvalues $\lambda_1 \neq \lambda_2$ be given. On account of $\lambda_1, \lambda_2 \in \mathbb{R}$ we infer

$$\lambda_1(u_1, u_2) = (\lambda_1 u_1, u_2) = (\mathbb{K}u_1, u_2) = (u_1, \mathbb{K}u_2) = (u_1, \lambda_2 u_2) = \lambda_2(u_1, u_2)$$

or $(\lambda_1 - \lambda_2)(u_1, u_2) = 0$ and consequently $(u_1, u_2) = 0$. We namely deduce for all $u, v \in \mathcal{D}$:

$$\begin{aligned} (\mathbb{K}u, v) &= \int_G \overline{\left(\int_G K(x, y) u(y) dy \right)} v(x) dx = \int_G \int_G \overline{K(x, y) u(y)} v(x) dx dy \\ &= \int_G \int_G K(y, x) v(x) \overline{u(y)} dx dy = \int_G \overline{u(y)} \left(\int_G K(y, x) v(x) dx \right) dy \\ &= (u, \mathbb{K}v). \end{aligned}$$

q.e.d.

2 Singular Integral Equations

In Section 1 we have equivalently transformed eigenvalue problems for differential equations into so-called *integral equations of the first kind*

$$\int_G K(x, y)u(y) dy = \mu u(x), \quad x \in G, \quad (1)$$

with the singular kernels $K = K(x, y)$. Parallel to the swinging equation we take a bounded Dirichlet domain $G \subset \mathbb{R}^n$ satisfying the assumptions of the Gaussian integral theorem with the associate Green's function $H = H(x, y) \in \mathcal{S}_{n-2}(G)$ for the Laplace operator. Especially for the unit ball $B := \{x \in \mathbb{R}^n : |x| < 1\}$ we obtain as Green's function in the case $n = 2$:

$$G(\zeta, z) := \frac{1}{2\pi} \log \left| \frac{1 - \bar{z}\zeta}{\zeta - z} \right|, \quad (\zeta, z) \in B \otimes B, \quad (2)$$

and in the case $n \geq 3$:

$$G(x, y) := \frac{1}{(n-2)\omega_n} \left\{ \frac{1}{|y-x|^{n-2}} - \frac{1}{|x|^{n-2} \left| y - \frac{x}{|x|^2} \right|^{n-2}} \right\}, \quad (x, y) \in B \otimes B. \quad (3)$$

We now consider the Dirichlet problem

$$\begin{aligned} u &= u(x) \in C^2(G) \cap C^0(\bar{G}), \\ \Delta u(x) + \sum_{i=1}^n b_i(x)u_{x_i}(x) + c(x)u(x) &= f(x), \quad x \in G, \\ u(x) &= 0, \quad x \in \partial G. \end{aligned} \quad (4)$$

Here we assume the functions $b_i(x)$, $i = 1, \dots, n$, $c(x)$ and $f(x)$ to be Hölder continuous in $\bar{G}$. We transfer the equation (4) into an integral equation as follows: With the representation

$$-\Delta u(x) = \sum_{i=1}^n b_i(x)u_{x_i}(x) + c(x)u(x) - f(x) =: g(x), \quad x \in G, \quad (5)$$

we deduce (similarly to the oscillation equation)

$$u(x) = \int_G H(x, y) \left\{ \sum_{i=1}^n b_i(y)u_{x_i}(y) + c(y)u(y) - f(y) \right\} dy$$

and

$$\begin{aligned}
 u(x) - \int_G \left\{ (H(x, y)c(y)) u(y) + \sum_{i=1}^n (H(x, y)b_i(y)) u_{x_i}(y) \right\} dy \\
 = - \int_G H(x, y)f(y) dy \quad \text{for all } x \in G.
 \end{aligned} \tag{6}$$

We differentiate (6) with respect to x_j for $j = 1, \dots, n$ and obtain the additional n equations

$$\begin{aligned}
 u_{x_j}(x) - \int_G \left\{ (H_{x_j}(x, y)c(y)) u(y) + \sum_{i=1}^n (H_{x_j}(x, y)b_i(y)) u_{x_i}(y) \right\} dy \\
 = - \int_G H_{x_j}(x, y)f(y) dy, \quad x \in G, \quad j = 1, \dots, n.
 \end{aligned} \tag{7}$$

Setting

$$\begin{aligned}
 K_{00}(x, y) &:= H(x, y)c(y), & K_{0i}(x, y) &:= H(x, y)b_i(y), \\
 K_{j0}(x, y) &:= H_{x_j}(x, y)c(y), & K_{ji}(x, y) &:= H_{x_j}(x, y)b_i(y)
 \end{aligned}$$

for $i, j = 1, \dots, n$ and

$$f_0(x) := - \int_G H(x, y)f(y) dy, \quad f_j(x) := - \int_G H_{x_j}(x, y)f(y) dy$$

for $j = 1, \dots, n$, we arrive at the following

Theorem 2.1. *The solution $u = u(x)$ of (4) is transferred into the system of Fredholm's integral equations*

$$u_j(x) - \int_G \sum_{i=0}^n K_{ji}(x, y)u_i(y) dy = f_j(x), \quad x \in G, \quad j = 0, \dots, n \tag{8}$$

with the functions $u_0(x) := u(x)$ and $u_i(x) := u_{x_i}(x)$ for $i = 1, \dots, n$. Here the singular kernels $K_{ji}(x, y) \in \mathcal{S}_{n-1}(G)$ are real for $i, j = 0, \dots, n$. However, they are not symmetric in general.

Remark: In the special case $n = 2$, $G = B$, $b_1(x) \equiv 0 \equiv b_2(x)$ in B we can transfer the problem

$$\begin{aligned}
 u &= u(z) \in C^2(B) \cap C^0(\overline{B}), \\
 \Delta u(z) + c(z)u(z) &= f(z), \quad z \in B, \\
 u(z) &= 0, \quad z \in \partial B,
 \end{aligned} \tag{9}$$

into Fredholm's integral equation

$$u(z) - \int_B \frac{1}{2\pi} \log \left| \frac{1 - \bar{z}\zeta}{\zeta - z} \right| c(\zeta) u(\zeta) d\zeta = - \int_B \frac{1}{2\pi} \log \left| \frac{1 - \bar{z}\zeta}{\zeta - z} \right| f(\zeta) d\zeta, \quad z \in B. \quad (10)$$

Sometimes (10) is called an *integral equation of the second kind*. We remark that the integral kernel which appears is not symmetric in general.

For the L^p -spaces used in the following we refer the reader to Chapter 2, Section 7. On the bounded domain $G \subset \mathbb{R}^n$ we choose a singular kernel $K = K(x, y) \in \mathcal{S}_\alpha(G, \mathbb{C})$ with $\alpha \in [0, n)$. On the domain of definition

$$\mathcal{D} := \left\{ f : G \rightarrow \mathbb{C} \in C^0(G) : \sup_{x \in G} |f(x)| < +\infty \right\}$$

we consider the associate integral operator

$$\mathbb{K}f(x) := \int_G K(x, y) f(y) dy, \quad x \in G, \quad \text{for } f \in \mathcal{D}. \quad (11)$$

Choosing an exponent $p \in (1, \frac{n}{\alpha})$, we obtain a constant $C = C(c, \alpha, n, p) \in (0, +\infty)$ with the following property

$$\int_G |K(x, y)|^p dy \leq C \quad \text{for all } x \in G, \quad (12)$$

because of Section 1, Definition 1.8. When $q \in (\frac{n}{n-\alpha}, +\infty)$ denotes the conjugate exponent to p satisfying $\frac{1}{p} + \frac{1}{q} = 1$, Hölder's inequality from Theorem 7.4 in Chapter 2, Section 7 yields the following estimate:

$$\begin{aligned} |\mathbb{K}f(x)| &\leq \int_G |K(x, y)| |f(y)| dy \\ &\leq \left(\int_G |K(x, y)|^p dy \right)^{\frac{1}{p}} \left(\int_G |f(y)|^q dy \right)^{\frac{1}{q}} \\ &\leq C^{\frac{1}{p}} \|f\|_{L^q(G)}, \quad x \in G \end{aligned} \quad (13)$$

for all $f \in \mathcal{D}$. Here the symbol

$$\|f\|_p = \|f\|_{L^p(G)} := \left(\int_G |f(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < +\infty$$

denotes the L^p -norm on the Banach space

$$L^p(G) := \left\{ f : G \rightarrow \mathbb{C} \text{ measurable} : \|f\|_{L^p(G)} < +\infty \right\}$$

(compare Chapter 2, Sections 6 and 7). Furthermore, we introduce the C^0 -norm

$$\|f\|_{C^0(G)} := \sup_{x \in G} |f(x)|, \quad f \in \mathcal{D},$$

and (13) yields the estimate

$$\|\mathbb{K}f\|_{C^0(G)} \leq C\|f\|_{L^q(G)} \quad \text{for all } f \in \mathcal{D} \quad (14)$$

with a constant $C \in (0, +\infty)$. Therefore, $\mathbb{K} : \mathcal{D} \rightarrow C^0(G)$ represents a bounded linear operator, where $\mathcal{D}$ is endowed with the $L^q(G)$ -norm (see Chapter 2, Section 6). Parallel to Theorem 8.1 from Section 8 in Chapter 2, we can now continue $\mathbb{K}$ to the operator

$$\mathbb{K} : L^q(G) \rightarrow C^0(G) \quad (15)$$

on the Banach space $L^q(G)$. The set $C_0^\infty(G) \subset \mathcal{D}$ is dense in the space $L^q(G)$, and for each $f \in L^q(G)$ we have a sequence

$$\{f_j\}_{j=1,2,\dots} \subset C_0^\infty(G) \quad \text{satisfying} \quad \|f - f_j\|_{L^q(G)} \rightarrow 0 \quad (j \rightarrow \infty).$$

We then define

$$\mathbb{K}f := \lim_{j \rightarrow \infty} \mathbb{K}f_j \quad \text{in } C^0(G). \quad (16)$$

We summarize our considerations to the following

Theorem 2.2. *The integral operator $\mathbb{K} : \mathcal{D} \rightarrow C^0(G)$ with the singular kernel $K \in \mathcal{S}_\alpha(G)$ and $\alpha \in [0, n)$ can be uniquely continued to the bounded linear operator $\mathbb{K} : L^q(G) \rightarrow C^0(G)$ satisfying*

$$\|\mathbb{K}f\|_{C^0(G)} \leq C(q)\|f\|_{L^q(G)}, \quad f \in L^q(G) \quad (17)$$

for each $q \in (\frac{n}{n-\alpha}, +\infty)$, according to (16). Here we have chosen the constant $C = C(q) \in (0, +\infty)$ appropriately.

Remark: In the case $n \geq 3$, Green's function of the Laplace operator $H = H(x, y)$ belongs to the class $\mathcal{S}_{n-2}(G)$ which means $\alpha = n-2$ and $q \in (\frac{n}{2}, +\infty)$. Therefore, the associate singular integral operator

$$\mathbb{H} : L^q(G) \rightarrow C^0(G)$$

is even defined on the Hilbert space $L^2(G)$ for $n = 3$. In the case $n > 3$, Green's function $\mathbb{H}$ cannot be continued onto the Hilbert space $L^2(G)$.

For the orders $\alpha \in [0, n)$ and $\beta \in [0, n)$ let $K = K(x, y) \in \mathcal{S}_\alpha(G, \mathbb{C})$ and $L = L(y, z) \in \mathcal{S}_\beta(G, \mathbb{C})$ denote two singular kernels with the associate integral operators

$$\begin{aligned} \mathbb{K}f(x) &:= \int_G K(x, y)f(y) dy, \quad x \in G; \quad f \in \mathcal{D}, \\ \mathbb{L}f(y) &:= \int_G L(y, z)f(z) dz, \quad y \in G; \quad f \in \mathcal{D}. \end{aligned} \quad (18)$$

With the aid of Fubini's theorem from Chapter 2, Section 5 we now calculate for all $f \in \mathcal{D}$ and all $x \in G$ as follows:

$$\begin{aligned}
 \mathbb{K} \circ \mathbb{L}f(x) &= \mathbb{K} \left(\int_G L(y, z) f(z) dz \right) \Big|_x \\
 &= \int_G K(x, y) \left(\int_G L(y, z) f(z) dz \right) dy \\
 &= \int_G \int_G K(x, y) L(y, z) f(z) dz dy \\
 &= \int_G \left(\int_G K(x, y) L(y, z) dy \right) f(z) dz \\
 &= \int_G M(x, z) f(z) dz = \mathbb{M}f(x), \quad x \in G.
 \end{aligned} \tag{19}$$

Here we have as the product kernel

$$M(x, z) = \int_G K(x, y) L(y, z) dy, \quad (x, z) \in G \otimes G. \tag{20}$$

Proposition 2.3. *We have the regularity result $M = M(x, z) \in C^0(G \otimes G, \mathbb{C})$.*

Proof: We take the point $(x^0, z^0) \in G \otimes G$ such that $x^0, z^0 \in G$ and $x^0 \neq z^0$ holds true. Then we choose the number $0 < \delta < \frac{1}{4}|x^0 - z^0|$ sufficiently small and define the sets

$$B_\delta := \left\{ y \in G : |y - x^0| \leq 2\delta \text{ or } |y - z^0| \leq 2\delta \right\},$$

$$G_\delta := G \setminus B_\delta = \left\{ y \in G : |y - x^0| > 2\delta \text{ and } |y - z^0| > 2\delta \right\}.$$

Given the quantity $\varepsilon > 0$, we find a number $\delta = \delta(\varepsilon) > 0$ with the property

$$\int_{B_\delta} |K(x, y) L(y, z)| dy \leq \varepsilon \tag{21}$$

for all $x, z \in G$ with $|x - x^0| \leq \delta$ and $|z - z^0| \leq \delta$, taking $K \in \mathcal{S}_\alpha$ and $L \in \mathcal{S}_\beta$ into account. Furthermore, we have a number $\eta \in (0, \delta]$ such that

$$\left| K(x, y) L(y, z) - K(x^0, y) L(y, z^0) \right| \leq \varepsilon \tag{22}$$

holds true for all $y \in G_\delta$ and $x, z \in G$ with $|x - x^0| \leq \eta$ and $|z - z^0| \leq \eta$. Finally, we obtain the following estimate

$$\begin{aligned}
 |M(x, z) - M(x^0, z^0)| &\leq \int_{G_\delta} |K(x, y)L(y, z) - K(x^0, y)L(y, z^0)| dy \\
 &\quad + \int_{B_\delta} |K(x, y)L(y, z) - K(x^0, y)L(y, z^0)| dy \\
 &\leq \varepsilon|G| + 2\varepsilon
 \end{aligned}$$

for all $x, z \in G$ with $|x - x^0| \leq \eta$ and $|z - z^0| \leq \eta$. Therefore, the regularity result $M = M(x, z) \in C^0(G \otimes G)$ is correct. q.e.d.

Proposition 2.4. *If $\alpha + \beta < n$ holds true, we have $M \in \mathcal{S}_0(G, \mathbb{C})$.*

Proof: We have to prove only that the kernel M is bounded. Without loss of generality, we can assume $\alpha > 0$ and $\beta > 0$. Taking $(x, z) \in G \otimes G$, we estimate with the aid of Hölder's inequality as follows:

$$\begin{aligned}
 |M(x, z)| &\leq \int_G |K(x, y)||L(y, z)| dy \leq c_1 c_2 \int_G \frac{1}{|x - y|^\alpha} \frac{1}{|y - z|^\beta} dy \\
 &\leq c_1 c_2 \left(\int_G \frac{1}{|x - y|^{\alpha+\beta}} dy \right)^{\frac{\alpha}{\alpha+\beta}} \left(\int_G \frac{1}{|y - z|^{\alpha+\beta}} dy \right)^{\frac{\beta}{\alpha+\beta}} \\
 &\leq c_1 c_2 C \quad \text{for all } (x, z) \in G \otimes G.
 \end{aligned}$$

Here we observe $C := \sup_{x \in G} \int_G \frac{1}{|x - y|^{\alpha+\beta}} dy < +\infty$, since $\alpha + \beta < n$ holds true. q.e.d.

Proposition 2.5. *In the case $\alpha + \beta > n$, we have the regularity result $M \in \mathcal{S}_{\alpha+\beta-n}(G, \mathbb{C})$.*

Proof: We set $R := \text{diam } G \in (0, +\infty)$, and for the points $x, z \in G$ satisfying $x \neq z$ we define the quantity $\delta := |x - z| \in (0, R)$. Then we calculate

$$\begin{aligned}
 |M(x, z)| &\leq \int_G |K(x, y)||L(y, z)| dy \leq c \int_G \frac{1}{|x - y|^\alpha} \cdot \frac{1}{|y - z|^\beta} dy \\
 &= c \int_{\substack{y \in G \\ |y-x| \leq \frac{1}{2}\delta}} \frac{1}{|x - y|^\alpha} \frac{1}{|y - z|^\beta} dy + c \int_{\substack{y \in G \\ \frac{1}{2}\delta \leq |y-x| \leq 2\delta}} \frac{1}{|x - y|^\alpha} \frac{1}{|y - z|^\beta} dy \\
 &\quad + c \int_{\substack{y \in G \\ |y-x| \geq 2\delta}} \frac{1}{|x - y|^\alpha} \frac{1}{|y - z|^\beta} dy
 \end{aligned}$$

with a constant $c \in (0, +\infty)$.

Taking the point $y \in G$ with $|y - x| \leq \frac{1}{2}\delta$, we estimate as follows:

$$|y - z| \geq |z - x| - |x - y| \geq \delta - \frac{1}{2}\delta = \frac{1}{2}\delta.$$

Taking the point $y \in G$ with $|y - x| \geq 2\delta$, we obtain

$$|y - z| \geq |y - x| - |x - z| = |y - x| - \delta \geq |y - x| - \frac{1}{2}|y - x| = \frac{1}{2}|y - x|.$$

Consequently, we see

$$\begin{aligned} |M(x, z)| &\leq \frac{c}{(\frac{1}{2}\delta)^\beta} \int_{y: |y-x| \leq \frac{1}{2}\delta} \frac{1}{|y-x|^\alpha} dy + \frac{c}{(\frac{1}{2}\delta)^\alpha} \int_{y: |y-x| \leq 2\delta} \frac{1}{|y-z|^\beta} dy \\ &\quad + \frac{c}{(\frac{1}{2})^\beta} \int_{y: |y-x| \geq 2\delta} \frac{1}{|y-x|^{\alpha+\beta}} dy \\ &\leq \frac{c}{(\frac{1}{2}\delta)^\beta} \int_{y: |y-x| \leq \frac{1}{2}\delta} \frac{1}{|y-x|^\alpha} dy + \frac{c}{(\frac{1}{2}\delta)^\alpha} \int_{y: |y-z| \leq 3\delta} \frac{1}{|y-z|^\beta} dy \\ &\quad + \frac{c}{(\frac{1}{2})^\beta} \int_{y: |y-x| \geq 2\delta} \frac{1}{|y-x|^{\alpha+\beta}} dy. \end{aligned} \tag{23}$$

We now substitute

$$y = x + \varrho\xi, \quad dy = \omega_n \varrho^{n-1} d\varrho, \quad \varrho \in (0, \frac{1}{2}\delta), \quad \xi \in S^{n-1},$$

and calculate

$$\begin{aligned} \int_{y: |y-x| \leq \frac{1}{2}\delta} \frac{1}{|y-x|^\alpha} dy &= \int_0^{\frac{1}{2}\delta} \varrho^{-\alpha} \varrho^{n-1} \omega_n d\varrho = \omega_n \int_0^{\frac{1}{2}\delta} \varrho^{n-\alpha-1} d\varrho \\ &= \frac{\omega_n}{n-\alpha} \left[\varrho^{n-\alpha} \right]_0^{\frac{1}{2}\delta} = \frac{\omega_n}{n-\alpha} \left(\frac{1}{2}\delta \right)^{n-\alpha}. \end{aligned} \tag{24}$$

Analogously, we get

$$\int_{y: |y-z| \leq 3\delta} \frac{1}{|y-z|^\beta} dy = \frac{\omega_n}{n-\beta} (3\delta)^{n-\beta}. \tag{25}$$

With the aid of the substitution above, we deduce

$$\begin{aligned}
 \int_{y:|y-x|\geq 2\delta} \frac{1}{|y-x|^{\alpha+\beta}} dy &= \omega_n \int_{2\delta}^{+\infty} \varrho^{-\alpha-\beta} \varrho^{n-1} d\varrho \\
 &= \omega_n \frac{1}{n-(\alpha+\beta)} \left[\varrho^{n-\alpha-\beta} \right]_{2\delta}^{+\infty} \\
 &= \frac{\omega_n}{\alpha+\beta-n} (2\delta)^{n-\alpha-\beta}.
 \end{aligned} \tag{26}$$

Combining (23), (24), (25), and (26) we finally obtain the estimate

$$\begin{aligned}
 |M(x, z)| &\leq c \left\{ \left(\frac{1}{2} \right)^{n-\alpha-\beta} \frac{\omega_n}{n-\alpha} + 3^{n-\beta} \left(\frac{1}{2} \right)^{-\alpha} \frac{\omega_n}{n-\beta} + \frac{2^{n-\alpha}\omega_n}{\alpha+\beta-n} \right\} \delta^{n-\alpha-\beta} \\
 &= \frac{C(n, \alpha, \beta)}{|x-z|^{\alpha+\beta-n}} \quad \text{for all } x, z \in G \text{ with } x \neq z.
 \end{aligned} \tag{27}$$

Therefore, the statement $M \in \mathcal{S}_{\alpha+\beta-n}(G, \mathbb{C})$ follows. q.e.d.

We summarize our arguments to the subsequent

Theorem 2.6. (I. Schur)

To the given orders $\alpha \in [0, n)$, $\beta \in [0, n)$ let $K = K(x, y) \in \mathcal{S}_\alpha(G, \mathbb{C})$, $L = L(y, z) \in \mathcal{S}_\beta(G, \mathbb{C})$ denote singular kernels with the associate integral operators $\mathbb{K}$, $\mathbb{L}$. Then the composition

$$\mathbb{K} \circ \mathbb{L} f(x) = \int_G M(x, z) f(z) dz, \quad x \in G, \quad f \in \mathcal{D}$$

represents a singular integral operator as well, where its product kernel

$$M(x, z) = \int_G K(x, y) L(y, z) dy, \quad (x, z) \in G \otimes G$$

satisfies the following regularity properties:

$$M = M(x, y) \in \begin{cases} \mathcal{S}_0(G, \mathbb{C}), & \text{if } \alpha + \beta < n \\ \mathcal{S}_{\alpha+\beta-n}(G, \mathbb{C}), & \text{if } \alpha + \beta > n \end{cases}.$$

Theorem 2.7. (Iterated kernels)

Let $K = K(x, y) \in \mathcal{S}_\alpha(G, \mathbb{C})$ denote a singular kernel of the order $0 < \alpha < n$ with the associate integral operator $\mathbb{K}$. Then we have a positive integer $k = k(K) \in \mathbb{N}$ and a kernel $L = L(x, y) \in \mathcal{S}_0(G, \mathbb{C})$ with the associate integral operator $\mathbb{L}$ such that

$$\mathbb{K}^k f = \mathbb{L} f \quad \text{for all } f \in \mathcal{D}.$$

Proof: We choose $\beta \in (\alpha, n)$ satisfying

$$\beta \neq \frac{m}{m+1}n \quad \text{for all } m \in \mathbb{N}.$$

This implies

$$\beta + m(\beta - n) \neq 0 \quad \text{for all } m \in \mathbb{N}.$$

With the aid of the theorem by I. Schur we now consider the iterated kernels:

$$\begin{aligned} K &\in \mathcal{S}_\alpha \subset \mathcal{S}_\beta, \quad K^2 = K \circ K \in \mathcal{S}_{\beta+\beta-n} = \mathcal{S}_{\beta+1(\beta-n)}, \\ K^3 &= K \circ K \circ K \in \mathcal{S}_{\beta+2(\beta-n)}, \quad \dots, \quad K^k = \underbrace{K \circ \dots \circ K}_k \in \mathcal{S}_{\beta+(k-1)(\beta-n)}. \end{aligned}$$

We now determine the number $k \in \mathbb{N}$ such that

$$\beta + (k-2)(\beta - n) > 0 \quad \text{and} \quad \beta + (k-1)(\beta - n) < 0$$

is satisfied, and we infer

$$\{\beta + (k-2)(\beta - n)\} + \beta = \beta + (k-1)(\beta - n) + n < n.$$

Theorem 2.6 finally yields $K^k \in \mathcal{S}_0(G, \mathbb{C})$. q.e.d.

An outlook on the treatment of the eigenvalue problem for the n-dimensional oscillation equation (Weyl's eigenvalue problem): Parallel to Theorem 1.9 from Section 1 we use the domain of definition

$$\mathcal{D}_0 := \left\{ u = u(x) \in C^2(G) \cap C^0(\overline{G}) : u(x) = 0 \text{ on } \partial G \right\}$$

and consider the eigenvalue problem for the n-dimensional oscillation equation

$$-\Delta u(x) = \lambda u(x), \quad x \in G; \quad u \in \mathcal{D}_0 \setminus \{0\}, \quad \lambda \in \mathbb{R}. \quad (28)$$

In Section 9 we show the property $\lambda > 0$. Then the differential equation (28) can be transferred into the singular integral equation

$$\mathbb{H}u(x) := \int_G H(x, y)u(y) dy = \frac{1}{\lambda}u(x), \quad x \in G, \quad (29)$$

with the singular kernel $H = H(x, y) \in \mathcal{S}_{n-2}(G)$ which is symmetric. Due to Theorem 2.7 we now choose a number $k \in \mathbb{N}$ satisfying

$$\mathbb{H}^k u = \mathbb{K}u = \int_G K(x, y)u(y) dy, \quad u \in \mathcal{D}_0,$$

with a kernel $K = K(x, y) \in \mathcal{S}_0(G)$. The eigenvalue problem (29) is transferred into the equation

$$\mathbb{K}u = \mathbb{H}^k u = \frac{1}{\lambda^k} u(x), \quad x \in G. \quad (30)$$

Now we can continue the operator $\mathbb{K} : L^q(G) \rightarrow C^0(G)$ for each exponent $q > 1$, due to Theorem 2.2. With (30) we obtain an eigenvalue problem on the Hilbert space $L^2(G) \subset L^q(G)$, if $q \in (1, 2]$ holds true. Therefore, it suffices in the following considerations to investigate eigenvalue problems for operators in Hilbert spaces.

3 The Abstract Hilbert Space

We now continue the considerations from Section 6 in Chapter 2.

Postulate (A): $\mathcal{H}$ is a linear space. This means $\mathcal{H}$ is an additive Abelian group with 0 as its neutral element:

$$x, y \in \mathcal{H} \Rightarrow x + y \in \mathcal{H}, \quad x = 0 \in \mathcal{H}.$$

Furthermore, we have a scalar multiplication in $\mathcal{H}$: With the number $\lambda \in \mathbb{C}$ and the element $x \in \mathcal{H}$ the statement $\lambda x \in \mathcal{H}$ is correct, and the axioms for vector spaces are valid.

Postulate (B): In $\mathcal{H}$ we have defined the inner product

$$\begin{aligned} \mathcal{H} \times \mathcal{H} &\rightarrow \mathbb{C} \\ (x, y) &\mapsto (x, y)_{\mathcal{H}} \end{aligned}$$

with the following properties:

- (a) $(x, \alpha y)_{\mathcal{H}} = \alpha(x, y)_{\mathcal{H}}$ for all $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{C}$
- (b) $(x, y)_{\mathcal{H}} = \overline{(y, x)_{\mathcal{H}}}$ for all $x, y \in \mathcal{H}$ (Hermitian character)
- (c) $(x_1 + x_2, y)_{\mathcal{H}} = (x_1, y)_{\mathcal{H}} + (x_2, y)_{\mathcal{H}}$ for all $x_1, x_2, y \in \mathcal{H}$
- (d) $(x, x)_{\mathcal{H}} \geq 0$ for all $x \in \mathcal{H}$ and $(x, x)_{\mathcal{H}} = 0 \Leftrightarrow x = 0$ (positive-definiteness)

Postulate (C): For each positive integer $n \in \mathbb{N}$ we have n linear independent elements $x_1, \dots, x_n \in \mathcal{H}$, which means

$$\alpha_1, \dots, \alpha_n \in \mathbb{C}, \quad \sum_{i=1}^n \alpha_i x_i = 0 \quad \Rightarrow \quad \alpha_1 = \dots = \alpha_n = 0.$$

Definition 3.1. If the set $\mathcal{H}'$ satisfies the Postulates (A), (B), and (C) then $\mathcal{H}'$ is named a pre-Hilbert-space.

Example 3.2. Let $G \subset \mathbb{R}^n$ denote a bounded open set and define

$$\mathcal{H}' := \left\{ f : G \rightarrow \mathbb{C} \in C^0(G) : \sup_{x \in G} |f(x)| < +\infty \right\}.$$

With the inner product

$$(f, g) := \int_G \overline{f(x)} g(x) dx, \quad f, g \in \mathcal{H}', \quad (1)$$

the vector space $\mathcal{H}'$ becomes a pre-Hilbert-space.

Theorem 3.3. *In pre-Hilbert-spaces $\mathcal{H}'$ we have the following calculus rules for the inner product $(\cdot, \cdot)$:*

a) *For all $x, y, y_1, y_2 \in \mathcal{H}'$, $\alpha \in \mathbb{C}$ we have*

$$(\alpha x, y) = \overline{\alpha}(x, y), \quad (x, y_1 + y_2) = (x, y_1) + (x, y_2).$$

Consequently, the bilinear form $(\cdot, \cdot)$ is antilinear in the first and linear in the second component.

b) *The Cauchy-Schwarz inequality is satisfied:*

$$|(x, y)| \leq \sqrt{(x, x)} \sqrt{(y, y)} \quad \text{for all } x, y \in \mathcal{H}'.$$

c) *Setting $\|x\| := \sqrt{(x, x)}$ with $x \in \mathcal{H}'$, the pre-Hilbert-space $\mathcal{H}'$ becomes a normed space. This means*

$$\|x\| = 0 \Leftrightarrow x = 0,$$

$$\|x + y\| \leq \|x\| + \|y\| \quad \text{for all } x, y \in \mathcal{H}',$$

$$\|\lambda x\| = |\lambda| \|x\| \quad \text{for all } x \in \mathcal{H}', \lambda \in \mathbb{C},$$

$$\|x - y\| \geq |\|x\| - \|y\|| \quad \text{for all } x, y \in \mathcal{H}'.$$

d) *The inner product is continuous on $\mathcal{H}'$ in the following sense: From the assumptions*

$$x_n \rightarrow x \ (n \rightarrow \infty) \quad \text{with } \{x_n\}_{n=1,2,\dots} \subset \mathcal{H}' \quad \text{and } x \in \mathcal{H}'$$

and

$$y_n \rightarrow y \ (n \rightarrow \infty) \quad \text{with } \{y_n\}_{n=1,2,\dots} \subset \mathcal{H}' \quad \text{and } y \in \mathcal{H}'$$

we infer

$$(x_n, y_n) \rightarrow (x, y) \ (n \rightarrow \infty).$$

Here the symbol $x_n \rightarrow x \ (n \rightarrow \infty)$ indicates that $\|x_n - x\| \rightarrow 0 \ (n \rightarrow \infty)$ is satisfied.

Proof:

a) We calculate

$$(\alpha x, y) = \overline{(y, \alpha x)} = \overline{\alpha(y, x)} = \overline{\alpha}(x, y)$$

and

$$\begin{aligned} (x, y_1 + y_2) &= \overline{(y_1 + y_2, x)} = \overline{(y_1, x) + (y_2, x)} \\ &= \overline{(y_1, x)} + \overline{(y_2, x)} = (x, y_1) + (x, y_2). \end{aligned}$$

b) and c) are contained in Section 6 of Chapter 2; more precisely in Theorem 6.7 with its proof and the Remark following Definition 6.1.

d) The subsequent estimate yields this statement:

$$\begin{aligned} |(x_n, y_n) - (x, y)| &\leq |(x_n, y_n) - (x_n, y)| + |(x_n, y) - (x, y)| \\ &= |(x_n, y_n - y)| + |(x_n - x, y)| \\ &\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

q.e.d.

Postulate (D): $\mathcal{H}$ is complete. This means each sequence $\{x_n\}_{n=1,2,\dots} \subset \mathcal{H}$ satisfying $\|x_n - x_m\| \rightarrow 0$ ($n, m \rightarrow \infty$) possesses a limit element $x \in \mathcal{H}$ such that

$$\lim_{n \rightarrow \infty} \|x_n - x\| = 0.$$

Definition 3.4. If $\mathcal{H}$ satisfies the Postulates (A), (B), (C), and (D) we name $\mathcal{H}$ a Hilbert space.

Remark: The Hilbert space $\mathcal{H}$ becomes a Banach space via the norm given in Theorem 3.3,c).

Definition 3.5. The Hilbert space $\mathcal{H}$ is separable, if the following Postulate (E) holds true additionally:

Postulate (E): There exists a sequence $\{x_n\}_{n=1,2,\dots} \subset \mathcal{H}$ which is dense in $\mathcal{H}$: This means for all $x \in \mathcal{H}$ and every $\varepsilon > 0$ we have an index $n \in \mathbb{N}$ satisfying $\|x - x_n\| < \varepsilon$.

Example 3.6. Hilbert's sequential space

We endow the set of sequences

$$l_2 := \left\{ x = (x_1, x_2, \dots) \in \mathbb{C} \times \mathbb{C} \times \dots : \sum_{k=1}^{\infty} |x_k|^2 < +\infty \right\}$$

with the inner product

$$(x, y) := \sum_{k=1}^{\infty} \overline{x_k} y_k \in \mathbb{C}$$

and we obtain a separable Hilbert space.

Example 3.7. Let $G \subset \mathbb{R}^n$ be a bounded open set, and by

$$L^2(G) := \left\{ f : G \rightarrow \mathbb{C} \text{ measurable} : \int_G |f(x)|^2 dx < +\infty \right\}$$

we denote the Lebesgue space of the square-integrable functions with the inner product

$$(f, g) := \int_G \overline{f(x)} g(x) dx \quad \text{for } f, g \in L^2(G).$$

Then $\mathcal{H} = L^2(G)$ represents a separable Hilbert space. The pre-Hilbert-space $\mathcal{H}'$ described in Example 3.2 lies dense in $\mathcal{H}$ (compare Chapter 2, Section 7).

Parallel to the transition from rational numbers $\mathbb{Q}$ to real numbers $\mathbb{R}$ we prove the following result using the ideas of D. Hilbert, which were presented in his famous book on *The Foundations of Geometry*.

Theorem 3.8. (Hilbert's fundamental theorem)

Each pre-Hilbert-space $\mathcal{H}'$ can be completed to a Hilbert space $\mathcal{H}$ such that $\mathcal{H}'$ lies dense in $\mathcal{H}$. We name $\mathcal{H}$ the abstract completion of $\mathcal{H}'$. When $\mathcal{H}'$ satisfies the Postulate (E), then the abstract completion $\mathcal{H}$ is a separable Hilbert space.

Proof: Let $\mathcal{H}'$ be a pre-Hilbert-space. We then consider the Cauchy sequences $\{f'_n\}_{n=1,2,\dots} \subset \mathcal{H}'$ and $\{g'_n\}_{n=1,2,\dots} \subset \mathcal{H}'$. We call them equivalent if

$$f'_n - g'_n \rightarrow 0 \quad (n \rightarrow \infty)$$

is satisfied. Now we set

$$\mathcal{H} := \left\{ f = [f'_n]_{n=1,2,\dots} : \begin{array}{l} [f'_n] \text{ is the equivalence class} \\ \text{of the Cauchy sequences } \{f'_n\}_{n=1,2,\dots} \subset \mathcal{H}' \end{array} \right\}.$$

For $f = [f'_n]_n \in \mathcal{H}$ and $g = [g'_n]_n \in \mathcal{H}$ we evidently have the statement

$$[f'_n]_n = [g'_n]_n \quad \Leftrightarrow \quad \|f'_n - g'_n\| \rightarrow 0 \quad (n \rightarrow \infty).$$

To Postulate (A): On $\mathcal{H}$ we define a vector space structure as follows: For $\alpha, \beta \in \mathbb{C}$ and $f = [f'_n]_n \in \mathcal{H}$, $g = [g'_n]_n \in \mathcal{H}$ we set

$$\alpha f + \beta g := [\alpha f'_n + \beta g'_n]_n.$$

The null element is the equivalence class of all zero sequences in $\mathcal{H}'$:

$$0 = [f'_n]_n \quad \text{with} \quad \{f'_n\}_{n=1,2,\dots} \subset \mathcal{H}' \quad \text{and} \quad \|f'_n\| \rightarrow 0 \quad (n \rightarrow \infty).$$

To *Postulate (B)*: For the elements $f = [f'_n]_n \in \mathcal{H}$ and $g = [g'_n]_n \in \mathcal{H}$ we define the inner product

$$(f, g) := \lim_{n \rightarrow \infty} (f'_n, g'_n).$$

On account of

$$\begin{aligned} |(f'_n, g'_n) - (f'_m, g'_m)| &\leq |(f'_n - f'_m, g'_n)| + |(f'_m, g'_n - g'_m)| \\ &\leq \|f'_n - f'_m\| \|g'_n\| + \|f'_m\| \|g'_n - g'_m\| \rightarrow 0 \quad (n, m \rightarrow \infty), \end{aligned}$$

the limit given above exists. One easily verifies that the so-defined inner product satisfies the Postulate (B).

To the *Postulates (C) and (E)*: Let $\tilde{\mathcal{H}}$ denote the set of all $f \in \mathcal{H}$ satisfying $f = [f', f', \dots]$ and $f' \in \mathcal{H}'$. Then the vector spaces $\mathcal{H}'$ and $\tilde{\mathcal{H}}$ are isomorphic, and consequently $\mathcal{H}'$ is embedded into $\mathcal{H}$. Now $\tilde{\mathcal{H}}$ is dense in $\mathcal{H}$: Taking $f = [f'_n]_n \in \mathcal{H}$ we set $\tilde{f}_m = [f'_m, f'_m, \dots] \in \tilde{\mathcal{H}}$ and see

$$\|f - \tilde{f}_m\| = \lim_{n \rightarrow \infty} \|f'_n - f'_m\| \rightarrow 0 \quad (m \rightarrow \infty).$$

Evidently, the Postulate (C) remains valid for $\mathcal{H}$. In the case that $\mathcal{H}'$ additionally satisfies the Postulate (E), this holds true for $\mathcal{H}$ as well.

To *Postulate (D)*: Let $\{f_n\}_{n=1,2,\dots} \subset \mathcal{H}$ with

$$\|f_n - f_m\| \rightarrow 0 \quad (n, m \rightarrow \infty)$$

be chosen. Since $\tilde{\mathcal{H}}$ lies dense in $\mathcal{H}$ we have a sequence $\{\tilde{f}_n\}_{n=1,2,\dots} \subset \tilde{\mathcal{H}}$ satisfying

$$\|f_n - \tilde{f}_n\| \leq \frac{1}{n}, \quad n = 1, 2, \dots$$

Here we have $\tilde{f}_n = [f'_n, f'_n, \dots]$ with $f'_n \in \mathcal{H}'$. We now set $f := [f'_n]_{n=1,2,\dots}$ and show that $f \in \mathcal{H}$ and $\|f - f_n\| \rightarrow 0 \quad (n \rightarrow \infty)$ are correct. At first, we estimate

$$\begin{aligned} \|f'_n - f'_m\| &= \|\tilde{f}_n - \tilde{f}_m\| \leq \|\tilde{f}_n - f_n\| + \|f_n - f_m\| + \|f_m - \tilde{f}_m\| \\ &\leq \frac{1}{n} + \|f_n - f_m\| + \frac{1}{m} \rightarrow 0 \quad (n, m \rightarrow \infty). \end{aligned}$$

Now we have

$$\|f - f_m\| \leq \|f - \tilde{f}_m\| + \|\tilde{f}_m - f_m\| \leq \|f - \tilde{f}_m\| + \frac{1}{m},$$

and note that $\tilde{f}_m = [f'_m, f'_m, \dots]$ and $f = [f'_1, f'_2, \dots]$. Then we infer

$$\|f - \tilde{f}_m\| = \lim_{n \rightarrow \infty} \|f'_n - f'_m\| \leq \varepsilon_m$$

with the numbers $\varepsilon_m > 0$ and $m \in \mathbb{N}$ satisfying $\varepsilon_m \rightarrow 0$ ($m \rightarrow \infty$). We summarize our considerations to

$$\|f - f_m\| \leq \varepsilon_m + \frac{1}{m} \rightarrow 0 \quad (m \rightarrow \infty). \quad \text{q.e.d.}$$

Remark: We can complete the pre-Hilbert-space $\mathcal{H}'$ from the Example 3.2 abstractly to a Hilbert space $\mathcal{H}$ with the aid of Theorem 3.8. Alternatively, we can concretely complete $\mathcal{H}'$ to the Hilbert space

$$L^2(G, \mathbb{C}) := \left\{ f : G \rightarrow \mathbb{C} \text{ measurable} : \int_G |f(x)|^2 dx < +\infty \right\},$$

whose inner product is given in (1).

Definition 3.9. A sequence of elements $\{\varphi_1, \varphi_2, \dots\} \subset \mathcal{H}'$ in a pre-Hilbert-space $\mathcal{H}'$ is called orthonormal if and only if

$$(\varphi_i, \varphi_j) = \delta_{ij} \quad \text{for all } i, j \in \mathbb{N}$$

is correct. We name the orthonormal system $\{\varphi_k\}_{k=1,2,\dots}$ complete - briefly we speak of a c.o.n.s. - if each element $f \in \mathcal{H}'$ satisfies the completeness relation

$$\|f\|^2 = \sum_{k=1}^{\infty} |(\varphi_k, f)|^2. \quad (2)$$

This definition is justified by the following theorem, whose proof is contained in the Propositions 6.17, 6.18 and Theorem 6.19 at the end of Section 6 in Chapter 2.

Theorem 3.10. Let $\{\varphi_k\}_{k=1,2,\dots} \subset \mathcal{H}'$ represent an orthonormal system. For all $f \in \mathcal{H}'$ we then have Bessel's inequality

$$\sum_{k=1}^{\infty} |(\varphi_k, f)|^2 \leq \|f\|^2. \quad (3)$$

An element $f \in \mathcal{H}'$ satisfies the equation

$$\sum_{k=1}^{\infty} |(\varphi_k, f)|^2 = \|f\|^2 \quad (4)$$

if and only if

$$\lim_{N \rightarrow \infty} \left\| f - \sum_{k=1}^N (\varphi_k, f) \varphi_k \right\| = 0 \quad (5)$$

holds true. The latter statement means that $f \in \mathcal{H}'$ can be represented by the Fourier series

$$\sum_{k=1}^{\infty} (\varphi_k, f) \varphi_k$$

converging with respect to the norm $\|\cdot\|$ in the Hilbert space.

Example 3.11. With the Fourier series and the spherical harmonic functions in Section 4 and Section 5 from Chapter 5, respectively, we obtain two c.o.n.s. in the adequate Hilbert spaces.

Theorem 3.12. *An orthonormal system $\{\varphi_k\}_{k=1,2,\dots}$ in the pre-Hilbert-space $\mathcal{H}'$ is complete if and only if the relation $(\varphi_k, x) = 0$, $k = 1, 2, \dots$ with $x \in \mathcal{H}'$ implies the identity $x = 0$.*

Proof:

‘ $\Rightarrow$ ’ Let $x \in \mathcal{H}'$ and $(\varphi_k, x) = 0$ hold true for all $k \in \mathbb{N}$. Then the completeness relation yields

$$\|x\|^2 = \sum_{k=1}^{\infty} |(\varphi_k, x)|^2 = 0$$

as well as $\|x\| = 0$ and consequently $x = 0 \in \mathcal{H}'$.

‘ $\Leftarrow$ ’ Let $\{\varphi_k\}_{k=1,2,\dots}$ be an orthonormal system such that the statement $(\varphi_k, x) = 0$ for all $k \in \mathbb{N}$ implies $x = 0$. For arbitrary $y \in \mathcal{H}$ we set

$$x := y - \sum_{k=1}^{\infty} (\varphi_k, y) \varphi_k$$

and we calculate

$$(\varphi_l, x) = (\varphi_l, y) - \left(\varphi_l, \sum_{k=1}^{\infty} (\varphi_k, y) \varphi_k \right) = (\varphi_l, y) - (\varphi_l, y) = 0$$

for all $l \in \mathbb{N}$. This implies $x = 0$ and consequently

$$y = \sum_{k=1}^{\infty} (\varphi_k, y) \varphi_k.$$

Therefore, the system $\{\varphi_k\}_{k=1,2,\dots} \subset \mathcal{H}'$ is complete due to Theorem 3.10. q.e.d.

Theorem 3.13. *Let $\mathcal{H}$ denote a separable Hilbert space.*

a) *Then there exists a c.o.n.s. $\{\varphi_k\}_{k=1,2,\dots} \subset \mathcal{H}$.*

b) For two arbitrary elements $x, y \in \mathcal{H}$ we have the Parseval equation

$$(x, y) = \sum_{k=1}^{\infty} \overline{(\varphi_k, x)} (\varphi_k, y). \quad (6)$$

c) The Hilbert space $\mathcal{H}$ is isomorphic to the Hilbert sequential space l_2 via the mapping

$$\Phi: \mathcal{H} \rightarrow l_2, \quad x \mapsto (x_1, x_2, \dots) \quad \text{with} \quad x_k := (\varphi_k, x).$$

By the prescription

$$x = \sum_{k=1}^{\infty} x_k \varphi_k \quad \text{with} \quad (x_1, x_2, \dots) \in l_2$$

the mapping inverse to Φ is given.

Proof:

a) Since $\mathcal{H}$ is separable, we have a sequence $\{g_1, g_2, \dots\} \subset \mathcal{H}$ which is dense in $\mathcal{H}$. We eliminate the linear dependent functions from $\{g_1, g_2, \dots\}$, and construct a system of linear independent functions $\{f_1, f_2, \dots\}$ in $\mathcal{H}$ with the following property:

$$[g_1, \dots, g_n] \subset [f_1, \dots, f_p] \quad \text{for all} \quad p \geq n \geq 1, \quad (7)$$

denoting with $[g_1, \dots, g_n]$ and $[f_1, \dots, f_p]$ the $\mathbb{C}$ -linear spaces spanned by the elements $g_1, \dots, g_n$ and $f_1, \dots, f_p$, respectively; here $n, p \in \mathbb{N}$ holds true. Now we apply the *orthonormalizing procedure of E. Schmidt* to the system of functions $\{f_k\}_{k=1,2,\dots}$:

$$\begin{aligned} \varphi_1 &:= \frac{1}{\|f_1\|} f_1, & \varphi_2 &:= \frac{f_2 - (\varphi_1, f_2) \varphi_1}{\|f_2 - (\varphi_1, f_2) \varphi_1\|}, \quad \dots, \\ \varphi_n &:= \frac{f_n - \sum_{j=1}^{n-1} (\varphi_j, f_n) \varphi_j}{\left\| f_n - \sum_{j=1}^{n-1} (\varphi_j, f_n) \varphi_j \right\|}, & n &= 1, 2, \dots \end{aligned}$$

We evidently have $(\varphi_j, \varphi_k) = \delta_{jk}$ for $j, k = 1, 2, \dots$ and

$$[g_1, \dots, g_n] \subset [\varphi_1, \dots, \varphi_p] \quad \text{for all} \quad p \geq n \geq 1. \quad (8)$$

When $f \in \mathcal{H}$ and $\varepsilon > 0$ are given, we have an index $n \in \mathbb{N}$ satisfying $\|f - g_n\| \leq \varepsilon$. Due to (8) we find $p \geq n$ numbers $c_1, \dots, c_p \in \mathbb{C}$ with

$$\left\| f - \sum_{k=1}^p c_k \varphi_k \right\| \leq \varepsilon.$$

Because of the minimal property of the Fourier coefficients (compare Chapter 2, Section 6, Corollary to Proposition 6.17) we can still choose n and p such that

$$\left\| f - \sum_{k=1}^p (\varphi_k, f) \varphi_k \right\| \leq \varepsilon \quad \text{for all } \varepsilon > 0.$$

Observing the limit process $\varepsilon \downarrow 0$ we deduce

$$f = \sum_{k=1}^{\infty} (\varphi_k, f) \varphi_k,$$

and $\{\varphi_k\}_{k=1,2,\dots}$ is a c.o.n.s.

b) For two elements $x, y \in \mathcal{H}$ with the representations

$$x = \sum_{k=1}^{\infty} (\varphi_k, x) \varphi_k, \quad y = \sum_{l=1}^{\infty} (\varphi_l, y) \varphi_l$$

we evaluate the inner product

$$\begin{aligned} (x, y) &= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n (\varphi_k, x) \varphi_k, \sum_{l=1}^n (\varphi_l, y) \varphi_l \right) \\ &= \lim_{n \rightarrow \infty} \sum_{k,l=1}^n \overline{(\varphi_k, x)} (\varphi_l, y) (\varphi_k, \varphi_l) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \overline{(\varphi_k, x)} (\varphi_k, y) \\ &= \sum_{k=1}^{\infty} \overline{(\varphi_k, x)} (\varphi_k, y). \end{aligned}$$

c) Here nothing has to be shown any more.

q.e.d.

Definition 3.14. We name $\mathcal{M} \subset \mathcal{H}$ a linear subspace of the Hilbert space $\mathcal{H}$, if for arbitrary elements $f, g \in \mathcal{M}$ and all numbers $\alpha, \beta \in \mathbb{C}$ we obtain the inclusion

$$\alpha f + \beta g \in \mathcal{M}.$$

A linear subspace $\mathcal{M} \subset \mathcal{H}$ is called closed, if each Cauchy sequence

$$\{f_n\}_{n=1,2,\dots} \subset \mathcal{M}$$

fulfills

$$f := \lim_{n \rightarrow \infty} f_n \in \mathcal{M}.$$

With a linear subspace $\mathcal{M} \subset \mathcal{H}$ we denote by

$$\overline{\mathcal{M}} := \left\{ f \in \mathcal{H} : \begin{array}{l} \text{There exists a Cauchy sequence } \{f_n\}_{n=1,2,\dots} \subset \mathcal{M} \\ \text{satisfying } f = \lim_{n \rightarrow \infty} f_n \end{array} \right\}$$

the closure of $\mathcal{M}$.

Example 3.15. The space $C_0^\infty(G) =: \mathcal{M} \subset \mathcal{H} := L^2(G)$ is a nonclosed linear subspace, and we have $\overline{\mathcal{M}} = \mathcal{H}$.

Definition 3.16. We call $\mathcal{H}$ a unitary space, if the following Postulate (C') is satisfied in addition to the Postulates (A) and (B).

Postulate (C'): With an integer $n \in \mathbb{N}$ we have $\dim \mathcal{H} = n$.

Remarks:

1. In an n -dimensional unitary space $\mathcal{H}$ we have n linearly independent elements $\{f_1, \dots, f_n\}$, and each $g \in \mathcal{H}$ can be represented in the form

$$g = \sum_{k=1}^n c_k f_k \quad \text{with} \quad c_1, \dots, c_n \in \mathbb{C}.$$

2. A unitary space $\mathcal{H}$ possesses an orthonormal basis $\{\varphi_1, \dots, \varphi_n\}$ with $n = \dim \mathcal{H}$ satisfying

$$f = \sum_{k=1}^n (\varphi_k, f) \varphi_k \quad \text{for all} \quad f \in \mathcal{H}.$$

3. Each unitary space $\mathcal{H}$ endowed with the inner product

$$(x, y) := \sum_{k=1}^n \overline{x_k} y_k, \quad x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n) \in \mathbb{C}$$

is isomorphic to $\mathbb{C}^n$, where $n = \dim \mathcal{H}$ holds true.

4. Each unitary space is complete.

Noticing the Definitions 3.14 and 3.16, we easily prove the following

Theorem 3.17. Let $\mathcal{H}$ denote a Hilbert space, and $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$. Then $\mathcal{M}$ represents either a Hilbert space or a unitary space. In the case that $\mathcal{H}$ is separable the same holds true for $\mathcal{M}$.

Definition 3.18. Let $\mathcal{M}$ be a linear subspace of $\mathcal{H}$. Then we define the orthogonal space of $\mathcal{M}$ in $\mathcal{H}$ setting

$$\mathcal{M}^\perp := \left\{ g \in \mathcal{H} : (g, f) = 0 \text{ for all } f \in \mathcal{M} \right\}.$$

Remark: On account of the continuity for the inner product, the orthogonal space $\mathcal{M}^\perp \subset \mathcal{H}$ is closed.

From Theorem 6.9 on the Orthogonal projection in Chapter 2, Section 6 we take over the proof of the following

Theorem 3.19. (Projection theorem)

Let $\mathcal{M}$ denote a closed linear subspace in $\mathcal{H}$. Then each element $x \in \mathcal{H}$ can be uniquely represented in the form $x = x_1 + x_2$ with $x_1 \in \mathcal{M}$ and $x_2 \in \mathcal{M}^\perp$. We then obtain the decomposition

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp.$$

We still note the subsequent

Theorem 3.20. Let $\mathcal{M}$ be a linear subspace in a Hilbert space $\mathcal{H}$. Then $\mathcal{M}$ lies dense in $\mathcal{H}$ if and only if the following implication is correct:

$$\varphi \in \mathcal{H} : (f, \varphi) = 0 \text{ for all } f \in \mathcal{M} \quad \Rightarrow \quad \varphi = 0. \quad (9)$$

Proof: We have the orthogonal decomposition $\mathcal{H} = \overline{\mathcal{M}} \oplus \mathcal{M}^\perp$ via the Projection theorem. Now the subspace $\mathcal{M}$ lies dense in $\mathcal{H}$ if the statement $\overline{\mathcal{M}} = \mathcal{H}$ and consequently $\mathcal{M}^\perp = \{0\}$ is correct. The latter statement coincides with the implication (9). q.e.d.

4 Bounded Linear Operators in Hilbert Spaces

We begin with the fundamental

Definition 4.1. On the Hilbert space $\mathcal{H}$ the mapping $A : \mathcal{H} \rightarrow \mathbb{C}$ is a bounded linear functional, if the following conditions are fulfilled:

- a) $A(\alpha f + \beta g) = \alpha A f + \beta A g$ for all $f, g \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$,
- b) $|A f| \leq c \|f\|$ for all $f \in \mathcal{H}$, with a constant $c \in [0, +\infty)$.

Due to Theorem 6.11 of Section 6 in Chapter 2, the following three statements are equivalent for a linear functional:

- (i) A is bounded,
- (ii) A is continuous at one point,
- (iii) A is continuous at all points of the Hilbert space.

We define the *norm of the bounded linear functional* A by

$$\|A\| := \sup_{x \in \mathcal{H}, \|x\| \leq 1} |A x| = \sup_{x \in \mathcal{H}, \|x\| = 1} |A x| < +\infty.$$

In Chapter 2, Section 6, Theorem 6.14 we have proved the following statement:

Theorem 4.2. (Representation theorem of Fréchet and Riesz)

Each bounded linear functional $A : \mathcal{H} \rightarrow \mathbb{C}$ on the Hilbert space $\mathcal{H}$ can be represented in the form

$$A f = (g, f) \quad \text{for all } f \in \mathcal{H} \quad (1)$$

with a uniquely determined generating element $g \in \mathcal{H}$.

Definition 4.3. Let $\mathcal{D}$ denote a linear subspace of the Hilbert space $\mathcal{H}$. A linear operator T consists of a function $T : \mathcal{D} \rightarrow \mathcal{H}$ with the following property

$$T(c_1 u_1 + c_2 u_2) = c_1 T(u_1) + c_2 T(u_2) \quad \text{for all } u_1, u_2 \in \mathcal{D} \quad \text{and } c_1, c_2 \in \mathbb{C}.$$

Definition 4.4. A linear operator $T : \mathcal{D} \rightarrow \mathcal{H}$ is called bounded, if we have a number $c \in [0, +\infty)$ such that

$$\|Tu\| \leq c\|u\| \quad \text{for all } u \in \mathcal{D} \quad (2)$$

holds true. Then the norm of T is defined by

$$\|T\| := \sup_{u \in \mathcal{D}, u \neq 0} \frac{\|Tu\|}{\|u\|} = \sup_{u \in \mathcal{D}, \|u\| \leq 1} \|Tu\| = \sup_{u \in \mathcal{D}, \|u\|=1} \|Tu\|. \quad (3)$$

Remark: The Example 1.2 in Section 1 presents an unbounded operator with $T := -\Delta$.

Definition 4.5. Let $\mathcal{D}_T, \mathcal{D}_{\tilde{T}}$ denote two linear subspaces of the Hilbert space $\mathcal{H}$. Then the mapping

$$\tilde{T} : \mathcal{D}_{\tilde{T}} \rightarrow \mathcal{H}$$

is called the extension of the bounded linear operator $T : \mathcal{D}_T \rightarrow \mathcal{H}$, if the following properties are satisfied

- a) $\mathcal{D}_T \subset \mathcal{D}_{\tilde{T}}$,
- b) $\tilde{T}u = Tu$ for all $u \in \mathcal{D}_T$.

We then write $T \subset \tilde{T}$.

For bounded operators it suffices to define them on dense subspaces of Hilbert spaces due to the following

Theorem 4.6. (Extension theorem)

Let $T : \mathcal{D} \rightarrow \mathcal{H}$ denote a bounded linear operator, and the linear subspace $\mathcal{D} \subset \mathcal{H}$ lies dense in the Hilbert space $\mathcal{H}$. Then there exists a uniquely determined bounded extension $\tilde{T} \supset T$ satisfying $\mathcal{D}_{\tilde{T}} = \mathcal{H}$ and $\|\tilde{T}\| = \|T\|$.

Proof:

1. We define $\tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$ as follows: Taking $f \in \mathcal{H}$ we have a sequence $\{f_n\}_{n=1,2,\dots} \subset \mathcal{D}_T$ satisfying $f_n \rightarrow f$ ($n \rightarrow \infty$) in $\mathcal{H}$. Now $\{Tf_n\}_{n=1,2,\dots}$ gives a Cauchy sequence in $\mathcal{H}$ on account of

$$\|Tf_n - Tf_m\| = \|T(f_n - f_m)\| \leq \|T\|\|f_n - f_m\| \rightarrow 0 \quad (n, m \rightarrow \infty).$$

Therefore the limit $\lim_{n \rightarrow \infty} Tf_n$ exists, and we set

$$\tilde{T}f := \lim_{n \rightarrow \infty} Tf_n, \quad f \in \mathcal{H}.$$

This notion is uniquely determined: Taking namely a further sequence $\{f'_n\}_{n=1,2,\dots} \subset \mathcal{D}_T$ satisfying $f'_n \rightarrow f$ ($n \rightarrow \infty$) in $\mathcal{H}$, we observe

$$\|Tf_n - Tf'_n\| \leq \|T\| \|f_n - f'_n\| \leq \|T\| (\|f_n - f\| + \|f - f'_n\|) \rightarrow 0 \quad (n \rightarrow \infty).$$

Finally, we note that $\tilde{T}f = Tf$ for all $f \in \mathcal{D}_T$.

2. Now the relation

$$\|\tilde{T}\| := \sup_{f \in \mathcal{H}, \|f\| \leq 1} \|\tilde{T}f\| = \sup_{f \in \mathcal{D}_T, \|f\| \leq 1} \|\tilde{T}f\| = \sup_{f \in \mathcal{D}_T, \|f\| \leq 1} \|Tf\| = \|T\|$$

is correct. Furthermore, the operator $\tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$ is linear: For two elements

$$f = \lim_{n \rightarrow \infty} f_n, \quad g = \lim_{n \rightarrow \infty} g_n \quad \text{from } \mathcal{H}$$

with $\{f_n\}_n \subset \mathcal{D}_T$ and $\{g_n\}_n \subset \mathcal{D}_T$, we have the equation

$$\begin{aligned} \tilde{T}(\alpha f + \beta g) &= \tilde{T}\left(\lim_{n \rightarrow \infty} (\alpha f_n + \beta g_n)\right) = \lim_{n \rightarrow \infty} (\alpha Tf_n + \beta Tg_n) \\ &= \alpha \lim_{n \rightarrow \infty} Tf_n + \beta \lim_{n \rightarrow \infty} Tg_n = \alpha \tilde{T}f + \beta \tilde{T}g \end{aligned}$$

for arbitrary $\alpha, \beta \in \mathbb{C}$, on account of the continuity of $\tilde{T}$.

3. If $\hat{T}, \tilde{T} : \mathcal{H} \rightarrow \mathcal{H}$ are two extensions of $\mathcal{D}_T$ on $\mathcal{H}$, we have

$$(\tilde{T} - \hat{T})(f) = 0 \quad \text{for all } f \in \mathcal{D}_T.$$

Since $\mathcal{D}_T \subset \mathcal{H}$ lies dense and $(\tilde{T} - \hat{T}) : \mathcal{H} \rightarrow \mathcal{H}$ is continuous, we infer

$$(\tilde{T} - \hat{T})(f) = 0 \quad \text{for all } f \in \mathcal{H}$$

and consequently $\tilde{T} = \hat{T}$.

q.e.d.

Theorem 4.7. *Let $T : \mathcal{H} \rightarrow \mathcal{H}$ denote a bounded linear operator in the Hilbert space $\mathcal{H}$. Then we have a uniquely determined linear operator $T^* : \mathcal{H} \rightarrow \mathcal{H}$ such that*

$$(Tf, g) = (f, T^*g) \quad \text{for all } f, g \in \mathcal{H} \quad (4)$$

is correct. Furthermore, we have

$$\|T^*\| = \|T\| \quad \text{and} \quad T^{**} = T,$$

which means that the operation $$ is an involution.*

Definition 4.8. *The operator T^* is named the adjoint operator of T .*

Definition 4.9. *A bounded linear operator H is called Hermitian if $H^* = H$ holds true, which means*

$$(Hx, y) = (x, Hy) \quad \text{for all } x, y \in \mathcal{H}.$$

Proof of Theorem 4.7:

-Uniqueness: Let T_1 and T_2 be two adjoint operators to T : Then (4) yields

$$(f, T_1 g) = (T f, g) = (f, T_2 g) \quad \text{for all } f, g \in \mathcal{H}$$

as well as $(f, (T_1 - T_2)g) = 0$ for all $f, g \in \mathcal{H}$, and consequently $T_1 = T_2$.

-Existence: For a fixed $g \in \mathcal{H}$ we consider the bounded linear functional

$$A_g(f) := (g, T f), \quad f \in \mathcal{H}.$$

This functional is bounded according to

$$|A_g(f)| \leq \|g\| \|T f\| \leq (\|g\| \|T\|) \|f\| \quad \text{for all } f \in \mathcal{H}.$$

We apply the Representation theorem of Fréchet-Riesz: For each $g \in \mathcal{H}$ we have an element $g^* \in \mathcal{H}$ with the property

$$(g, T f) = A_g(f) = (g^*, f) \quad \text{for all } f \in \mathcal{H}.$$

We now set $T^* g := g^*$, and the so-defined mapping $T^* : \mathcal{H} \rightarrow \mathcal{H}$ has the property (4).

-Linearity: We take the elements $g_1, g_2 \in \mathcal{H}$ and the numbers $c_1, c_2 \in \mathbb{C}$. With the aid of (4) we then calculate

$$\begin{aligned} (T^*(c_1 g_1 + c_2 g_2), f) &= (c_1 g_1 + c_2 g_2, T f) \\ &= \bar{c}_1 (g_1, T f) + \bar{c}_2 (g_2, T f) \\ &= \bar{c}_1 (T^* g_1, f) + \bar{c}_2 (T^* g_2, f) \\ &= (c_1 T^* g_1 + c_2 T^* g_2, f) \quad \text{for all } f \in \mathcal{H}. \end{aligned}$$

-Boundedness: At first, we note that

$$(T f, g) = (f, T^* g) = (T^{**} f, g) \quad \text{for all } f, g \in \mathcal{H}.$$

Therefore, T is an involution. With the aid of (4) we obtain the following estimate

$$\|(f, T^* g)\| \leq \|T f\| \|g\| \leq \|T\| \|f\| \|g\| \quad \text{for all } f, g \in \mathcal{H}.$$

Inserting $f = T^* g$, we obtain

$$\|T^* g\|^2 \leq \|T\| \|T^* g\| \|g\|$$

and consequently

$$\|T^* g\| \leq \|T\| \|g\|,$$

which means

$$\|T^*\| \leq \|T\|.$$

Since T is an involution, we infer

$$\|T\| = \|T^{**}\| \leq \|T^*\|.$$

Consequently, the relation $\|T\| = \|T^*\|$ is correct.

q.e.d.

Example 4.10. Let the following cube in $\mathbb{R}^n$ be given, namely

$$Q := \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x_j| \leq R \text{ for } j = 1, \dots, n \right\}$$

whose sides have the length $2R \in (0, +\infty)$. Then we consider a *Hilbert-Schmidt integral kernel*

$$K = K(x, y) : Q \times Q \rightarrow \mathbb{C} \in L^2(Q \times Q, \mathbb{C}). \quad (5)$$

On account of

$$\int_{Q \times Q} |K(x, y)|^2 dx dy < \infty$$

we have a null-set $N \subset Q$ with the following property: For all $x \in Q \setminus N$ the function $y \mapsto K(x, y)$ is measurable on Q and $\int_Q |K(x, y)|^2 dy < \infty$ is satisfied.

Furthermore, we have

$$\int_Q \left(\int_Q |K(x, y)|^2 dy \right) dx = \int_{Q \times Q} |K(x, y)|^2 dx dy =: \|K\|^2 < \infty$$

because of the Fubini-Tonelli theorem. Taking $f \in L^2(Q, \mathbb{C})$ we define the *Hilbert-Schmidt operator*

$$\mathbb{K}f(x) = \begin{cases} \int_Q K(x, y)f(y) dy, & x \in Q \setminus N \\ 0, & x \in N \end{cases}. \quad (6)$$

The Hölder inequality yields

$$|\mathbb{K}f(x)|^2 \leq \left(\int_Q |K(x, y)|^2 dy \right) \|f\|^2$$

for all $x \in Q \setminus N$ and integration with respect to $x \in Q$ gives

$$\int_Q |\mathbb{K}f(x)|^2 dx \leq \left(\int_{Q \times Q} |K(x, y)|^2 dx dy \right) \|f\|^2 = \|K\|^2 \|f\|^2,$$

and finally

$$\|\mathbb{K}f\| \leq \|K\| \|f\| \quad \text{for all } f \in L^2(Q, \mathbb{C}). \quad (7)$$

Therefore, we obtain the following

Theorem 4.11. *The Hilbert-Schmidt operator $\mathbb{K} : \mathcal{H} \rightarrow \mathcal{H}$ from (6) with the integral kernel (5) represents a bounded linear operator on the Hilbert space $\mathcal{H} = L^2(Q, \mathbb{C})$, and we have the estimate*

$$\|\mathbb{K}\| \leq \|K\|.$$

Remarks:

1. The singular kernels

$$K = K(x, y) \in \mathcal{S}_\alpha(G, \mathbb{C}) \quad \text{with} \quad \alpha \in [0, n)$$

generate special Hilbert-Schmidt operators. The statements from Section 2, Theorem 2.6 and Theorem 2.7, valid for these special operators, will later be utilized to obtain regularity results concerning the solutions of the integral equation.

2. The kernel

$$K^*(x, y) := \overline{K(y, x)} \in L^2(Q \times Q, \mathbb{C})$$

generates the adjoint operator $\mathbb{K}^*$ belonging to the Hilbert-Schmidt operator $\mathbb{K}$.

3. The operator $\mathbb{K}$ is Hermitian if and only if the following identity is satisfied:

$$K(x, y) = \overline{K(y, x)} \quad \text{a. e. in} \quad Q \times Q.$$

We now shall investigate the inverse of a linear operator.

Definition 4.12. Let $T : \mathcal{D}_T \rightarrow \mathcal{H}$ denote a linear operator on the subset $\mathcal{D}_T \subset \mathcal{H}$ of the Hilbert space $\mathcal{H}$ with the range $\mathcal{W}_T := T(\mathcal{D}_T) \subset \mathcal{H}$. Furthermore, let the mapping

$$x \mapsto Tx, \quad x \in \mathcal{D}_T$$

be injective. Setting $f := T^{-1}g$ the inverse $T^{-1} : \mathcal{W}_T \rightarrow \mathcal{D}_T \subset \mathcal{H}$ of the operator T is then defined if $Tf = g$ holds true. We note that

$$\mathcal{D}_{T^{-1}} = \mathcal{W}_T, \quad \mathcal{W}_{T^{-1}} = \mathcal{D}_T.$$

We immediately obtain the following

Theorem 4.13. The operator $T^{-1} : \mathcal{W}_T \rightarrow \mathcal{D}_T$ is linear and does exist if and only if the equation

$$Tx = 0, \quad x \in \mathcal{D}_T$$

possesses only the trivial solution $x = 0$.

Theorem 4.14. (O. Toeplitz)

Let $T : \mathcal{H} \rightarrow \mathcal{H}$ denote a bounded linear operator in the Hilbert space $\mathcal{H}$. Then the operator T possesses a bounded inverse in $\mathcal{H}$ - namely $T^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ - if and only if the following conditions are satisfied:

- a) For all $x \in \mathcal{H}$ we have $\|Tx\| \geq d\|x\|$ with a bound $d \in (0, +\infty)$.
- b) The homogeneous equation $T^*x = 0$ admits only the trivial solution $x = 0$.

Proof:

‘ $\Rightarrow$ ’ We assume that the bounded inverse $T^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ exists. Then we have a number $c > 0$ satisfying

$$\|T^{-1}x\| \leq c\|x\| \quad \text{for all } x \in \mathcal{H}.$$

With $x := Tf$ we infer

$$\|Tf\| \geq \frac{1}{c}\|f\|.$$

Therefore, the condition a) is fulfilled with $d := \frac{1}{c}$.

If $z \in \mathcal{H}$ is a solution of $T^*z = 0$, we deduce

$$(Tx, z) = (x, T^*z) = (x, 0) = 0 \quad \text{for all } x \in \mathcal{H}.$$

Inserting the element $x = T^{-1}z$, we obtain $z = 0$.

‘ $\Leftarrow$ ’ At first, we show that $\mathcal{W}_T \subset \mathcal{H}$ is closed: Let $\{y_n\}_{n=1,2,\dots} \subset \mathcal{W}_T$ denote an arbitrary sequence with $y_n \rightarrow y$ ($n \rightarrow \infty$) in $\mathcal{H}$. We set $y_n = Tx_n$, $n = 1, 2, \dots$, and with the aid of a) we get the inequality

$$\|x_n - x_m\| \leq \frac{1}{d}\|y_n - y_m\| \rightarrow 0 \quad (m, n \rightarrow \infty).$$

This implies $x_n \rightarrow x$ ($n \rightarrow \infty$), and the continuity of T yields

$$Tx = T(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} y_n = y \in \mathcal{W}_T.$$

Consequently, $\mathcal{W}_T$ is closed in $\mathcal{H}$, and we have the orthogonal decomposition

$$\mathcal{H} = \mathcal{W}_T \oplus \mathcal{W}_T^\perp.$$

Now we take $z \in \mathcal{W}_T^\perp$ and obtain

$$0 = (z, Tx) = (T^*z, x) \quad \text{for all } x \in \mathcal{H}$$

and finally $T^*z = 0$. The condition b) therefore yields $z = 0$ and consequently $\mathcal{H} = \mathcal{W}_T$, which means T is surjective. The injectivity of T follows immediately from a). Consequently, T^{-1} exists and is bounded due to a) with

$$\|T^{-1}\| \leq \frac{1}{d}.$$

q.e.d.

Remark: If $H : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded Hermitian operator satisfying

$$\|Hx\| \geq d\|x\| \quad \text{for all } x \in \mathcal{H}$$

with the number $d \in (0, +\infty)$, Theorem 4.14 implies the existence of the bounded inverse $H^{-1} : \mathcal{H} \rightarrow \mathcal{H}$.

Theorem 4.15. *Let $T : \mathcal{H} \rightarrow \mathcal{H}$ denote a bounded linear operator in the Hilbert space $\mathcal{H}$. Furthermore, let the bounded inverse $T^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ be defined. Then the operator T^* possesses an inverse $(T^*)^{-1}$, which is defined and bounded in $\mathcal{H}$. Furthermore, we have $(T^{-1})^* = (T^*)^{-1}$.*

Proof: Because of the assumptions we see

$$\|T^{-1}x\| \leq c\|x\| \quad \text{for all } x \in \mathcal{H}.$$

When we insert the element $x = T^{-1}y$ into the relation

$$(Tx, y) = (x, T^*y) \quad \text{for all } y \in \mathcal{H}$$

we obtain

$$\|y\|^2 \leq \|T^{-1}y\| \|T^*y\| \leq c\|y\| \|T^*y\|$$

and consequently

$$\|T^*y\| \geq \frac{1}{c}\|y\| \quad \text{for all } y \in \mathcal{H}.$$

Theorem 4.7 and the relation

$$(T^*)^*f = T^{**}f = Tf$$

imply, with $(T^*)^*f = 0$ then $f = 0$ holds true. Theorem 4.14 shows that the inverse

$$(T^*)^{-1} : \mathcal{H} \rightarrow \mathcal{H}$$

exists, and we have

$$\|(T^*)^{-1}\| \leq \|T^{-1}\|.$$

Let the elements $f, g \in \mathcal{H}$ be chosen arbitrarily. With $x = T^{-1}f$ and $y = (T^*)^{-1}g$ we then obtain the relation

$$(f, (T^*)^{-1}g) = (Tx, y) = (x, T^*y) = (T^{-1}f, g) = (f, (T^{-1})^*g).$$

Consequently, the identity $(T^*)^{-1} = (T^{-1})^*$ is correct. q.e.d.

In the Hilbert space $\mathcal{H}$ we consider a closed linear nonvoid subspace $\mathcal{M} \subset \mathcal{H}$, and the Projection theorem yields the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Noting that

$$f = f_1 + f_2 \in \mathcal{H} \quad \text{with } f_1 \in \mathcal{M}, f_2 \in \mathcal{M}^\perp$$

holds true, the following definition for a projector P is sensible:

$$P : \mathcal{H} \rightarrow \mathcal{M} \quad \text{via } f = f_1 + f_2 \mapsto Pf := f_1.$$

We consider

$$\|Pf\|^2 = \|f_1\|^2 \leq \|f\|^2 \quad \text{for all } f \in \mathcal{H}$$

and

$$\|Pf\| = \|f\| \quad \text{for all } f \in \mathcal{M}.$$

The norm of the projector consequently satisfies

$$\|P\| = 1.$$

Furthermore, we observe

$$P^2 f = P \circ Pf = Pf \quad \text{and} \quad P^2 = P \quad \text{in } \mathcal{H},$$

and we conclude

$$\begin{aligned} (Pf, g) &= (f_1, g_1 + g_2) = (f_1, g_1) \\ &= (f_1 + f_2, g_1) = (f, Pg) \quad \text{for all } f, g \in \mathcal{H}, \end{aligned}$$

which means $P = P^*$.

Definition 4.16. A bounded linear operator $P : \mathcal{H} \rightarrow \mathcal{H}$ is a projection operator or a projector if the following holds true:

- a) P is Hermitian, which means $P = P^*$;
- b) $P^2 = P$.

Theorem 4.17. Let $P : \mathcal{H} \rightarrow \mathcal{H}$ denote a projector. Then the set

$$\mathcal{M} := \left\{ g \in \mathcal{H} : g = Pf \text{ with } f \in \mathcal{H} \right\}$$

is a closed linear subspace in $\mathcal{H}$. Furthermore, we have

$$f = Pf + (f - Pf) \in \mathcal{M} \oplus \mathcal{M}^\perp.$$

Proof:

1. We show that $\mathcal{M} = P(\mathcal{H})$ is closed: Let $\{g_n\}_{n=1,2,\dots} \subset \mathcal{M}$ be a sequence with $g_n \rightarrow g$ ($n \rightarrow \infty$) in $\mathcal{H}$. On account of $g_n = Pf_n$ with $f_n \in \mathcal{H}$ we infer

$$Pg_n = P^2 f_n = Pf_n = g_n.$$

Since P is continuous, $Pg = g$ follows and consequently $g \in \mathcal{M}$.

2. We take $f \in \mathcal{H}$, set $f_1 := Pf$ and $f_2 := f - Pf$, and observe $f_1 \in \mathcal{M}$. Furthermore, all $h \in \mathcal{M}$ satisfy

$$(f_2, h) = (f - Pf, h) = (f - Pf, Ph) = (Pf - P^2 f, h) = 0.$$

Consequently, $f_2 = f - Pf \in \mathcal{M}^\perp$ is correct.

q.e.d.

Remark: In the Hilbert space $\mathcal{H}$ let the linear subspace $\mathcal{M} \subset \mathcal{H}$ be given. The sequence $\{\varphi_j\}_{j=1,2,\dots}$ is assumed to constitute a c.o.n.s. in $\mathcal{M}$. Then we have

$$P_{\mathcal{M}}f = \sum_{j=1}^{j_0} (\varphi_j, f) \varphi_j, \quad f \in \mathcal{H}$$

with $j_0 \in \mathbb{N} \cup \{\infty\}$.

In physics the energy of a system is measured with the aid of *bilinear forms*. Linear operators are then attributed to the latter.

Definition 4.18. A complex-valued function

$$B(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$$

is named a bilinear form if

$$\begin{aligned} B(f, c_1 g_1 + c_2 g_2) &= c_1 B(f, g_1) + c_2 B(f, g_2) && \text{for all } f, g_1, g_2 \in \mathcal{H} \\ B(c_1 f_1 + c_2 f_2, g) &= \bar{c}_1 B(f_1, g) + \bar{c}_2 B(f_2, g) && \text{for all } f_1, f_2, g \in \mathcal{H} \end{aligned} \quad (8)$$

and all $c_1, c_2 \in \mathbb{C}$ holds true. The bilinear form is Hermitian if

$$B(f, g) = \overline{B(g, f)} \quad \text{for all } f, g \in \mathcal{H} \quad (9)$$

is correct, and we name B symmetric if

$$B(f, g) = B(g, f) \quad \text{for all } f, g \in \mathcal{H} \quad (10)$$

holds true. For real-valued bilinear forms the conditions (9) and (10) are equivalent. The bilinear form B is bounded if we have a constant $c \in [0, +\infty)$ with the property

$$|B(f, g)| \leq c \|f\| \|g\| \quad \text{for all } f, g \in \mathcal{H}. \quad (11)$$

A Hermitian bilinear form is strictly positive-definite if we have a constant $c \in (0, +\infty)$ such that

$$B(f, f) \geq c \|f\|^2 \quad \text{for all } f \in \mathcal{H} \quad (12)$$

is satisfied.

Remarks:

1. Alternatively, one calls (12) the *coercivity condition*.
2. For a given bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$, we obtain with

$$B(f, g) := (Tf, g), \quad f, g \in \mathcal{H}$$

a bilinear form. This is bounded on account of

$$|B(f, g)| \leq \|Tf\| \|g\| \leq \|T\| \|f\| \|g\|, \quad f, g \in \mathcal{H}.$$

If T is Hermitian, the bilinear form is Hermitian as well since we have

$$B(f, g) = (Tf, g) = (f, Tg) = \overline{(Tg, f)} = \overline{B(g, f)}, \quad f, g \in \mathcal{H}.$$

We now address the inverse question.

Theorem 4.19. (Representation theorem for bilinear forms)

For each bounded bilinear form $B = B(f, g)$ with $f, g \in \mathcal{H}$, there exists a uniquely determined bounded linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ satisfying

$$B(f, g) = (Tf, g) \quad \text{for all } f, g \in \mathcal{H}. \quad (13)$$

If B is Hermitian then T is Hermitian as well.

Proof: For a fixed element $f \in \mathcal{H}$ we obtain with

$$L_f(g) := B(f, g), \quad g \in \mathcal{H}$$

a bounded linear functional on $\mathcal{H}$. Due to the representation theorem of Fréchet-Riesz, we have an element $f^* \in \mathcal{H}$ with the property

$$(f^*, g) = B(f, g) = L_f(g) \quad \text{for all } g \in \mathcal{H}. \quad (14)$$

Now f^* is uniquely determined by f , and we set

$$Tf := f^*, \quad f \in \mathcal{H}.$$

1. The operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is linear: We calculate

$$\begin{aligned} (T(c_1 f_1 + c_2 f_2), g) &= B(c_1 f_1 + c_2 f_2, g) = \bar{c}_1 B(f_1, g) + \bar{c}_2 B(f_2, g) \\ &= \bar{c}_1 (Tf_1, g) + \bar{c}_2 (Tf_2, g) = (c_1 Tf_1 + c_2 Tf_2, g) \end{aligned}$$

for all $f_1, f_2, g \in \mathcal{H}$ and all $c_1, c_2 \in \mathbb{C}$.

2. Since the bilinear form $B(f, g) = (Tf, g)$ is bounded, we have

$$|(Tf, g)| \leq c \|f\| \|g\| \quad \text{for all } f, g \in \mathcal{H}.$$

With $g = \frac{Tf}{\|Tf\|}$ we easily comprehend the inequality

$$\|Tf\| \leq c \|f\| \quad \text{for all } f \in \mathcal{H},$$

and we conclude

$$\|T\| \leq c < +\infty.$$

3. If B is Hermitian, we see

$$(Tf, g) = B(f, g) = \overline{B(g, f)} = \overline{(Tg, f)} = (f, Tg) \quad \text{for all } f, g \in \mathcal{H}.$$

Therefore, the operator T is Hermitian.

q.e.d.

Theorem 4.20. (Lax, Milgram)

Let $B : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ denote a Hermitian bilinear form, which is bounded due to

$$|B(f, g)| \leq c^+ \|f\| \|g\| \quad \text{for all } f, g \in \mathcal{H} \quad (15)$$

and satisfies the following coercivity condition

$$B(f, f) \geq c^- \|f\|^2 \quad \text{for all } f \in \mathcal{H}. \quad (16)$$

Here the constants $0 < c^- \leq c^+ < +\infty$ have been chosen adequately. Then we have a bounded Hermitian operator $T : \mathcal{H} \rightarrow \mathcal{H}$ satisfying $\|T\| \leq c^+$ and

$$B(f, g) = (Tf, g) \quad \text{for all } f, g \in \mathcal{H}. \quad (17)$$

This operator possesses a bounded inverse $T^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ which is Hermitian and subject to

$$\|T^{-1}\| \leq \frac{1}{c^-}.$$

Proof: From Theorem 4.19 we obtain a Hermitian operator $T : \mathcal{H} \rightarrow \mathcal{H}$ with $\|T\| \leq c^+$ and the property (17). Together with (8) we arrive at

$$c^- \|f\|^2 \leq B(f, f) = (Tf, f) \leq \|Tf\| \|f\| \quad \text{for all } f \in \mathcal{H}$$

and consequently

$$\|Tf\| \geq c^- \|f\| \quad \text{for all } f \in \mathcal{H}. \quad (18)$$

Due to the Theorem of Toeplitz, the operator T possesses a bounded inverse $T^{-1} : \mathcal{H} \rightarrow \mathcal{H}$, which is Hermitian because of Theorem 4.15. Finally, the relation (18) implies

$$\|T^{-1}\| \leq \frac{1}{c^-}. \quad \text{q.e.d.}$$

Remarks:

1. The Theorems 4.19 and 4.20 remain valid for real bilinear forms, if we replace *Hermitian* with *symmetric*.
2. Theorem 4.20 provides the basic result for the weak solvability of elliptic differential equations, as we shall see in Section 3 of Chapter 10.
3. In its full strength, the Theorem of Lax-Milgram even holds true for real, not necessarily symmetric, bilinear forms. However, we only need the symmetric situation, when we apply our Theorem 4.20 to elliptic p.d.e.s – without first order terms.

5 Unitary Operators

Definition 5.1. Let $\mathcal{H}$ and $\mathcal{H}'$ denote two Hilbert spaces with the inner products (x, y) and $(x, y)'$. Then the linear operator $V : \mathcal{H} \rightarrow \mathcal{H}'$ is called isometric if the following holds true:

$$(Vf, Vg)' = (f, g) \quad \text{for all } f, g \in \mathcal{H}. \quad (1)$$

Remarks:

1. With the isometric operator $V : \mathcal{H} \rightarrow \mathcal{H}'$ we calculate

$$\begin{aligned} \|Vf - Vg\|'^2 &= \|V(f - g)\|'^2 = (V(f - g), V(f - g))' \\ &= (f - g, f - g) = \|f - g\|^2 \quad \text{for all } f, g \in \mathcal{H}. \end{aligned} \quad (2)$$

Therefore, the relation $f \neq g$ implies $Vf \neq Vg$, and consequently the operator V is injective.

2. The operator V is bounded. Noting that

$$\|Vf\|' = \sqrt{(Vf, Vf)'} = \sqrt{(f, f)} = \|f\| \quad \text{for all } f \in \mathcal{H},$$

we infer

$$\|V\| = 1. \quad (3)$$

3. We have $\mathcal{D}_V = \mathcal{H}$ for the domain of definition of an isometric operator V , and the range $\mathcal{W}_V \subset \mathcal{H}'$ is closed. We take a sequence $g_n = Vf_n \in \mathcal{W}_V$, $n = 1, 2, \dots$ satisfying $g_n \rightarrow g$ ($n \rightarrow \infty$) and observe that $\{f_n\}_{n=1,2,\dots} \subset \mathcal{H}$ is a Cauchy sequence due to (2), namely

$$\|f_n - f_m\|^2 = \|g_n - g_m\|'^2 \rightarrow 0 \quad (n, m \rightarrow \infty).$$

This implies $f_n \rightarrow f \in \mathcal{H}$ ($n \rightarrow \infty$) and furthermore

$$g = \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} Vf_n = V(\lim_{n \rightarrow \infty} f_n) = Vf \in \mathcal{W}_V,$$

since V is continuous. Consequently, $\mathcal{W}_V \subset \mathcal{H}$ is closed.

4. In the case $\dim \mathcal{H} = \dim \mathcal{H}' < +\infty$, the injectivity implies the surjectivity. For infinite-dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ this is not true, as illustrated by the following example:

Example 5.2. We consider the so-called *shift-operator* in Hilbert's sequential space $\mathcal{H} := l_2 =: \mathcal{H}'$:

$$\begin{aligned} V : \mathcal{H} &\rightarrow \mathcal{H}', \\ (x_1, x_2, \dots) &\mapsto (0, \dots, 0, x_1, x_2, \dots). \end{aligned}$$

The operator V is evidently isometric, however, V is not surjective.

Definition 5.3. An isometric operator $V : \mathcal{H} \rightarrow \mathcal{H}'$ is called unitary if $V : \mathcal{H} \rightarrow \mathcal{H}'$ is surjective, more precisely $V(\mathcal{H}) = \mathcal{H}'$.

Remark: For a unitary operator $U : \mathcal{H} \rightarrow \mathcal{H}'$ there exists its inverse $U^{-1} : \mathcal{H}' \rightarrow \mathcal{H}$, and we have the identity

$$(U^{-1}f, U^{-1}g) = (U \circ U^{-1}f, U \circ U^{-1}g)' = (f, g)' \quad \text{for all } f, g \in \mathcal{H}' \quad (4)$$

on account of (1). Therefore, the inverse U^{-1} is unitary as well.

Definition 5.4. Let $\mathcal{H}$ and $\mathcal{H}'$ denote two Hilbert spaces with the inner products (x, y) and $(x, y)'$. Furthermore, T and T' are two linear operators in $\mathcal{H}$ and $\mathcal{H}'$, respectively. Then the operators T and T' are named unitary equivalent, if there exists a unitary operator $U : \mathcal{H} \rightarrow \mathcal{H}'$ satisfying

$$T' = U \circ T \circ U^{-1}. \quad (5)$$

Theorem 5.5. A bounded linear operator $V : \mathcal{H} \rightarrow \mathcal{H}$ is unitary if and only if

$$V^* \circ V = V \circ V^* = \mathbb{E} \quad (6)$$

is correct. Here the symbol $\mathbb{E} : \mathcal{H} \rightarrow \mathcal{H}$ denotes the identity operator.

Proof:

‘ $\Rightarrow$ ’ At first, we remark that $V : \mathcal{H} \rightarrow \mathcal{H}$ is isometric if and only if

$$(V^* \circ Vf, g) = (Vf, Vg) = (f, g) = (\mathbb{E}f, g) \quad \text{for all } f, g \in \mathcal{H}$$

or equivalently

$$V^* \circ V = \mathbb{E}$$

is valid.

If V is unitary, we have the existence of $V^{-1} : \mathcal{H} \rightarrow \mathcal{H}$. From the last relation we infer $V^* = V^{-1}$ and therefore $V \circ V^* = \mathbb{E}$.

‘ $\Leftarrow$ ’ Now let the identity (6) be satisfied for $V : \mathcal{H} \rightarrow \mathcal{H}$. Then we infer $V^{-1} = V^*$. In particular, the operator V is surjective and isometric as well according to the following relation:

$$(f, g) = (V^* \circ Vf, g) = (Vf, Vg) \quad \text{for all } f, g \in \mathcal{H}.$$

q.e.d.

Now we shall prove the Theorem of Fourier-Plancherel (see Theorem 3.1 in Section 3 of Chapter 6), which we have already applied there. At first, we present the transition from Fourier series to the Fourier integral: Taking $c > 0$ arbitrarily, the functions

$$\left\{ \frac{1}{\sqrt{2c}} e^{-\frac{\pi}{c} i k x} \right\}_{k \in \mathbb{Z}}$$

constitute a complete orthonormal system of functions on the interval $[-c, +c]$. For all

$$f \in L^2([-c, +c], \mathbb{R}) \cap C_0^0((-c, +c), \mathbb{R})$$

the completeness relation yields the following identity:

$$\begin{aligned} \int_{-c}^{+c} |f(x)|^2 dx &= \sum_{k=-\infty}^{+\infty} \left| \int_{-c}^{+c} \frac{1}{\sqrt{2c}} e^{\frac{\pi}{c} i k x} f(x) dx \right|^2 \\ &= \sum_{k=-\infty}^{+\infty} \frac{1}{2c} \left| \int_{-c}^{+c} e^{\frac{\pi}{c} i k y} f(y) dy \right|^2. \end{aligned}$$

We set

$$g(x) := \frac{1}{\sqrt{2\pi}} \int_{-c}^{+c} e^{i x y} f(y) dy, \quad x \in \mathbb{R},$$

and $x_k = \frac{\pi}{c} k$ for $k \in \mathbb{Z}$. Then we obtain $x_k - x_{k-1} = \frac{\pi}{c}$ and

$$\frac{1}{2c} \left| \int_{-c}^{+c} e^{\frac{\pi}{c} i k y} f(y) dy \right|^2 = \frac{1}{2c} \left| \sqrt{2\pi} g(x_k) \right|^2 = \frac{\pi}{c} |g(x_k)|^2 = |g(x_k)|^2 (x_k - x_{k-1})$$

for all $k \in \mathbb{Z}$. For all $c > 0$, the following identity holds true:

$$\int_{-c}^{+c} |f(x)|^2 dx = \sum_{k=-\infty}^{+\infty} |g(x_k)|^2 (x_k - x_{k-1}).$$

The transition to the limit $c \rightarrow +\infty$ yields

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |g(x)|^2 dx. \quad (7)$$

We expect the operator

$$Tf(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i x y} f(y) dy, \quad x \in \mathbb{R} \quad (8)$$

to be unitary on the space $L^2(\mathbb{R})$.

More generally, we define *Fourier's integral operator* on the Euclidean space $\mathbb{R}^n$ as follows:

$$Tf(x) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{i(x \cdot y)} f(y) dy, \quad x \in \mathbb{R}^n. \quad (9)$$

We shall prove that $T : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is unitary.

At first, we determine T from (8) explicitly for the *characteristic function*

$$\varphi_{a,b}(x) = \varphi(a, b, x) = \begin{cases} 1, & a \leq x \leq b \\ 0, & x < a \text{ or } x > b \end{cases}. \quad (10)$$

We calculate

$$T\varphi_{a,b}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i0y} \varphi_{a,b}(y) dy = \frac{b-a}{\sqrt{2\pi}}, \quad (11)$$

and for $x \neq 0$ we evaluate

$$\begin{aligned} T\varphi_{a,b}(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixy} \varphi_{a,b}(y) dy = \frac{1}{\sqrt{2\pi}} \int_a^b e^{ixy} dy \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ixy}}{ix} \right]_{y=a}^{y=b} = \frac{1}{\sqrt{2\pi}} \frac{e^{ibx} - e^{iax}}{ix}. \end{aligned} \quad (12)$$

Proposition 5.6. *For Cauchy's principal values*

$$\psi(h) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{e^{ihx} - 1}{x^2} dx := \lim_{\varepsilon \downarrow 0} \left\{ \frac{1}{\pi} \int_{-\infty}^{-\varepsilon} \frac{e^{ihx} - 1}{x^2} dx + \frac{1}{\pi} \int_{+\varepsilon}^{+\infty} \frac{e^{ihx} - 1}{x^2} dx \right\}$$

we have

$$\psi(h) = -|h|, \quad h \in \mathbb{R}.$$

Proof: Taking an arbitrary $h \geq 0$, we consider the holomorphic function

$$\begin{aligned} f(z) &:= \frac{e^{ihz} - 1}{z^2} = \frac{1 + ihz + \frac{1}{2}(ihz)^2 + \dots - 1}{z^2} \\ &= \frac{ih}{z} + \dots, \quad z \in \mathbb{C} \setminus \{0\}. \end{aligned} \quad (13)$$

For $0 < \varepsilon < R < +\infty$ we utilize the domain

$$G_{\varepsilon,R} := \left\{ z \in \mathbb{C} : \varepsilon < |z| < R, \operatorname{Re} z > 0 \right\}$$

with the positive-oriented boundary

$$[-R, -\varepsilon] \cup K_\varepsilon \cup [+ \varepsilon, +R] \cup K_R = \partial G_{\varepsilon, R}.$$

Here we have defined the semicircle

$$K_R : z = Re^{i\varphi}, \quad 0 \leq \varphi \leq \pi,$$

and the following semicircle

$$-K_\varepsilon : z = \varepsilon e^{i\varphi}, \quad 0 \leq \varphi \leq \pi$$

negatively traversed. Since f is a holomorphic function in $\overline{G_{\varepsilon, R}}$, Cauchy's integral theorem yields the following identity:

$$\begin{aligned} 0 = \int_{\partial G_{\varepsilon, R}} f(z) dz &= \int_{-R}^{-\varepsilon} \frac{e^{ihx} - 1}{x^2} dx + \int_{+\varepsilon}^{+R} \frac{e^{ihx} - 1}{x^2} dx \\ &\quad + \int_{K_R} \frac{e^{ihz} - 1}{z^2} dz - \int_{-K_\varepsilon} \frac{e^{ihz} - 1}{z^2} dz \end{aligned} \quad (14)$$

for all $0 < \varepsilon < R < +\infty$. With the aid of (13), let us calculate

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \int_{-K_\varepsilon} \frac{e^{ihz} - 1}{z^2} dz &= \lim_{\varepsilon \downarrow 0} \int_{-K_\varepsilon} \left(\frac{ih}{z} + \dots \right) dz = \lim_{\varepsilon \downarrow 0} \int_{-K_\varepsilon} \frac{ih}{z} dz \\ &= \lim_{\varepsilon \downarrow 0} \int_0^\pi \frac{ih}{\varepsilon e^{i\varphi}} i\varepsilon e^{i\varphi} d\varphi = -h\pi. \end{aligned} \quad (15)$$

Futhermore, we deduce

$$\begin{aligned} \int_{K_R} \frac{e^{ihz} - 1}{z^2} dz &= \int_0^\pi \frac{\exp \{ ih(\cos \varphi + i \sin \varphi)R \} - 1}{R^2 e^{2i\varphi}} iRe^{i\varphi} d\varphi \\ &= \frac{i}{R} \int_0^\pi e^{-i\varphi} \left\{ e^{ihR \cos \varphi} e^{-hR \sin \varphi} - 1 \right\} d\varphi, \end{aligned}$$

and we estimate for all $R > 0$ as follows:

$$\begin{aligned} \left| \int_{K_R} \frac{e^{ihz} - 1}{z^2} dz \right| &\leq \frac{1}{R} \int_0^\pi 1 \cdot \{ 1 \cdot e^{-hR \sin \varphi} + 1 \} d\varphi \\ &\leq \frac{2\pi}{R} \rightarrow 0 \quad (R \rightarrow +\infty). \end{aligned}$$

This implies

$$\lim_{R \rightarrow +\infty} \int_{K_R} \frac{e^{ihz} - 1}{z^2} dz = 0 \quad \text{for all } h \geq 0. \quad (16)$$

In (14) we observe the transition $\varepsilon \downarrow 0$ and $R \uparrow +\infty$. With the aid of (15) and (16), we obtain the identity

$$0 = \int_{-\infty}^{+\infty} \frac{e^{ihx} - 1}{x^2} dx + h\pi$$

and consequently

$$\psi(h) = -h \quad \text{for all } h \geq 0. \quad (17)$$

Via the substitution $y = -x$, we evaluate

$$\begin{aligned} \psi(-h) &= \lim_{\varepsilon \downarrow 0} \left\{ \frac{1}{\pi} \int_{-\infty}^{-\varepsilon} \frac{e^{ih(-x)} - 1}{x^2} dx + \frac{1}{\pi} \int_{\varepsilon}^{+\infty} \frac{e^{ih(-x)} - 1}{x^2} dx \right\} \\ &= \lim_{\varepsilon \downarrow 0} \left\{ \frac{1}{\pi} \int_{\varepsilon}^{+\infty} \frac{e^{ihy} - 1}{y^2} dy + \frac{1}{\pi} \int_{-\infty}^{-\varepsilon} \frac{e^{ihy} - 1}{y^2} dy \right\} \\ &= \psi(h) \quad \text{for all } h \in \mathbb{R}. \end{aligned}$$

Finally, we obtain the following identity from (17):

$$\psi(h) = -|h| \quad \text{for all } h \in \mathbb{R}. \quad \text{q.e.d.}$$

Proposition 5.7. *With respect to the inner product in $L^2(\mathbb{R}, \mathbb{C})$, we see*

$$(T\varphi_{a,b}, T\varphi_{c,d}) = \begin{cases} 0, & \text{if } -\infty < a < b \leq c < d < +\infty \\ b - a, & \text{if } -\infty < a = c < b = d < +\infty \end{cases}. \quad (18)$$

Proof: We utilize (12) and calculate

$$\begin{aligned} (T\varphi_{a,b}, T\varphi_{c,d}) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{(e^{-ibx} - e^{-iax})(e^{idx} - e^{icx})}{x^2} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i(d-b)x} - e^{i(c-b)x} - e^{i(d-a)x} + e^{i(c-a)x}}{x^2} dx. \end{aligned}$$

If $-\infty < a < b \leq c < d < +\infty$ is fulfilled, Proposition 5.6 implies

$$\begin{aligned}
(T\varphi_{a,b}, T\varphi_{c,d}) &= \frac{1}{2} \left\{ \psi(d-b) - \psi(c-b) - \psi(d-a) + \psi(c-a) \right\} \\
&= \frac{1}{2} \left\{ b-d+c-b+d-a+a-c \right\} = 0.
\end{aligned}$$

If $-\infty < a = c < b = d < +\infty$ holds true, we obtain

$$\begin{aligned}
(T\varphi_{a,b}, T\varphi_{c,d}) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1 - e^{i(c-b)x} - e^{i(d-a)x} + 1}{x^2} dx \\
&= -\frac{1}{2} \left\{ \psi(c-b) + \psi(d-a) \right\} = -\frac{1}{2} \{c-b+a-d\} \\
&= -\frac{1}{2} \{2a-2b\} = b-a.
\end{aligned}$$

q.e.d.

Let the rectangle

$$Q := \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : a_\alpha \leq x_\alpha \leq b_\alpha \text{ for } \alpha = 1, \dots, n \right\}$$

in $\mathbb{R}^n$ be given. We decompose the interval $[a_\alpha, b_\alpha]$ into the parts

$$a_\alpha = x_\alpha^{(0)} < x_\alpha^{(1)} < \dots < x_\alpha^{(m_\alpha)} = b_\alpha \quad \text{for } \alpha = 1, \dots, n$$

and set

$$I_\alpha^{(k_\alpha)} = [x_\alpha^{(k_\alpha-1)}, x_\alpha^{(k_\alpha)}]$$

for $1 \leq k_\alpha \leq m_\alpha$ and $\alpha = 1, \dots, n$. Finally, we define the following rectangles for $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ with $1 \leq k_\alpha \leq m_\alpha$:

$$I^{(k)} = I_1^{(k_1)} \times \dots \times I_n^{(k_n)} \subset \mathbb{R}^n.$$

We define the characteristic function of the set $I^{(k)}$ by

$$\varphi_{I^{(k)}}(x) = \begin{cases} 1, & x \in I^{(k)} \\ 0, & x \in \mathbb{R}^n \setminus I^{(k)} \end{cases}$$

and similarly

$$\varphi_{I_\alpha^{(k_\alpha)}}(x) = \begin{cases} 1, & x \in I_\alpha^{(k_\alpha)} \\ 0, & x \in \mathbb{R} \setminus I_\alpha^{(k_\alpha)} \end{cases} \quad \text{for } \alpha = 1, \dots, n.$$

Then we see

$$\varphi_{I^{(k)}}(x) = \varphi_{I_1^{(k_1)}}(x_1) \cdot \dots \cdot \varphi_{I_n^{(k_n)}}(x_n), \quad x \in \mathbb{R}^n. \quad (19)$$

Now we calculate

$$\begin{aligned}
T\varphi_{I^{(k)}}(x) &= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{i(x \cdot y)} \varphi_{I^{(k)}}(y) dy \\
&= \left(\int_{-\infty}^{+\infty} \frac{e^{ix_1 y_1}}{\sqrt{2\pi}} \varphi_{I_1^{(k_1)}}(y_1) dy_1 \right) \cdots \left(\int_{-\infty}^{+\infty} \frac{e^{ix_n y_n}}{\sqrt{2\pi}} \varphi_{I_n^{(k_n)}}(y_n) dy_n \right) \quad (20) \\
&= T\varphi_{I_1^{(k_1)}}(x_1) \cdots T\varphi_{I_n^{(k_n)}}(x_n).
\end{aligned}$$

For the admissible multi-indices $k = (k_1, \dots, k_n)$ and $l = (l_1, \dots, l_n)$ we deduce

$$\begin{aligned}
&(T\varphi_{I^{(k)}}, T\varphi_{I^{(l)}}) \\
&= \int_{\mathbb{R}^n} \overline{(T\varphi_{I_1^{(k_1)}}(x_1) \cdots T\varphi_{I_n^{(k_n)}}(x_n))} (T\varphi_{I_1^{(l_1)}}(x_1) \cdots T\varphi_{I_n^{(l_n)}}(x_n)) dx \\
&= (T\varphi_{I_1^{(k_1)}}, T\varphi_{I_1^{(l_1)}}) \cdots (T\varphi_{I_n^{(k_n)}}, T\varphi_{I_n^{(l_n)}}) \\
&= |I_1^{(k_1)}| \cdots |I_n^{(k_n)}| \delta_{k_1 l_1} \cdots \delta_{k_n l_n} \\
&= |I^{(k)}| \delta_{k_1 l_1} \cdots \delta_{k_n l_n}
\end{aligned}$$

and consequently

$$(T\varphi_{I^{(k)}}, T\varphi_{I^{(l)}}) = \begin{cases} |I^{(k)}|, & k = l \\ 0, & k \neq l \end{cases} \quad (21)$$

Here we have set

$$|I_\alpha^{(k_\alpha)}| = x_\alpha^{(k_\alpha)} - x_\alpha^{(k_\alpha-1)} \quad \text{and} \quad |I^{(k)}| = |I_1^{(k_1)}| \cdots |I_n^{(k_n)}|.$$

We summarize our considerations to the following

Proposition 5.8. *Let $\varphi_k := \varphi_{I^{(k)}}$ with $k = (k_1, \dots, k_n)$ and $1 \leq k_\alpha \leq m_\alpha$ for $\alpha = 1, \dots, n$ be chosen. Then we have the inclusion*

$$T\varphi_k \in L^2(\mathbb{R}^n)$$

and furthermore the equation

$$(T\varphi_k, T\varphi_l) = (\varphi_k, \varphi_l) \quad \text{holds true for all admissible } k, l. \quad (22)$$

We now consider the linear subspace $\mathcal{D} \subset L^2(\mathbb{R}^n)$ of the step functions in $\mathbb{R}^n$. They consist of all those functions f satisfying the following conditions:

1. Outside of a rectangle $Q \subset \mathbb{R}^n$ the relation $f(x) = 0$ is correct.

2. There exists a decomposition $Q = \bigcup_k I^{(k)}$ of the rectangle as above, and we have the representation

$$f(x) = \sum_k c_k \varphi_k$$

with the coefficients $c_k \in \mathbb{C}$ and the characteristic functions $\varphi_k := \varphi_{I^{(k)}}$.

Proposition 5.9. *For all step functions $f, g \in \mathcal{D}$ we have*

$$(Tf, Tg) = (f, g).$$

Proof: We choose a rectangle $Q \supset \text{supp}(f) \cup \text{supp}(g)$ and find a canonical decomposition of Q , such that

$$f = \sum_k c_k \varphi_k, \quad g = \sum_l d_l \varphi_l.$$

Therefore, we obtain

$$\begin{aligned} (Tf, Tg) &= \left(\sum_k c_k T\varphi_k, \sum_l d_l T\varphi_l \right) = \sum_{k,l} \overline{c_k} d_l (T\varphi_k, T\varphi_l) \\ &= \sum_{k,l} \overline{c_k} d_l (\varphi_k, \varphi_l) = (f, g). \end{aligned}$$

q.e.d.

We now consider the integral operator

$$Sf := \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-i(x \cdot y)} f(y) dy, \quad f \in \mathcal{D},$$

and note that

$$Sf = \overline{T\overline{f}}, \quad f \in \mathcal{D}. \quad (23)$$

Since $T : \mathcal{D} \rightarrow L^2(\mathbb{R}^n)$ represents a linear bounded operator, this is the case for S as well. Furthermore, the operator S is isometric according to

$$(Sf, Sg) = (\overline{T\overline{f}}, \overline{T\overline{g}}) = \overline{(T\overline{f}, T\overline{g})} = \overline{(\overline{f}, \overline{g})} = (f, g)$$

for all $f, g \in \mathcal{D}$.

Intermediate statement: We have the identity

$$(Tf, g) = (f, Sg) \quad \text{for all } f, g \in \mathcal{D}.$$

Proof: We now calculate

$$\begin{aligned}
(Tf, g) &= \int_Q \left(\frac{1}{\sqrt{2\pi}^n} \int_Q e^{-i(x \cdot y)} \bar{f}(y) dy \right) g(x) dx \\
&= \frac{1}{\sqrt{2\pi}^n} \int_{Q \times Q} e^{-i(x \cdot y)} \bar{f}(y) g(x) dy dx \\
&= \int_Q \bar{f}(y) \left(\frac{1}{\sqrt{2\pi}^n} \int_Q e^{-i(x \cdot y)} g(x) dx \right) dy \\
&= (f, Sg).
\end{aligned}$$

We sum up our considerations to the following

Proposition 5.10. *Let $\mathcal{D}$ denote the set of all step functions in $\mathbb{R}^n$. Then the operators*

$$T, S : \mathcal{D} \rightarrow L^2(\mathbb{R}^n)$$

are isometric, and S is adjoint to T which means

$$(Tf, Tg) = (f, g) = (Sf, Sg) \quad (24)$$

as well as

$$(Tf, g) = (f, Sg) \quad (25)$$

for all $f, g \in \mathcal{D}$.

Now the linear space $\mathcal{D} \subset L^2(\mathbb{R}^n)$ lies dense, and consequently for each $f \in L^2(\mathbb{R}^n)$ we have a sequence

$$\{f_k\}_{k=1,2,\dots} \subset \mathcal{D} \quad \text{with} \quad \|f_k - f\|_{L^2(\mathbb{R}^n)} \rightarrow 0 \quad (k \rightarrow \infty).$$

Therefore, we can uniquely extend the bounded operators T, S of $\mathcal{D}$ onto $L^2(\mathbb{R}^n)$ as follows:

$$Tf := \lim_{k \rightarrow \infty} Tf_k, \quad Sf := \lim_{k \rightarrow \infty} Sf_k. \quad (26)$$

The relations (24) and (25) yield

$$S \circ T = T^* \circ T = \mathbb{E} = S^* \circ S = T \circ S \quad \text{on } \mathcal{D},$$

and we infer

$$S \circ T = \mathbb{E} = T \circ S \quad \text{on } L^2(\mathbb{R}^n). \quad (27)$$

Consequently, the operator $T : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is unitary and satisfies

$$T^* = S = T^{-1}.$$

From (26) we shall derive a direct representation of T and S as follows: We choose

$$f \in L_0^2(\mathbb{R}^n) := \left\{ g \in L^2(\mathbb{R}^n) : \text{supp}(g) \subset \mathbb{R}^n \text{ is compact} \right\}.$$

For $f \in L_0^2(\mathbb{R}^n)$ we have a sequence $\{f_k\}_{k=1,2,\dots} \subset \mathcal{D}$ such that

$$\text{supp}(f), \text{supp}(f_k) \subset Q, \quad k = 1, 2, \dots$$

- where $Q \subset \mathbb{R}^n$ is a fixed rectangle - and

$$\|f - f_k\|_{L^2(Q)} \rightarrow 0 \quad (k \rightarrow \infty)$$

holds true. For all $x \in \mathbb{R}^n$ we obtain the estimate

$$\begin{aligned} \left| Tf_k(x) - \frac{1}{\sqrt{2\pi}^n} \int_Q e^{i(x \cdot y)} f(y) dy \right| &= \frac{1}{\sqrt{2\pi}^n} \left| \int_Q e^{i(x \cdot y)} (f_k(y) - f(y)) dy \right| \\ &\leq \frac{1}{\sqrt{2\pi}^n} \int_Q |f_k(y) - f(y)| dy \\ &\leq \frac{1}{\sqrt{2\pi}^n} \sqrt{|Q|} \|f_k - f\|_{L^2(Q)} \rightarrow 0 \quad (k \rightarrow \infty) \end{aligned} \quad (28)$$

and consequently

$$\left\| Tf_k(x) - \frac{1}{\sqrt{2\pi}^n} \int_Q e^{i(x \cdot y)} f(y) dy \right\|_{L^2(\mathbb{R}^n)} \rightarrow 0 \quad (k \rightarrow \infty). \quad (29)$$

Together with (26) we obtain

$$\begin{aligned} &\left\| Tf(x) - \frac{1}{\sqrt{2\pi}^n} \int_Q e^{i(x \cdot y)} f(y) dy \right\|_{L^2(\mathbb{R}^n)} \\ &\leq \|Tf - Tf_k\|_{L^2(\mathbb{R}^n)} + \left\| Tf_k(x) - \frac{1}{\sqrt{2\pi}^n} \int_Q e^{i(x \cdot y)} f(y) dy \right\|_{L^2(\mathbb{R}^n)} \\ &\rightarrow 0 \quad (k \rightarrow \infty). \end{aligned}$$

For all $f \in L_0^2(\mathbb{R}^n)$ therefore the relation

$$Tf(x) = \frac{1}{\sqrt{2\pi}^n} \int_Q e^{i(x \cdot y)} f(y) dy \quad \text{a. e. in } \mathbb{R}^n \quad (30)$$

is correct. Taking $f \in L^2(\mathbb{R}^n)$ arbitrarily, we choose a sequence of rectangles

$$Q_1 \subset Q_2 \subset \dots \quad \text{with} \quad \bigcup_{n=1}^{\infty} Q_n = \mathbb{R}^n$$

and set

$$f_k(x) = \begin{cases} f(x), & x \in Q_k \\ 0, & x \in \mathbb{R}^n \setminus Q_k \end{cases}.$$

Then the relation $\|f_k - f\|_{L^2(\mathbb{R}^n)} \rightarrow 0$ for $(k \rightarrow \infty)$ is valid, and (26) yields

$$Tf = \lim_{k \rightarrow \infty} Tf_k = \text{l.i.m.} \frac{1}{\sqrt{2\pi}^n} \int_{Q_k} e^{i(x \cdot y)} f(y) dy, \quad (31)$$

$$Sf = \lim_{k \rightarrow \infty} Sf_k = \text{l.i.m.} \frac{1}{\sqrt{2\pi}^n} \int_{Q_k} e^{-i(x \cdot y)} f(y) dy. \quad (32)$$

Here the symbol l.i.m. denotes the limit for $k \rightarrow \infty$ in the quadratic means, more precisely in the $L^2(\mathbb{R}^n)$ -norm.

We summarize our considerations to the following

Theorem 5.11. (Fourier, Plancherel)

The Fourier integral operator $T : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ exists according to (31) and is unitary. Moreover, the adjoint integral operator $S : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ from (32) is unitary as well, and we have the identities

$$S \circ T = T \circ S = \mathbb{E} \quad \text{on } L^2(\mathbb{R}^n).$$

6 Completely Continuous Operators in Hilbert Spaces

We owe the following notion of convergence to David Hilbert:

Definition 6.1. *In the Hilbert space $\mathcal{H}$ a sequence $\{x_n\}_{n=1,2,\dots} \subset \mathcal{H}$ is called weakly convergent towards an element $x \in \mathcal{H}$, symbolically $x_n \rightharpoonup x$ ($n \rightarrow \infty$), if the relation*

$$\lim_{n \rightarrow \infty} (x_n, y) = (x, y) \quad \text{for all } y \in \mathcal{H}$$

is satisfied.

Example 6.2. Let $\{\varphi_i\}_{i=1,2,\dots}$ denote an orthonormal system in the Hilbert space $\mathcal{H}$, and we observe

$$\|\varphi_i - \varphi_j\| = \sqrt{(\varphi_i - \varphi_j, \varphi_i - \varphi_j)} = \sqrt{(\varphi_i, \varphi_i) + (\varphi_j, \varphi_j)} = \sqrt{2}$$

for all $i, j \in \mathbb{N}$ with $i \neq j$. Consequently, $\{\varphi_i\}_{i=1,2,\dots}$ does not contain a subsequence which represents a Cauchy sequence with respect to the Hilbert space norm. Because of Bessel's inequality, all $f \in \mathcal{H}$ are subject to the estimate

$$\sum_{i=1}^{\infty} |(\varphi_i, f)|^2 \leq \|f\|^2 < +\infty,$$

and we infer

$$\lim_{i \rightarrow \infty} (\varphi_i, f) = 0 = (0, f) \quad \text{for all } f \in \mathcal{H}.$$

Thus we obtain $\varphi_i \rightarrow 0$ ($i \rightarrow \infty$) and note

$$\|0\| \leq \liminf_{i \rightarrow \infty} \|\varphi_i\| = 1.$$

Theorem 6.3. (Principle of uniform boundedness)

On the Hilbert space $\mathcal{H}$ let a sequence of the following bounded linear functionals $A_n : \mathcal{H} \rightarrow \mathbb{C}$ with $n \in \mathbb{N}$ be given, such that each element $f \in \mathcal{H}$ possesses a constant $c_f \in [0, +\infty)$ with the property

$$|A_n f| \leq c_f, \quad n = 1, 2, \dots \quad (1)$$

Then we have a constant $\alpha \in [0, +\infty)$ satisfying

$$\|A_n\| \leq \alpha \quad \text{for all } n \in \mathbb{N}. \quad (2)$$

Proof:

1. Let $A : \mathcal{H} \rightarrow \mathbb{C}$ denote a bounded linear functional such that

$$|Af| \leq c \quad \text{for all } f \in \mathcal{H} \quad \text{with} \quad \|f - f_0\| \leq \varepsilon.$$

Here we have chosen an element $f_0 \in \mathcal{H}$ as well as a quantity $\varepsilon > 0$ and a constant $c \geq 0$. Then we have the estimate

$$\|A\| \leq \frac{2c}{\varepsilon}.$$

Setting $x := \frac{1}{\varepsilon}(f - f_0)$, we infer $\|x\| \leq 1$ and

$$|Ax| = \left| \frac{1}{\varepsilon}Af - \frac{1}{\varepsilon}Af_0 \right| \leq \frac{1}{\varepsilon}(|Af| + |Af_0|) \leq \frac{2c}{\varepsilon}$$

and finally $\|A\| \leq \frac{2c}{\varepsilon}$.

2. If the statement (2) does not hold true, part 1 of this proof together with the continuity of the functionals $\{A_n\}_n$ enables us to construct a sequence of balls

$$\Sigma_n := \left\{ f \in \mathcal{H} : \|f - f_n\| \leq \varepsilon_n \right\}, \quad n \in \mathbb{N}$$

satisfying $\Sigma_1 \supset \Sigma_2 \supset \dots$ with $\varepsilon_n \downarrow 0$ ($n \rightarrow \infty$) and a subsequence $1 \leq n_1 < n_2 < \dots$ such that

$$|A_{n_j} x| \geq j \quad \text{for all } x \in \Sigma_j \quad \text{and } j = 1, 2, \dots \quad (3)$$

is correct. Evidently, the relation (3) yields a contradiction to (1). q.e.d.

Theorem 6.4. (Weak convergence criterion)

Let the sequence $\{x_n\}_{n=1,2,\dots} \subset \mathcal{H}$ be given in a Hilbert space such that all elements $y \in \mathcal{H}$ possess the limit

$$\lim_{n \rightarrow \infty} (x_n, y).$$

Then the sequence $\{x_n\}_n$ is bounded in $\mathcal{H}$ and weakly convergent towards an element $x \in \mathcal{H}$, which means $x_n \rightharpoonup x$ ($n \rightarrow \infty$).

Proof: We consider the bounded linear functionals

$$A_n(y) := (x_n, y), \quad y \in \mathcal{H}$$

with the norms $\|A_n\| = \|x_n\|$ for $n = 1, 2, \dots$. Since the limits

$$\lim_{n \rightarrow \infty} A_n(y) =: A(y)$$

exist for all $y \in \mathcal{H}$ by assumption, Theorem 6.3 gives us a constant $c \in [0, +\infty)$ with $\|x_n\| = \|A_n\| \leq c$ for all $n \in \mathbb{N}$. This implies $\|A\| \leq c$, and the representation theorem of Fréchet-Riesz yields the existence of exactly one element $x \in \mathcal{H}$ satisfying

$$A(y) = (x, y), \quad y \in \mathcal{H}$$

for the bounded linear functional A . We obtain

$$\lim_{n \rightarrow \infty} (x_n, y) = \lim_{n \rightarrow \infty} A_n(y) = A(y) = (x, y) \quad \text{for all } y \in \mathcal{H}$$

which means $x_n \rightharpoonup x$ ($n \rightarrow \infty$). q.e.d.

Though it is not possible in general to select a subsequence convergent with respect to the norm out of a bounded sequence in a Hilbert space (compare the Example 6.2 above), we can prove the following fundamental result (compare Theorem 8.9 in Chapter 2, Section 8 for the special case $\mathcal{H} = L^2(X)$):

Theorem 6.5. (Hilbert's selection theorem)

Each bounded sequence $\{x_n\}_{n=1,2,\dots} \subset \mathcal{H}$ in a Hilbert space $\mathcal{H}$ contains a weakly convergent subsequence $\{x_{n_k}\}_{k=1,2,\dots}$.

Proof:

1. The sequence $\{x_n\}_{n=1,2,\dots}$ is bounded and we have a constant $c \in [0, +\infty)$, such that

$$\|x_n\| \leq c, \quad n = 1, 2, \dots \tag{4}$$

is correct. On account of

$$|(x_1, x_n)| \leq c \|x_1\| \quad \text{for all } n \in \mathbb{N}$$

we find a subsequence $\{x_n^{(1)}\}_n \subset \{x_n\}_n$, such that $\lim_{n \rightarrow \infty} (x_1, x_n^{(1)})$ exists. Noting that

$$|(x_2, x_n^{(1)})| \leq c \|x_2\| \quad \text{for all } n \in \mathbb{N}$$

we select a further subsequence $\{x_n^{(2)}\}_n \subset \{x_n^{(1)}\}_n$ whose limit exists, namely $\lim_{n \rightarrow \infty} (x_2, x_n^{(2)})$. The continuation of this procedure gives us a chain of subsequences

$$\{x_n\}_n \supset \{x_n^{(1)}\}_n \supset \{x_n^{(2)}\}_n \supset \dots \supset \{x_n^{(k)}\}_n,$$

such that the limits

$$\lim_{n \rightarrow \infty} (x_i, x_n^{(k)})$$

exist for $i = 1, \dots, k$. With the aid of Cantor's diagonal procedure we get the sequence $x_k' := x_k^{(k)}$. Then the sequence $\{(x_i, x_k')\}_{k=1,2,\dots}$ is convergent for all $i \in \mathbb{N}$. Denoting the linear subspace of all finite linear combinations by $\mathcal{M}$, namely

$$x = \sum_{i=1}^{N(x)} \alpha_i x_i, \quad \alpha_i \in \mathbb{C}, \quad N(x) \in \mathbb{N},$$

the following limits exist:

$$\lim_{k \rightarrow \infty} (x, x_k') \quad \text{for all } x \in \mathcal{M}. \quad (5)$$

2. Now we make the transition to the linear closed subspace $\mathcal{M} \subset \overline{\mathcal{M}} \subset \mathcal{H}$, and the following limits exist as well:

$$\lim_{k \rightarrow \infty} (y, x_k') \quad \text{for all } y \in \overline{\mathcal{M}}. \quad (6)$$

Here we note that we can extend the bounded linear functional

$$A(y) := \lim_{k \rightarrow \infty} (x_k', y) = \overline{\lim_{k \rightarrow \infty} (y, x_k')}, \quad y \in \mathcal{M}$$

continuously onto the closure $\overline{\mathcal{M}}$. Due to the Projection theorem each element $y \in \mathcal{H}$ can be represented in the form $y = y_1 + y_2$ with $y_1 \in \overline{\mathcal{M}}$ and $y_2 \in \overline{\mathcal{M}}^\perp$. This implies the existence of

$$\lim_{k \rightarrow \infty} (y, x_k') = \lim_{k \rightarrow \infty} (y_1 + y_2, x_k') = \lim_{k \rightarrow \infty} (y_1, x_k')$$

for all $y \in \mathcal{H}$. Consequently, the sequence $\{x_k'\}_{k=1,2,\dots}$ in the Hilbert space $\mathcal{H}$ converges weakly.

q.e.d.

Remarks to the weak convergence:

1. If the sequence $x_n \rightarrow x$ ($n \rightarrow \infty$) converges strongly, which means

$$\lim_{n \rightarrow \infty} \|x_n - x\| = 0,$$

then it converges weakly $x_n \rightharpoonup x$ ($n \rightarrow \infty$) as well. For arbitrary elements $y \in \mathcal{H}$ we namely deduce

$$|(x_n, y) - (x, y)| = |(x_n - x, y)| \leq \|x_n - x\| \|y\| \rightarrow 0 \quad (n \rightarrow \infty).$$

2. The norm is lower-semi-continuous with respect to weak convergence. This means

$$x_n \rightharpoonup x \quad (n \rightarrow \infty) \quad \Rightarrow \quad \liminf_{n \rightarrow \infty} \|x_n\| \geq \|x\|, \quad x_n, x \in \mathcal{H}$$

for a real Hilbert space $\mathcal{H}$. We namely observe

$$\begin{aligned} \|x_n\|^2 - \|x\|^2 &= (x_n, x_n) - (x, x) = (x_n - x, x_n + x) \\ &= (x_n - x, x_n - x) + 2(x - x_n, x), \quad n = 1, 2, \dots, \end{aligned}$$

and consequently

$$\liminf_{n \rightarrow \infty} \|x_n\|^2 - \|x\|^2 = \liminf_{n \rightarrow \infty} \|x_n - x\|^2 + 2 \liminf_{n \rightarrow \infty} (x - x_n, x) \geq 0,$$

and finally

$$\liminf_{n \rightarrow \infty} \|x_n\| \geq \|x\|.$$

3. From $x_n \rightharpoonup x$ ($n \rightarrow \infty$) and $y_n \rightarrow y$ ($n \rightarrow \infty$) we infer $(x_n, y_n) \rightarrow (x, y)$ ($n \rightarrow \infty$). Here we utilize the estimate

$$\begin{aligned} |(x_n, y_n) - (x, y)| &= |(x_n, y_n) - (x_n, y) + (x_n, y) - (x, y)| \\ &\leq |(x_n, y_n - y)| + |(x_n - x, y)| \\ &\leq \|y_n - y\| \|x_n\| + |(x_n - x, y)| \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Definition 6.6. A subset $\Sigma \subset \mathcal{H}$ of a Hilbert space is named precompact, if each sequence $\{y_n\}_{n=1,2,\dots} \subset \Sigma$ contains a strongly convergent subsequence $\{y_{n_k}\}_{k=1,2,\dots} \subset \{y_n\}_n$, which means

$$\lim_{k,l \rightarrow \infty} \|y_{n_k} - y_{n_l}\| = 0.$$

Definition 6.7. A linear operator $K : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is called completely continuous or alternatively compact, if the following set

$$\Sigma := \left\{ y = Kx : x \in \mathcal{H}_1 \text{ with } \|x\|_1 \leq r \right\} \subset \mathcal{H}_2$$

is precompact, with a certain radius $r \in (0, +\infty)$ given. This means that each sequence $\{x_n\}_{n=1,2,\dots} \subset \mathcal{H}_1$ with $\|x_n\|_1 \leq r$ for $n \in \mathbb{N}$ contains a subsequence $\{x_{n_k}\}_{k=1,2,\dots}$ such that $\{Kx_{n_k}\}_{k=1,2,\dots} \subset \mathcal{H}_2$ converges strongly.

Remarks:

1. It suffices to choose $r = 1$ in Definition 6.7.
2. A completely continuous linear operator $K : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is bounded. If this were not the case, there would exist a sequence $\{x_n\}_{n=1,2,\dots} \subset \mathcal{H}_1$ with $\|x_n\|_1 = 1$ for all $n \in \mathbb{N}$ and $\|Kx_n\|_2 \rightarrow +\infty$ ($n \rightarrow \infty$). Therefore, we cannot select a convergent subsequence from $\{Kx_n\}_{n=1,2,\dots}$ in the Hilbert space $\mathcal{H}_2$. This yields a contradiction to Definition 6.7.

Theorem 6.8. *Let $K : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ denote a linear operator between the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The operator K is completely continuous if and only if for each weakly convergent sequence $x_n \rightharpoonup x$ ($n \rightarrow \infty$) in $\mathcal{H}_1$ the statement*

$$Kx_n \rightarrow Kx \quad (n \rightarrow \infty) \quad \text{in } \mathcal{H}_2$$

follows. Consequently, the operator K is completely continuous if and only if each weakly convergent sequence in $\mathcal{H}_1$ is transferred into a strongly convergent sequence in $\mathcal{H}_2$.

Proof:

‘ $\Leftarrow$ ’ Let $\{x_n\}_{n=1,2,\dots} \subset \mathcal{H}_1$ be a sequence with $\|x_n\|_1 \leq 1$, $n \in \mathbb{N}$. Hilbert’s selection theorem yields a subsequence $\{x_{n_k}\}_{k=1,2,\dots} \subset \{x_n\}_n$ satisfying

$$x_{n_k} \rightharpoonup x \quad (k \rightarrow \infty) \quad \text{in } \mathcal{H}_1.$$

By assumption

$$y_{n_k} \rightarrow Kx \quad (k \rightarrow \infty)$$

is fulfilled for the subsequence $y_{n_k} := Kx_{n_k}$, $k = 1, 2, \dots$. Consequently, the operator $K : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is completely continuous.

‘ $\Rightarrow$ ’ Now let K be completely continuous, and $\{x_n\}_{n=1,2,\dots} \subset \mathcal{H}_1$ denotes a sequence satisfying $x_n \rightharpoonup x = 0$ ($n \rightarrow \infty$). We then have to prove: $Kx_n \rightarrow Kx = 0$ ($n \rightarrow \infty$) in $\mathcal{H}_2$. If the latter statement were false, there would exist a number $d > 0$ and a subsequence $\{x_n'\}_n \subset \{x_n\}_n$ satisfying

$$\|Kx_n'\| \geq d > 0 \quad \text{for all } n \in \mathbb{N}.$$

Since the operator K is completely continuous, we have a further subsequence

$$\{x_n''\}_n \subset \{x_n'\}_n \quad \text{with} \quad Kx_n'' \rightarrow y \quad (n \rightarrow \infty).$$

Therefore, we obtain with the statement

$$0 < d^2 \leq (y, y) = \lim_{n \rightarrow \infty} (y, Kx_n'') = \lim_{n \rightarrow \infty} (K^*y, x_n'') = (K^*y, 0) = 0$$

a contradiction.

q.e.d.

Remarks about completely continuous operators:

1. If $K : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded linear operator with a finite-dimensional range $\mathcal{W}_K := K(\mathcal{H})$, then K is completely continuous.
2. If $T_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and $T_2 : \mathcal{H}_2 \rightarrow \mathcal{H}_3$ are two bounded linear operators, and T_1 or T_2 is completely continuous, then the operator

$$T := T_2 \circ T_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_3$$

is completely continuous as well. If the operator T_1 is completely continuous for instance, then the weakly convergent sequence $x_n \rightharpoonup x$ ($n \rightarrow \infty$) in $\mathcal{H}_1$ is transferred into the strongly convergent sequence $T_1 x_n \rightarrow T_1 x$ ($n \rightarrow \infty$) in $\mathcal{H}_2$. Since T_2 is continuous, we infer

$$Tx_n = T_2 \circ T_1 x_n \rightarrow T_2 \circ T_1 x = Tx \quad (n \rightarrow \infty) \quad \text{in } \mathcal{H}_3.$$

3. The operator $K : \mathcal{H} \rightarrow \mathcal{H}$ is completely continuous on the Hilbert space $\mathcal{H}$ if and only if the adjoint operator $K^* : \mathcal{H} \rightarrow \mathcal{H}$ is completely continuous. *Proof:* Let $K : \mathcal{H} \rightarrow \mathcal{H}$ be completely continuous, then the operator $K \circ K^*$ is completely continuous as well. From an arbitrary sequence

$$\{x_n\}_{n=1,2,\dots} \subset \mathcal{H} \quad \text{with} \quad \|x_n\| \leq 1, \quad n \in \mathbb{N}$$

we can extract a subsequence $\{x_{n_k}\}_{k=1,2,\dots}$ such that $\{K \circ K^* x_{n_k}\}_k$ converges strongly in $\mathcal{H}$. We infer

$$\begin{aligned} \|K^* x_{n_k} - K^* x_{n_l}\|^2 &= \|K^*(x_{n_k} - x_{n_l})\|^2 \\ &= (K^*(x_{n_k} - x_{n_l}), K^*(x_{n_k} - x_{n_l})) \\ &= (K \circ K^*(x_{n_k} - x_{n_l}), x_{n_k} - x_{n_l}) \\ &\leq \|K \circ K^*(x_{n_k} - x_{n_l})\| \|x_{n_k} - x_{n_l}\| \rightarrow 0 \quad (k, l \rightarrow \infty). \end{aligned}$$

Consequently, the sequence $\{K^* x_{n_k}\}_{k=1,2,\dots}$ converges in $\mathcal{H}$, and the operator K^* is completely continuous.

The inverse direction can be seen via the identity $K = (K^*)^*$.

4. Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a completely continuous Hermitian linear operator on the Hilbert space $\mathcal{H}$. Then the associate *bilinear form*

$$\alpha(x, y) := (Ax, y) = (x, Ay), \quad (x, y) \in \mathcal{H} \times \mathcal{H}$$

is continuous with respect to weak convergence. This means, with $x_n \rightharpoonup x$ ($n \rightarrow \infty$) and $y_n \rightharpoonup y$ ($n \rightarrow \infty$) in $\mathcal{H}$ we have the limit relation

$$\alpha(x_n, y_n) \rightarrow \alpha(x, y) \quad (n \rightarrow \infty).$$

Proof: Remark 3 concerning the weak convergence in combination with Theorem 6.8 yield this statement.

Definition 6.9. Let $\mathcal{H}$ denote a separable Hilbert space with two c.o.n.s.

$$\varphi = \{\varphi_i\}_{i=1,2,\dots} \quad \text{and} \quad \psi = \{\psi_i\}_{i=1,2,\dots}.$$

The linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ has a finite square-norm if

$$N(T; \varphi, \psi) := \sqrt{\sum_{i,k=1}^{\infty} |(T\varphi_i, \psi_k)|^2} < +\infty$$

holds true.

Proposition 6.10. Let $T : \mathcal{H} \rightarrow \mathcal{H}$ denote a linear operator as in Definition 6.9 with $N(T; \varphi, \psi) < +\infty$. Then we have

$$\|T\| \leq N(T; \varphi, \psi) = \sqrt{\sum_{i=1}^{\infty} \|T\varphi_i\|^2}. \quad (7)$$

Proof: At first, we observe

$$N(T; \varphi, \psi)^2 = \sum_{i,k=1}^{\infty} |(T\varphi_i, \psi_k)|^2 = \sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} |(T\varphi_i, \psi_k)|^2 \right) = \sum_{i=1}^{\infty} \|T\varphi_i\|^2.$$

With the series

$$f = \sum_{i=1}^{\infty} c_i \varphi_i \in \mathcal{H}$$

we calculate

$$Tf = \sum_{i=1}^{\infty} c_i T\varphi_i$$

and consequently

$$\|Tf\| \leq \sum_{i=1}^{\infty} |c_i| \|T\varphi_i\|.$$

This implies

$$\|Tf\| \leq \sqrt{\sum_{i=1}^{\infty} |c_i|^2} \sqrt{\sum_{i=1}^{\infty} \|T\varphi_i\|^2} = \|f\| N(T; \varphi, \psi) \quad \text{for all } f \in \mathcal{H}$$

and therefore $\|T\| \leq N(T; \varphi, \psi)$.

q.e.d.

Proposition 6.11. We consider with

$$\varphi = \{\varphi_i\}_{i=1,2,\dots}, \quad \varphi' = \{\varphi'_i\}_{i=1,2,\dots}, \quad \psi = \{\psi_i\}_{i=1,2,\dots}, \quad \psi' = \{\psi'_i\}_{i=1,2,\dots}$$

four complete orthonormal systems in $\mathcal{H}$. Then the identity

$$N(T; \varphi, \psi) = N(T; \varphi', \psi') =: N(T)$$

holds true. Furthermore, the equality $N(T) = N(T^*)$ is correct.

Proof: We calculate as follows:

$$\begin{aligned}
 N(T; \varphi, \psi)^2 &= \sum_{i,k=1}^{\infty} |(T\varphi_i, \psi_k)|^2 = \sum_{i=1}^{\infty} \|T\varphi_i\|^2 \\
 &= \sum_{i,k=1}^{\infty} |(\psi'_k, T\varphi_i)|^2 = \sum_{i,k=1}^{\infty} |(T^*\psi'_k, \varphi_i)|^2 \\
 &= \sum_{k=1}^{\infty} \|T^*\psi'_k\|^2 = \sum_{i,k=1}^{\infty} |(T^*\psi'_k, \varphi'_i)|^2 \\
 &= \sum_{i,k=1}^{\infty} |(\psi'_k, T\varphi'_i)|^2 = \sum_{i,k=1}^{\infty} |(T\varphi'_i, \psi'_k)|^2 = N(T; \varphi', \psi').
 \end{aligned}$$

From the identity above we also infer $N(T) = N(T^*)$. q.e.d.

Remark: Proposition 6.11 implies that the square-norm is independent of the chosen c.o.n.s.

Example 6.12. On the rectangle

$$Q := \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for } i = 1, \dots, n \right\}$$

let the kernel $K = K(x, y) : Q \times Q \rightarrow \mathbb{C} \in L^2(Q \times Q, \mathbb{C})$ with

$$\int_{Q \times Q} |K(x, y)|^2 dx dy < +\infty \quad (8)$$

be given. As in the Example 4.10 from Section 4 we define the Hilbert-Schmidt operator

$$\mathbb{K}f(x) := \int_Q K(x, y)f(y) dy \quad \text{for almost all } x \in Q.$$

On account of Theorem 4.11 from Section 4, the linear operator $\mathbb{K} : L^2(Q) \rightarrow L^2(Q)$ is bounded by $\|\mathbb{K}\| \leq \|K\|_{L^2(Q \times Q)}$.

Statement: The Hilbert-Schmidt operator $\mathbb{K}$ has the finite square-norm

$$N(\mathbb{K}) = \sqrt{\int_{Q \times Q} |K(x, y)|^2 dx dy} < +\infty. \quad (9)$$

Proof: Let $\{\varphi_i(x)\}_{i=1,2,\dots}$ constitute a c.o.n.s. in $L^2(Q)$. Then we set

$$\psi_i(x) = \int_Q K(x, y) \varphi_i(y) dy = \mathbb{K} \varphi_i(x) \quad \text{a. e. in } Q \quad \text{for } i = 1, 2, \dots$$

We calculate

$$\begin{aligned} \sum_{i=1}^{\infty} |\psi_i(x)|^2 &= \sum_{i=1}^{\infty} \left| \int_Q K(x, y) \varphi_i(y) dy \right|^2 \\ &= \sum_{i=1}^{\infty} \left| \overline{(K(x, \cdot), \varphi_i)} \right|^2 = \int_Q |K(x, y)|^2 dy, \end{aligned}$$

and Fubini's theorem yields

$$\begin{aligned} \int_{Q \times Q} |K(x, y)|^2 dx dy &= \int_Q \sum_{i=1}^{\infty} |\psi_i(x)|^2 dx = \sum_{i=1}^{\infty} \int_Q |\psi_i(x)|^2 dx \\ &= \sum_{i=1}^{\infty} \|\mathbb{K} \varphi_i\|^2 = N(\mathbb{K})^2. \end{aligned} \quad \text{q.e.d.}$$

Central significance possesses the subsequent

Theorem 6.13. *On the separable Hilbert space $\mathcal{H}$ let $T : \mathcal{H} \rightarrow \mathcal{H}$ denote a linear operator with finite square-norm $N(T) < +\infty$. Then the operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is completely continuous.*

Proof: Let the sequence $f_n \rightharpoonup f = 0$ ($n \rightarrow \infty$) be weakly convergent. If $\{\varphi_i\}_{i=1,2,\dots}$ represents a c.o.n.s. in $\mathcal{H}$, we have the expansion

$$f_n = \sum_{i=1}^{\infty} c_n^i \varphi_i$$

with

$$\lim_{n \rightarrow \infty} c_n^i = 0 \quad \text{for } i = 1, 2, \dots \quad (10)$$

and

$$\sum_{i=1}^{\infty} |c_n^i|^2 \leq M^2 \quad \text{for } n = 1, 2, \dots \quad (11)$$

Due to Proposition 6.10, the operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is continuous. Thus we infer

$$T f_n = \sum_{i=1}^{\infty} c_n^i T \varphi_i$$

and

$$\begin{aligned}
 \|Tf_n\| &\leq \left\| \sum_{i=1}^N c_n^i T\varphi_i \right\| + \left\| \sum_{i=N+1}^{\infty} c_n^i T\varphi_i \right\| \\
 &\leq \left\| \sum_{i=1}^N c_n^i T\varphi_i \right\| + \sqrt{\sum_{i=1}^{\infty} |c_n^i|^2} \sqrt{\sum_{i=N+1}^{\infty} \|T\varphi_i\|^2}.
 \end{aligned}$$

With the aid of (11), we obtain

$$\|Tf_n\| \leq \left\| \sum_{i=1}^N c_n^i T\varphi_i \right\| + M \sqrt{\sum_{i=N+1}^{\infty} \|T\varphi_i\|^2}, \quad n = 1, 2, \dots \quad (12)$$

Given the quantity $\varepsilon > 0$, we choose an integer $N = N(\varepsilon) \in \mathbb{N}$ so large that

$$M \sqrt{\sum_{i=N+1}^{\infty} \|T\varphi_i\|^2} \leq \varepsilon$$

is attained. Observing (10), we then can choose a number $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\left\| \sum_{i=1}^{N(\varepsilon)} c_n^i T\varphi_i \right\| \leq \varepsilon \quad \text{for all } n \geq n_0$$

is correct. Altogether, we obtain the estimate

$$\|Tf_n\| \leq 2\varepsilon \quad \text{for all } n \geq n_0$$

with the quantity $\varepsilon > 0$ given. Therefore, $Tf_n \rightarrow 0$ ($n \rightarrow \infty$) holds true. q.e.d.

Remark: Due to Theorem 6.13, the Hilbert-Schmidt operators are completely continuous.

Definition 6.14. *With the completely continuous operator $K : \mathcal{H} \rightarrow \mathcal{H}$ on the Hilbert space $\mathcal{H}$ we associate the so-called Fredholm operator*

$$T := \mathbb{E} - K : \mathcal{H} \rightarrow \mathcal{H} \quad \text{via} \quad Tx := \mathbb{E}x - Kx = x - Kx, \quad x \in \mathcal{H}.$$

Using the theorem of F. Riesz we now prove the important

Theorem 6.15. (Fredholm)

Let $K : \mathcal{H} \rightarrow \mathcal{H}$ denote a completely continuous linear operator on the Hilbert space $\mathcal{H}$ with the associate Fredholm operator $T := \mathbb{E} - K$. Then we have the following statements:

i) *The null-spaces*

$$\mathcal{N}_T := \left\{ x \in \mathcal{H} : Tx = 0 \right\}$$

of the operator T and

$$\mathcal{N}_{T^*} := \left\{ x \in \mathcal{H} : T^*x = 0 \right\}$$

of $T^* = \mathbb{E} - K^*$ fulfill the identity

$$\omega := \dim \mathcal{N}_T = \dim \mathcal{N}_{T^*} \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \quad (13)$$

ii) The operator equation

$$x - Kx = Tx = y, \quad x \in \mathcal{H} \quad (14)$$

is solvable for the right-hand side $y \in \mathcal{H}$ if and only if $y \in \mathcal{N}_{T^*}^\perp$ is correct, which means

$$(y, z) = 0 \quad \text{for all } z \in \mathcal{N}_{T^*} \quad (15)$$

is satisfied.

iii) If $\omega = 0$ holds true, the bounded inverse operator $T^{-1} : \mathcal{H} \rightarrow \mathcal{H}$ exists.

Proof:

1. At first, we show $\dim \mathcal{N}_T < +\infty$. If this were violated, there would exist an orthonormal system $\{\varphi_i\}_{i=1,2,\dots}$ satisfying

$$0 = T\varphi_i = \varphi_i - K\varphi_i, \quad i = 1, 2, \dots$$

Since the operator K is completely continuous, we can select a subsequence $\{\varphi_{i_j}\}_{j=1,2,\dots} \subset \{\varphi_i\}_i$ such that $\varphi_{i_j} \rightarrow \varphi$ ($j \rightarrow \infty$) in $\mathcal{H}$. This contradicts the statement

$$\|\varphi_i - \varphi_j\| = \sqrt{2} \quad \text{for all } i, j \in \mathbb{N} \text{ with } i \neq j.$$

Therefore, we see $\dim \mathcal{N}_T \in \mathbb{N}_0$. With K the operator K^* is completely continuous as well, and we comprehend $\dim \mathcal{N}_{T^*} \in \mathbb{N}_0$.

We now decompose $\mathcal{H}$ into the closed linear subspaces

$$\mathcal{H} = \mathcal{N}_T^\perp \oplus \mathcal{N}_T. \quad (16)$$

Furthermore, we assume

$$\dim \mathcal{N}_T \leq \dim \mathcal{N}_{T^*}.$$

If this were not the case, we could replace T by T^* and T^* by $T^{**} = T$. Finally, we set

$$\mathcal{W}_T := T(\mathcal{H}) \quad \text{and} \quad \mathcal{W}_{T^*} := T^*(\mathcal{H}).$$

2. We now have $y \in \mathcal{W}_T^\perp$ if and only if

$$0 = (y, Tx) = (T^*y, x) \quad \text{for all } x \in \mathcal{H}$$

is correct, and therefore $T^*y = 0$ or equivalently $y \in \mathcal{N}_{T^*}$ holds true. This implies

$$\mathcal{N}_{T^*} = \mathcal{W}_T^\perp \quad \text{or equivalently} \quad \mathcal{W}_T = \mathcal{N}_{T^*}^\perp. \quad (17)$$

In particular, the range of T is closed in $\mathcal{H}$. Let us consider the orthonormal basis

$$\{\varphi_1, \dots, \varphi_d\} \subset \mathcal{N}_T$$

in $\mathcal{N}_T$ and the orthonormal basis

$$\{\psi_1, \dots, \psi_{d^*}\} \subset \mathcal{N}_{T^*}$$

in $\mathcal{N}_{T^*}$ satisfying $0 \leq d \leq d^* < +\infty$. We modify the operator T and obtain a Fredholm operator

$$Sx := Tx - \sum_{i=1}^d (\varphi_i, x) \psi_i, \quad x \in \mathcal{H}. \quad (18)$$

On account of (16) and (17), the null-space of the operator S evidently satisfies

$$\mathcal{N}_S = \{x \in \mathcal{H} : Sx = 0\} = \{0\}.$$

Theorem 4.6 from Chapter 7, Section 4 of F. Riesz implies that the mapping $S : \mathcal{H} \rightarrow \mathcal{H}$ is surjective. Consequently, $d^* = d$ and $\dim \mathcal{N}_T = \dim \mathcal{N}_{T^*}$ are correct. Furthermore, the mapping $T : \mathcal{N}_T^\perp \rightarrow \mathcal{N}_{T^*}^\perp$ is invertible. In the special case

$$\omega = \dim \mathcal{N}_T = \dim \mathcal{N}_{T^*} = 0$$

the theorem of F. Riesz quoted above guarantees the existence of the bounded inverse operator on the entire Hilbert space $\mathcal{H}$.

q.e.d.

Remark: Theorem 6.15 is especially valid for linear operators $K : \mathcal{H} \rightarrow \mathcal{H}$ on the separable Hilbert space $\mathcal{H}$ with finite square-norm $N(K) < +\infty$. We can estimate the dimension of the null-space for the operator T according to

$$\dim \mathcal{N}_T \leq N(K)^2. \quad (19)$$

If $\{\varphi_1, \dots, \varphi_d\}$ namely denotes an orthonormal system in $\mathcal{N}_T$, we enlarge it to a c.o.n.s. $\{\varphi_i\}_{i=1,2,\dots}$ in $\mathcal{H}$ and obtain

$$\begin{aligned}
N(K)^2 &= \sum_{i=1}^{\infty} \|K\varphi_i\|^2 \geq \sum_{i=1}^d \|K\varphi_i\|^2 \\
&= \sum_{i=1}^d \|\varphi_i\|^2 = d = \dim \mathcal{N}_T.
\end{aligned}$$

We collect our results for Hilbert-Schmidt operators in the following

Theorem 6.16. (D. Hilbert, E. Schmidt)

On the rectangle $Q = [a_1, b_1] \times \dots \times [a_n, b_n]$ let the integral kernel

$$K = K(x, y) : Q \times Q \rightarrow \mathbb{C} \in L^2(Q \times Q)$$

be given. We consider the linear subspaces of $L^2(Q)$ satisfying

$$\begin{aligned}
\mathcal{N} : \quad & \int_Q K(x, y) f(y) dy = f(x), \quad f \in L^2(Q), \\
\mathcal{N}^* : \quad & \int_Q K^*(x, y) \psi(y) dy = \psi(x), \quad \psi \in L^2(Q),
\end{aligned}$$

with $K^*(x, y) := \overline{K(y, x)}$ for $(x, y) \in Q \times Q$, and the following statement

$$\dim \mathcal{N} = \dim \mathcal{N}^* \leq \int_{Q \times Q} |K(x, y)|^2 dx dy < +\infty \quad (20)$$

holds true. The right-hand side $f(x) \in L^2(Q)$ given, the integral equation

$$u(x) - \int_Q K(x, y) u(y) dy = f(x), \quad u \in L^2(Q),$$

can be solved if and only if

$$\int_Q \overline{f(x)} \psi(x) dx = 0 \quad \text{for all } \psi \in \mathcal{N}^*$$

is satisfied.

Proof: The Hilbert-Schmidt operator

$$\mathbb{K}f(x) := \int_Q K(x, y) f(y) dy, \quad f \in L^2(Q)$$

has the finite square-norm

$$N(\mathbb{K}) = \sqrt{\int_{Q \times Q} |K(x, y)|^2 dx dy} < +\infty.$$

Due to Theorem 6.13, the operator is completely continuous with the adjoint operator

$$\mathbb{K}^* f(x) := \int_Q K^*(x, y) f(y) dy, \quad f \in L^2(Q).$$

From Theorem 6.15 and the subsequent remark we infer the statements given. q.e.d.

In the bounded domain $G \subset \mathbb{R}^n$ we consider the weakly singular kernels

$$K = K(x, y) \in \mathcal{S}_\alpha(G, \mathbb{C})$$

from Section 1, Definition 1.8 with $\alpha \in [0, n)$ and their associate integral operators

$$\begin{aligned} \mathbb{K}f(x) &:= \int_G K(x, y) f(y) dy, \quad x \in G, \\ \text{for } f &\in \mathcal{D} := C^0(G, \mathbb{C}) \cap L^\infty(G, \mathbb{C}). \end{aligned} \tag{21}$$

Proposition 6.17. *Let the kernel $K = K(x, y) \in \mathcal{S}_\alpha(G, \mathbb{C})$ with the properties*

$$\begin{aligned} \int_G |K(x, y)| dy &\leq M, \quad x \in G, \\ \int_G |K(x, y)| dx &\leq N, \quad y \in G, \end{aligned} \tag{22}$$

be given. Then the operator $\mathbb{K} : \mathcal{H} \rightarrow \mathcal{H}$ can be extended from $\mathcal{D}$ onto the Hilbert space $\mathcal{H} = L^2(G, \mathbb{C})$, and the following estimate holds true:

$$\|\mathbb{K}\| \leq \sqrt{MN}.$$

Proof: We estimate for arbitrary functions $f, g \in \mathcal{D}$ as follows:

$$\begin{aligned}
|(g, \mathbb{K}f)| &= \left| \int_G \overline{g(x)} \left(\int_G K(x, y) f(y) dy \right) dx \right| \\
&\leq \int_{G \otimes G} |K(x, y)| |g(x)| |f(y)| dx dy \\
&= \int_{G \otimes G} \left(|K(x, y)|^{\frac{1}{2}} |g(x)| \right) \left(|K(x, y)|^{\frac{1}{2}} |f(y)| \right) dx dy \\
&\leq \left(\int_{G \otimes G} |K(x, y)| |g(x)|^2 dx dy \right)^{\frac{1}{2}} \left(\int_{G \otimes G} |K(x, y)| |f(y)|^2 dx dy \right)^{\frac{1}{2}} \\
&= \left(\int_G |g(x)|^2 \left(\int_G |K(x, y)| dy \right) dx \right)^{\frac{1}{2}} \left(\int_G |f(y)|^2 \left(\int_G |K(x, y)| dx \right) dy \right)^{\frac{1}{2}} \\
&\leq \sqrt{MN} \sqrt{\int_G |g(x)|^2 dx} \sqrt{\int_G |f(y)|^2 dy} = \sqrt{MN} \|g\| \|f\|.
\end{aligned}$$

Consequently, the operator $\mathbb{K} : \mathcal{H} \rightarrow \mathcal{H}$ is defined with $\|\mathbb{K}\| \leq \sqrt{MN}$. q.e.d.

We define the auxiliary function

$$\Theta(t) := \begin{cases} 0, & 0 \leq t \leq 1 \\ t - 1, & 1 \leq t \leq 2 \\ 1, & 2 \leq t \end{cases}.$$

With $K = K(x, y) \in \mathcal{S}_\alpha(G, \mathbb{C})$ and $\delta \in (0, \delta_0)$, let us consider the continuous kernels

$$K_\delta(x, y) := K(x, y) \Theta\left(\frac{|x - y|}{\delta}\right), \quad (x, y) \in G \times G, \quad (23)$$

together with their associate integral operators $\mathbb{K}_\delta$. For all $\delta \in (0, \delta_0)$ the operator $\mathbb{K}_\delta : \mathcal{H} \rightarrow \mathcal{H}$ is completely continuous, due to Theorem 6.13. With the aid of Proposition 6.17, we easily deduce the limit relation

$$\|\mathbb{K}_\delta - \mathbb{K}\| \rightarrow 0 \quad (\delta \rightarrow 0). \quad (24)$$

The complete continuity of $\mathbb{K}$ is inferred from the following

Proposition 6.18. *On the Hilbert space $\mathcal{H}$ let the sequence of completely continuous operators $A_j : \mathcal{H} \rightarrow \mathcal{H}$ for $j = 1, 2, \dots$ be given, converging due to $\|A_j - A\| \rightarrow 0$ ($j \rightarrow \infty$) towards the bounded linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$. Then the limit operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is completely continuous.*

Proof: Taking the sequence $\{x_k\}_{k=1,2,\dots} \subset \mathcal{H}$ with $\|x_k\| \leq 1$ for all $k \in \mathbb{N}$, we can select a subsequence $\{x_k^{(1)}\}_k \subset \{x_k\}_k$ such that $\{A_1 x_k^{(1)}\}_{k=1,2,\dots} \subset \mathcal{H}$

converges. Furthermore, we have a subsequence $\{x_k^{(2)}\}_k \subset \{x_k^{(1)}\}_k$ such that $\{A_2 x_k^{(2)}\}_k \subset \mathcal{H}$ converges. In this way we successively select subsequences

$$\{x_k^{(1)}\} \supset \{x_k^{(2)}\} \supset \dots$$

and go over to the diagonal sequence $x'_k := x_k^{(k)}$ for $k = 1, 2, \dots$. We then show the sequence $\{Ax'_k\}_{k=1,2,\dots}$ to be convergent in $\mathcal{H}$ as well: At first, we estimate

$$\begin{aligned} \|Ax'_k - Ax'_l\| &\leq \|Ax'_k - A_j x'_k\| + \|A_j x'_k - A_j x'_l\| + \|A_j x'_l - Ax'_l\| \\ &\leq \|A - A_j\| \|x'_k\| + \|A_j x'_k - A_j x'_l\| + \|A_j - A\| \|x'_l\|. \end{aligned} \quad (25)$$

With a given quantity $\varepsilon > 0$, we now choose an index $j \in \mathbb{N}$ so large that $\|A - A_j\| \leq \varepsilon$ is correct. Furthermore, we chose an integer $N = N(\varepsilon) \in \mathbb{N}$ satisfying

$$\|A_j x'_k - A_j x'_l\| \leq \varepsilon \quad \text{for all } k, l \geq N.$$

From the relation (25) we obtain the inequality

$$\|Ax'_k - Ax'_l\| \leq 2\varepsilon + \varepsilon = 3\varepsilon \quad \text{for all } k, l \geq N(\varepsilon).$$

Therefore, the sequence $\{Ax'_k\}_{k=1,2,\dots}$ converges in $\mathcal{H}$. q.e.d.

Theorem 6.19. (Weakly singular integral equations)

On the bounded domain $G \subset \mathbb{R}^n$ let the weakly singular kernel $K = K(x, y)$ of the class $\mathcal{S}_\alpha(G, \mathbb{C})$ with $\alpha \in [0, n)$ and the integral operator $\mathbb{K}$ be given. Then the null-spaces

$$\begin{aligned} \mathcal{N} : \quad & \int_G K(x, y) \varphi(y) dy = \varphi(x), \quad x \in G; \quad \varphi \in \mathcal{D} \\ \mathcal{N}^* : \quad & \int_G \overline{K(x, y)} \psi(x) dx = \psi(y), \quad y \in G; \quad \psi \in \mathcal{D} \end{aligned}$$

satisfy the identity $\dim \mathcal{N} = \dim \mathcal{N}^ < +\infty$. The integral equation*

$$u(x) - \int_G K(x, y) u(y) dy = f(x), \quad x \in G; \quad u \in \mathcal{D}, \quad (26)$$

can be solved for the given right-hand side $f \in \mathcal{D}$ if and only if the following condition holds true:

$$\int_G \overline{\psi(x)} f(x) dx = 0 \quad \text{for all } \psi \in \mathcal{N}^*. \quad (27)$$

Proof: Because of Proposition 6.18 the integral operator $\mathbb{K} : \mathcal{H} \rightarrow \mathcal{H}$ is completely continuous, and Fredholm's Theorem 6.15 can be applied in the Hilbert space $\mathcal{H} = L^2(G, \mathbb{C})$. We still have to show that (26) is solvable in $\mathcal{D}$. Therefore, let $u \in \mathcal{H}$ be a solution of the integral equation

$$\mathbb{E}u - \mathbb{K}u = f \quad (28)$$

with the continuous right-hand side $f \in \mathcal{D}$. Because of the Theorem by I. Schur on iterated kernels (compare Section 2, Theorem 2.7) there exists an integer $k \in \mathbb{N}$, such that $\mathbb{K}^k = \mathbb{L}$ with a bounded kernel $L = L(x, y) \in \mathcal{S}_0(G, \mathbb{C})$ is correct. From Theorem 2.2 in Section 2 the property $\mathbb{L}u \in \mathcal{D}$ is satisfied. Via (28) we now obtain the following identity:

$$\mathbb{E}u - \mathbb{L}u = \mathbb{E}u - \mathbb{K}^k u = (\mathbb{E} + \mathbb{K} + \dots + \mathbb{K}^{k-1})f =: g. \quad (29)$$

Due to Section 2, Theorem 2.2 we have $\mathbb{E} + \mathbb{K} + \dots + \mathbb{K}^{k-1} : \mathcal{D} \rightarrow \mathcal{D}$ and consequently $g \in \mathcal{D}$ holds true. Finally, we obtain

$$u = g + \mathbb{L}u \in \mathcal{D}.$$

q.e.d.

7 Spectral Theory for Completely Continuous Hermitian Operators

At first, we consider the following

Example 7.1. On the Hilbert space $\mathcal{H} = L^2((0, 1), \mathbb{C})$ with the inner product

$$(f, g) = \int_0^1 \overline{f(x)}g(x) dx, \quad f, g \in \mathcal{H}$$

we define the linear operator

$$Af(x) := xf(x), \quad x \in (0, 1); \quad f = f(x) \in \mathcal{H}.$$

On account of

$$\|Af\|^2 = \int_0^1 x^2 \overline{f(x)}f(x) dx \leq \int_0^1 |f(x)|^2 dx = \|f\|^2,$$

the operator A is bounded by $\|A\| \leq 1$. Furthermore, the operator A is symmetric according to

$$(Af, g) = \int_0^1 x \overline{f(x)} g(x) dx = (f, Ag) \quad \text{for all } f, g \in \mathcal{H}.$$

We now claim that A does not possess eigenvalues. From the identity $Af = \lambda f$ we namely infer

$$(x - \lambda)f(x) = 0 \quad \text{a. e. in } (0, 1)$$

and consequently $f(x) = 0$ a. e. in $(0, 1)$ which implies $f = 0 \in \mathcal{H}$.

Theorem 7.2. *Let $A : \mathcal{H} \rightarrow \mathcal{H}$ denote a completely continuous Hermitian operator on the Hilbert space $\mathcal{H}$. Then we have an element $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$ and a number $\lambda \in \mathbb{R}$ with $|\lambda| = \|A\|$ such that*

$$A\varphi = \lambda\varphi.$$

Consequently, the numbers $\|A\|$ or $-\|A\|$ are eigenvalues of the operator A . Furthermore, we have the following estimate:

$$|(x, Ax)| \leq |\lambda|(x, x) \quad \text{for all } x \in \mathcal{H}. \quad (1)$$

Proof:

1. At first, we show

$$\|A\| = \sup_{x \in \mathcal{H}, \|x\|=1} |(Ax, x)|. \quad (2)$$

From the estimate

$$|(Ax, x)| \leq \|Ax\| \|x\| \leq \|A\| \|x\|^2 = \|A\|$$

for all $x \in \mathcal{H}$ with $\|x\| = 1$ we infer

$$\sup_{x \in \mathcal{H}, \|x\|=1} |(Ax, x)| \leq \|A\|.$$

In order to show the reverse inequality, we choose an arbitrary $\alpha \in [0, +\infty)$ satisfying

$$|(Ax, x)| \leq \alpha \|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

With arbitrary elements $f, g \in \mathcal{H}$ we calculate

$$(A(f+g), f+g) - (A(f-g), f-g) = 2\{(Af, g) + (Ag, f)\} = 4 \operatorname{Re}(Af, g)$$

and consequently

$$\begin{aligned} 4|\operatorname{Re}(Af, g)| &\leq |(A(f+g), f+g)| + |(A(f-g), f-g)| \\ &\leq \alpha\{\|f+g\|^2 + \|f-g\|^2\} = 2\alpha\{\|f\|^2 + \|g\|^2\}. \end{aligned}$$

We now replace

$$f = \sqrt{\frac{\|y\|}{\|x\|}} x, \quad g = e^{i\varphi} \sqrt{\frac{\|x\|}{\|y\|}} y$$

with a suitable angle $\varphi \in [0, 2\pi)$, such that the inequality

$$4|(Ax, y)| \leq 2\alpha \left\{ \frac{\|y\|}{\|x\|} \|x\|^2 + \frac{\|x\|}{\|y\|} \|y\|^2 \right\} = 4\alpha \|x\| \|y\|$$

follows and equivalently

$$|(Ax, y)| \leq \alpha \|x\| \|y\| \quad \text{for all } x, y \in \mathcal{H}.$$

Inserting the element $y = Ax$, we obtain

$$\|Ax\|^2 \leq \alpha \|x\| \|Ax\| \quad \text{or equivalently} \quad \|Ax\| \leq \alpha \|x\|$$

for all $x \in \mathcal{H}$, and therefore $\|A\| \leq \alpha$. Finally, we see

$$\sup_{x \in \mathcal{H}, \|x\|=1} |(Ax, x)| = \inf \left\{ \alpha \in [0, +\infty) : \begin{array}{l} |(Ax, x)| \leq \alpha \|x\|^2 \\ \text{for all } x \in \mathcal{H} \end{array} \right\} \geq \|A\|.$$

2. We now consider the variational problem

$$\|A\| = \sup_{x \in \mathcal{H} \setminus \{0\}} \frac{|(Ax, x)|}{\|x\|^2} = \sup_{x \in \mathcal{H}, \|x\|=1} |(Ax, x)|, \quad (3)$$

and without loss of generality we assume $A \neq 0$. Let $\{x_n\}_{n=1,2,\dots} \subset \mathcal{H}$ denote a sequence with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ satisfying

$$|(Ax_n, x_n)| \rightarrow \|A\| \quad (n \rightarrow \infty).$$

Then we have a subsequence $\{x'_n\}_{n=1,2,\dots} \subset \{x_n\}_{n=1,2,\dots}$ and an element $x \in \mathcal{H}$ with $\|x\| \leq 1$, such that $x'_n \rightharpoonup x$ ($n \rightarrow \infty$) and

$$(Ax'_n, x'_n) \rightarrow \lambda \in \{-\|A\|, \|A\|\}$$

hold true. Since the bilinear form $(y, z) \mapsto (Ay, z)$ is weakly continuous, we infer

$$0 \neq \lambda = \lim_{n \rightarrow \infty} (Ax'_n, x'_n) = (Ax, x)$$

and therefore $x \neq 0$. Now the condition $\|x\| = 1$ holds true: If $\|x\| < 1$ were correct, we would obtain

$$\frac{|(Ax, x)|}{\|x\|^2} > \frac{|\lambda|}{1} = \|A\|$$

contradicting (3).

3. Without loss of generality we now assume $\lambda = +\|A\|$; and the element $x \in \mathcal{H}$ satisfying $\|x\| = 1$ may solve the variational problem (3) from part 2 of our proof. We therefore have

$$(Ax, x) = \lambda\|x\|^2.$$

Taking an arbitrary element $y \in \mathcal{H}$, there exists a quantity $\varepsilon_0 = \varepsilon_0(y) > 0$ such that all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ fulfill

$$(A(x + \varepsilon y), x + \varepsilon y) \leq \lambda(x + \varepsilon y, x + \varepsilon y)$$

and consequently

$$(Ax, x) + \varepsilon\{(Ax, y) + (Ay, x)\} \leq \lambda\|x\|^2 + \varepsilon\lambda\{(x, y) + (y, x)\} + o(\varepsilon).$$

This implies

$$\varepsilon \operatorname{Re}(Ax - \lambda x, y) \leq o(\varepsilon),$$

and consequently $\operatorname{Re}(Ax - \lambda x, y) \leq o(1)$ for all $y \in \mathcal{H}$. Therefore, the relation

$$\operatorname{Re}(Ax - \lambda x, y) = 0 \quad \text{for all } y \in \mathcal{H}$$

has to be fulfilled and especially

$$Ax = \lambda x.$$

q.e.d.

Theorem 7.3. (Spectral theorem of F. Rellich)

Let the completely continuous Hermitian operator $A : \mathcal{H} \rightarrow \mathcal{H}$ be given on the Hilbert space $\mathcal{H}$ satisfying $A \neq 0$. Then we have a finite or countably infinite system of orthonormal elements $\{\varphi_i\}_{i=1,2,\dots}$ in $\mathcal{H}$ such that

- a) The elements φ_i are eigenfunctions to the eigenvalues $\lambda_i \in \mathbb{R}$ ordered as follows:

$$\|A\| = |\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \dots > 0,$$

more precisely

$$A\varphi_i = \lambda_i\varphi_i, \quad i = 1, 2, \dots$$

If the set $\{\varphi_i\}_i$ is infinite, we have the asymptotic behavior

$$\lim_{i \rightarrow \infty} \lambda_i = 0.$$

- b) For all $x \in \mathcal{H}$ we have the representations

$$Ax = \sum_{i=1,2,\dots} \lambda_i(\varphi_i, x)\varphi_i \quad \text{and} \quad (x, Ax) = \sum_{i=1,2,\dots} \lambda_i|(\varphi_i, x)|^2.$$

Remark: This theorem remains true for not separable Hilbert spaces. If the system $\{\varphi_i\}_{i=1,\dots,N}$ is finite, the series above reduce to sums.

Proof of Theorem 7.3: On account of $\|A\| > 0$, Theorem 7.2 yields the existence of an element $\varphi_1 \in \mathcal{H}$ with $\|\varphi_1\| = 1$ satisfying

$$A\varphi_1 = \lambda_1\varphi_1, \quad \lambda_1 \in \{-\|A\|, +\|A\|\}.$$

Furthermore, we have

$$|(Ax, x)| \leq |\lambda_1|(x, x) \quad \text{for all } x \in \mathcal{H}.$$

We now assume that we have already found $m \geq 1$ orthonormal eigenelements $\varphi_1, \dots, \varphi_m$ with the associate eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ satisfying the property a). Then we consider the completely continuous Hermitian operator

$$B_mx = Ax - \sum_{i=1}^m \lambda_i(\varphi_i, x)\varphi_i.$$

Case 1: We have $B_m = 0$. Then the following representation holds true:

$$Ax = \sum_{i=1}^m \lambda_i(\varphi_i, x)\varphi_i.$$

Case 2: We have $B_m \neq 0$. Due to Theorem 7.2 we have an element $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$, such that $B_m\varphi = \lambda\varphi$ and consequently

$$A\varphi - \sum_{i=1}^m \lambda_i(\varphi_i, \varphi)\varphi_i = \lambda\varphi$$

is satisfied with $|\lambda| = \|B_m\| > 0$. Multiplication by φ_k , with an index $k \in \{1, \dots, m\}$, from the left yields

$$\begin{aligned} \lambda(\varphi_k, \varphi) &= (\varphi_k, A\varphi) - \lambda_k(\varphi_k, \varphi) = (A\varphi_k, \varphi) - \lambda_k(\varphi_k, \varphi) \\ &= \lambda_k(\varphi_k, \varphi) - \lambda_k(\varphi_k, \varphi) = 0, \quad k = 1, \dots, m. \end{aligned}$$

Therefore, the system $\{\varphi_1, \dots, \varphi_m, \varphi\}$ is orthonormal as well; we set $\varphi_{m+1} := \varphi$ and $\lambda_{m+1} := \lambda \neq 0$. Now we deduce $|\lambda_{m+1}| \leq |\lambda_m|$:

By construction the following estimate

$$|(x, B_mx)| \leq |\lambda_m|(x, x) \quad \text{for all } x \in \mathcal{H}$$

holds true, and for $x = \varphi_{m+1}$ we obtain

$$|\lambda_m| \geq |(\varphi_{m+1}, B_m\varphi_{m+1})| = |(\varphi_{m+1}, \lambda_{m+1}\varphi_{m+1})| = |\lambda_{m+1}|.$$

We now assume that the procedure above does not end. Since the elements $\{\varphi_i\}_i$ are orthonormal, we infer $\varphi_i \rightarrow 0$ ($i \rightarrow \infty$) and the complete continuity of the operator A yields

$$|\lambda_i| = \|A\varphi_i\| \rightarrow 0 \quad (i \rightarrow \infty).$$

On account of $\|B_m\| = |\lambda_{m+1}|$ we obtain the statement

$$\left\| A - \sum_{i=1}^m \lambda_i(\varphi_i, \cdot)\varphi_i \right\| = |\lambda_{m+1}| \rightarrow 0 \quad (m \rightarrow \infty) \quad (4)$$

and consequently

$$Ax = \sum_{i=1}^{\infty} \lambda_i(\varphi_i, x)\varphi_i, \quad x \in \mathcal{H}.$$

Therefore, all $y = Ax$ with $x \in \mathcal{H}$ can be represented in the form

$$y = \sum_{i=1}^{\infty} (\varphi_i, y)\varphi_i,$$

and the system $\{\varphi_i\}_{i=1,2,\dots}$ is complete in $\overline{\mathcal{W}_A} = \overline{A(\mathcal{H})}$. q.e.d.

Theorem 7.4. *The Hermitian operator $A : \mathcal{H} \rightarrow \mathcal{H}$ with finite square-norm $N(A) < +\infty$ satisfying $A \neq 0$ is defined on the separable Hilbert space $\mathcal{H}$. The operator A may possess a countably infinite system of orthonormal eigenelements $\{\varphi_i\}_{i=1,2,\dots}$ and associate eigenvalues $\{\lambda_i\}_{i=1,2,\dots}$ with the properties a) and b) from Theorem 7.3. We set*

$$A_n f := Af - \sum_{i=1}^n \lambda_i(\varphi_i, f)\varphi_i, \quad n = 1, 2, \dots$$

Then the sequence of square-norms

$$N(A_n)^2 = \sum_{i=n+1}^{\infty} \lambda_i^2, \quad n = 1, 2, \dots$$

is a zero sequence.

Proof: Noting that $N(A) < +\infty$, the operator $A : \mathcal{H} \rightarrow \mathcal{H}$ is completely continuous and Theorem 7.3 gives us the representation

$$y = \sum_{i=1}^{\infty} (\varphi_i, y)\varphi_i \quad \text{for all } y \in \mathcal{W}_A.$$

We observe the decomposition $\mathcal{H} = \overline{\mathcal{W}_A} \oplus \mathcal{N}_A$. The relation $y \in \mathcal{N}_A$ or equivalently $Ay = 0$ holds true if and only if

$$0 = (Ay, x) = (y, Ax) \quad \text{for all } x \in \mathcal{H}$$

is satisfied, which means $\mathcal{N}_A = \mathcal{W}_A^\perp$.

Let now $\{\psi_i\}_{i=1,2,\dots}$ represent a c.o.n.s. in $\mathcal{N}_A$. Then the set $\{\varphi_i\}_i \cup \{\psi_i\}_i$ constitutes a c.o.n.s. in $\mathcal{H}$. This allows us to evaluate

$$N(A)^2 = \sum_{i=1}^{\infty} \|A\varphi_i\|^2 + \sum_{i=1}^{\infty} \|A\psi_i\|^2 = \sum_{i=1}^{\infty} \lambda_i^2 < +\infty$$

and finally

$$N(A_n)^2 = \sum_{i=1}^{\infty} \|A_n\varphi_i\|^2 + \sum_{i=1}^{\infty} \|A_n\psi_i\|^2 = \sum_{i=n+1}^{\infty} \lambda_i^2 \rightarrow 0 \quad (n \rightarrow \infty). \quad \text{q.e.d.}$$

We specialize Theorem 7.3 and Theorem 7.4 to Hilbert-Schmidt operators and obtain the following

Theorem 7.5. (Spectral theorem of D. Hilbert and E. Schmidt)

On the rectangle $Q \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ let the integral kernel $K = K(x, y) : Q \times Q \rightarrow \mathbb{C} \in L^2(Q \times Q)$ be given satisfying

$$\int_{Q \times Q} |K(x, y)|^2 dx dy > 0$$

and

$$\overline{K(y, x)} = K(x, y) \quad \text{for almost all } (x, y) \in Q \times Q. \quad (5)$$

Then we have a finite or countably infinite system of eigenfunctions

$$\{\varphi_i(x)\}_{i=1,2,\dots} \subset L^2(Q, \mathbb{C})$$

with the associate eigenvalues $\{\lambda_i\}_{i=1,2,\dots} \subset \mathbb{R}$, such that the following integral-eigenvalue-equation

$$\int_Q K(x, y) \varphi_i(y) dy = \lambda_i \varphi_i(x) \quad \text{for almost all } x \in Q \quad (6)$$

is satisfied with $i = 1, 2, \dots$. The eigenvalues have the properties

$$|\lambda_1| \geq |\lambda_2| \geq \dots > 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \lambda_i = 0. \quad (7)$$

Furthermore, we have the eigenvalue expansions

$$\int_{Q \times Q} |K(x, y)|^2 dx dy = \sum_{i=1}^{\infty} \lambda_i^2 < +\infty \quad (8)$$

and

$$\int_{Q \times Q} \left| K(x, y) - \sum_{i=1}^n \lambda_i \varphi_i(x) \overline{\varphi_i(y)} \right|^2 dx dy = \sum_{i=n+1}^{\infty} \lambda_i^2 \rightarrow 0 \quad (n \rightarrow \infty). \quad (9)$$

8 The Sturm-Liouville Eigenvalue Problem

We need the following result:

Theorem 8.1. (Eigenvalue problem for weakly singular integral operators)

Let the weakly singular kernel $K = K(x, y) \in \mathcal{S}_\alpha(G, \mathbb{C})$ with $\alpha \in [0, n)$ be given on the bounded domain $G \subset \mathbb{R}^n$, and we have $K(x, y) \not\equiv 0$ and

$$K(x, y) = \overline{K(y, x)} \quad \text{for all } (x, y) \in G \otimes G.$$

We denote the associate integral operator by $\mathbb{K}$, and define as our domain of definition

$$\mathcal{D} := \left\{ f \in C^0(G, \mathbb{C}) : \sup_{x \in G} |f(x)| < +\infty \right\}.$$

Statements: Then we have a finite or countably infinite orthonormal system of eigenfunctions $\{\varphi_i\}_{i \in I} \subset \mathcal{D}$ with their eigenvalues $\lambda_i \in \mathbb{R} \setminus \{0\}$ for $i \in I$ satisfying

$$\int_G K(x, y) \varphi_i(y) dy = \lambda_i \varphi_i(x), \quad x \in G, \quad i \in I. \quad (1)$$

If $I = \{1, 2, \dots\}$ is countably infinite, we have

$$\lim_{i \rightarrow \infty} \lambda_i = 0, \quad (2)$$

and each function $g = \mathbb{K}f$ with $f \in \mathcal{D}$ can be approximated in the square-means according to

$$\lim_{n \rightarrow \infty} \int_G \left| g(x) - \sum_{i=1}^n g_i \varphi_i(x) \right|^2 dx = 0. \quad (3)$$

Here we have set

$$g_i := \int_G \overline{\varphi_i(x)} g(x) dx, \quad i \in I \quad (4)$$

for the Fourier coefficients. When we assume $\alpha \in [0, \frac{n}{2})$ in addition, the function $g = \mathbb{K}f$ with $f \in \mathcal{D}$ can be expanded into the following uniformly convergent series:

$$g(x) = \sum_{i=1}^{\infty} g_i \varphi_i(x), \quad x \in G. \quad (5)$$

Proof: As it has been elaborated in the proof of Theorem 6.19 from Section 6, the operator $\mathbb{K} : \mathcal{H} \rightarrow \mathcal{H}$ is completely continuous on the Hilbert space $\mathcal{H} = L^2(G, \mathbb{C})$. Furthermore, we have the regularity result

$$\mathbb{E}u - \mathbb{K}u = v \quad \text{with } u \in \mathcal{H} \text{ and } v \in \mathcal{D} \quad \Rightarrow \quad u \in \mathcal{D}. \quad (6)$$

Due to Rellich's spectral theorem from Section 7, Theorem 7.3 the operator $\mathbb{K}$ possesses a finite or countably infinite system of orthonormal eigenfunctions $\{\varphi_i\}_{i=1,2,\dots} \subset \mathcal{H}$. Then we have the identities

$$\int_G K(x, y) \varphi_i(y) dy = \lambda_i \varphi_i(x), \quad x \in G$$

for all $i = 1, 2, \dots$ with $|\lambda_1| \geq |\lambda_2| \geq \dots > 0$ and $\lambda_i \rightarrow 0$ ($i \rightarrow \infty$), in the case that infinitely many eigenfunctions exist. Because of the regularity result (6) we see $\varphi_i \in \mathcal{D}$ for $i = 1, 2, \dots$. Furthermore, the function $g = \mathbb{K}f$ with an arbitrary element $f \in \mathcal{D}$ satisfies the following identity in the Hilbert space:

$$\begin{aligned} g = \mathbb{K}f &= \sum_{i=1,2,\dots} \lambda_i (\varphi_i, f) \varphi_i = \sum_{i=1,2,\dots} (\mathbb{K}\varphi_i, f) \varphi_i \\ &= \sum_{i=1,2,\dots} (\varphi_i, \mathbb{K}f) \varphi_i = \sum_{i=1,2,\dots} (\varphi_i, g) \varphi_i \end{aligned}$$

or equivalently (3), where the Fourier coefficients g_i are defined in (4). When we assume $\alpha \in [0, \frac{n}{2})$ in addition, the linear operator $\mathbb{K} : L^2(G) \rightarrow C^0(G)$ is bounded by

$$\|\mathbb{K}f\|_{C^0(G)} \leq C \|f\|_{L^2(G)} \quad \text{for all } f \in \mathcal{D}$$

due to Theorem 2.2 from Section 2. The series $\sum_{i=1,2,\dots} (\varphi_i, f) \varphi_i$, convergent in the Hilbert space $\mathcal{H} = L^2(G, \mathbb{C})$, is consequently transferred by the operator $\mathbb{K}$ into the uniformly convergent series

$$\sum_{i=1,2,\dots} \lambda_i (\varphi_i, f) \varphi_i = \mathbb{K} \left(\sum_{i=1,2,\dots} (\varphi_i, f) \varphi_i \right) = g.$$

q.e.d.

Theorem 8.2. (Expansion theorem for kernels)

Let $K = K(x, y) : G \times G \rightarrow \mathbb{C}$ denote a Hermitian integral kernel of the class $\mathcal{S}_0(G, \mathbb{C})$, which is continuous on $G \times G$. For the associate integral operator $\mathbb{K}$ we assume

$$(f, \mathbb{K}f) \geq 0 \quad \text{for all } f \in \mathcal{D}. \quad (7)$$

Then we have a representation by the uniformly convergent series in each compact set $\Gamma \subset G$ as follows:

$$K(x, y) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x) \overline{\varphi_i(y)}, \quad (x, y) \in \Gamma \times \Gamma. \quad (8)$$

Proof:

1. We show firstly that

$$K(x, x) \geq 0 \quad \text{for all } x \in G \quad (9)$$

is fulfilled. Here we utilize the function $\varphi = \varphi(y) : \mathbb{R}^n \rightarrow [0, +\infty) \in C^0(\mathbb{R}^n)$ satisfying

$$\varphi(y) = 0, \quad |y| \geq 1, \quad \text{and} \quad \int_{\mathbb{R}^n} \varphi(y) dy = 1.$$

The quantity $\delta > 0$ given arbitrarily, we consider the approximate point distributions about $x \in G$, namely

$$f_\delta(y) := \frac{1}{\delta^n} \varphi\left(\frac{1}{\delta}(y - x)\right), \quad y \in \mathbb{R}^n.$$

We insert the function $f_\delta \in \mathcal{D}$ into (7) and obtain

$$\begin{aligned} 0 \leq (f_\delta, \mathbb{K} f_\delta) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_\delta(y) K(y, z) f_\delta(z) dy dz \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, x) f_\delta(y) f_\delta(z) dy dz \\ &\quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (K(y, z) - K(x, x)) f_\delta(y) f_\delta(z) dy dz \\ &= K(x, x) + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (K(y, z) - K(x, x)) f_\delta(y) f_\delta(z) dy dz. \end{aligned}$$

Since the second term on the right-hand side vanishes for $\delta \rightarrow 0$, we infer the inequality (9).

2. We now show

$$0 \leq \sum_{i=1}^{\infty} \lambda_i |\varphi_i(x)|^2 \leq K(x, x) < +\infty \quad \text{for all } x \in G. \quad (10)$$

Let us define the integral kernel

$$K_N(x, y) := K(x, y) - \sum_{i=1}^N \lambda_i \varphi_i(x) \overline{\varphi_i(y)}$$

with the associate integral operator $\mathbb{K}_N$. The latter satisfies

$$(f, \mathbb{K}_N f) = \sum_{i=N+1}^{\infty} \lambda_i |\langle \varphi_i, f \rangle|^2 \geq 0 \quad \text{for all } f \in \mathcal{D}.$$

From part 1 of our proof we obtain

$$K_N(x, x) \geq 0 \quad \text{for all } x \in G$$

or equivalently

$$K(x, x) \geq \sum_{i=1}^N \lambda_i \varphi_i(x) \overline{\varphi_i(x)}, \quad x \in G$$

for all $N \in \mathbb{N}$, which implies (10).

3. Let the point $x \in G$ be chosen as fixed. With an arbitrary quantity $\varepsilon > 0$, we then can estimate

$$\begin{aligned} \sum_{i=N+1}^{\infty} |\lambda_i \varphi_i(x) \overline{\varphi_i(y)}| &\leq \sqrt{\sum_{i=N+1}^{\infty} \lambda_i |\varphi_i(x)|^2} \sqrt{\sum_{i=N+1}^{\infty} \lambda_i |\varphi_i(y)|^2} \\ &\leq \varepsilon \sqrt{K(y, y)} \leq \varepsilon \cdot \text{const}, \quad y \in G, \end{aligned}$$

for all $N \geq N_0(\varepsilon)$. Therefore, we have the following statement for each fixed $x \in G$:

$$\text{The series } \Phi(y) := \sum_{i=1}^{\infty} \lambda_i \varphi_i(x) \overline{\varphi_i(y)} \text{ converges uniformly in } G. \quad (11)$$

4. Due to Theorem 7.5 from Section 7, we have the relation

$$K(x, y) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x) \overline{\varphi_i(y)} \quad (12)$$

in the $L^2(G \times G, \mathbb{C})$ -sense. Choosing the point $x \in G$ and the function $f \in C_0^0(G)$ arbitrarily, we obtain the following identity via (11) and (12):

$$\begin{aligned} \int_G K(x, y) f(y) dy &= \lim_{N \rightarrow \infty} \int_G \left(\sum_{i=1}^N \lambda_i \varphi_i(x) \overline{\varphi_i(y)} \right) f(y) dy \\ &= \int_G \left(\sum_{i=1}^{\infty} \lambda_i \varphi_i(x) \overline{\varphi_i(y)} \right) f(y) dy. \end{aligned}$$

This implies

$$\int_G \left(K(x, y) - \sum_{i=1}^{\infty} \lambda_i \varphi_i(x) \overline{\varphi_i(y)} \right) f(y) dy = 0 \quad \text{for all } f \in C_0^0(G).$$

Since $K \in C^0(G \times G)$ holds true, and the series in the integrand is continuous with respect to $y \in G$, we deduce the pointwise identity

$$K(x, y) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x) \overline{\varphi_i(y)} \quad \text{for all } x, y \in G. \quad (13)$$

5. Especially for $x = y$, we infer the following identity from (13):

$$K(x, x) = \sum_{i=1}^{\infty} \lambda_i |\varphi_i(x)|^2, \quad x \in G.$$

Because of Dini's theorem, the series converges uniformly on each compact set $\Gamma \subset G$. Finally, we obtain the following inequality for arbitrary $\varepsilon > 0$ and suitable $N \geq N_0(\varepsilon)$, namely

$$\left| \sum_{i=N+1}^{\infty} \lambda_i \varphi_i(x) \overline{\varphi_i(y)} \right| \leq \sqrt{\sum_{i=N+1}^{\infty} \lambda_i |\varphi_i(x)|^2} \sqrt{\sum_{i=N+1}^{\infty} \lambda_i |\varphi_i(y)|^2} \leq \varepsilon^2$$

for all $(x, y) \in \Gamma \times \Gamma$.

q.e.d.

Theorem 8.3. (The Sturm-Liouville eigenvalue problem)

We prescribe $a, b \in \mathbb{R}$ with $a < b$ and the coefficient functions

$$p = p(x) \in C^1([a, b], (0, +\infty)), \quad q = q(x) \in C^0([a, b], \mathbb{R}),$$

and consider the Sturm-Liouville operator $\mathbb{L} : C^2([a, b], \mathbb{C}) \rightarrow C^0([a, b], \mathbb{C})$ defined by

$$\mathbb{L}u(x) := -(p(x)u'(x))' + q(x)u(x), \quad x \in [a, b].$$

Furthermore, we use the real boundary operators $\mathbb{B}_j : C^2([a, b], \mathbb{C}) \rightarrow \mathbb{C}$ for $j = 1, 2$ defined by

$$\mathbb{B}_1 u := c_1 u(a) + c_2 u'(a) \quad \text{with} \quad c_1^2 + c_2^2 > 0$$

and

$$\mathbb{B}_2 u := d_1 u(b) + d_2 u'(b) \quad \text{with} \quad d_1^2 + d_2^2 > 0.$$

Finally, we fix the domain of definition

$$\mathcal{D} := \left\{ u \in C^2([a, b], \mathbb{C}) : \mathbb{B}_1 u = 0 = \mathbb{B}_2 u \right\}.$$

Statements: Then we have a sequence $\{\lambda_i\}_{i=1,2,\dots} \subset \mathbb{R}$ of eigenvalues satisfying

$$-\infty < \lambda_1 \leq \lambda_2 \leq \dots \quad \text{and} \quad \lim_{i \rightarrow \infty} \lambda_i = +\infty$$

and the associate eigenfunctions $\{\varphi_i\}_{i=1,2,\dots} \subset \mathcal{D}$ with the following properties:

a) We have

$$\mathbb{L}\varphi_i = \lambda_i \varphi_i, \quad \int_a^b \overline{\varphi_i(x)} \varphi_j(x) dx = \delta_{ij} \quad \text{for all } i, j \in \mathbb{N},$$

and the identity

$$\sum_{i=1}^{\infty} \left| \int_a^b \overline{\varphi_i(x)} f(x) dx \right|^2 = \int_a^b |f(x)|^2 dx \quad \text{for all } f \in \mathcal{D}$$

is satisfied.

- b) Each function $g \in \mathcal{D}$ can be expanded into the uniformly convergent series on the interval $[a, b]$ as follows:

$$g(x) = \sum_{i=1}^{\infty} g_i \varphi_i(x), \quad x \in [a, b], \quad \text{with } g_i := \int_a^b \overline{\varphi_i(x)} g(x) dx, \quad i \in \mathbb{N}.$$

- c) If the property $\lambda_i \neq 0$ for all $i \in \mathbb{N}$ is satisfied, the following series

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} \varphi_i(x) \overline{\varphi_i(y)}, \quad (x, y) \in [a, b] \times [a, b]$$

converges uniformly towards the Green function K of $\mathbb{L}$ under the boundary conditions $\mathbb{B}_1 = 0 = \mathbb{B}_2$.

Proof: We continue our considerations from Section 1 concerning the Sturm-Liouville eigenvalue problem.

1. All eigenvalues of $\mathbb{L}$ are real. Since the coefficient functions p and q are real, we obtain the following statement from Proposition 1.3 in Section 1 by separation into the real and imaginary parts:

$$\int_a^b \overline{\mathbb{L}u(x)} v(x) dx = \int_a^b u(x) \overline{\mathbb{L}v(x)} dx \quad \text{for all } u, v \in \mathcal{D}. \quad (14)$$

We calculate

$$\begin{aligned} \overline{\lambda_i} \int_a^b |\varphi_i(x)|^2 dx &= \int_a^b \overline{\lambda_i \varphi_i(x)} \varphi_i(x) dx = \int_a^b \overline{\mathbb{L} \varphi_i} \varphi_i dx \\ &= \int_a^b \overline{\varphi_i} \mathbb{L} \varphi_i dx = \int_a^b \overline{\varphi_i} \lambda_i \varphi_i dx \\ &= \lambda_i \int_a^b |\varphi_i(x)|^2 dx, \quad i=1, 2, \dots \end{aligned}$$

This implies

$$\lambda_i = \overline{\lambda_i} \quad \text{for all } i \in \mathbb{N}.$$

2. We now prove that the sequence of eigenvalues is bounded from below. Here we consider the class of admissible functions

$$\mathcal{D}_0 := \left\{ u = u(x) \in C^2([a, b], \mathbb{C}) : u(a) = 0 = u(b) \right\}.$$

If $u \in \mathcal{D}_0$ is a solution of $\mathbb{L}u = \lambda u$, we infer

$$\lambda \geq q_* := \inf_{a \leq x \leq b} q(x). \quad (15)$$

With the aid of partial integration, we evaluate

$$\begin{aligned} \lambda \int_a^b |u(x)|^2 dx &= \lambda \int_a^b \overline{u(x)} u(x) dx = \int_a^b \overline{\mathbb{L}u(x)} u(x) dx \\ &= \int_a^b \left\{ p(x) |u'(x)|^2 + q(x) |u(x)|^2 \right\} dx \\ &\geq q_* \int_a^b |u(x)|^2 dx. \end{aligned}$$

We now show indirectly that the operator $\mathbb{L}$ on $\mathcal{D}$ possesses at most two eigenvalues smaller than q_* . On the contrary, we assume that we had three eigenfunctions $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{D}$ satisfying

$$\mathbb{L}\varphi_i = \lambda_i \varphi_i, \quad i = 1, 2, 3 \quad \text{and} \quad \lambda_1 \leq \lambda_2 \leq \lambda_3 < q_*.$$

Then we can find numbers $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ with $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = 1$ such that

$$v := \sum_{i=1}^3 \alpha_i \varphi_i \in \mathcal{D}_0$$

is correct. On account of (15) we see

$$\begin{aligned} q_* \int_a^b |v(x)|^2 dx &\leq \int_a^b \overline{\mathbb{L}v(x)} v(x) dx = \int_a^b \overline{\left(\sum_{i=1}^3 \lambda_i \alpha_i \varphi_i \right)} \left(\sum_{j=1}^3 \alpha_j \varphi_j \right) dx \\ &= \int_a^b \sum_{i=1}^3 \lambda_i |\alpha_i|^2 |\varphi_i(x)|^2 dx \leq \lambda_3 \int_a^b \sum_{i=1}^3 |\alpha_i|^2 |\varphi_i(x)|^2 dx \\ &= \lambda_3 \int_a^b |v(x)|^2 dx. \end{aligned}$$

We obtain $\lambda_3 \geq q_*$ in contradiction to $\lambda_3 < q_*$.

3. We name $\lambda_1 \in \mathbb{R}$ the least eigenvalue of $\mathbb{L}$ on $\mathcal{D}$, existing due to part 2 of our proof. Then we obtain in

$$\tilde{\mathbb{L}} := \mathbb{L} - \lambda_1 \mathbb{E} + \mathbb{E}$$

a Sturm-Liouville operator with the eigenvalues $\tilde{\lambda}_k \geq 1$, $k = 1, 2, \dots$

Due to Theorem 1.6 in Section 1, the operator $\mathbb{L}$ on $\mathcal{D}$ possesses a symmetric Green's function

$$K = K(x, y), (x, y) \in [a, b] \times [a, b] \quad \text{of the class} \quad C^0([a, b] \times [a, b], \mathbb{R}).$$

We now take Theorem 1.7 from Section 1 into account and utilize Theorem 8.1 for the given integral equation. Then we obtain a sequence of eigenfunctions

$$\{\varphi_i\}_{i=1,2,\dots} \subset \mathcal{D} \text{ satisfying } \mathbb{L}\varphi_i = \lambda_i \varphi_i \ (i \in \mathbb{N}) \text{ and } \lambda_1 \leq \lambda_2 \leq \dots \rightarrow +\infty.$$

From Theorem 8.1 and Theorem 8.2 we immediately infer all the statements above. q.e.d.

9 Weyl's Eigenvalue Problem for the Laplace Operator

We need the following generalization of the Gaussian integral theorem, which does not require regularity assumptions for the boundary of the domain with vanishing boundary values:

Proposition 9.1. (Giesecke, E. Heinz)

- I. Let the bounded domain $G \subset \mathbb{R}^n$ be given, in which $N \in \mathbb{N}_0$ mutually disjoint balls

$$K_j := \left\{ x \in \mathbb{R}^n : |x - x^{(j)}| \leq r_j \right\}, \quad j = 1, 2, \dots, N$$

with their radii $r_j > 0$ and their centers $x^{(j)}$ are contained. We set

$$G' := G \setminus \{x^{(1)}, \dots, x^{(N)}\} \quad \text{and} \quad G'' := G \setminus \bigcup_{j=1}^N K_j.$$

The topological closure of the set G'' is denoted by $\overline{G''}$.

- II. For the two functions $u, v \in C^2(G') \cap C^0(\overline{G''})$ we assume

$$u|_{\partial G} = 0 = v|_{\partial G}; \quad \int_{G''} \left\{ |\Delta u(x)| + |\Delta v(x)| \right\} dx < +\infty.$$

Statement: *Then the following identity*

$$\int_{G''} (v \Delta u + \nabla v \cdot \nabla u) dx = - \sum_{j=1}^N \int_{|x-x^{(j)}|=r_j} v \frac{\partial u}{\partial \nu_j} d\Omega_j \quad (1)$$

holds true. Here the symbol ν_j denotes the exterior normal to K_j and $d\Omega_j$ the surface element on the spheres $\{x : |x - x^{(j)}| = r_j\} = \partial K_j$ for $j = 1, \dots, N$.

From Proposition 9.1 we immediately infer the subsequent

Proposition 9.2. *With the assumptions from Proposition 9.1 we have Green's identity*

$$\int_{G''} (v \Delta u - u \Delta v) dx = - \sum_{j=1}^N \int_{|x-x^{(j)}|=r_j} \left(v \frac{\partial u}{\partial \nu_j} - u \frac{\partial v}{\partial \nu_j} \right) d\Omega_j. \quad (2)$$

Proof of Proposition 9.1:

1. At first, we assume that $v \in C_0^0(G)$ is satisfied in addition to the assumptions above, and we consider the vector-field $f = v \nabla u$. Then the Gaussian integral theorem yields the identity

$$\int_{G''} (v \Delta u + \nabla v \cdot \nabla u) dx = - \sum_{j=1}^N \int_{|x-x^{(j)}|=r_j} v \frac{\partial u}{\partial \nu_j} d\Omega_j. \quad (3)$$

We approximate an arbitrary function $v \in C^2(G') \cap C^0(\overline{G''})$ by a sequence $\{v_k\}_{k=1,2,\dots}$ as follows: Let $\{w_k(t)\}_{k=1,2,\dots} \subset C^\infty(\mathbb{R}, [0, 1])$ denote a sequence of functions with the properties

$$w_k(t) = \begin{cases} 1, & |t| \geq \frac{1}{k} \\ 0, & |t| \leq \frac{1}{2k} \end{cases}, \quad k = 1, 2, \dots$$

The functions

$$\varphi_k(t) := \int_0^t w_k(s) ds, \quad t \in \mathbb{R}$$

then satisfy

$$\varphi_k(0) = 0, \quad \varphi'_k(t) = w_k(t), \quad k = 1, 2, \dots,$$

and we estimate

$$|\varphi_k(t) - t| = \left| \int_0^t (w_k(s) - 1) ds \right| \leq \frac{2}{k}, \quad k = 1, 2, \dots$$

Now we define the sequence

$$v_k(x) := \varphi_k(v(x)), \quad x \in \overline{G''}, \quad k = 1, 2, \dots \quad (4)$$

and consider the relation

$$|v_k(x) - v(x)| = |\varphi_k(v(x)) - v(x)| \leq \frac{2}{k} \rightarrow 0 \quad (k \rightarrow \infty)$$

for all $x \in G''$, which implies

$$v_k(x) \rightarrow v(x) \quad (k \rightarrow \infty) \quad \text{uniformly in } \overline{G''}. \quad (5)$$

2. We prove that

$$E := \left\{ x \in G'' : v(x) = 0, \nabla v(x) \neq 0 \right\}$$

represents a Lebesgue null-set: Here we choose the point $z \in E$ arbitrarily. Taking the quantity $\varepsilon > 0$ sufficiently small, the set

$$E \cap \left\{ x \in G'' : |x - z| < \varepsilon \right\}$$

constitutes a graph, and is consequently a Lebesgue null-set due to the theorem on implicit functions. We exhaust the set G'' with the aid of the cube decomposition. For each point $z \in E$ we consider a sufficiently small cube $z \in W \subset G''$ such that $W \cap E$ is a Lebesgue null-set. Now the set E consists of a countable union of those sets $W \cap E$, and the σ -additivity of the Lebesgue measure yields the statement above.

3. For all points $x \in G'' \setminus E$ we deduce

$$\nabla v_k(x) = \varphi'_k(v(x)) \nabla v(x) = w_k(v(x)) \nabla v(x) \rightarrow \nabla v(x) \quad (k \rightarrow \infty),$$

which holds true a. e. in G'' , due to part 2 of our proof. We insert $v = v_k$ into (3) and obtain

$$\int_{G''} (v_k \Delta u + \nabla v_k \cdot \nabla u) dx = - \sum_{j=1}^N \int_{|x-x^{(j)}|=r_j} v_k \frac{\partial u}{\partial \nu_j} d\Omega_j, \quad k \in \mathbb{N}.$$

Then we observe the passage to the limit $k \rightarrow \infty$ and see

$$\begin{aligned} \int_{G''} v(x) \Delta u(x) dx + \lim_{k \rightarrow \infty} \int_{G''} (\nabla v_k(x) \cdot \nabla u(x)) dx \\ = - \sum_{j=1}^N \int_{|x-x^{(j)}|=r_j} v(x) \frac{\partial u(x)}{\partial \nu_j} d\Omega_j. \end{aligned} \quad (6)$$

Inserting $v(x) = u(x)$ into (6) and noting that

$$\nabla v_k(x) = w_k(u(x)) \nabla u(x)$$

holds true, we see

$$\begin{aligned} \int_{G''} u(x) \Delta u(x) dx + \lim_{k \rightarrow \infty} \int_{G''} w_k(u(x)) |\nabla u(x)|^2 dx \\ = - \sum_{j=1}^N \int_{|x-x^{(j)}|=r_j} u(x) \frac{\partial u(x)}{\partial \nu_j} d\Omega_j. \end{aligned} \quad (7)$$

Fatou's theorem yields

$$\int_{G''} |\nabla u(x)|^2 dx < +\infty \quad \text{and} \quad \int_{G''} |\nabla v(x)|^2 dx < +\infty. \quad (8)$$

On account of

$$\begin{aligned} |\nabla v_k(x) \cdot \nabla u(x)| &= |w_k(v(x))| |\nabla v(x) \cdot \nabla u(x)| \\ &\leq \frac{1}{2} (|\nabla u(x)|^2 + |\nabla v(x)|^2), \quad x \in G'', \end{aligned}$$

we have an integrable majorant for the limit in (6). The identity (1) then follows by Lebesgue's convergence theorem. q.e.d.

We now continue the considerations from Section 1 concerning the eigenvalue problem of the n -dimensional oscillation equation : Let $G \subset \mathbb{R}^n$ denote a bounded Dirichlet domain. On the linear space

$$\mathcal{E} := \left\{ u = u(x) \in C^2(G) \cap C^0(\overline{G}) : u|_{\partial G} = 0 \right\}$$

we consider *Weyl's eigenvalue problem*

$$-\Delta u(x) = \lambda u(x), \quad x \in G \quad \text{with} \quad u \in \mathcal{E} \setminus \{0\} \quad \text{and} \quad \lambda \in \mathbb{R}. \quad (9)$$

Proposition 9.3. *All eigenvalues λ of (9) have the property $\lambda > 0$.*

Proof: With $u \in \mathcal{E} \setminus \{0\}$ we consider a solution of (9) belonging to the eigenvalue $\lambda \in \mathbb{R}$. Then we infer

$$\int_G |\Delta u(x)| dx = |\lambda| \int_G |u(x)| dx < +\infty.$$

We apply Proposition 9.1 with $v = u$ and obtain

$$\int_G |\nabla u(x)|^2 dx = - \int_G u(x) \Delta u(x) dx = \lambda \int_G |u(x)|^2 dx$$

or equivalently

$$\lambda = \frac{\int_G |\nabla u(x)|^2 dx}{\int_G |u(x)|^2 dx} > 0.$$

q.e.d.

Remark: The *Rayleigh quotient* appears in the last formula.

We do not mention the case $n = 2$ particularly, and we shall utilize Green's function in the dimensions $n = 3, 4, \dots$ as follows:

$$H(x, y) = \frac{1}{(n-2)\omega_n} \frac{1}{|y-x|^{n-2}} + h(x, y), \quad (x, y) \in G \otimes G. \quad (10)$$

A solution u of (9) evidently belongs to the space

$$\mathcal{D} := \left\{ u = u(x) \in C^0(G) : \sup_{x \in G} |u(x)| < +\infty \right\} = C^0(G) \cap L^\infty(G)$$

and satisfies the integral-equation problem

$$u(x) = \lambda \int_G H(x, y) u(y) dy, \quad x \in G \quad \text{with} \quad u \in \mathcal{D} \setminus \{0\} \text{ and } \lambda \in \mathbb{R}. \quad (11)$$

We deduce the latter statements as in Section 1 (compare Theorem 1.9 there), using the Propositions 9.1 and 9.2 above. We have already shown the symmetry of Green's function and controlled the growth condition:

$$\begin{aligned} H &= H(x, y) \in \mathcal{S}_{n-2}(G), \\ 0 &\leq H(x, y) = H(y, x), \quad (x, y) \in G \otimes G. \end{aligned} \quad (12)$$

Now we shall prove that a solution u of (11) solves (9) as well.

Proposition 9.4. *Let the function $u = u(x) \in \mathcal{D}$ be given, and the parameter integral*

$$v(x) := \int_G H(x, y) u(y) dy, \quad x \in G$$

be defined. Then we have the properties $v \in C^0(\overline{G})$ and $v|_{\partial G} = 0$.

Proof: With the aid of the auxiliary function

$$\Theta(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ t-1, & 1 \leq t \leq 2 \\ 1, & 2 \leq t \end{cases}$$

we define the continuous integral kernel

$$\begin{aligned} H_\delta(x, y) &:= H(x, y) \Theta\left(\frac{|x - y|}{\delta}\right) \\ &= \Theta\left(\frac{|x - y|}{\delta}\right) \left(\frac{1}{(n-2)\omega_n} |y - x|^{2-n} + h(x, y) \right), \quad \delta > 0. \end{aligned}$$

For all $\delta > 0$, the parameter integral

$$v_\delta(x) := \int_G H_\delta(x, y) u(y) dy, \quad x \in \overline{G}$$

is continuous on $\overline{G}$, and we observe the boundary condition

$$v_\delta|_{\partial G} = 0.$$

Furthermore, we have the following inequality

$$\begin{aligned} |v_\delta(x) - v(x)| &\leq \int_G \left| \Theta\left(\frac{|x - y|}{\delta}\right) - 1 \right| |H(x, y)| |u(y)| dy \\ &\leq \int_{y: |y-x| \leq 2\delta} \frac{c}{|y - x|^{n-2}} dy \leq \gamma(\delta) \rightarrow 0 \quad (\delta \downarrow 0) \end{aligned} \tag{13}$$

for all $x \in G$. This implies

$$v_\delta(x) \rightarrow v(x) \quad (\delta \downarrow 0) \quad \text{uniformly in } G,$$

and therefore $v \in C^0(\overline{G})$ and $v|_{\partial G} = 0$ holds true. q.e.d.

Proposition 9.5. *Each solution u of (11) satisfies $u \in C^2(G)$ and fulfills the eigenvalue equation*

$$-\Delta u(x) = \lambda u(x), \quad x \in G.$$

Proof: We take an arbitrary point $z \in G$, and then choose a quantity $\varepsilon > 0$ so small that the inclusion

$$K_\varepsilon(z) := \left\{ x \in \mathbb{R}^n : |x - z| \leq \varepsilon \right\} \subset G$$

is valid. From the integral equation (11) we infer

$$\begin{aligned} u(x) &= \lambda \int_{K_\varepsilon(z)} \frac{1}{(n-2)\omega_n} \frac{1}{|y - x|^{n-2}} u(y) dy \\ &\quad + \lambda \int_{G \setminus K_\varepsilon(z)} \frac{1}{(n-2)\omega_n} \frac{1}{|y - x|^{n-2}} u(y) dy + \int_G h(x, y) u(y) dy \\ &= \lambda \int_{K_\varepsilon(z)} \frac{1}{(n-2)\omega_n} \frac{1}{|y - x|^{n-2}} u(y) dy + \psi_{z, \varepsilon}(x), \quad x \in K_\varepsilon^\circ(z). \end{aligned} \tag{14}$$

Here the function $\psi_{z,\varepsilon}(x)$ is harmonic in $K_\varepsilon^\circ(z)$. We can differentiate the equation (14) once (but not twice!) and obtain, via the Gaussian integral theorem, the following identity

$$\begin{aligned}
 \nabla u(x) &= \lambda \int_{K_\varepsilon(z)} \frac{1}{(n-2)\omega_n} \left(\nabla_x \frac{1}{|y-x|^{n-2}} \right) u(y) dy + \nabla \psi_{z,\varepsilon}(x) \\
 &= -\lambda \int_{K_\varepsilon(z)} \frac{1}{(n-2)\omega_n} \left(\nabla_y \frac{1}{|y-x|^{n-2}} \right) u(y) dy + \nabla \psi_{z,\varepsilon}(x) \\
 &= -\lambda \int_{K_\varepsilon(z)} \frac{1}{(n-2)\omega_n} \nabla_y \left(\frac{u(y)}{|y-x|^{n-2}} \right) dy \\
 &\quad + \lambda \int_{K_\varepsilon(z)} \frac{1}{(n-2)\omega_n} \frac{\nabla u(y)}{|y-x|^{n-2}} u(y) dy + \nabla \psi_{z,\varepsilon}(x) \\
 &= -\lambda \int_{\partial K_\varepsilon(z)} \frac{1}{(n-2)\omega_n} \frac{u(y)}{|y-x|^{n-2}} \nu(y) d\Omega(y) \\
 &\quad + \lambda \int_{K_\varepsilon(z)} \frac{1}{(n-2)\omega_n} \frac{\nabla u(y)}{|y-x|^{n-2}} dy + \nabla \psi_{z,\varepsilon}(x)
 \end{aligned} \tag{15}$$

for all $x \in K_\varepsilon^\circ(z)$. Here the symbol $\nu(y)$ denotes the exterior normal to the ball $K_\varepsilon(z)$ and $d\Omega(y)$ the surface element on the sphere $\partial K_\varepsilon(z)$.

From (15) we infer the statement

$$u \in C^2(G), \tag{16}$$

since the point $z \in G$ could be chosen arbitrarily. We differentiate (15) once more, choose $x = z$, and evaluate the limit $\varepsilon \downarrow 0$. Thus we obtain

$$\begin{aligned}
 \Delta u(z) &= \lim_{\varepsilon \downarrow 0} \left\{ -\lambda \int_{\partial K_\varepsilon(z)} \frac{1}{(n-2)\omega_n} \left(\nabla_x \frac{u(y)}{|y-x|^{n-2}} \right) \Big|_{x=z} \nu(y) d\Omega(y) \right\} \\
 &\quad + \lim_{\varepsilon \downarrow 0} \left\{ \lambda \int_{K_\varepsilon(z)} \frac{1}{(n-2)\omega_n} \left(\nabla_x \frac{\nabla u(y)}{|y-x|^{n-2}} \right) \Big|_{x=z} dy \right\} \\
 &= -\lambda u(z) + 0 = -\lambda u(z) \quad \text{for all } z \in G.
 \end{aligned} \tag{17}$$

q.e.d.

We summarize our considerations to the following

Proposition 9.6. *The function u solves the eigenvalue problem (9) if and only if the function u solves the eigenvalue problem (11).*

Theorem 9.7. (H. Weyl)

On each bounded Dirichlet domain $G \subset \mathbb{R}^n$ with $n = 2, 3, \dots$, the Laplace

operator possesses a c.o.n.s. of eigenfunctions $\varphi_k \in \mathcal{E}$, $k = 1, 2, \dots$. This means

$$-\Delta\varphi_k(x) = \lambda_k\varphi_k(x), \quad x \in G, \quad k = 1, 2, \dots, \quad (18)$$

and the eigenvalues have the properties

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \rightarrow +\infty. \quad (19)$$

Proof: Equivalently to (9) we consider the integral eigenvalue problem (11)

$$\int_G H(x, y)u(y) dy = \mu u(x), \quad x \in G, \quad \mu = \frac{1}{\lambda}$$

with the symmetric weakly singular kernel $H(x, y)$ from (10) and (12). The statements of the theorem can now be inferred from Section 8, Theorem 8.1. q.e.d.

Remarks:

1. In the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, we can even uniformly expand each function $f \in \mathcal{E}$ into the series of eigenfunctions for the Laplace operator.
2. The least eigenvalue λ_1 for the Laplacian on the bounded domain $G \subset \mathbb{R}^n$ satisfies

$$\lambda_1(G) = \inf_{\varphi \in W_0^{1,2}(G) \cap G^0(\overline{G}), \varphi \neq 0} \frac{\int_G |\nabla \varphi(x)|^2 dx}{\int_G |\varphi(x)|^2 dx}. \quad (20)$$

Here we refer the reader to the Sobolev spaces in the Sections 1 and 2 of Chapter 10. From the relation (20) we immediately infer the *monotonicity property of the least eigenvalue*:

$$G \subset G_* \quad \Rightarrow \quad \lambda_1(G) \geq \lambda_1(G_*). \quad (21)$$

With the aid of a regularity theorem for weak solutions of the Laplace equation one proves the *strict monotonicity property*:

$$G \subset\subset G_* \quad \Rightarrow \quad \lambda_1(G) > \lambda_1(G_*). \quad (22)$$

In this context we refer the reader to [CH], Band II, Kapitel VI.

3. Comparing sufficiently regular domains $G \subset \mathbb{R}^n$ with the ball of the same volume $K \subset \mathbb{R}^n$ - which means $|K| = |G|$ - we have the estimate

$$\lambda_1(G) \geq \lambda_1(K). \quad (23)$$

Here the equality is attained only in the case that G is already a ball in $\mathbb{R}^n$. This *Theorem of Faber and Krahn* is based on the isoperimetric inequality in $\mathbb{R}^n$ and had already been conjectured by Rayleigh in his book *Theory of the Sound*. In the case $n = 2$ we recommend the following paper by E. KRAHN: *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises*. Mathematische Annalen **94** (1924), 97-100.

4. If the function $\varphi \in \mathcal{E}$ is a solution of (9) for the eigenvalue $\lambda \in \mathbb{R}$, we infer

$$\lambda = \lambda_1 \quad \Leftrightarrow \quad \varphi(x) \neq 0 \quad \text{for all } x \in G. \quad (24)$$

Therefore, the eigenfunction to the least eigenvalue λ_1 has no zeroes in G .

5. About the eigenfunctions for higher eigenvalues and their nodal domains almost no results are available (compare [CH]).
6. Endowing the domain $G \subset \mathbb{R}^n$ with the elliptic Riemannian metric

$$ds^2 = \sum_{i,j=1}^n g_{ij}(x) dx_i dx_j,$$

we propose the integral equation method in order to treat the eigenvalue problem of the Laplace-Beltrami operator

$$\Delta = \frac{1}{\sqrt{g(x)}} \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{g(x)} \sum_{j=1}^n g^{ij}(x) \frac{\partial}{\partial x_j} \right) \quad (25)$$

with $((g^{kl})_{kl} = (g_{ij})_{ij}^{-1}$ and $g = \det(g_{ij})_{ij}$. In this context we need the generalized Green's function for elliptic operators in divergence form which is weakly singular again. Here we refer the reader to our approach in Chapter 10, Sections 9 and 10, or to the following original paper by M. GRÜTER, K. O. WIDMAN: *The Green function for uniformly elliptic equations*. Manuscripta mathematica **37** (1982), 303–342.

7. Theorems of Faber-Krahn type are valid even for the operators (25). Here we recommend the monographs by G. POLYA: *Isoperimetric Inequalities*, Princeton University Press, Princeton N.J., 1944 and by C. BANDLE: *Isoperimetric Inequalities in Mathematical Physics*, Pitman, 1984.
8. The spectral theory for unbounded operators is presented e.g. in the Kapitel IV: *Selbstadjungierte Operatoren im Hilbertraum* of the monograph by H. TRIEBEL: *Höhere Analysis*. Verlag der Wissenschaften, Berlin, 1972.
9. A simple proof of the spectral theorem for selfadjoint operators has been discovered by H. LEINFELDER: *A geometric proof of the spectral theorem for unbounded selfadjoint operators*. Mathematische Annalen **242** (1979), 85–96.

10 Some Historical Notices to Chapter 8

The investigation of eigenvalue problems for ordinary differential operators started in 1837; then C.F. Sturm invented his well-known comparison theorem, essential for the stability question of geodesics. C.G. Jacobi (1804–1851) created the general stability theory for one-dimensional variational problems. In order to study the stability question for parametric minimal surfaces, H.A. Schwarz investigated eigenvalue problems for the two-dimensional Laplacian already in the *Festschrift dedicated to his academic teacher Karl Weierstrass* from 1885.

D. Hilbert created the theory of integral equations in the years 1904–1910, solving linear systems of infinitely many equations. This theory may be seen as one of Hilbert’s greatest achievements, and it was substantially further developed by his students H. Weyl and E. Schmidt.

We have presented H. Weyl’s approach to the eigenvalue problem of the n -dimensional Laplacian via the integral equation method in this chapter. In his famous textbook together with Hilbert, R. Courant solved eigenvalue problems for partial differential equations alternatively by direct variational methods. His student F. Rellich then created a spectral theory for abstract operators in Hilbert spaces, as well as K. Friedrichs.

In the meantime, physicists became intensively interested in eigenvalue problems for partial differential equations; these are situated in the center of *Quantum Mechanics* – evolving in the 1930s. Their source of information were mainly the textbooks *Methoden der Mathematischen Physik* [CH] by R. Courant and D. Hilbert.

Figure 2.3 PORTRAIT OF R. COURANT (1888–1972)
taken from page 240 of the biography by *C. Reid: Hilbert*, Springer-Verlag, Berlin... (1970).



Partial Differential Equations 2

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