
Preface – Volume 2

In this second volume we continue our textbook PARTIAL DIFFERENTIAL EQUATIONS OF GEOMETRY AND PHYSICS. Originating from both disciplines, we shall treat central questions such as eigenvalue problems, curvature estimates, and boundary value problems, for instance. With the title of our textbook we want to emphasize both the pure and applied aspects of partial differential equations. It turns out that the concepts of solutions are permanently extended in the theory of partial differential equations. Here the classical methods do not lose their significance. Besides the n -dimensional theory, we want to present the two-dimensional theory – so important to our geometric intuition.

We shall solve the differential equations by the continuity method, the variational method or the topological method. The continuity method may be preferred from a geometric point of view, since the stability of the solution is investigated there. The variational method is very attractive from the point of view in physics; however, difficult regularity questions for the weak solution appear with this method. The topological method controls the whole set of solutions during the deformation of the problem, and it does not depend on uniqueness as does the variational method.

The original version of this textbook, *Friedrich Sauvigny: Partielle Differentialgleichungen der Geometrie und der Physik 2 – Funktionalanalytische Lösungsmethoden – Unter Berücksichtigung der Vorlesungen von E. Heinz*, appeared in 2005 from *Springer-Verlag*. A translated and expanded version of this monograph followed in 2006 as *Springer-Universitext*, namely *Friedrich Sauvigny: Partial Differential Equations 2*, and we are now presenting a second edition of this textbook. We have carefully revised the original Volume 2 and added the Chapter 13 on *Boundary Value Problems from Differential Geometry*. The topics of this new Chapter 13 are listed in the table of contents, and we include a table of contents for our revised Volume 1 for the convenience of the reader.

In Chapter 7 we consider, in general, nonlinear operators in Banach spaces. With the aid of Brouwer's degree of mapping from Chapter 3 we prove Schauder's fixed point theorem in Section 1, and we supplement Banach's fixed point theorem. In Section 2 we define the Leray-Schauder degree for mappings in Banach spaces by a suitable approximation, and we prove its fundamental properties in Section 3. In this section we refer to the lecture [H4] by my academic teacher, Professor Dr. E. Heinz in Göttingen.

By transition to linear operators in Banach spaces, we prove the fundamental solution-theorem of F. Riesz via the Leray-Schauder degree. At the end of this chapter we derive the Hahn-Banach continuation theorem by Zorn's lemma (compare [HS]).

In Chapter 8 on *Linear Operators in Hilbert Spaces*, we transform the eigenvalue problems of Sturm-Liouville and of H. Weyl for differential operators into integral equations in Section 1. Then we consider weakly singular integral operators in Section 2 and prove a theorem of I. Schur on iterated kernels. In Section 3 we further develop the results from Chapter 2, Section 6 on the Hilbert space and present the abstract completion of pre-Hilbert-spaces. Bounded linear operators in Hilbert spaces are treated in Section 4: The continuation theorem, Adjoint and Hermitian operators, Hilbert-Schmidt operators, Inverse operators, Bilinear forms and the theorem of Lax-Milgram are presented. In Section 5 we study the transformation of Fourier-Plancherel as a unitary operator on the Hilbert space $L^2(\mathbb{R}^n)$.

Completely continuous, respectively compact operators are studied in Section 6 together with weak convergence. The operators with finite square norms represent an important example. The solution-theorem of Fredholm on operator equations in Hilbert spaces is deduced from the corresponding result of F. Riesz in Banach spaces. We apply these results particularly to weakly singular integral operators.

In Section 7 we prove the spectral theorem of F. Rellich on completely continuous and Hermitian operators by variational methods. Then we address the Sturm-Liouville eigenvalue problem in Section 8 and expand the relevant integral kernels into their eigenfunctions. Following ideas of H. Weyl, we treat the eigenvalue problem for the Laplacian on domains in $\mathbb{R}^n$ by the integral equation method in Section 9. In this chapter as well, we take a lecture of Professor Dr. E. Heinz into consideration (compare [H3]). For the study of eigenvalue problems we recommend the classical treatise [CH] of R. Courant and D. Hilbert.

We have been guided into functional analysis with the aid of problems concerning differential operators in mathematical physics (compare [He1] and [He2]). The usual contents of functional analysis can be taken from Chapter 2, Sections 6-8, Chapter 7, and Chapter 8. Additionally, we investigated the solvability of nonlinear operator equations in Banach spaces. For the spec-

tral theorem of unbounded, self-adjoint operators we refer the reader to the literature.

In our compendium we shall directly construct classical solutions of boundary and initial value problems for linear and nonlinear partial differential equations with the aid of functional analytic methods. The appropriate a priori estimates with respect to the Hölder norm allow us to establish the existence of solutions in classical function spaces.

In Chapter 9, Sections 1-3, we essentially follow the book of I. N. Vekua [V] and solve the Riemann-Hilbert boundary value problem by the integral-equation method. Using the lecture [H6], we present Schauder's continuity method in the Sections 4-7 in order to solve boundary value problems for linear elliptic differential equations with n independent variables. Therefore, we completely prove the Schauder estimates.

In Chapter 10 on *Weak Solutions of Elliptic Differential Equations*, we profit from the *Grundlehren* [GT] Chapters 7 and 8 of D. Gilbarg and N. S. Trudinger. Here, we additionally recommend the textbook [Jo] of J. Jost and the compendium [E] by L. C. Evans.

We introduce Sobolev spaces in Section 1 and prove the embedding theorems in Section 2. Having established the existence of weak solutions in Section 3, we show the boundedness of weak solutions by Moser's iteration method in Section 4. Then we investigate the Hölder continuity of weak solutions in the interior and at the boundary; see the Sections 5-7. Restricting ourselves to interesting classes of equations, we can illustrate the methods of proof in a transparent way. Finally, we apply the results to equations in divergence form; see the Sections 8, 9, and 10.

In Chapter 11, Sections 1-2, we concisely lay the foundations of differential geometry (compare [BL]) and of the calculus of variations. Then we discuss the theory of characteristics for nonlinear hyperbolic differential equations in two variables (compare [CH], [G], [H5]) in Section 3 and Section 4. In particular, we solve the Cauchy initial value problem via Banach's fixed point theorem. In Section 6 we present H. Lewy's ingenious proof for the analyticity theorem of S. Bernstein. Here we would like to refer the reader to the textbook by P. Garabedian [G] as well.

On the basis of Chapter 4 from Volume 1, *Generalized Analytic Functions*, we treat *Nonlinear Elliptic Systems* in Chapter 12. Having presented Jäger's maximum principle in Section 1, we develop the general theory in the Sections 2-5 from the fundamental treatise of E. Heinz [H7] on nonlinear elliptic systems. An existence theorem for nonlinear elliptic systems is situated in the center, which is attained via the Leray-Schauder degree. In Section 6 we derive a well-known curvature estimate for minimal surfaces and obtain a Bernstein theorem. Within Section 8 we introduce conformal parameters into a nonanalytic Riemannian metric by a nonlinear continuity method. We establish the necessary a priori estimates in Section 7 which extend to the boundary.

In Chapter 13 we apply the theory of the nonlinear elliptic systems to *Boundary Value Problems from Differential Geometry*. At first, we solve the Dirichlet problem for nonparametric equations of prescribed mean curvature by the uniformization method in Section 1. Then we present in Section 2 and Section 3 a mathematical model for winding staircases: We solve a mixed boundary value problem for minimal graphs over a Riemannian surface, the so-called *etale plane*, via a nonlinear continuity method. In Section 4 we solve Plateau's problem for surfaces of constant mean curvature by the variational method.

Then we present various answers to the following central question: *In which class of closed surfaces is the sphere the only one with constant mean curvature?* In Section 5 we give an alternative proof for H. Hopf's celebrated answer to this question.

In the final Section 7, we also study the elliptic Monge-Ampère differential equation with the aid of nonlinear elliptic systems. For this fully nonlinear, elliptic p.d.e., H. Lewy and E. Heinz provided a transparent approach by controlling the associate uniformizing mapping in Section 6. We speak of Lewy-Heinz systems on account of their stupendous investigations; however, a special case of these systems already appear within the Darboux lectures [D] for the hyperbolic case. Our results on the Monge-Ampère equation will enable us to solve Weyl's embedding problem for C^3 -metrics of positive Gaussian curvature on the sphere.

For further studies on *the Monge-Ampère equation* we would like to recommend the book [Sc] by F. Schulz. If our readers would like to study the theory of minimal surfaces more intensively, we recommend the three new volumes [DHS], [DHT1], and [DHT2] by U. Dierkes, S. Hildebrandt, A. Tromba and the present author within the *Springer-Grundlehren*.

We follow the Total Order Code of the Universitext series, however, adapt this to the present extensive contents. Since we see our books as an entity, we count the chapters throughout our two volumes from 1-13. In each section individually, we count the equations and refer to them by a single number; when we refer to an equation in another section of the same chapter, say the m -th section, we have to add *Section m* ; when we refer to an equation in the m -th section of another chapter, say the l -th chapter, we have to add *Section m in Chapter l* .

We assemble *definitions, theorems, propositions, examples* to the expression *environment*, which is borrowed from the underlying TEX-file. Individually in each section, we count these environments consecutively by the number n , and denote the n -th environment within the m -th section by *Environment $m.n$* . Thus we have attributed a pair of integers $m.n$ to all definitions, theorems, propositions, and examples, which is unique within each chapter and easy to find. Referring to these environments throughout both books, we proceed as described above for the equations.

We add Figure 1.1 – Figure 1.9 to our Volume 1 and Figure 2.1 – Figure 2.11 to our Volume 2, which mostly represent portraits of mathematicians. This small photo collection of some scientists, who have contributed to the theory of partial differential equations, already shows that our area is situated in the center of modern mathematics and possesses profound interrelations with geometry and physics.

This textbook PARTIAL DIFFERENTIAL EQUATIONS has been developed from lectures that I have been giving in the Brandenburgische Technische Universität at Cottbus from the winter semester 1992/93 to the present semester. The monograph builds, in part, upon the lectures (see [H1] – [H6]) of Professor Dr. Dr.h.c. E. Heinz, whom I was fortunate to know as his student in Göttingen from 1971 to 1978 and as postdoctoral researcher in his *Oberseminar* from 1983 to 1989. As an assistant in Aachen from 1978 to 1983, I very much appreciated the elegant lecture cycles of Professor Dr. G. Hellwig (see [He1] – [He3]). Since my research fellowship at the University of Bonn in 1989/90, an intensive scientific collaboration with Professor Dr. Dr.h.c.mult. S. Hildebrandt has developed and continues to this day (see [DHS] and [DHT2] with the list of references therein). All three of these excellent representatives of mathematics will forever have my sincere gratitude and my deep respect!

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