2
Holomorphic Functions

In this chapter we introduce the notion of a differentiable function, or of a holomorphic function. It turns out that differentiability is characterized by a pair of (partial differential) equations—the Cauchy–Riemann equations. We also introduce the notion of the integral along a path and we study its relation to the notion of a holomorphic function. Finally, we introduce the index of a closed path, we obtain Cauchy's integral formula for a holomorphic function, and we discuss the relation between integrals and homotopy.

2.1 Limits and Continuity

Let $f\colon \Omega \to \mathbb{C}$ be a complex-valued function in a set $\Omega \subset \mathbb{C}$. We first introduce the notion of limit.

Definition 2.1

We say that the *limit* of f at a point $z_0 \in \Omega$ exists, and that it is given by $w \in \mathbb{C}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\bigl|f(z) - w\bigr| < \varepsilon \quad \text{whenever } |z - z_0| < \delta.$$

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In this case we write

$$\lim_{z\to z_0} f(z) = w.$$

Now we introduce the notion of continuity.

Definition 2.2

We say that f is *continuous* at a point $z_0 \in \Omega$ if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

Otherwise, the function f is said to be *discontinuous* at z_0. We also say that f is *continuous* in Ω if it is continuous at all points of Ω.

Example 2.3

For the function $f(z) = |z|$, we have

$$\bigl|f(z) - f(z_0)\bigr| = \bigl||z| - |z_0|\bigr| \le |z - z_0|.$$

This implies that $|f(z) - f(z_0)| < \delta$ whenever $|z - z_0| < \delta$, and hence, the function f is continuous in $\mathbb{C}$.

Example 2.4

For the function $f(z) = z^2$, we have

$$\begin{aligned}\bigl|f(z) - f(z_0)\bigr| &= \bigl|(z - z_0)(z + z_0)\bigr| \\ &= |z - z_0| \cdot |z - z_0 + 2z_0| \\ &\le |z - z_0|\bigl(|z - z_0| + 2|z_0|\bigr) \\ &< \delta(\delta + 2|z_0|)\end{aligned}$$

whenever $|z - z_0| < \delta$. Since $\delta(\delta + 2|z_0|) \to 0$ when $\delta \to 0$, the function f is continuous in $\mathbb{C}$.

Example 2.5

Now we show that the function $f(z) = \log z$ is discontinuous at all points $z = -x + i0$ with $x > 0$. For $w \in \mathbb{C}$ in the second quadrant, we have

$$\log w = \log|w| + i \arg w$$

with $\arg w \in [\pi/2, \pi]$. On the other hand, for $w \in \mathbb{C}$ in the third quadrant and outside the half-line $\mathbb{R}^-$, the same formula holds, but now with $\arg w \in (-\pi, -\pi/2]$. Letting $w \to z$ in the second and third quadrants, we obtain respectively

$$\log w \to \log x + i\pi$$

and

$$\log w \to \log x - i\pi.$$

Since the right-hand sides are different, the logarithm has no limit at points of $\mathbb{R}^-$. Therefore, the function f is discontinuous at all points of $\mathbb{R}^-$. On the other hand, one can show that it is continuous in $\mathbb{C} \setminus \mathbb{R}_0^-$ (see Exercise 2.25).

2.2 Differentiability

Now we consider a function $f\colon \Omega \to \mathbb{C}$ in an open set $\Omega \subset \mathbb{C}$, that is, in an open set $\Omega \subset \mathbb{R}^2$.

Definition 2.6

We say that f is *differentiable* at a point $z_0 \in \Omega$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, the number $f'(z_0)$ is called the *derivative* of f at z_0.

We also introduce the notion of a holomorphic function.

Definition 2.7

When f is differentiable at all points of Ω we say that f is *holomorphic* in Ω.

Example 2.8

We show that the function $f(z) = z^2$ is holomorphic in $\mathbb{C}$. Indeed,

$$\begin{aligned}\lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} &= \lim_{z \to z_0} \frac{(z - z_0)(z + z_0)}{z - z_0} \\ &= \lim_{z \to z_0} (z + z_0) = 2z_0,\end{aligned}$$

and thus $(z^2)' = 2z$. One can show by induction that

$$\left(z^n\right)' = nz^{n-1}$$

for every $n \in \mathbb{N}$ (with the convention that $0^0 = 1$).

Example 2.9

Now we consider the function $f(z) = \overline{z}$. Given $h = re^{i\theta}$, we have

$$\begin{aligned}\frac{f(z+h) - f(z)}{h} &= \frac{\overline{z} + \overline{h} - \overline{z}}{h} \\ &= \frac{\overline{h}}{h} = e^{-2i\theta}. \end{aligned} \tag{2.1}$$

Since $e^{-2i\theta}$ varies with θ, one cannot take the limit in (2.1) when $r \to 0$. Hence, the function f is differentiable at no point.

Example 2.10

For the function $f(z) = |z|^2$, given $h = re^{i\theta}$ we have

$$\begin{aligned}\frac{f(z+h) - f(z)}{h} &= \frac{(z+h)(\overline{z} + \overline{h}) - z\overline{z}}{h} \\ &= \frac{z\overline{h} + \overline{z}h + h\overline{h}}{h} \\ &= \frac{z\overline{h}}{h} + \overline{z} + \overline{h} \\ &= \frac{zre^{-i\theta}}{re^{i\theta}} + \overline{z} + re^{-i\theta} \\ &= ze^{-2i\theta} + \overline{z} + re^{-i\theta} \to ze^{-2i\theta} + \overline{z} \end{aligned} \tag{2.2}$$

when $r \to 0$. For $z \neq 0$, since the limit in (2.2) varies with θ, the function f is not differentiable at z. On the other hand,

$$\frac{f(z) - f(0)}{z - 0} = \frac{|z|^2}{z} = \frac{z\overline{z}}{z} = \overline{z} \to 0$$

when $z \to 0$. Therefore, f is only differentiable at the origin, and $f'(0) = 0$.

The following properties are obtained as in $\mathbb{R}$, and thus their proofs are omitted.

Proposition 2.11

Given holomorphic functions $f, g\colon \Omega \to \mathbb{C}$, we have:

1. $(f+g)' = f' + g'$;
2. $(fg)' = f'g + fg'$;
3. $(f/g)' = (f'g - fg')/g^2$ at all points where $g \neq 0$.

Proposition 2.12

Given holomorphic functions $f\colon \Omega \to \mathbb{C}$ and $g\colon \Omega' \to \mathbb{C}$, with $g(\Omega') \subset \Omega$, we have

$$(f \circ g)' = (f' \circ g)g'.$$

Now we show that any differentiable function is continuous.

Proposition 2.13

If f is differentiable at z_0, then f is continuous at z_0.

Proof

For $z \neq z_0$, we have

$$f(z) - f(z_0) = \frac{f(z) - f(z_0)}{z - z_0}(z - z_0),$$

and thus,

$$\begin{aligned}
\lim_{z \to z_0} f(z) &= \lim_{z \to z_0} \big[f(z) - f(z_0)\big] + f(z_0) \\
&= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0) + f(z_0) \\
&= f'(z_0) \cdot 0 + f(z_0) = f(z_0).
\end{aligned}$$

This yields the desired property. □

We also describe a necessary condition for the differentiability of a function $f\colon \Omega \to \mathbb{C}$ at a given point. We always write

$$f(x + iy) = u(x, y) + iv(x, y),$$

where u and v are real-valued functions.

Theorem 2.14 (Cauchy–Riemann equations)

If f is differentiable at $z_0 = x_0 + iy_0$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.3}$$

at (x_0, y_0). Moreover, the derivative of f at z_0 is given by

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \tag{2.4}$$

Proof

Writing $f'(z_0) = a + ib$, we obtain

$$\begin{aligned} f'(z_0)(z - z_0) &= (a + ib)\big[(x - x_0) + i(y - y_0)\big] \\ &= \big[a(x - x_0) - b(y - y_0)\big] + i\big[b(x - x_0) + ia(y - y_0)\big] \\ &= C(x - x_0, y - y_0), \end{aligned}$$

where

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

and hence,

$$\begin{aligned} f(z) - f(z_0) - f'(z_0)(z - z_0) = \big(u(x, y), v(x, y)\big) &- \big(u(x_0, y_0), v(x_0, y_0)\big) \\ &- C(x - x_0, y - y_0). \end{aligned}$$

For $z \neq z_0$, we have

$$\begin{aligned} \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{|z - z_0|} &= \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0} \cdot \frac{z - z_0}{|z - z_0|} \\ &= \left(\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right)\frac{z - z_0}{|z - z_0|}, \end{aligned}$$

and since

$$\left|\frac{z - z_0}{|z - z_0|}\right| = \frac{|z - z_0|}{|z - z_0|} = 1,$$

we obtain

$$\frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{|z - z_0|} \to 0$$

when $z \to z_0$. Since

$$|z - z_0| = \|(x - x_0, y - y_0)\|,$$

this is the same as

$$\frac{(u(x,y), v(x,y)) - (u(x_0, x_0), v(x_0, y_0)) - C(x - x_0, y - y_0)}{\|(x - x_0, y - y_0)\|} \to 0$$

when $(x,y) \to (x_0, y_0)$. It thus follows from the notion of differentiability in $\mathbb{R}^2$ that the function $F\colon \Omega \to \mathbb{R}^2$ given by

$$F(x,y) = \big(u(x,y), v(x,y)\big) \tag{2.5}$$

is differentiable at (x_0, y_0), with derivative

$$\begin{aligned} DF(x_0, y_0) &= \begin{pmatrix} \frac{\partial u}{\partial x}(x_0, y_0) & \frac{\partial u}{\partial y}(x_0, y_0) \\ \frac{\partial v}{\partial x}(x_0, y_0) & \frac{\partial v}{\partial y}(x_0, y_0) \end{pmatrix} \\ &= C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \end{aligned}$$

This shows that the identities in (2.3) are satisfied. □

The equations in (2.3) are called the *Cauchy–Riemann equations*.

Example 2.15

Let

$$f(x + iy) = u(x,y) + iv(x,y)$$

be a holomorphic function in $\mathbb{C}$ with $u(x,y) = x^2 - xy - y^2$. By Theorem 2.14, the Cauchy–Riemann equations are satisfied. Since

$$\frac{\partial u}{\partial x} = 2x - y,$$

it follows from the first equation in (2.3) that

$$\frac{\partial v}{\partial y} = 2x - y.$$

Therefore,

$$v(x,y) = 2xy - \frac{y^2}{2} + C(x)$$

for some function C. Taking derivatives, we obtain

$$\frac{\partial u}{\partial y} = -x - 2y \quad \text{and} \quad -\frac{\partial v}{\partial x} = -2y - C'(x).$$

Hence,

$$-x - 2y = -2y - C'(x),$$

and $C'(x) = x$. We conclude that $C(x) = x^2/2 + c$ for some constant $c \in \mathbb{R}$, and hence,

$$v(x,y) = \frac{x^2}{2} + 2xy - \frac{y^2}{2} + c.$$

We thus have

$$f(x+iy) = \bigl(x^2 - xy - y^2\bigr) + i\left(\frac{x^2}{2} + 2xy - \frac{y^2}{2} + c\right).$$

Rearranging the terms, we obtain

$$\begin{aligned} f(x+iy) &= \bigl[\bigl(x^2 - y^2\bigr) + i2xy\bigr] + \left[-xy + i\left(\frac{x^2}{2} - \frac{y^2}{2}\right)\right] + ic \\ &= z^2 + \frac{i}{2}\bigl[\bigl(x^2 - y^2\bigr) + i2xy\bigr] + ic \\ &= z^2 + \frac{i}{2}z^2 + ic = \left(1 + \frac{i}{2}\right)z^2 + ic. \end{aligned}$$

In particular, $f'(z) = (2+i)z$.

Example 2.16

We show that a holomorphic function $f = u + iv$ cannot have $u(x,y) = x^2 + y^2$ as its real part. Otherwise, by the first Cauchy–Riemann equation, we would have

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y},$$

and thus, $v(x,y) = 2xy + C(x)$ for some function C. But then

$$\frac{\partial u}{\partial x} = 2y \quad \text{and} \quad \frac{\partial v}{\partial x} = 2y + C'(x),$$

and by the second Cauchy–Riemann equation we would also have

$$2y = -\bigl(2y + C'(x)\bigr).$$

Therefore, $C'(x) = -4y$, but this identity cannot hold for every $x, y \in \mathbb{R}$. For example, taking derivatives with respect to y we would obtain $0 = -4$, which is impossible.

As an illustration of the former concepts, in the remainder of this section we shall describe conditions for a holomorphic function to be constant.

Given a set $A \subset \mathbb{C}$, we denote by $\overline{A}$ the *closure* of A. This is the smallest closed subset of $\mathbb{C} = \mathbb{R}^2$ containing A. It is also the set of points $a \in \mathbb{C}$ such that

$$\big\{z \in \mathbb{C} : |z - a| < r\big\} \cap A \neq \emptyset$$

for every $r > 0$. In spite of the notation, the notion of closure should not be confused with the notion of the conjugate of a complex number. Now we recall the notion of a connected set.

Definition 2.17

A set $\Omega \subset \mathbb{C}$ is said to be *disconnected* if there exist nonempty sets $A, B \subset \mathbb{C}$ such that

$$\Omega = A \cup B \quad \text{and} \quad \overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

A set $\Omega \subset \mathbb{C}$ is said to be *connected* if it is not disconnected.

Finally, we introduce the notion of a connected component.

Definition 2.18

Given $\Omega \subset \mathbb{C}$, we say that a connected set $A \subset \Omega$ is a *connected component* of Ω if any connected set $B \subset \Omega$ containing A is equal to A.

We note that if a set $\Omega \subset \mathbb{C}$ is connected, then it is its own unique connected component.

Now we show that in any connected open set, a holomorphic function with zero derivative is constant.

Proposition 2.19

If f is a holomorphic function in a connected open set Ω and $f' = 0$ in Ω, then f is constant in Ω.

Proof

By (2.4), we have

$$f'(x+iy)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=0.$$

Together with the Cauchy–Riemann equations, this yields

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0.$$

Now let us consider points $x+iy$ and $x+iy'$ in Ω such that the line segment between them is contained in Ω. By the Mean value theorem, we obtain

$$u(x,y)-u(x,y')=\frac{\partial u}{\partial y}(x,z)(y-y')=0,$$

where z is some point between y and y'. Analogously,

$$v(x,y)-v(x,y')=\frac{\partial v}{\partial y}(x,w)(y-y')=0,$$

where w is some point between y and y'. This shows that

$$f(x+iy)=f(x+iy'). \tag{2.6}$$

One can show in a similar manner that if $x+iy'$ and $x'+iy'$ are points in Ω such that the line segment between them is contained in Ω, then

$$f(x+iy')=f(x'+iy'). \tag{2.7}$$

Now we consider an open rectangle $R\subset\Omega$ with horizontal and vertical sides. Given $x+iy, x'+iy'\in R$, the point $x+iy'$ is also in R, as well as the vertical segment between $x+iy$ and $x+iy'$, and the horizontal segment between $x+iy'$ and $x'+iy'$ (each of these segments can be a single point). It follows from (2.6) and (2.7) that

$$f(x+iy)=f(x+iy')=f(x'+iy').$$

This shows that f is constant in R. Finally, we consider sequences $\mathcal{R}_R=(R_n)_{n\in\mathbb{N}}$ of open rectangles in Ω, with horizontal and vertical sides, such that $R_1=R$ and $R_n\cap R_{n+1}\neq\emptyset$ for each $n\in\mathbb{N}$. We also consider the set

$$U_R=\bigcup_{\mathcal{R}_R}\bigcup_{n=1}^{\infty}R_n.$$

Clearly, U_R is open (since it is a union of open sets), and f is constant in U_R, since it is constant in each union $\bigcup_{n=1}^{\infty}R_n$.

We show that $U_R = \Omega$. On the contrary, let us assume that $\Omega \setminus U_R \neq \emptyset$. We note that

$$(\overline{U_R} \cap \Omega) \setminus U_R \neq \emptyset,$$

since otherwise $\overline{U_R} \cap \Omega = U_R$, and hence,

$$\Omega = U_R \cup (\Omega \setminus U_R),$$

with

$$\overline{U_R} \cap (\Omega \setminus U_R) = (\overline{U_R} \cap \Omega) \setminus U_R = \emptyset$$

and

$$U_R \cap \overline{\Omega \setminus U_R} = U_R \cap (\overline{\Omega} \setminus U_R) = \emptyset$$

(since U_R is open); that is, Ω would be disconnected. Let us then take $z \in (\overline{U_R} \cap \Omega) \setminus U_R$ and a rectangle $S \subset \Omega$ with horizontal and vertical sides such that $z \in S$. Then $S \cap U_R \neq \emptyset$ and thus, S is an element of some sequence $\mathcal{R}_R$. This implies that $S \subset U_R$ and hence $z \in U_R$, which yields a contradiction. Therefore, $U_R = \Omega$ and f is constant in Ω. □

We also describe some applications of Proposition 2.19.

Example 2.20

We show that for a holomorphic function $f = u + iv$ in a connected open set, if u is constant or v is constant, then f is also constant. Indeed, if u is constant, then

$$\begin{aligned} f'(x+iy) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 0, \end{aligned}$$

and it follows from Proposition 2.19 that f is constant. Similarly, if v is constant, then

$$\begin{aligned} f'(x+iy) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} = 0, \end{aligned}$$

and again it follows from Proposition 2.19 that f is constant.

Example 2.21

Now we show that for a holomorphic function $f = u + iv$ in a connected open set, if $|f|$ is constant, then f is constant. We first note that by hypothesis $|f|^2 = u^2 + v^2$ is also constant. If the constant is zero, then $u = v = 0$ and hence, $f = u + iv = 0$. Now we assume that $|f|^2 = c$ for some constant $c \neq 0$. Then $u^2 + v^2 = c$, and taking derivatives with respect to x and y, we obtain

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

and

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

Using the Cauchy–Riemann equations, one can rewrite these two identities in the matrix form

$$\begin{pmatrix} u & v \\ v & -u \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix} = 0. \tag{2.8}$$

Since the determinant of the 2×2 matrix in (2.8) is $-(u^2 + v^2) = -c \neq 0$, the unique solution is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0,$$

and thus,

$$f'(x + iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0.$$

It follows again from Proposition 2.19 that f is constant.

2.3 Differentiability Condition

The following example shows that for a function f to be differentiable at a given point it is not sufficient that the Cauchy–Riemann equations are satisfied at that point.

Example 2.22

We show that the function $f(x + iy) = \sqrt{|xy|}$ is not differentiable at the origin. Given

$$h = re^{i\theta} = r\cos\theta + ir\sin\theta,$$

we have

$$\begin{aligned}\frac{f(h)-f(0)}{h-0} &= \frac{\sqrt{|(r\cos\theta)(r\sin\theta)|}}{re^{i\theta}}\\ &= \frac{r\sqrt{|\cos\theta\sin\theta|}}{re^{i\theta}}\\ &= \sqrt{|\cos\theta\sin\theta|}e^{-i\theta}.\end{aligned}$$

Since the last expression depends on θ, one cannot take the limit when $r\to 0$. Therefore, f is not differentiable at the origin. On the other hand, we have

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x-0} = 0$$

and

$$\frac{\partial u}{\partial y}(0,0) = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y-0} = 0,$$

as well as

$$\frac{\partial v}{\partial x}(0,0) = \frac{\partial v}{\partial y}(0,0) = 0,$$

since $v=0$. Hence, the Cauchy–Riemann equations are satisfied at the origin.

Now we give a necessary and sufficient condition for the differentiability of a function f in some open set.

Theorem 2.23

Let $u,v\colon \Omega\to\mathbb{C}$ be C^1 functions in an open set $\Omega\subset\mathbb{C}$. Then the function $f=u+iv$ is holomorphic in Ω if and only if the Cauchy–Riemann equations are satisfied at all points of Ω.

Proof

By Theorem 2.14, if f is holomorphic in Ω, then the Cauchy–Riemann equations are satisfied at all points of Ω.

Now we assume that the Cauchy–Riemann equations are satisfied in Ω. This implies that

$$\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix} = \begin{pmatrix} a & -b\\ b & a\end{pmatrix}$$

at every point of Ω, for some constants a and b possibly depending on the point. On the other hand, since u and v are of class C^1, the function $F=(u,v)$ in (2.5) is differentiable in Ω. It follows from the proof of Theorem 2.14 that f is differentiable at z_0, with

$$f'(z_0)=a+ib,$$

if and only if F is differentiable at (x_0,y_0), with

$$DF(x_0,y_0)=\begin{pmatrix} \frac{\partial u}{\partial x}(x_0,y_0) & \frac{\partial u}{\partial y}(x_0,y_0) \\ \frac{\partial v}{\partial x}(x_0,y_0) & \frac{\partial v}{\partial y}(x_0,y_0) \end{pmatrix}=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

This shows that the function f is differentiable at all points of Ω. ☐

Example 2.24

Let us consider the function $f(z)=e^z$. We have

$$u(x,y)=e^x\cos y \quad \text{and} \quad v(x,y)=e^x\sin y,$$

and both functions are of class C^1 in the open set $\mathbb{R}^2=\mathbb{C}$. Since

$$\frac{\partial u}{\partial x}=e^x\cos y, \qquad \frac{\partial v}{\partial y}=e^x\cos y,$$

and

$$\frac{\partial u}{\partial y}=-e^x\sin y, \qquad -\frac{\partial v}{\partial x}=-e^x\sin y,$$

the Cauchy–Riemann equations are satisfied in $\mathbb{R}^2$. By Theorem 2.23, we conclude that the function f is differentiable in $\mathbb{C}$. Moreover, it follows from (2.4) that

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=e^x\cos y+ie^x\sin y=e^z,$$

that is, $(e^z)'=e^z$.

Example 2.25

For the cosine and sine functions, we have respectively

$$\begin{aligned}(\cos z)'&=\left(\frac{e^{iz}+e^{-iz}}{2}\right)'=\frac{ie^{iz}-ie^{-iz}}{2}\\&=-\frac{e^{iz}-e^{-iz}}{2i}=-\sin z\end{aligned}$$

and

$$(\sin z)' = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)' = \frac{ie^{iz} + ie^{-iz}}{2i}$$
$$= \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

Example 2.26

Now we find all points at which the function

$$f(x+iy) = xy + ixy$$

is differentiable. We first note that

$$u(x,y) = v(x,y) = xy$$

is of class C^1 in $\mathbb{R}^2$. On the other hand, the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

take the form

$$y = x \quad \text{and} \quad x = -y.$$

The unique solution is $x = y = 0$. By Theorem 2.14, we conclude that the function f is differentiable at no point of $\mathbb{C} \setminus \{0\}$. But since $\{0\}$ is not an open set, one cannot apply Theorem 2.23 to decide whether f is differentiable at the origin. Instead, we have to use the definition of derivative, that is, we must verify whether the limit

$$\lim_{(x,y)\to(0,0)} \frac{f(x+iy) - f(0)}{x+iy-0} = \lim_{(x,y)\to(0,0)} \frac{xy(1+i)}{x+iy}$$

exists. It follows from (1.6) that

$$\left|\frac{xy(1+i)}{x+iy}\right| \le \frac{|x| \cdot |y|\sqrt{2}}{|x+iy|} \le \sqrt{2}|x+iy| \to 0$$

when $(x,y) \to (0,0)$, and hence, f is differentiable at the origin, with $f'(0) = 0$.

Example 2.27

Let us consider the function $\log z$. It follows from (1.14) that

$$\left(e^{\log z}\right)' = 1.$$

Hence, if $\log z$ is differentiable at z, then it follows from the formula for the derivative of a composition in Proposition 2.12 that

$$e^{\log z}(\log z)' = 1.$$

Therefore,

$$(\log z)' = \frac{1}{e^{\log z}} = \frac{1}{z}.$$

Now we show that $\log z$ is differentiable (at least) in the open set $\mathbb{R}^+ \times \mathbb{R}$. For this we recall the formula

$$\log z = \frac{1}{2}\log\bigl(x^2 + y^2\bigr) + i\tan^{-1}\frac{y}{x}$$

obtained in Example 1.35 for $x > 0$. We note that the functions

$$u(x,y) = \frac{1}{2}\log\bigl(x^2 + y^2\bigr) \quad \text{and} \quad v(x,y) = \tan^{-1}\frac{y}{x}$$

are of class C^1 in $\mathbb{R}^+ \times \mathbb{R}$. Since

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \qquad \frac{\partial v}{\partial y} = \frac{1/x}{1 + (y/x)^2} = \frac{x}{x^2 + y^2},$$

and

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}, \qquad -\frac{\partial v}{\partial x} = -\frac{-y/x^2}{1 + (y/x)^2} = \frac{y}{x^2 + y^2},$$

it follows from Theorem 2.23 that the function $\log z$ is holomorphic in $\mathbb{R}^+ \times \mathbb{R}$.

2.4 Paths and Integrals

In order to define the integral of a complex function, we first introduce the notion of a path.

Definition 2.28

A continuous function $\gamma\colon [a,b] \to \Omega \subset \mathbb{C}$ is called a *path* in Ω, and its image $\gamma([a,b])$ is called a *curve* in Ω (see Figure 2.1).

We note that the same curve can be the image of several paths.

Now we define two operations. The first is the inverse of a path.

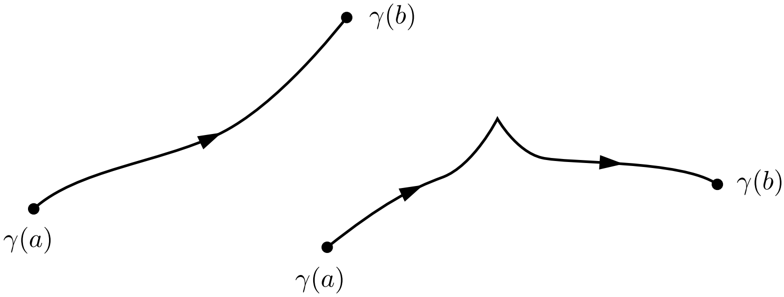

Figure 2.1 Paths and curves

Definition 2.29

Given a path $\gamma\colon [a,b] \to \Omega$, we define the path $-\gamma\colon [a,b] \to \Omega$ by

$$(-\gamma)(t) = \gamma(a+b-t)$$

for each $t \in [a,b]$ (see Figure 2.2).

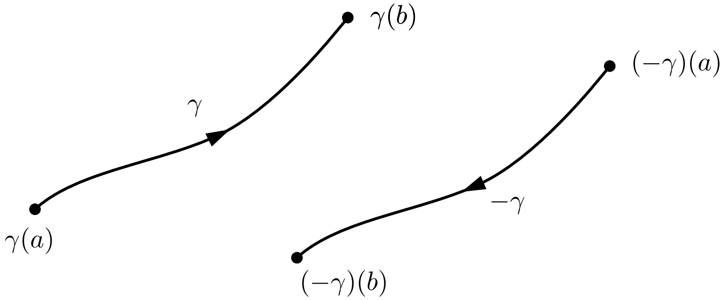

Figure 2.2 Paths γ and $-\gamma$

The second operation is the sum of paths.

Definition 2.30

Given paths $\gamma_1\colon [a_1,b_1] \to \Omega$ and $\gamma_2\colon [a_2,b_2] \to \Omega$ such that $\gamma_1(b_1) = \gamma_2(a_2)$, we define the path $\gamma_1 + \gamma_2\colon [a_1, b_1 + b_2 - a_2] \to \Omega$ by

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a_1,b_1], \\ \gamma_2(t - b_1 + a_2) & \text{if } t \in [b_1, b_1 + b_2 - a_2] \end{cases}$$

(see Figure 2.3).

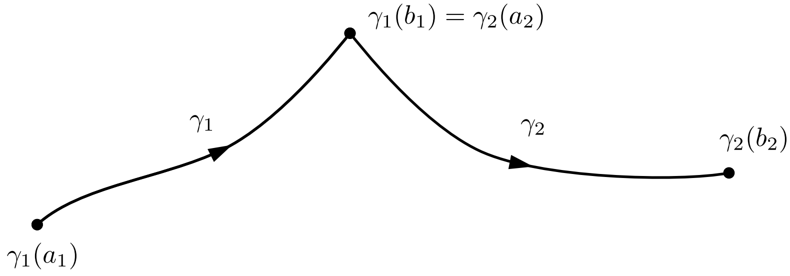

Figure 2.3 Path $\gamma_1 + \gamma_2$

We also consider the notions of a regular path and a piecewise regular path.

Definition 2.31

A path $\gamma\colon [a,b] \to \Omega$ is said to be *regular* if it is of class C^1 and $\gamma'(t) \neq 0$ for every $t \in [a,b]$, taking the right-sided derivative at a and the left-sided derivative at b.

More precisely, the path $\gamma\colon [a,b] \to \Omega$ is regular if there exists a path $\alpha\colon (c,d) \to \Omega$ of class C^1 in some open interval (c,d) containing $[a,b]$ such that $\alpha(t) = \gamma(t)$ and $\alpha'(t) \neq 0$ for every $t \in [a,b]$.

Definition 2.32

A path $\gamma\colon [a,b] \to \Omega$ is said to be *piecewise regular* if there exists a partition of $[a,b]$ into a finite number of subintervals $[a_j, b_j]$ (intersecting at most at their endpoints) such that each path $\gamma_j\colon [a_j,b_j] \to \Omega$ defined by $\gamma_j(t) = \gamma(t)$ for $t \in [a_j, b_j]$ is regular, taking the right-sided derivative at a_j and the left-sided derivative at b_j.

We have the following result.

Proposition 2.33

If the path $\gamma\colon [a,b] \to \mathbb{C}$ is piecewise regular, then

$$L_\gamma := \int_a^b \left|\gamma'(t)\right| dt < \infty. \tag{2.9}$$

Proof

Since γ is piecewise regular, the function $t \mapsto |\gamma'(t)|$ is continuous in each interval $[a_j, b_j]$ in Definition 2.32. Therefore, it is Riemann-integrable in each of these intervals, and thus also in their union, which is equal to $[a,b]$. □

The number L_γ is called the *length* of the path γ.

Example 2.34

Let $\gamma\colon [0,1] \to \mathbb{C}$ be the path given by $\gamma(t) = t(1+i)$ (see Figure 2.4). We have

$$L_\gamma = \int_0^1 \left|\gamma'(t)\right| dt = \int_0^1 |1+i|\, dt = \sqrt{2}.$$

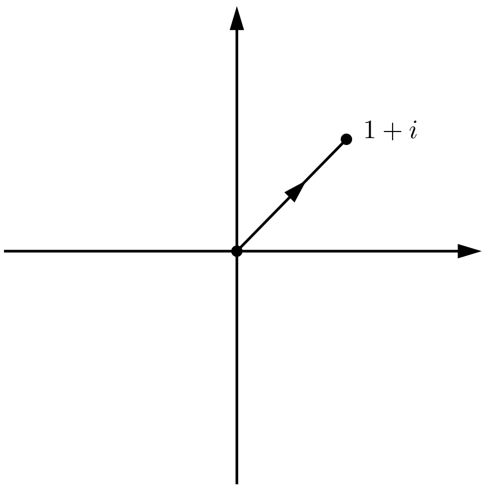

Figure 2.4 The path γ in Example 2.34

Example 2.35

Let $\gamma\colon [0,2\pi] \to \mathbb{C}$ be the path given by $\gamma(t) = re^{it}$ (see Figure 2.5). We have

$$\begin{aligned} L_\gamma &= \int_0^{2\pi} \left|\gamma'(t)\right| dt = \int_0^{2\pi} \left|rie^{it}\right| dt \\ &= \int_0^{2\pi} r\,dt = 2\pi r, \end{aligned}$$

since $|i| = 1$ and

$$\left|e^{it}\right| = |\cos t + i \sin t| = \sqrt{\cos^2 t + \sin^2 t} = 1.$$

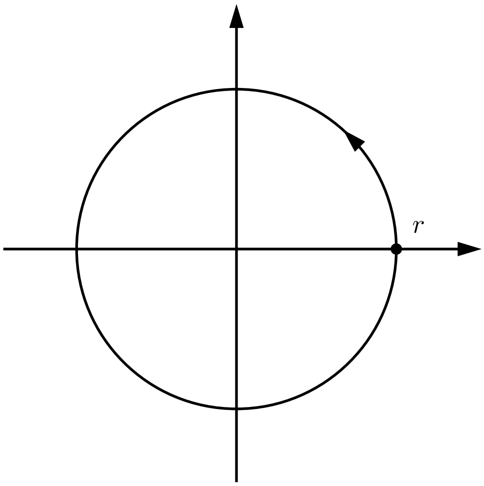

Figure 2.5 The path γ in Example 2.35

Now we introduce the notion of the integral along a path.

Definition 2.36

Let $f\colon \Omega \to \mathbb{C}$ be a continuous function and let $\gamma\colon [a,b] \to \Omega$ be a piecewise regular path. We define the *integral* of f along γ by

$$\begin{aligned} \int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b \operatorname{Re}\bigl[f(\gamma(t))\gamma'(t)\bigr]\,dt + i\int_a^b \operatorname{Im}\bigl[f(\gamma(t))\gamma'(t)\bigr]\,dt. \end{aligned}$$

We also write

$$\int_\gamma f = \int_\gamma f(z)\,dz.$$

We note that under the hypotheses of Definition 2.36, the functions

$$t \mapsto \operatorname{Re}\bigl[f(\gamma(t))\gamma'(t)\bigr] \quad \text{and} \quad t \mapsto \operatorname{Im}\bigl[f(\gamma(t))\gamma'(t)\bigr]$$

are Riemann-integrable in $[a,b]$, and thus the integral $\int_\gamma f$ is well defined.

Example 2.37

We compute the integral $\int_\gamma \operatorname{Re} z\,dz$ along the paths $\gamma_1, \gamma_2 \colon [0,1] \to \mathbb{C}$ given by

$$\gamma_1(t) = t(1+i) \quad \text{and} \quad \gamma_2(t) = t^2(1+i).$$

We have

$$\begin{aligned}\int_{\gamma_1} \operatorname{Re} z\,dz &= \int_0^1 \operatorname{Re}\bigl[t(1+i)\bigr] \cdot \bigl[t(1+i)\bigr]'\,dt \\ &= \int_0^1 t \cdot (1+i)\,dt \\ &= \frac{t^2}{2}(1+i)\Big|_{t=0}^{t=1} = \frac{1+i}{2}\end{aligned}$$

and

$$\begin{aligned}\int_{\gamma_2} \operatorname{Re} z\,dz &= \int_0^1 \operatorname{Re}\bigl[t^2(1+i)\bigr] \cdot \bigl[t^2(1+i)\bigr]'\,dt \\ &= \int_0^1 t^2 \cdot 2t(1+i)\,dt \\ &= \int_0^1 2t^3(1+i)\,dt \\ &= \frac{t^4}{2}(1+i)\Big|_{t=0}^{t=1} = \frac{1+i}{2}.\end{aligned}$$

Example 2.38

Now we compute the integral $\int_\gamma \operatorname{Im} z\,dz$ along the path $\gamma\colon [0,\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{it}$. We have $\operatorname{Im}\gamma(t) = \sin t$, and hence,

$$\begin{aligned}\int_\gamma \operatorname{Im} z\,dz &= \int_0^\pi \sin t \cdot ie^{it}\,dt \\ &= \int_0^\pi \frac{e^{it}-e^{-it}}{2i} ie^{it}\,dt \\ &= \int_0^\pi \frac{1}{2}\big(e^{2it}-1\big)\,dt = \left(\frac{1}{4i}e^{2it} - \frac{1}{2}t\right)\bigg|_{t=0}^{t=\pi} \\ &= \frac{1}{4i}\big(e^{2\pi i}-1\big) - \frac{1}{2}(\pi - 0) = 0 - \frac{\pi}{2} = -\frac{\pi}{2}.\end{aligned}$$

The integral has the following properties.

Proposition 2.39

If $f, g\colon \Omega \to \mathbb{C}$ are continuous functions and $\gamma\colon [a,b] \to \Omega$ is a piecewise regular path, then:

1. for any $c, d \in \mathbb{C}$, we have
$$\int_\gamma (cf + dg) = c\int_\gamma f + d\int_\gamma g;$$
2. $$\int_{-\gamma} f = -\int_\gamma f;$$
3. for any piecewise regular path $\alpha\colon [p,q] \to \Omega$ with $\alpha(p) = \gamma(b)$, we have
$$\int_{\gamma+\alpha} f = \int_\gamma f + \int_\alpha f.$$

Proof

For the second property, we note that

$$(-\gamma)'(t) = -\gamma'(a+b-t),$$

and thus,

$$\begin{aligned}\int_{-\gamma} f &= \int_a^b f\big((-\gamma)(t)\big)(-\gamma)'(t)\,dt\\ &= \int_a^b -f\big(\gamma(a+b-t)\big)\gamma'(a+b-t)\,dt.\end{aligned}$$

Making the change of variables $a+b-t=s$, we finally obtain

$$\begin{aligned}\int_{-\gamma} f &= \int_b^a f(\gamma(s))\gamma'(s)\,ds\\ &= -\int_a^b f(\gamma(s))\gamma'(s)\,ds\\ &= -\int_\gamma f.\end{aligned}$$

The remaining properties follow immediately from the definitions. □

We also describe two additional properties. For the first one we need the notion of equivalent paths.

Definition 2.40

Two paths $\gamma_1\colon [a_1,b_1]\to\mathbb{C}$ and $\gamma_2\colon [a_2,b_2]\to\mathbb{C}$ are said to be *equivalent* if there exists a differentiable function $\phi\colon [a_2,b_2]\to[a_1,b_1]$ with $\phi'>0$, $\phi(a_2)=a_1$, and $\phi(b_2)=b_1$, such that $\gamma_2=\gamma_1\circ\phi$.

We can now formulate the following result.

Proposition 2.41

If $f\colon \Omega\to\mathbb{C}$ is a continuous function, and γ_1 and γ_2 are equivalent piecewise regular paths in Ω, then

$$\int_{\gamma_1} f = \int_{\gamma_2} f.$$

Proof

We have

$$\int_{\gamma_2} f = \int_{a_2}^{b_2} f(\gamma_2(t))\gamma_2'(t)\,dt$$
$$= \int_{a_2}^{b_2} f\big((\gamma_1 \circ \phi)(t)\big)\gamma_1'(\phi(t))\phi'(t)\,dt.$$

Making the change of variables $s = \phi(t)$, we obtain

$$\int_{\gamma_2} f = \int_{a_1}^{b_1} f(\gamma_1(s))\gamma_1'(s)\,ds = \int_{\gamma_1} f,$$

which yields the desired identity. □

Finally, we obtain an upper bound for the modulus of the integral.

Proposition 2.42

If $f\colon \Omega \to \mathbb{C}$ is a continuous function and $\gamma\colon [a,b] \to \Omega$ is a piecewise regular path, then

$$\left|\int_\gamma f\right| \le \int_a^b \big|f(\gamma(t))\gamma'(t)\big|\,dt$$
$$\le L_\gamma \sup\big\{\big|f(\gamma(t))\big| : t \in [a,b]\big\}.$$

Proof

Writing $\int_\gamma f = re^{i\theta}$, we obtain

$$\left|\int_\gamma f\right| = r = \int_\gamma e^{-i\theta} f$$
$$= \int_a^b e^{-i\theta} f(\gamma(t))\gamma'(t)\,dt$$
$$= \int_a^b \operatorname{Re}\big[e^{-i\theta} f(\gamma(t))\gamma'(t)\big]\,dt + i\int_a^b \operatorname{Im}\big[e^{-i\theta} f(\gamma(t))\gamma'(t)\big]\,dt.$$

Since $|\int_\gamma f|$ is a real number, it follows from (1.6) that

$$\begin{aligned}\left|\int_\gamma f\right| &= \int_a^b \operatorname{Re}\left[e^{-i\theta} f(\gamma(t))\gamma'(t)\right] dt \\ &\le \int_a^b \left|e^{-i\theta} f(\gamma(t))\gamma'(t)\right| dt.\end{aligned}$$

Moreover, since $|e^{-i\theta}| = 1$, we obtain

$$\begin{aligned}\left|\int_\gamma f\right| &\le \int_a^b \left|f(\gamma(t))\gamma'(t)\right| dt \\ &\le \int_a^b \left|\gamma'(t)\right| dt \cdot \sup\left\{\left|f(\gamma(t))\right| : t \in [a,b]\right\} \\ &= L_\gamma \sup\left\{\left|f(\gamma(t))\right| : t \in [a,b]\right\}.\end{aligned}$$

This yields the desired inequalities. □

Example 2.43

Let us consider the integral

$$\int_\gamma z(z-1)\, dz$$

along the path $\gamma\colon [0,\pi] \to \mathbb{C}$ given by $\gamma(t) = 2e^{it}$. We have

$$L_\gamma = \int_0^\pi \left|2ie^{it}\right| dt = 2\pi.$$

By Proposition 2.42, since $|\gamma(t)| = 2$ for every $t \in [0,\pi]$, we obtain

$$\begin{aligned}\left|\int_\gamma f\right| &\le L_\gamma \sup\left\{|z(z-1)| : z \in \gamma\big([0,\pi]\big)\right\} \\ &\le 2\pi \sup\left\{\left|z^2\right| + |z| : z \in \gamma\big([0,\pi]\big)\right\} \\ &= 2\pi(4+2) = 12\pi.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_\gamma f &= \int_0^\pi \left[\gamma(t)^2 - \gamma(t)\right]\gamma'(t)\, dt \\ &= \left(\frac{\gamma(t)^3}{3} - \frac{\gamma(t)^2}{2}\right)\Bigg|_{t=0}^{t=\pi} = -\frac{16}{3}.\end{aligned}$$

2.5 Primitives

The concept of primitive is useful for the computation of integrals. Let us consider a function $f\colon \Omega \to \mathbb{C}$ in an open set $\Omega \subset \mathbb{C}$.

Definition 2.44

A function $F\colon \Omega \to \mathbb{C}$ is said to be a *primitive* of f in the set Ω if F is holomorphic in Ω and $F' = f$ in Ω.

We first show that in connected open sets all primitives differ by a constant.

Proposition 2.45

If F and G are primitives of f in some connected open set $\Omega \subset \mathbb{C}$, then $F - G$ is constant in Ω.

Proof

We have

$$(F - G)' = F' - G' = f - f = 0$$

in Ω. Hence, it follows from Proposition 2.19 that $F - G$ is constant in Ω. $\square$

Primitives can be used to compute integrals as follows.

Proposition 2.46

If F is a primitive of a continuous function $f\colon \Omega \to \mathbb{C}$ in an open set $\Omega \subset \mathbb{C}$ and $\gamma\colon [a, b] \to \Omega$ is a piecewise regular path, then

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a)).$$

Proof

For $j = 1, \ldots, n$, let $[a_j, b_j]$, with $b_1 = a_2$, $b_2 = a_3, \ldots,$ $b_{n-1} = a_n$, be the subintervals of $[a, b]$ where γ is regular. We note that the function

$$t \mapsto f(\gamma(t))\gamma'(t)$$

is continuous in each interval $[a_j, b_j]$. Therefore,

$$\begin{aligned}\int_\gamma f &= \sum_{j=1}^n \int_{\gamma_j} f = \sum_{j=1}^n \int_{a_j}^{b_j} f(\gamma(t))\gamma'(t)\,dt \\ &= \sum_{j=1}^n \int_{a_j}^{b_j} F'(\gamma(t))\gamma'(t)\,dt = \sum_{j=1}^n \int_{a_j}^{b_j} (F\circ\gamma)'(t)\,dt \\ &= \sum_{j=1}^n \big[F(\gamma(b_j)) - F(\gamma(a_j))\big] = F(\gamma(b)) - F(\gamma(a)).\end{aligned}$$

This yields the desired identity. □

Example 2.47

We consider the integral $\int_\gamma (z^3+1)\,dz$ along the path $\gamma\colon [0,\pi]\to\mathbb{C}$ given by $\gamma(t) = e^{it}$. Since

$$\left(\frac{z^4}{4}+z\right)' = z^3+1,$$

the function $F(z) = z^4/4 + z$ is a primitive of z^3+1 in $\mathbb{C}$. Therefore,

$$\begin{aligned}\int_\gamma \big(z^3+1\big)\,dz &= F(\gamma(\pi)) - F(\gamma(0)) \\ &= \left(\frac{1}{4}-1\right) - \left(\frac{1}{4}+1\right) = -2.\end{aligned}$$

We also consider paths with the same initial and final points.

Definition 2.48

A path $\gamma\colon [a,b]\to\mathbb{C}$ is said to be *closed* if $\gamma(a) = \gamma(b)$ (see Figure 2.6).

The following property is an immediate consequence of Proposition 2.46.

Proposition 2.49

If $f\colon \Omega\to\mathbb{C}$ is a continuous function having a primitive in the open set $\Omega\subset\mathbb{C}$ and $\gamma\colon [a,b]\to\Omega$ is a closed piecewise regular path, then

$$\int_\gamma f = 0.$$

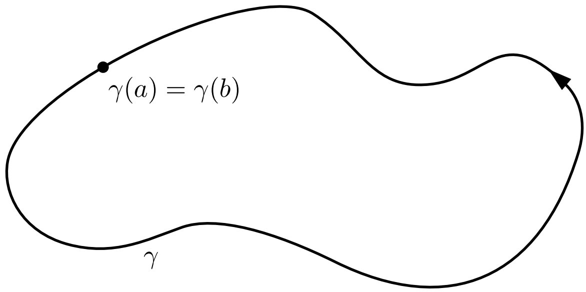

Figure 2.6 A closed path

Now we show that any holomorphic function has primitives. We recall that a set $\Omega \subset \mathbb{C}$ is said to be *convex* if

$$tz + (1-t)w \in \Omega$$

for every $z, w \in \Omega$ and $t \in [0,1]$.

Theorem 2.50

If $f\colon \Omega \to \mathbb{C}$ is a holomorphic function in a convex open set $\Omega \subset \mathbb{C}$, then f has a primitive in Ω.

More generally, we have the following result.

Theorem 2.51

If $f\colon \Omega \to \mathbb{C}$ is a continuous function in a convex open set $\Omega \subset \mathbb{C}$ and there exists $p \in \Omega$ such that f is holomorphic in $\Omega \setminus \{p\}$, then f has a primitive in Ω.

Proof

Take $a \in \Omega$. For each $z \in \Omega$, we consider the path $\gamma_z\colon [0,1] \to \Omega$ given by

$$\gamma_z(t) = a + t(z-a) \tag{2.10}$$

(we recall that Ω is convex). We also consider the function $F\colon \Omega \to \mathbb{C}$ defined by

$$F(z) = \int_{\gamma_z} f. \tag{2.11}$$

Lemma 2.52

We have

$$F(z+h)-F(z)=\int_\alpha f, \tag{2.12}$$

where the path $\alpha\colon [0,1]\to\mathbb{C}$ is given by $\alpha(t)=z+th$.

Proof of the lemma

Let Δ be the triangle whose boundary $\partial\Delta$ is the image of the closed path $\gamma_z+\alpha+(-\gamma_{z+h})$. We note that identity (2.12) is equivalent to

$$\begin{aligned}\int_{\partial\Delta} f &= \int_{\gamma_z} f+\int_\alpha f-\int_{\gamma_{z+h}} f\\ &= F(z)+\int_\alpha f-F(z+h)=0.\end{aligned} \tag{2.13}$$

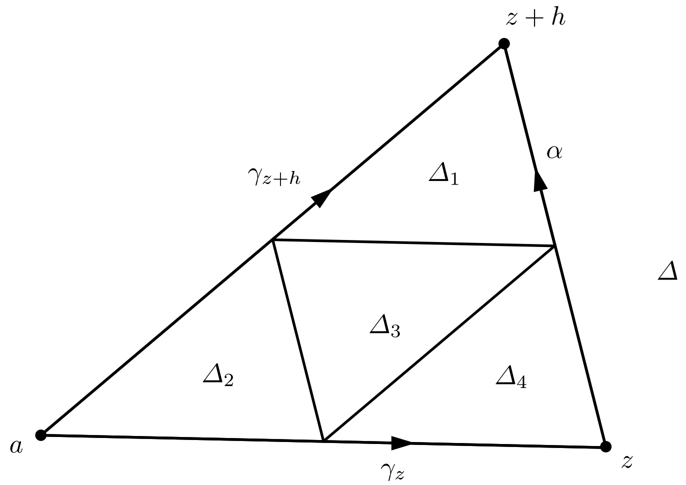

Figure 2.7 Triangles Δ_1, Δ_2, Δ_3 and Δ_4

We first assume that $p\notin\Delta$. We divide the triangle Δ into 4 triangles, say Δ_1, Δ_2, Δ_3 and Δ_4, by adding line segments between the midpoints of the sides of Δ (see Figure 2.7). Then

$$c:=\int_{\partial\Delta} f=\sum_{i=1}^4\int_{\partial\Delta_i} f,$$

in view of the fact that the integrals along common sides of the triangles Δ_i cancel out, since they have opposite signs. We note that there exists i such that

$$\left|\int_{\partial\Delta_i} f\right| \geq \frac{|c|}{4},$$

since otherwise we would have

$$\left|\sum_{i=1}^{4}\int_{\partial\Delta_i} f\right| < \sum_{i=1}^{4}\frac{|c|}{4} = |c|.$$

One can repeat the argument with this triangle Δ_i in order to obtain a sequence of triangles $\Delta(n) \subset \Delta(n-1)$ such that $\Delta(n)$ is one of the 4 triangles obtained from dividing $\Delta(n-1)$, and

$$\left|\int_{\partial\Delta(n)} f\right| \geq \frac{|c|}{4^n}. \tag{2.14}$$

On the other hand, since f is holomorphic in Δ, for each point $z_0 \in \Delta$, given $\varepsilon > 0$ we have

$$\big|f(z) - f(z_0) - f'(z_0)(z - z_0)\big| < \varepsilon|z - z_0|$$

whenever $|z - z_0|$ is sufficiently small. Since the perimeter of $\Delta(n)$ is

$$L_{\partial\Delta(n)} = 2^{-n}L_{\partial\Delta},$$

where $L_{\partial\Delta}$ is the perimeter of $\partial\Delta$, we obtain

$$\left|\int_{\partial\Delta(n)} \big[f(z) - f(z_0) - f'(z_0)(z - z_0)\big]\,dz\right| \leq \varepsilon L^2_{\partial\Delta(n)} = \varepsilon 4^{-n} L^2_{\partial\Delta} \tag{2.15}$$

for any sufficiently large n. Moreover, since the function $-f(z_0) - f'(z_0)(z - z_0)$ has the primitive $-f(z_0)z - f'(z_0)(z - z_0)^2/2$, we have

$$\int_{\partial\Delta_n} \big[-f(z_0) - f'(z_0)(z - z_0)\big]\,dz = 0,$$

and it follows from (2.14) and (2.15) that

$$|c| \leq 4^n\left|\int_{\partial\Delta(n)} f\right| \leq \varepsilon L^2_{\partial\Delta}.$$

Letting $\varepsilon \to 0$ we conclude that

$$c = \int_{\partial\Delta} f = 0,$$

which establishes (2.13).

Now we assume that $p \in \varDelta$. We note that it is sufficient to consider the case when p is a vertex. Otherwise, being p_1, p_2, p_3 the vertices of $\varDelta$, one can consider the three triangles determined by p_i, p_j, p with $i \neq j$. When p belongs to a side of $\varDelta$, one of these triangles reduces to a line segment (see Figure 2.8).

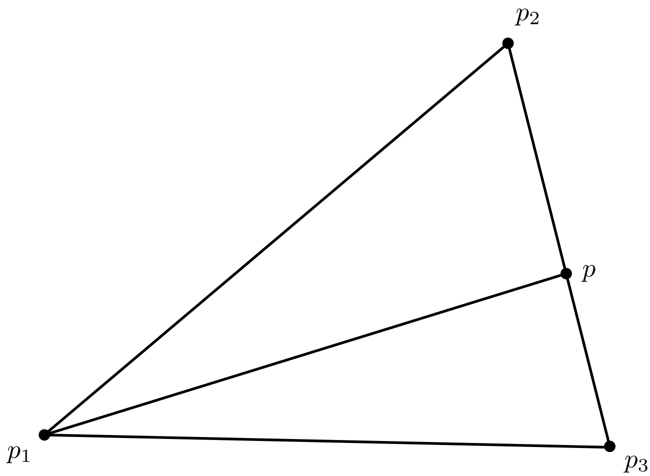

Figure 2.8 Case when p belongs to a side of $\varDelta$

When $p = p_3$ is a vertex of $\varDelta$, it is sufficient to consider triangles determined by points q_1 and q_2 in the sides containing p (see Figure 2.9). Indeed, by the previous argument, the triangles $\varDelta_1$ and $\varDelta_2$ respectively with vertices p_1, p_2, q_1 and p_1, q_1, q_2 have zero integral, that is,

$$\int_{\partial \varDelta_1} f = \int_{\partial \varDelta_2} f = 0.$$

Now let $\varDelta'$ be the triangle determined by q_1, q_2 and p. Letting $q_1 \to p$ and $q_2 \to p$, we conclude that

$$\left| \int_{\partial \varDelta'} f \right| \le L_{\partial \varDelta'} \sup\left\{ \left| f(z) \right| : z \in \varDelta' \right\} \to 0,$$

since $L_{\partial \varDelta'} \to 0$. This completes the proof of the lemma. $\square$

We are now ready to show that F is a primitive of f. It follows from

$$\int_\alpha f(z)\, d\zeta = \int_0^1 f(z) h \, dt = f(z) h$$

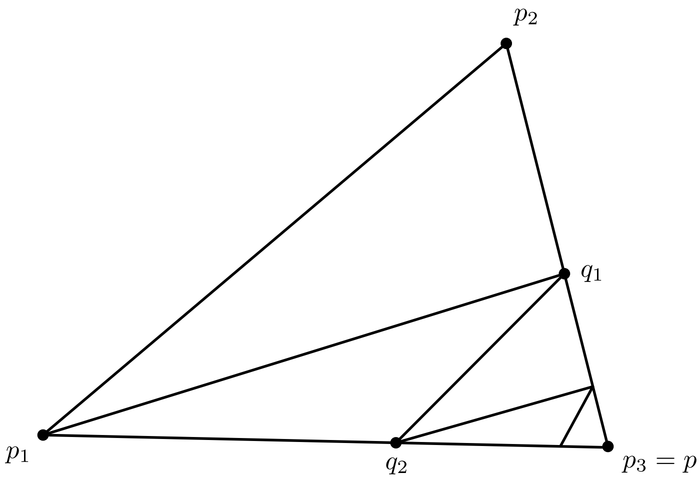

Figure 2.9 Case when p is a vertex of Δ

together with Lemma 2.52 that

$$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_\alpha \bigl[f(\zeta)-f(z)\bigr]\,d\zeta.$$

Since f is continuous, given $\varepsilon > 0$, we have

$$\bigl|f(\zeta)-f(z)\bigr| < \varepsilon$$

whenever $|\zeta - z|$ is sufficiently small. Therefore,

$$\begin{aligned}\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| &\le \frac{1}{|h|}\left|\int_\alpha \bigl[f(\zeta)-f(z)\bigr]\,d\zeta\right| \\ &\le \frac{\varepsilon L_\alpha}{|h|} = \varepsilon\end{aligned}$$

whenever $|h|$ is sufficiently small (since $|\zeta - z| \le |h|$). Letting $\varepsilon \to 0$ we thus obtain $F'(z) = f(z)$, and F is a primitive of f in Ω. □

Example 2.53

For the path γ_z in (2.10) we have $\gamma_z'(t) = z - a$, and by (2.11) a primitive of f is given by

$$F(z) = \int_0^1 f\bigl(a + t(z-a)\bigr)(z-a)\,dt. \tag{2.16}$$

In particular, when $0 \in \Omega$, taking $a = 0$ we obtain

$$F(z) = z \int_0^1 f(tz)\,dt. \tag{2.17}$$

Example 2.54

We have

$$\begin{aligned}\lim_{z\to 0} \frac{\sin z}{z} &= \lim_{z\to 0} \frac{\sin z - \sin 0}{z - 0}\\ &= (\sin z)'\big|_{z=0}\\ &= \cos 0 = 1.\end{aligned}$$

Hence, the function

$$f(z) = \begin{cases} (\sin z)/z & \text{if } z \neq 0,\\ 1 & \text{if } z = 0\end{cases}$$

is continuous in $\mathbb{C}$ and holomorphic in $\mathbb{C} \setminus \{0\}$. It thus follows from Theorem 2.51 that f has a primitive in $\mathbb{C}$. Moreover, by (2.17), a primitive is given by

$$F(z) = z \int_0^1 \frac{\sin(tz)}{tz}\,dt = \int_0^1 \frac{\sin(tz)}{t}\,dt.$$

The following result is an immediate consequence of Theorem 2.51 and Proposition 2.49.

Theorem 2.55 (Cauchy's theorem)

If $f\colon \Omega \to \mathbb{C}$ is a continuous function in a convex open set $\Omega \subset \mathbb{C}$ and there exists $p \in \Omega$ such that f is holomorphic in $\Omega \setminus \{p\}$, then

$$\int_\gamma f = 0$$

for any closed piecewise regular path γ in Ω.

2.6 Index of a Closed Path

Now we introduce the notion of the index of a closed path.

Definition 2.56

Given a closed piecewise regular path $\gamma\colon [a,b] \to \mathbb{C}$, we define the *index* of a point $z \in \mathbb{C} \setminus \gamma([a,b])$ with respect to γ by

$$\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{dw}{w-z}.$$

Example 2.57

Let $\gamma\colon [0,2\pi] \to \mathbb{C}$ be the path given by $\gamma(t) = a + re^{it}$. Then

$$\begin{aligned}\operatorname{Ind}_\gamma(a) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{rie^{it}}{re^{it}}\, dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} i\, dt = 1.\end{aligned}$$

The following result specifies the values that the index can take.

Theorem 2.58

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed piecewise regular path and let $\Omega = \mathbb{C} \setminus \gamma([a,b])$. Then:

1. $\operatorname{Ind}_\gamma(z) \in \mathbb{Z}$ for each $z \in \Omega$;
2. the function $z \mapsto \operatorname{Ind}_\gamma(z)$ is constant in each connected component of Ω;
3. $\operatorname{Ind}_\gamma(z) = 0$ for each z in the unbounded connected component of Ω.

Proof

We define a function $\phi\colon [a,b] \to \mathbb{C}$ by

$$\phi(s) = \exp\left(\int_a^s \frac{\gamma'(t)}{\gamma(t)-z}\, dt\right).$$

We have

$$\phi'(s) = \phi(s) \frac{\gamma'(s)}{\gamma(s)-z}$$

in each subinterval $[a_j, b_j]$ of $[a,b]$ where γ is regular. Then

$$\left(\frac{\phi(s)}{\gamma(s)-z}\right)' = \frac{\phi'(s)(\gamma(s)-z) - \gamma'(s)\phi(s)}{(\gamma(s)-z)^2} = 0,$$

and for each j there exists $c_j \in \mathbb{C}$ such that

$$\frac{\phi(s)}{\gamma(s)-z} = c_j$$

for every $s \in [a_j, b_j]$. But since γ and ϕ are continuous functions, we conclude that there exists $c \in \mathbb{C}$ such that

$$\frac{\phi(s)}{\gamma(s)-z} = c$$

for every $s \in [a, b]$. In particular,

$$\frac{\phi(s)}{\gamma(s)-z} = \frac{\phi(a)}{\gamma(a)-z} = \frac{1}{\gamma(a)-z},$$

that is,

$$\phi(s) = \frac{\gamma(s)-z}{\gamma(a)-z}.$$

Letting $s = b$, since γ is a closed path, we obtain

$$\phi(b) = \frac{\gamma(b)-z}{\gamma(a)-z} = 1,$$

that is,

$$\begin{aligned} \phi(b) &= \exp\left(\int_a^b \frac{\gamma'(t)}{\gamma(t)-z}\, dt\right) \\ &= \exp\bigl(2\pi i \operatorname{Ind}_\gamma(z)\bigr) = 1. \end{aligned} \tag{2.18}$$

We note that

$$e^{2\pi i\alpha} = 1 \quad \Leftrightarrow \quad \alpha \in \mathbb{Z},$$

since $e^{2\pi i\alpha} = \cos(2\pi\alpha) + i\sin(2\pi\alpha)$. It then follows from (2.18) that $\operatorname{Ind}_\gamma(z) \in \mathbb{Z}$.

For the second property, we first note that

$$\begin{aligned} \bigl|\operatorname{Ind}_\gamma(z) - \operatorname{Ind}_\gamma(w)\bigr| &= \left|\frac{1}{2\pi i}\int_\gamma \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right) d\zeta\right| \\ &= \frac{1}{2\pi}\left|\int_\gamma \frac{z-w}{(\zeta-z)(\zeta-w)}\, d\zeta\right| \\ &\le \frac{L_\gamma}{2\pi} \sup\left\{\frac{|z-w|}{|(\gamma(t)-z)(\gamma(t)-w)|} : t \in [a,b]\right\}. \end{aligned} \tag{2.19}$$

For w sufficiently close to z, we have

$$\left|\gamma(t)-w\right| \geq \left|\gamma(t)-z\right| - |z-w|,$$

and thus,

$$\begin{aligned}\frac{1}{|(\gamma(t)-z)(\gamma(t)-w)|} &\leq \frac{1}{|\gamma(t)-z|(|\gamma(t)-z|-|z-w|)}\\ &\leq \frac{1}{A(A-|z-w|)}\end{aligned}$$

for every $z \in \mathbb{C} \setminus \gamma([a,b])$, where

$$A = \inf\left\{\left|\gamma(t)-z\right| : t \in [a,b]\right\} > 0.$$

Hence, it follows from (2.19) that

$$\left|\operatorname{Ind}_\gamma(z) - \operatorname{Ind}_\gamma(w)\right| \leq \frac{L_\gamma}{2\pi} \frac{|z-w|}{A(A-|z-w|)},$$

and letting $w \to z$ we obtain

$$\lim_{w\to z} \operatorname{Ind}_\gamma(w) = \operatorname{Ind}_\gamma(z). \tag{2.20}$$

Since the index takes only integer values, it follows from the continuity in (2.20) that the function $z \mapsto \operatorname{Ind}_\gamma(z)$ is constant in each connected component of Ω (we note that since Ω is open, each connected component of Ω is an open set).

For the last property, we note that

$$\begin{aligned}\left|\operatorname{Ind}_\gamma(z)\right| &= \left|\frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)-z}\,dt\right|\\ &\leq \frac{1}{2\pi} L_\gamma \sup\left\{\frac{|\gamma'(t)|}{|\gamma(t)-z|} : t \in [a,b]\right\}\\ &\leq \frac{1}{2\pi} L_\gamma \frac{\sup\{\gamma'(t) : t \in [a,b]\}}{|z| - \sup\{\gamma(t) : t \in [a,b]\}},\end{aligned} \tag{2.21}$$

since

$$\left|\gamma(t)-z\right| \geq |z| - \left|\gamma(t)\right|$$

whenever $|z|$ is sufficiently large. In particular, it follows from (2.21) that $|\operatorname{Ind}_\gamma(z)| < 1$ for any sufficiently large $|z|$. Since the index takes only integer values, we obtain $\operatorname{Ind}_\gamma(z) = 0$. It follows again from the continuity in (2.20) that the index is zero in the unbounded connected component of Ω. □

Example 2.59

For each $n \in \mathbb{N}$, let $\gamma\colon [0, 2\pi n] \to \mathbb{C}$ be the path given by $\gamma(t) = a + re^{it}$, looping n times around the point a in the positive direction. Then

$$\begin{aligned}\operatorname{Ind}_\gamma(a) &= \frac{1}{2\pi i}\int_0^{2\pi n} \frac{\gamma'(t)}{\gamma(t) - a}\,dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi n} \frac{rie^{it}}{re^{it}}\,dt = n.\end{aligned}$$

It follows from Theorem 2.58 that

$$\operatorname{Ind}_\gamma(z) = \begin{cases} n & \text{if } |z-a| < r, \\ 0 & \text{if } |z-a| > r. \end{cases}$$

2.7 Cauchy's Integral Formula

Now we establish Cauchy's integral formula for a holomorphic function. In particular, it guarantees that any holomorphic function is uniquely determined by its values along closed paths.

Theorem 2.60

If $f\colon \Omega \to \mathbb{C}$ is a holomorphic function in a convex open set $\Omega \subset \mathbb{C}$ and $\gamma\colon [a, b] \to \Omega$ is a closed piecewise regular path, then

$$f(z)\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z}\,dw \tag{2.22}$$

for every $z \in \Omega \setminus \gamma([a, b])$.

Proof

Let us consider the function $g\colon \Omega \to \mathbb{C}$ defined by

$$g(w) = \begin{cases} (f(w) - f(z))/(w - z) & \text{if } w \in \Omega \setminus \{z\}, \\ f'(z) & \text{if } w = z. \end{cases}$$

Clearly, g is continuous in Ω and holomorphic in $\Omega \setminus \{z\}$. It then follows from Theorem 2.55 that

$$\begin{aligned}
0 &= \int_\gamma g \\
&= \int_\gamma \frac{f(w)-f(z)}{w-z}\,dw \\
&= \int_\gamma \frac{f(w)}{w-z}\,dw - f(z)\int_\gamma \frac{dw}{w-z} \\
&= \int_\gamma \frac{f(w)}{w-z}\,dw - f(z)2\pi i \operatorname{Ind}_\gamma(z).
\end{aligned}$$

This yields the desired identity. □

Example 2.61

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function in $\mathbb{C}$ and let $\gamma\colon [0,2\pi] \to \mathbb{C}$ be the path given by $\gamma(t) = z + re^{it}$. Then $\operatorname{Ind}_\gamma(z) = 1$, and by Theorem 2.60 we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+re^{it})}{re^{it}} rie^{it}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f\bigl(z+re^{it}\bigr)\,dt.
\end{aligned}$$

2.8 Integrals and Homotopy of Paths

In this section we show that the integral of a holomorphic function does not change with homotopies of the path. We first recall the notion of homotopy.

Definition 2.62

Two closed paths $\gamma_1, \gamma_2\colon [a,b] \to \Omega$ are said to be *homotopic* in Ω if there exists a continuous function $H\colon [a,b]\times[0,1] \to \Omega$ such that (see Figure 2.10):

1. $H(t,0) = \gamma_1(t)$ and $H(t,1) = \gamma_2(t)$ for every $t \in [a,b]$;
2. $H(a,s) = H(b,s)$ for every $s \in [0,1]$.

Then the function H is called a *homotopy* between γ_1 and γ_2.

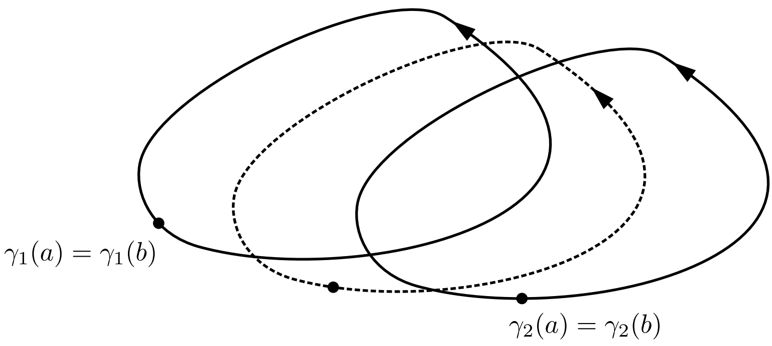

Figure 2.10 Homotopy of paths

We then have the following result.

Theorem 2.63

If $f\colon \Omega \to \mathbb{C}$ is a holomorphic function in an open set $\Omega \subset \mathbb{C}$, and γ_1 and γ_2 are closed piecewise regular paths that are homotopic in Ω, then

$$\int_{\gamma_1} f = \int_{\gamma_2} f. \tag{2.23}$$

Proof

Let H be a homotopy between the paths γ_1 and γ_2. We note that H is uniformly continuous (since it is defined in a compact set). Hence, there exists $n \in \mathbb{N}$ such that

$$\left|H(t,s) - H(t',s')\right| < r$$

for every $(t,s), (t',s') \in [a,b] \times [0,1]$ with

$$|t-t'| < \frac{2(b-a)}{n} \quad \text{and} \quad |s-s'| < \frac{2}{n}. \tag{2.24}$$

Now we consider the points

$$p_{j,k} = H\left(a + \frac{j}{n}(b-a), \frac{k}{n}\right), \quad j,k = 0,\dots,n,$$

and the closed polygons $P_{j,k}$ defined by the points

$$p_{j,k}, \quad p_{j+1,k}, \quad p_{j+1,k+1} \quad \text{and} \quad p_{j,k+1},$$

in this order. It follows from (2.24) that these four points are contained in the ball $B_r(p_{j,k})$ of radius r centered at $p_{j,k}$, and since any ball is a convex set, we also have $P_{j,k} \subset B_r(p_{j,k})$. It then follows from Theorem 2.55 that

$$\int_{\partial P_{j,k}} f = 0, \tag{2.25}$$

where $\partial P_{j,k}$ is the path along the boundary of $P_{j,k}$.

Now we consider the closed polygons Q_k defined by the points

$$p_{0,k}, \quad p_{1,k}, \quad \ldots, \quad p_{n-1,k} \quad \text{and} \quad p_{n,k},$$

in this order, as well as the paths $\alpha_j\colon [j/n, (j+1)/n] \to \mathbb{C}$ and $\beta_j\colon [0,1] \to \mathbb{C}$ given respectively by $\alpha_j(t) = \gamma_1(t)$ and

$$\beta_j(t) = p_{j+1,0} + t(p_{j,0} - p_{j+1,0}).$$

Since $\alpha_j + \beta_j$ is a closed path in the ball $B_r(p_{j,0})$, it follows again from Theorem 2.55 that

$$\int_{\gamma_j} f = -\int_{\beta_j} f = \int_{-\beta_j} f$$

for $j = 0, \ldots, n-1$. Therefore,

$$\int_{\gamma_1} f = \sum_{j=0}^{n-1} \int_{\alpha_j} f = \sum_{j=0}^{n-1} \int_{-\beta_j} f = \int_{\partial Q_0} f. \tag{2.26}$$

One can show in a similar manner that

$$\int_{\gamma_2} f = \int_{\partial Q_n} f. \tag{2.27}$$

On the other hand, it follows from (2.25) that

$$\sum_{j=0}^{n-1} \int_{\partial P_{j,k}} f = 0. \tag{2.28}$$

We note that the path $\partial P_{j,k}$ includes the line segment from $p_{j+1,k}$ to $p_{j+1,k+1}$, in this direction, while $\partial P_{j+1,k}$ includes the same segment but in the opposite direction, and thus the corresponding terms cancel out in the sum in (2.28). Moreover, $\partial P_{0,k}$ includes the line segment from $p_{0,k+1}$ to $p_{0,k}$, in this direction, while $\partial P_{n-1,k}$ includes the same segment but in the opposite direction. In fact, since each path $t \mapsto H(t,s)$ is closed, we have $p_{n,k+1} = p_{0,k+1}$ and $p_{n,k} = p_{0,k}$.

Therefore,

$$0=\sum_{j=0}^{n-1}\int_{\partial P_{j,k}} f=\int_{\partial Q_k} f-\int_{\partial Q_{k+1}} f,$$

that is,

$$\int_{\partial Q_{k+1}} f=\int_{\partial Q_k} f,$$

for $k=0,1,\dots,n-1$. Identity (2.23) now follows readily from (2.26) and (2.27). □

The following result is an immediate consequence of Theorem 2.63.

Theorem 2.64

If $f\colon \Omega\to\mathbb{C}$ is a holomorphic function in an open set $\Omega\subset\mathbb{C}$, and γ is a closed piecewise regular path that is homotopic to a constant path in Ω, then

$$\int_\gamma f=0.$$

We also show that the index does not change with homotopies of the path.

Proposition 2.65

Let γ_1 and γ_2 be closed piecewise regular paths that are homotopic in Ω. Then for each $z\in\mathbb{C}\setminus\Omega$, we have

$$\operatorname{Ind}_{\gamma_1}(z)=\operatorname{Ind}_{\gamma_2}(z). \tag{2.29}$$

Proof

Let us take $z\in\mathbb{C}\setminus\Omega$. We note that the function

$$f(w)=\frac{1}{2\pi i}\cdot\frac{1}{w-z}$$

is holomorphic in $\mathbb{C}\setminus\{z\}$, and thus in particular in Ω. Since

$$\int_{\gamma_j} f=2\pi i\operatorname{Ind}_{\gamma_j}(z)$$

for $j=1,2$, identity (2.29) follows readily from Theorem 2.63. □

2.9 Harmonic Conjugate Functions

In this section we discuss the concept of harmonic conjugate functions. We recall that a function $u\colon \Omega \to \mathbb{C}$ with second derivatives in some open set $\Omega \subset \mathbb{C}$ is said to be *harmonic* in Ω if $\Delta u = 0$, where the Laplacian Δu is defined by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Definition 2.66

Two harmonic functions $u, v\colon \Omega \to \mathbb{C}$ in the open set $\Omega \subset \mathbb{C}$ are said to be *harmonic conjugate functions* in Ω if u and v satisfy the Cauchy–Riemann equations in Ω.

If the function $f = u + iv$ is holomorphic in an open set $\Omega \subset \mathbb{C}$, then the Cauchy–Riemann equations are satisfied, and

$$\Delta u = \Delta v = 0 \quad \text{in } \Omega$$

(see Problem 2.24). In fact, one can show that $\Delta u = \Delta v = 0$ even without assuming a priori that u and v are of class C^2 (see Exercise 4.36). Therefore, the real and imaginary parts of a holomorphic function are harmonic conjugate functions.

We show that any harmonic function of class C^2 in a simply connected open set has a harmonic conjugate. We first recall the notions of a path connected set and a simply connected set.

Definition 2.67

A set $\Omega \subset \mathbb{C}$ is said to be *path connected* if for each $z, w \in \Omega$ there exists a path $\gamma\colon [a,b] \to \Omega$ with $\gamma(a) = z$ and $\gamma(b) = w$.

In particular, a path connected set is necessarily connected.

Definition 2.68

A set $\Omega \subset \mathbb{C}$ is said to be *simply connected* if it is path connected and any closed path $\gamma\colon [a,b] \to \Omega$ is homotopic to a constant path in Ω.

We then have the following result.

Proposition 2.69

Let $u\colon \Omega \to \mathbb{C}$ be a function of class C^2 in a simply connected open set $\Omega \subset \mathbb{C}$. If $\Delta u = 0$, then there exists a function $v\colon \Omega \to \mathbb{C}$ of class C^2 with $\Delta v = 0$ such that u and v are harmonic conjugate functions. Moreover, the function v is unique up to a constant.

Proof

Since Ω is simply connected and u is of class C^2, it follows from Green's theorem that if α is a closed path in Ω without intersections, then

$$\begin{aligned}\int_\alpha -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy &= \int_U \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial x}\right)\right) dx\,dy \\ &= \int_U \Delta u\,dx\,dy = 0,\end{aligned} \tag{2.30}$$

where U is the open set whose boundary is the image of α. This shows that given $p \in \Omega$, one can define a function $v\colon \Omega \to \mathbb{C}$ by the line integral

$$v(x,y) = \int_\gamma -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy, \tag{2.31}$$

where $\gamma\colon [a,b] \to \Omega$ is any path between p and (x,y). Now we show that the Cauchy–Riemann equations are satisfied. It follows from (2.30) that

$$\begin{aligned}\frac{\partial v}{\partial x}(x,y) &= \lim_{h\to 0} \frac{v(x+h,y) - v(x,y)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \int_{\gamma_h} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy,\end{aligned}$$

where the path $\gamma_h\colon [0,1] \to \mathbb{R}^2$ is given by

$$\gamma_h(t) = (x + th, y).$$

Since

$$\int_{\gamma_h} -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy = \int_0^1 -\frac{\partial u}{\partial x}(x+th,y)h\,dt,$$

and the function $-\partial u/\partial x$ is continuous, we obtain

$$\frac{\partial v}{\partial x}(x,y) = \lim_{h\to 0} \int_0^1 -\frac{\partial u}{\partial x}(x+th,y)\,dt = -\frac{\partial u}{\partial x}(x,y).$$

One can show in a similar manner that

$$\frac{\partial v}{\partial y}(x,y)=\frac{\partial u}{\partial x}(x,y),$$

and hence, the Cauchy–Riemann equations are satisfied in Ω. Moreover, v is of class C^2 and thus $\Delta v=0$.

It remains to show that v is unique up to a constant. By Theorem 2.23, the function $f=u+iv$ is holomorphic in Ω. If w is another function of class C^2 with $\Delta w=0$ such that $f=u+iw$ is holomorphic in Ω, then

$$u+iv-(u+iw)=i(v-w)$$

is also holomorphic in Ω. Since Ω is connected (because it is simply connected) and $i(v-w)$ has constant real part, it follows from Example 2.20 that $v-w$ is constant. □

The following result can be established in a similar manner.

Proposition 2.70

Let $v\colon \Omega\to\mathbb{C}$ be a function of class C^2 in a simply connected open set $\Omega\subset\mathbb{C}$. If $\Delta v=0$, then there exists a function $u\colon \Omega\to\mathbb{C}$ of class C^2 with $\Delta u=0$ such that u and v are harmonic conjugate functions. Moreover, the function u is unique up to a constant.

Proof

Since Ω is simply connected and v is of class C^2, given $p\in\Omega$, one can define a function $u\colon \Omega\to\mathbb{C}$ by the line integral

$$u(x,y)=\int_\gamma \frac{\partial v}{\partial y}\,dx-\frac{\partial v}{\partial x}\,dy,$$

where $\gamma\colon [a,b]\to\Omega$ is any path between p and (x,y). We can now proceed in a similar manner to that in the proof of Proposition 2.69 to show that the Cauchy–Riemann equations are satisfied. □

We also give some examples.

Example 2.71

We consider the function $f = u + iv$ with real part $u(x, y) = x^2 - xy - y^2$ as in Example 2.15. Since u is of class C^2 and

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

in the simply connected open set $\Omega = \mathbb{C}$, by Proposition 2.69 there exists a function v of class C^2 such that $f = u + iv$ is holomorphic in $\mathbb{C}$. By (2.31), one can take

$$\begin{aligned} v(x, y) &= \int_\gamma -\frac{\partial u}{\partial y}\, dx + \frac{\partial u}{\partial x}\, dy \\ &= \int_\gamma (x + 2y)\, dx + (2x - y)\, dy, \end{aligned}$$

with the path $\gamma\colon [0, 1] \to \mathbb{C}$ given by $\gamma(t) = (tx, ty)$. We then obtain

$$\begin{aligned} v(x, y) &= \int_0^1 \big[(tx + 2ty)x + (2tx - ty)y\big]\, dt \\ &= \left(\frac{1}{2}t^2x^2 + t^2yx + t^2xy - \frac{1}{2}t^2y^2\right)\Big|_{t=0}^{t=1} \\ &= \frac{x^2}{2} + 2xy - \frac{y^2}{2}. \end{aligned}$$

Example 2.72

Now we consider the function $u(x, y) = x^2 + y^2$ as in Example 2.16. Since u is of class C^2 and $\Delta u = 4 \neq 0$, the function u is not the real part of any holomorphic function in an open set $\Omega \subset \mathbb{C}$.

Example 2.73

Let us consider the function $u(x, y) = ax^2 + by$, with $a, b \in \mathbb{R}$. Since u is of class C^2 and $\Delta u = 2a$, in order that u is the real part of a holomorphic function in some open set we must have $a = 0$. Moreover, it follows from Proposition 2.69 that if $a = 0$, then there exists a function v of class C^2 in $\mathbb{R}^2$ such that

$$f(x + iy) = u(x, y) + iv(x, y) = by + iv(x, y)$$

is holomorphic in $\mathbb{C}$. One can use the Cauchy–Riemann equations to determine v. Indeed, it follows from the equation $\partial u/\partial x = \partial v/\partial y$ that $\partial v/\partial y = 0$, and hence v does not depend on y. Moreover,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -b,$$

and thus $v(x,y) = -bx + c$ for some constant $c \in \mathbb{R}$.

2.10 Solved Problems and Exercises

Problem 2.1

Verify that the function $f(z) = z^2 - z$ is continuous in $\mathbb{C}$.

Solution

Writing

$$f(x+iy) = u(x,y) + iv(x,y), \tag{2.32}$$

with $x, y \in \mathbb{R}$, we obtain

$$u(x,y) = x^2 - y^2 - x \quad \text{and} \quad v(x,y) = 2xy - y.$$

Since u and v are continuous in $\mathbb{R}^2$, the function f is continuous in $\mathbb{C}$.

Problem 2.2

Use the Cauchy–Riemann equations to show that the function $f(z) = e^z + z$ is holomorphic in $\mathbb{C}$.

Solution

One can write the function f in the form (2.32), with

$$u(x,y) = e^x \cos y + x \quad \text{and} \quad v(x,y) = e^x \sin y + y.$$

The Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.33}$$

take the form

$$e^x \cos y + 1 = e^x \cos y + 1 \quad \text{and} \quad -e^x \sin y = -e^x \sin y,$$

and thus they are satisfied in $\mathbb{R}^2$. Since u and v are functions of class C^1 in the open set $\mathbb{R}^2$, it follows from Theorem 2.23 that f is holomorphic in $\mathbb{C}$.

Problem 2.3

Show that

$$\left(z^n\right)' = nz^{n-1} \tag{2.34}$$

for every $n \in \mathbb{N}$ and $z \in \mathbb{C}$ (with the convention that $0^0 = 1$).

Solution

Let $f_n(z) = z^n$. For $n = 1$ we have

$$f_1'(z_0) = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = 1,$$

which establishes (2.34). For $n > 1$, it follows from the identity

$$z^n - z_0^n = (z - z_0) \sum_{k=0}^{n-1} z^k z_0^{n-1-k}$$

that

$$f_n'(z_0) = \lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = \lim_{z \to z_0} \sum_{k=0}^{n-1} z^k z_0^{n-1-k} = nz_0^{n-1}.$$

Problem 2.4

Use the definition of derivative to verify that $|z|$ is not differentiable at $z = 0$.

Solution

Writing $z = |z|e^{i\theta}$, we obtain

$$\frac{|z| - |0|}{z - 0} = \frac{|z|}{z} = \frac{|z|}{|z|e^{i\theta}} = e^{-i\theta}.$$

Since $e^{-i\theta}$ depends on θ, one cannot take the limit when $z \to 0$, and hence f is not differentiable at the origin.

Problem 2.5

Find all points $z \in \mathbb{C}$ at which the function $|z|$ is differentiable.

Solution

We have $|x+iy| = u(x,y) + iv(x,y)$, where

$$u(x,y) = \sqrt{x^2+y^2} \quad \text{and} \quad v(x,y) = 0.$$

The Cauchy–Riemann equations in (2.33) are thus

$$\frac{x}{\sqrt{x^2+y^2}} = 0 \quad \text{and} \quad \frac{y}{\sqrt{x^2+y^2}} = 0.$$

We note that these have no solutions ($x = y = 0$ is not a solution, since one cannot divide by zero). Hence, by Theorem 2.14, the function $|z|$ has no points of differentiability.

Problem 2.6

Find all points of differentiability of the function $f(x+iy) = xy + iy$.

Solution

We write the function f in the form (2.32), with $u(x,y) = xy$ and $v(x,y) = y$. The Cauchy–Riemann equations in (2.33) are thus $y = 1$ and $x = 0$. Hence, by Theorem 2.14, the function f is not differentiable at any point of $\mathbb{C} \setminus \{i\}$. Since the set $\{i\}$ is not open, in order to determine whether f is differentiable at i we must use the definition of derivative, that is, we have to verify whether

$$\frac{f(x+iy) - f(i)}{x+iy-i} = \frac{xy+iy-i}{x+iy-i}$$

has a limit when $x + iy \to i$. Since

$$\begin{aligned}\frac{xy+iy-i}{x+iy-i} &= \frac{x(y-1)+x+i(y-1)}{x+i(y-1)} \\ &= \frac{x(y-1)}{x+i(y-1)} + 1\end{aligned}$$

and

$$\left|\frac{x(y-1)}{x+i(y-1)}\right| = \frac{|x| \cdot |y-1|}{|x+i(y-1)|} \le |x| \to 0$$

when $x + iy \to i$, we conclude that f is differentiable at i, with $f'(i) = 1$.

Problem 2.7

Find all constants $a, b \in \mathbb{R}$ such that the function

$$f(x+iy) = ax^2 + 2xy + by^2 + i\big(y^2 - x^2\big)$$

is holomorphic in $\mathbb{C}$.

Solution

We first note that taking $z = x + iy$, we have $z^2 = x^2 - y^2 + i2xy$, and thus,

$$-iz^2 = 2xy + i\big(y^2 - x^2\big).$$

Therefore,

$$f(z) = ax^2 + by^2 - iz^2.$$

Since the function $-iz^2$ is holomorphic in $\mathbb{C}$, it is sufficient to find constants $a, b \in \mathbb{R}$ such that the function

$$f(z) + iz^2 = ax^2 + by^2 + i0$$

is holomorphic in $\mathbb{C}$. By Theorem 2.23, this happens if and only if the Cauchy–Riemann equations in (2.33) are satisfied in $\mathbb{R}^2$, that is, if and only if

$$2ax = 0 \quad \text{and} \quad 2by = 0$$

for every $x, y \in \mathbb{R}$. Therefore, $a = b = 0$.

Problem 2.8

Find whether there exists $a \in \mathbb{R}$ such that the function

$$f(x+iy) = ax^2 + 2xy + i\big(x^2 - y^2 - 2xy\big)$$

is holomorphic in $\mathbb{C}$.

Solution

By Theorem 2.23, the function f is holomorphic in $\mathbb{C}$ if and only if the Cauchy–Riemann equations are satisfied in $\mathbb{R}^2$. In this case they take the form

$$2ax + 2y = -2y - 2x \quad \text{and} \quad 2x = -(2x - 2y),$$

or equivalently

$$(a+1)x = -2y \quad \text{and} \quad 2x = y. \tag{2.35}$$

We then obtain $(a+1)x = -4x$, and thus $a = -5$. Hence, the equations in (2.35) reduce to the identity $2x = y$, which does not hold for every x and y. Therefore, there exists no $a \in \mathbb{R}$ such that the function f is holomorphic in $\mathbb{C}$.

Problem 2.9

Let $f = u + iv$ be a holomorphic function in $\mathbb{C}$ with real part

$$u(x,y) = 2x^2 - 3xy - 2y^2.$$

Compute explicitly $f(z)$ and $f'(z)$.

Solution

Since f is holomorphic in $\mathbb{C}$, the Cauchy–Riemann equations in (2.33) are satisfied in $\mathbb{R}^2$. It follows from

$$\frac{\partial u}{\partial x} = 4x - 3y$$

and the first equation in (2.33) that

$$\frac{\partial v}{\partial y} = 4x - 3y.$$

Therefore,

$$v(x,y) = 4xy - \frac{3y^2}{2} + C(x)$$

for some differentiable function C. We then obtain

$$\frac{\partial u}{\partial y} = -3x - 4y \quad \text{and} \quad -\frac{\partial v}{\partial x} = -4y - C'(x),$$

and it follows from the second equation in (2.33) that $C'(x) = 3x$. Therefore,

$$C(x) = \frac{3x^2}{2} + c \quad \text{for some } c \in \mathbb{R},$$

and

$$v(x,y) = 4xy - \frac{3y^2}{2} + \frac{3x^2}{2} + c.$$

Hence,

$$\begin{aligned}
f(x+iy) &= \left(2x^2-3xy-2y^2\right)+i\left(4xy-\frac{3y^2}{2}+\frac{3x^2}{2}+c\right)\\
&= 2\left(x^2-y^2+2ixy\right)+\frac{3}{2}i\left(x^2-y^2+2ixy\right)+ic\\
&= 2z^2+\frac{3}{2}iz^2+ic=\left(2+\frac{3}{2}i\right)z^2+ic,
\end{aligned}$$

and thus $f'(z)=(4+3i)z$.

Problem 2.10

Find whether there exists a holomorphic function in $\mathbb{C}$ with real part x^2-y^2+y.

Solution

We note that the function $u(x,y)=x^2-y^2+y$ is of class C^2 in the simply connected open set $\mathbb{R}^2$. Since $\Delta u=0$, by Proposition 2.69 there exists a harmonic conjugate function, that is, a function v such that $f=u+iv$ is holomorphic in $\mathbb{C}$. In other words, there exists a holomorphic function in $\mathbb{C}$ with real part u.

Problem 2.11

Find whether there exists a holomorphic function f in $\mathbb{C}$ with real part $x-y+1$, and if so determine such a function.

Solution

In order to show that there exists such a function f it is sufficient to observe that $u(x,u)=x-y+1$ is of class C^2 in the simply connected open set $\mathbb{R}^2$ and that $\Delta u=0$. Indeed, by Proposition 2.69, this implies that u has a harmonic conjugate function.

Now we determine a holomorphic function

$$f(x+iy)=u(x,y)+iv(x,y)$$

with $u(x,y)=x-y+1$. The Cauchy–Riemann equations must be satisfied in $\mathbb{R}^2$. It follows from $\partial u/\partial x=1$ and the first equation in (2.33) that $\partial v/\partial y=1$. Hence,

$$v(x,y)=y+C(x)$$

for some differentiable function C. We then obtain

$$\frac{\partial u}{\partial y} = -1 \quad \text{and} \quad -\frac{\partial v}{\partial x} = -C'(x),$$

and hence $C'(x) = 1$. Therefore, $C(x) = x + c$ for some constant $c \in \mathbb{R}$, and

$$v(x, y) = y + x + c.$$

We conclude that

$$\begin{aligned} f(x+iy) &= (x - y + 1) + i(y + x + c) \\ &= (x + iy) + i(x + iy) + 1 + ic \\ &= (1 + i)z + 1 + ic. \end{aligned}$$

Problem 2.12

Find all values of $a, b \in \mathbb{R}$ for which the function $u(x, y) = ax^2 + xy + by^2$ is the real part of a holomorphic function in $\mathbb{C}$, and determine explicitly all such functions.

Solution

We write $f(x + iy) = u(x, y) + iv(x, y)$. In order that f is holomorphic in $\mathbb{C}$ the Cauchy–Riemann equations must be satisfied in $\mathbb{R}^2$. It follows from

$$\frac{\partial u}{\partial x} = 2ax + y$$

and the first equation in (2.33) that

$$\frac{\partial v}{\partial y} = 2ax + y.$$

Hence,

$$v(x, y) = 2axy + \frac{y^2}{2} + C(x)$$

for some differentiable function C. We obtain

$$\frac{\partial u}{\partial y} = x + 2by \quad \text{and} \quad -\frac{\partial v}{\partial x} = -2ay - C'(x),$$

and thus, $b = -a$ and $C'(x) = -x$. Therefore, $C(x) = -x^2/2 + c$ for some constant $c \in \mathbb{R}$, and

$$v(x, y) = 2axy + \frac{y^2}{2} - \frac{x^2}{2} + c.$$

We conclude that

$$\begin{aligned}
f(x+iy) &= \left(ax^2+xy-ay^2\right)+i\left(2axy+\frac{y^2}{2}-\frac{x^2}{2}+c\right)\\
&= a\left(x^2-y^2+2ixy\right)-\frac{i}{2}\left(x^2-y^2+2ixy\right)+ic\\
&= \left(a-\frac{i}{2}\right)z^2+ic,
\end{aligned}$$

with $a, c \in \mathbb{R}$.

Problem 2.13

Show that if $f, g\colon \Omega \to \mathbb{C}$ are holomorphic functions in an open set $\Omega \subset \mathbb{C}$, then

$$(f+g)' = f' + g' \quad \text{and} \quad (fg)' = f'g + fg'.$$

Solution

Since f and g are holomorphic in Ω, the derivatives

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \quad \text{and} \quad g'(z_0) = \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}$$

are well defined for each $z_0 \in \Omega$. Therefore,

$$\begin{aligned}
(f+g)'(z_0) &= \lim_{z\to z_0} \frac{f(z)+g(z)-f(z_0)-g(z_0)}{z-z_0}\\
&= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} + \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}\\
&= f'(z_0)+g'(z_0)
\end{aligned}$$

and

$$\begin{aligned}
(fg)'(z_0) &= \lim_{z\to z_0} \frac{f(z)g(z)-f(z_0)g(z_0)}{z-z_0}\\
&= \lim_{z\to z_0} \frac{(f(z)-f(z_0))g(z_0)+f(z)(g(z)-g(z_0))}{z-z_0}\\
&= \lim_{z\to z_0} \frac{(f(z)-f(z_0))g(z_0)}{z-z_0} + \lim_{z\to z_0} \frac{f(z)(g(z)-g(z_0))}{z-z_0}
\end{aligned}$$

$$= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} g(z_0) + \lim_{z\to z_0} f(z) \cdot \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}$$
$$= f'(z_0)g(z_0) + f(z_0)g'(z_0).$$

Problem 2.14

Show that if f and g are holomorphic functions in $\mathbb{C}$ with $f(z_0) = g(z_0) = 0$ and $g'(z_0) \neq 0$, then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

Solution

We have

$$\begin{aligned}
\lim_{z\to z_0} \frac{f(z)}{g(z)} &= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{g(z)-g(z_0)} \\
&= \lim_{z\to z_0} \frac{(f(z)-f(z_0))/(z-z_0)}{(g(z)-g(z_0))/(z-z_0)} \\
&= \frac{f'(z_0)}{g'(z_0)}.
\end{aligned}$$

Problem 2.15

Compute the length of the path $\gamma\colon [0,1] \to \mathbb{C}$ given by $\gamma(t) = e^{it}\cos t$ (see Figure 2.11).

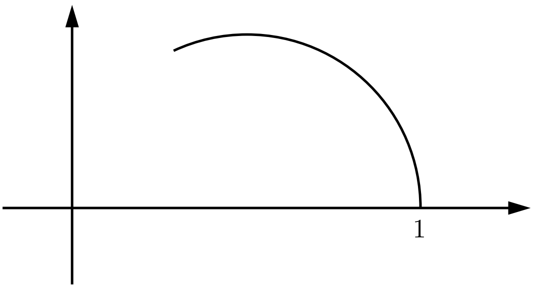

Figure 2.11 Curve defined by the path γ in Problem 2.15

Solution

The length of a piecewise regular path $\gamma\colon [a,b] \to \mathbb{C}$ is given by (2.9). We then have

$$\begin{aligned} L_\gamma &= \int_0^1 \left| ie^{it}\cos t - e^{it}\sin t \right| dt \\ &= \int_0^1 \left| e^{it} \right| \cdot \left| i\cos t - \sin t \right| dt \\ &= \int_0^1 1\, dt = 1. \end{aligned}$$

Problem 2.16

Compute the integral

$$\int_\gamma \left(z^2 - \overline{z}\right) dz$$

along the path $\gamma\colon [0,1] \to \mathbb{C}$ given by $\gamma(t) = e^{it}$.

Solution

We have

$$\int_\gamma \left(z^2 - \overline{z}\right) dz = \int_0^1 \left(\gamma(t)^2 - \overline{\gamma(t)}\right)\gamma'(t)\, dt,$$

and hence,

$$\begin{aligned} \int_\gamma \left(z^2 - \overline{z}\right) dz &= \int_0^1 \left(e^{2it} - e^{-it}\right) ie^{it}\, dt \\ &= i\int_0^1 \left(e^{3it} - 1\right) dt \\ &= \left(\frac{1}{3}e^{3it} - it\right)\Bigg|_{t=0}^{t=1} = \frac{1}{3}e^{3i} - i - \frac{1}{3}. \end{aligned}$$

Problem 2.17

For each $n \in \mathbb{Z}$, compute the integral

$$\int_\gamma \cos(nz)\, dz$$

along the path $\gamma\colon [0,1] \to \mathbb{C}$ given by $\gamma(t) = e^{\pi it}$.

Solution

Let $f_n(z) = \cos(nz)$. If $n = 0$, then $F_0(z) = z$ is a primitive of $f_0(z) = 1$, and thus,

$$\begin{aligned}\int_\gamma f_0(z)\,dz = \int_\gamma 1\,dz = F_0(\gamma(t))\Big|_{t=0}^{t=1} \\ = e^{\pi i t}\Big|_{t=0}^{t=1} = e^{\pi i} - 1 = -2.\end{aligned}$$

If $n \neq 0$, then $F_n(z) = \sin(nz)/n$ is a primitive of f_n, and thus,

$$\begin{aligned}\int_\gamma f_n(z)\,dz = F_n(\gamma(t))\Big|_{t=0}^{t=1} &= \frac{1}{n}\sin\big(n\gamma(t)\big)\Big|_{t=0}^{t=1} \\ &= \frac{1}{n}\sin\big(ne^{\pi i}\big) - \frac{1}{n}\sin\big(ne^{0}\big) \\ &= \frac{1}{n}\sin(-n) - \frac{1}{n}\sin n = -\frac{2}{n}\sin n.\end{aligned}$$

Problem 2.18

Compute the integral $\int_\gamma z\,dz$, where $\gamma\colon [a,b] \to \mathbb{C}$ is a path looping once along the boundary of the square defined by the condition $|x| + |y| \leq 3$ (see Figure 2.12), in the positive direction.

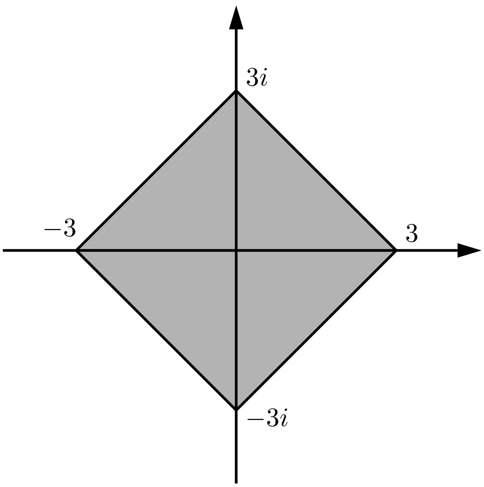

Figure 2.12 Square defined by the condition $|x| + |y| \leq 3$

Solution

We note that the function $f(z) = z$ is holomorphic in $\mathbb{C}$, and that the boundary of the square defined by the condition $|x| + |y| \leq 3$ is homotopic to the circle of radius 3 centered at 0. It thus follows from Theorem 2.63 that

$$\int_\gamma z\,dz = \int_\alpha z\,dz,$$

where the path $\alpha\colon [0,1] \to \mathbb{C}$ is given by $\alpha(t) = 3e^{2\pi it}$. Hence,

$$\begin{aligned}
\int_\gamma z\,dz &= \int_0^1 3e^{2\pi it} 6\pi i e^{2\pi it}\,dt \\
&= 18\pi i \int_0^1 e^{4\pi it}\,dt \\
&= \frac{9}{2}\big(e^{4\pi i} - 1\big) = 0.
\end{aligned}$$

Problem 2.19

For each $n \in \mathbb{N}$, show that

$$\int_0^{2\pi} (2\cos t)^{2n}\,dt = 2\pi \binom{2n}{n}. \tag{2.36}$$

Solution

Let us consider the integral

$$I = \int_\gamma \frac{1}{z}\left(z + \frac{1}{z}\right)^{2n} dz,$$

where the path $\gamma\colon [0,2\pi] \to \mathbb{C}$ is given by $\gamma(t) = e^{it}$. We have

$$\begin{aligned}
I &= \int_\gamma \frac{1}{z} \sum_{k=0}^{2n} \binom{2n}{k} z^k \big(z^{-1}\big)^{2n-k}\,dz \\
&= \sum_{k=0}^{n} \binom{2n}{k} \int_\gamma z^{2k-2n-1}\,dz.
\end{aligned} \tag{2.37}$$

Since

$$\int_\gamma z^p\,dz = \begin{cases} 2\pi i & \text{if } p = -1, \\ 0 & \text{if } p \in \mathbb{Z} \setminus \{-1\}, \end{cases}$$

the only nonzero term in (2.37) occurs when $2k-2n-1=-1$, that is, $k=n$, and we obtain

$$I=\binom{2n}{n}\int_\gamma z^{-1}\,dz=2\pi i\binom{2n}{n}. \tag{2.38}$$

On the other hand,

$$I=\int_0^{2\pi} e^{-it}\big(e^{it}+e^{-it}\big)^{2n} ie^{it}\,dt=i\int_0^{2\pi}(2\cos t)^{2n}\,dt. \tag{2.39}$$

Comparing (2.38) and (2.39), we obtain identity (2.36).

Problem 2.20

For the path $\gamma\colon [0,\pi]\to\mathbb{C}$ given by $\gamma(t)=e^{it}$, show that

$$\left|\int_\gamma \frac{e^z}{z}\,dz\right|\le \pi e. \tag{2.40}$$

Solution

The length of γ is given by

$$L_\gamma=\int_0^\pi \big|\gamma'(t)\big|\,dt=\int_0^\pi \big|ie^{it}\big|\,dt=\pi,$$

and thus, by Proposition 2.42,

$$\begin{aligned}\left|\int_\gamma \frac{e^z}{z}\,dz\right| &\le L_\gamma \sup\left\{\left|\frac{e^z}{z}\right| : z\in\gamma\big([0,\pi]\big)\right\}\\ &=\pi\sup\left\{\frac{|e^{\gamma(t)}|}{|\gamma(t)|} : t\in[0,\pi]\right\}.\end{aligned} \tag{2.41}$$

Since $|\gamma(t)|=1$ and

$$\big|e^{\gamma(t)}\big|=\big|e^{\cos t+i\sin t}\big|=\big|e^{\cos t}e^{i\sin t}\big|=e^{\cos t}\le e$$

for each $t\in[0,\pi]$, inequality (2.40) follows readily from (2.41).

Problem 2.21

Find all functions $u\colon \mathbb{R}^2\to\mathbb{R}$ of class C^1 such that

$$f(x+iy)=u(x,y)+iu(x,y) \tag{2.42}$$

is a holomorphic function in $\mathbb{C}$.

Solution

By Theorem 2.23, in order that f is holomorphic in $\mathbb{C}$, the Cauchy–Riemann equations in (2.33) must be satisfied in $\mathbb{R}^2$ with $u = v$, that is,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}.$$

In particular, we have

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y},$$

and thus,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0. \tag{2.43}$$

Since the open set $\mathbb{R}^2$ is connected, it follows from (2.43) that u is constant. Therefore, the holomorphic functions in $\mathbb{C}$ of the form (2.42) are the constant functions $a + ia$, with $a \in \mathbb{R}$.

Problem 2.22

Show that if f and $\overline{f}$ are holomorphic functions in $\mathbb{C}$, then f is constant in $\mathbb{C}$.

Solution

Writing the function f in the form (2.32), we obtain

$$\overline{f(x+iy)} = \overline{u(x,y) + iv(x,y)} = u(x,y) - iv(x,y).$$

Since f and $\overline{f}$ are holomorphic in $\mathbb{C}$, in addition to the Cauchy–Riemann equations in (2.33) for the function f, the Cauchy–Riemann equations for $\overline{f} = u - iv$ are also satisfied, that is,

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{2.44}$$

It follows from (2.33) and (2.44) that

$$\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y} \quad \text{and} \quad -\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x},$$

and hence,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0. \tag{2.45}$$

Since the open set $\mathbb{R}^2$ is connected, it follows from (2.45) that v is constant. It then follows from Example 2.20 that f is constant.

Problem 2.23

For the function $u\colon \mathbb{R}^2 \to \mathbb{R}$ given by $u(x,y) = e^x \sin y$:

1. find a function v such that $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic in $\mathbb{C}$ and $f(0) = -i$;
2. compute the integral $\int_\gamma (f(z)/z)\,dz$, where γ is the circle of radius 4 centered at the origin, looping three times in the negative direction.

Solution

1. In order that f is holomorphic in $\mathbb{C}$, the Cauchy–Riemann equations must be satisfied in $\mathbb{R}^2$, and thus,
$$e^x \sin y = \frac{\partial v}{\partial y} \quad \text{and} \quad e^x \cos y = -\frac{\partial v}{\partial x}. \tag{2.46}$$
It follows from the first equation that
$$v(x,y) = -e^x \cos y + C(x)$$
for some differentiable function C. Thus, it follows from the second equation in (2.46) that $e^x \cos y = e^x \cos y - C'(x)$, and hence, $C(x) = c$ for some constant $c \in \mathbb{R}$. We then obtain
$$f(x+iy) = e^x \sin y + i\bigl(-e^x \cos y + c\bigr) = -ie^z + ic,$$
and it follows from $f(0) = -i$ that $c = 0$. Hence, $f(z) = -ie^z$.
2. By Cauchy's integral formula in (2.22), since $\operatorname{Ind}_\gamma(0) = -3$, we obtain
$$\int_\gamma \frac{f(z)}{z}\,dz = 2\pi i f(0)\operatorname{Ind}_\gamma(0) = 2\pi i \cdot (-i) \cdot (-3) = -6\pi.$$

Problem 2.24

Let $f = u + iv$ be a holomorphic function in an open set $\Omega \subset \mathbb{C}$. Show that if u and v are of class C^2, then
$$\Delta u = \Delta v = 0 \quad \text{in } \Omega.$$

Solution

Since f is holomorphic in Ω, the Cauchy–Riemann equations are satisfied in Ω. Taking derivatives in these equations with respect to x and y we obtain respectively

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}, \tag{2.47}$$

and

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}. \tag{2.48}$$

On the other hand, since u and v are of class C^2, we have

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}.$$

Thus, combining the first equation in (2.47) with the second in (2.48), we obtain

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Analogously, combining the second equation in (2.47) with the first in (2.48), we obtain

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Problem 2.25

Let $f = u + iv$ be a holomorphic function in an open set $\Omega \subset \mathbb{C}$. Show that if u and v are of class C^2, then $\Delta(uv) = 0$ in Ω.

Solution

We obtain

$$\begin{aligned}\Delta(uv) &= \frac{\partial^2 (uv)}{\partial x^2} + \frac{\partial^2 (uv)}{\partial y^2} \\ &= \frac{\partial^2 u}{\partial x^2} v + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} v + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2} \\ &= (\Delta u)v + u\Delta v + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}.\end{aligned} \tag{2.49}$$

On the other hand, by Problem 2.24, we have $\Delta u = \Delta v = 0$ in Ω. Together with the Cauchy–Riemann equations, this implies that

$$\begin{aligned}\Delta(uv) &= 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \\ &= 2\frac{\partial v}{\partial y}\left(-\frac{\partial u}{\partial y}\right) + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0.\end{aligned}$$

Problem 2.26

Let $f = u + iv$ be a holomorphic function in an open set $\Omega \subset \mathbb{C}$. Show that if u and v are of class C^2, then $\Delta(u^2 + v^2) \geq 0$ in Ω.

Solution

By Problem 2.24, we have $\Delta u = \Delta v = 0$ in Ω. Setting $u = v$ in (2.49), we then obtain

$$\begin{aligned}\frac{1}{2}\Delta\big(u^2 + v^2\big) &= u\Delta u + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + v\Delta v + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \geq 0.\end{aligned}$$

Problem 2.27

Let f be a holomorphic function in some open set $\Omega \subset \mathbb{C}$ such that

$$\big|f(z) - 1\big| < 1 \quad \text{for } z \in \Omega. \tag{2.50}$$

Show that

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0$$

for any closed piecewise regular path γ in Ω.

Solution

It follows from (2.50) that f never vanishes in Ω. Therefore, the function $g\colon \Omega \to \mathbb{C}$ given by $g(z) = \log f(z)$ is well defined. It also follows from (2.50) that the image of f does not intersect the half-line $\mathbb{R}_0^- \subset \mathbb{C}$, and thus g is holomorphic in Ω. We then have

$$g'(z) = \frac{f'(z)}{f(z)},$$

and g is a primitive of f'/f. Since the path γ is closed, it follows from Proposition 2.49 that

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_\gamma g'(z)\,dz = 0.$$

Problem 2.28

Show that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rr\cos\theta+r^2}\,d\theta = 1, \quad 0<r<R.$$

Solution

We have

$$\begin{aligned}\frac{R+re^{i\theta}}{R-re^{i\theta}} &= \frac{(R+re^{i\theta})(R-re^{-i\theta})}{(R-re^{i\theta})(R-re^{-i\theta})} \\ &= \frac{R^2-r^2+2irR\sin\theta}{R^2-2rR\cos\theta+r^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2rR\cos\theta+r^2}\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{R+re^{i\theta}}{R-re^{i\theta}}\right)d\theta \\ &= \operatorname{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{R+re^{i\theta}}{R-re^{i\theta}}\,d\theta\right) \\ &= \operatorname{Re}\left(\frac{1}{2\pi i}\int_\gamma \frac{R+z}{z(R-z)}\,dz\right),\end{aligned}$$

where the path $\gamma\colon [0,2\pi]\to\mathbb{C}$ is given by $\gamma(\theta)=re^{i\theta}$. Moreover,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{R+z}{z(R-z)}\,dz &= \frac{1}{2\pi i}\int_\gamma \left(\frac{1}{z}+\frac{2}{R-z}\right)dz\\ &= \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz + \frac{1}{2\pi i}\int_\gamma \frac{2}{R-z}\,dz\\ &= 1+\frac{1}{2\pi i}\int_\gamma \frac{2}{R-z}\,dz.\end{aligned}$$

On the other hand, since the function $f(z) = 2/(R-z)$ is holomorphic for $|z| < R$, it follows from Cauchy's theorem (Theorem 2.55) that

$$\int_\gamma \frac{2}{R-z}\,dz = 0,$$

and hence,

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2rR\cos\theta+r^2}\,d\theta = \operatorname{Re}\left(1+\frac{1}{2\pi i}\int_\gamma \frac{2}{R-z}\,dz\right) = 1.$$

Problem 2.29

Verify that the function $f(z) = (z+1)\log z$ is continuous at $z=-1$.

Solution

Since $f(-1) = 0$, in order to verify that f is continuous at $z=-1$, one must show that

$$\lim_{z\to -1} f(z) = 0. \tag{2.51}$$

We first observe that since

$$\log z = \log|z| + i\arg z,$$

with $\arg z \in (-\pi,\pi]$, we have

$$|\log z| = \sqrt{(\log|z|)^2 + (\arg z)^2} \le \sqrt{(\log|z|)^2+\pi^2}.$$

Hence,

$$|\log z| \le \sqrt{1+\pi^2}$$

for $|z| < e$, and thus,

$$\big|f(z)\big| = |z+1| \cdot |\log z| \le |z+1|\sqrt{1+\pi^2} \to 0$$

when $z \to -1$ (we note that when we let $z \to -1$, one can always assume that $|z| < e$, since $|-1| < e$). This shows that (2.51) holds, and the function f is continuous at $z = -1$.

Problem 2.30

Find all continuous functions $f: \mathbb{C} \to \mathbb{C}$ such that $f(z)^2 = 1$ for $z \in \mathbb{C}$.

Solution

It follows from $f(z)^2 = 1$ that $f(z) = 1$ or $f(z) = -1$, for each $z \in \mathbb{C}$. We show that f takes only one of these values. Otherwise, there would exist $z_1, z_2 \in \mathbb{C}$ with $f(z_1) = 1$ and $f(z_2) = -1$, but by the continuity of f there would also exist a point z in the line segment between z_1 and z_2 with $f(z) \ne 1$ and $f(z) \ne -1$. But this contradicts the fact that f can only take the values 1 and -1. Therefore, either $f = 1$ or $f = -1$.

Problem 2.31

Compute the integral

$$\int_0^\infty \frac{\sin(t^2)}{t}\,dt.$$

Solution

Given $r, R > 0$, with $r < R$, we consider the path $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, where

$$\begin{aligned}
&\gamma_1 : [r,R] \to \mathbb{C} && \text{is given by } \gamma_1(t) = t,\\
&\gamma_2 : [0,\pi/2] \to \mathbb{C} && \text{is given by } \gamma_2(t) = Re^{it},\\
&\gamma_3 : [r,R] \to \mathbb{C} && \text{is given by } \gamma_3(t) = i(r+R-t),\\
&\gamma_4 : [0,\pi/2] \to \mathbb{C} && \text{is given by } \gamma_4(t) = e^{i(\pi/2-t)}
\end{aligned}$$

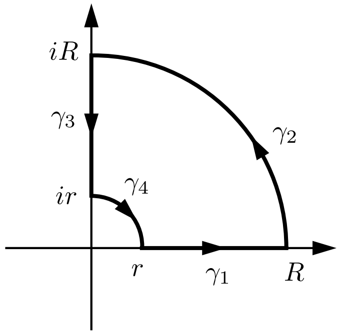

Figure 2.13 Path $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$

(see Figure 2.13). We also consider the function $f(z) = e^{iz^2}/z$. It follows from Cauchy's theorem (Theorem 2.55) that

$$\begin{aligned}
0 &= \int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f + \int_{\gamma_4} f \\
&= \int_r^R \frac{e^{it^2}}{t}\,dt + i\int_0^{\pi/2} e^{i(Re^{it})^2}\,dt \\
&\quad + \int_r^R \frac{e^{-i(r+R-t)^2}}{t}\,dt + i\int_0^{\pi/2} e^{i[re^{i(\pi/2-t)}]^2}\,dt \\
&= \int_r^R \frac{e^{it^2}}{t}\,dt + i\int_0^{\pi/2} e^{i(Re^{it})^2}\,dt \\
&\quad - \int_r^R \frac{e^{-it^2}}{t}\,dt - i\int_0^{\pi/2} e^{i(re^{it})^2}\,dt. \qquad (2.52)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left| i\int_0^{\pi/2} e^{i(Re^{it})^2}\,dt \right| &\le \int_0^{\pi/2} \left| e^{i(Re^{it})^2} \right| dt \\
&= \int_0^{\pi/2} e^{-R^2\sin(2t)}\,dt \\
&= 2\int_0^{\pi/4} e^{-R^2\sin(2t)}\,dt.
\end{aligned}$$

Now we consider the function

$$h(t) = \sin(2t) - 4t/\pi.$$

Since

$$h''(t) = -4\sin(2t) < 0 \quad \text{for } t \in (0, \pi/4),$$

the derivative

$$h'(t) = 2\cos(2t) - 4/\pi$$

is strictly decreasing in $[0, \pi/4]$. Hence, since $h'(0) > 0$ and $h'(\pi/4) < 0$, there exists a unique $s \in (0, \pi/4)$ such that h is increasing in $[0, s]$ and decreasing in $[s, \pi/4]$. Since $h(0) = h(\pi/4) = 0$, we conclude that $h(t) \geq 0$ for $t \in [0, \pi/4]$. Therefore,

$$\begin{aligned}
\left| i \int_0^{\pi/2} e^{i(Re^{it})^2}\, dt \right| &\leq 2 \int_0^{\pi/4} e^{-R^2 \sin(2t)}\, dt \\
&\leq 2 \int_0^{\pi/4} e^{-4R^2 t/\pi}\, dt \\
&= \frac{\pi}{2R^2}\left(1 - e^{-R^2}\right) \to 0
\end{aligned}$$

when $R \to +\infty$. It then follows from (2.52) that

$$\begin{aligned}
0 &= \int_r^R \frac{e^{it^2}}{t}\, dt + i \int_0^{\pi/2} e^{i(Re^{it})^2}\, dt - \int_r^R \frac{e^{-it^2}}{t}\, dt - i \int_0^{\pi/2} e^{i(re^{it})^2}\, dt \\
&\to \int_r^\infty \frac{e^{it^2} - e^{-it^2}}{t}\, dt - i \int_0^{\pi/2} e^{i(re^{it})^2}\, dt \\
&= 2i \int_r^\infty \frac{\sin(t^2)}{t}\, dt - i \int_0^{\pi/2} e^{i(re^{it})^2}\, dt
\end{aligned}$$

when $R \to +\infty$, and thus,

$$\int_r^\infty \frac{\sin(t^2)}{t}\, dt = \frac{1}{2} \int_0^{\pi/2} e^{i(re^{it})^2}\, dt. \tag{2.53}$$

Since the function e^{iz^2} is continuous, given $\varepsilon > 0$, there exists $r > 0$ such that $|e^{iz^2} - 1| < \varepsilon$ for every $z \in \mathbb{C}$ with $|z| \leq r$. Therefore,

$$\begin{aligned}
\left| \int_0^{\pi/2} e^{i(re^{it})^2}\, dt - \frac{\pi}{2} \right| &= \left| \int_0^{\pi/2} \left(e^{i(re^{it})^2} - 1\right) dt \right| \\
&\leq \int_0^{\pi/2} \left| e^{i(re^{it})^2} - 1 \right| dt \leq \frac{\varepsilon\pi}{2},
\end{aligned}$$

and it follows from (2.53) that

$$\begin{aligned}\left|\int_r^\infty \frac{\sin(t^2)}{t}\,dt - \frac{\pi}{4}\right| &= \frac{1}{2}\left|\int_0^{\pi/2} e^{i(re^{it})^2}\,dt - \frac{\pi}{2}\right| \\ &\le \frac{1}{2}\int_0^{\pi/2} \left|e^{i(re^{it})^2} - 1\right|\,dt \le \frac{\varepsilon\pi}{4}\end{aligned}$$

for any sufficiently small r. Letting $r\to 0$ and then $\varepsilon\to 0$, we conclude that

$$\int_0^\infty \frac{\sin(t^2)}{t}\,dt = \frac{\pi}{4}.$$

EXERCISES

2.1. Compute the limit, if it exists:
 (a) $\lim_{z\to 0}\frac{z}{\overline{z}}$;
 (b) $\lim_{z\to i}(\operatorname{Im} z - \operatorname{Re} z)$;
 (c) $\lim_{z\to 3} z^z$.

2.2. Verify that the functions $\operatorname{Re} z$, $\operatorname{Im} z$ and $|z|$ are continuous in $\mathbb{C}$.

2.3. Find whether the function $f(z) = z + \cos z$ is continuous in $\mathbb{C}$.

2.4. Determine the set of points $z\in\mathbb{C}$ where the function is continuous:
 (a) $x|z|$;
 (b) $\begin{cases} z^3/|z|^2 & \text{if } z\ne 0,\\ 0 & \text{if } z=0;\end{cases}$
 (c) $(z+1)\log z$.

2.5. Verify that the function $f(z) = (1-\log z)\log z$ is not continuous.

2.6. Determine the set of points $z\in\mathbb{C}$ where the function is differentiable:
 (a) $\operatorname{Re} z\cdot\operatorname{Im} z$;
 (b) $\operatorname{Re} z + \operatorname{Im} z$;
 (c) $z^2 - |z|^2$;
 (d) $|z|(z-1)$.

2.7. Determine the set of points $z\in\mathbb{C}$ where the function is differentiable:
 (a) $e^x\cos y - ie^x\sin y$;
 (b) $x^2y + ixy$;
 (c) $x(y-1) + ix^2(y-1)$.

2.8. Compute $(\log\log z)'$ and indicate its domain.

2.9. Let f be a holomorphic function in $\mathbb{C}$ with real part $xy - x^2 + y^2 - 1$ such that $f(0) = -1$. Find $f(z)$ explicitly and compute the second derivative $f''(z)$.

2.10. Find the constants $a, b \in \mathbb{R}$ for which the function u is the real part of a holomorphic function in $\mathbb{C}$:
(a) $u(x,y) = ax + by$;
(b) $u(x,y) = ax^2 - bxy$;
(c) $u(x,y) = ax^2 - by^2 + xy$;
(d) $u(x,y) = ax^2 + 3xy - by^4$;
(e) $u(x,y) = ax^2 + \cos x \cos y + by^2$.

2.11. For the values of $a, b \in \mathbb{R}$ obtained in Exercise 2.10, find a holomorphic function f in $\mathbb{C}$ with real part u.

2.12. Find whether there exists $a \in \mathbb{R}$ such that the function

$$f(x+iy) = ax^2 + 2xy + i\big(x^2 - y^2 - 2xy\big)$$

is holomorphic in $\mathbb{C}$.

2.13. Find all constants $a, b \in \mathbb{R}$ such that the function

$$f(x+iy) = ax^2 + 2xy + by^2 + i\big(y^2 - x^2\big)$$

is holomorphic in $\mathbb{C}$.

2.14. For each $a, b, c \in \mathbb{C}$, compute the integral

$$\int_\gamma \big(az^2 + bz + c\big)\, dz,$$

where the path $\gamma\colon [0,1] \to \mathbb{C}$ is given by $\gamma(t) = it$.

2.15. Compute the integral $\int_\gamma (3z^2 + 3)\, dz$ along a path $\gamma\colon [a,b] \to \mathbb{C}$ with $\gamma(a) = 3$ and $\gamma(b) = 2 + i$.

2.16. Compute the integral:
(a) $\int_\gamma z\overline{z}^2\, dz$ along the path $\gamma\colon [0,1] \to \mathbb{C}$ given by $\gamma(t) = 2e^{it}$;
(b) $\int_\gamma (e^z/z)\, dz$ along the path $\gamma\colon [0,2] \to \mathbb{C}$ given by $\gamma(t) = e^{2\pi i t}$.

2.17. Find a primitive of the function $y + e^x \cos y - i(x - e^x \sin y)$ in $\mathbb{C}$.

2.18. Identify each statement as true or false.
(a) The function $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = (|z|^2 - 2)\overline{z}$ is differentiable at $z = 0$.
(b) There exists a closed regular path γ such that $\int_\gamma \sin z\, dz \neq 0$.
(c) $\int_\gamma e^z\, dz = 0$ for some closed path γ whose image is the boundary of a square.
(d) There exists a holomorphic function f in $\mathbb{C} \setminus \{0\}$ such that $f'(z) = 1/z$ in $\mathbb{C} \setminus \{0\}$.
(e) If f is holomorphic in $\mathbb{C}$ and has real part $4xy + 2e^x \sin y$, then $f(z) = -2i(e^z + z^2)$.
(f) The largest open ball centered at the origin where the function $z^2 + z$ is one-to-one has radius 1.

(g) For the path $\gamma\colon [0,2\pi]\to\mathbb{C}$ given by $\gamma(t)=2e^{it}$, we have

$$\left|\int_\gamma \frac{\cos z}{z}\,dz\right| < 2\pi e.$$

2.19. Compute the derivative

$$\frac{d}{ds}\int_\gamma \left(s^2z+sz^2\right)dz$$

for each $s\in\mathbb{R}$, where the path $\gamma\colon [0,1]\to\mathbb{C}$ is given by $\gamma(t)=e^{\pi it}$.

2.20. Compute the derivative

$$\frac{d}{ds}\int_\gamma \frac{e^{s(z+1)}}{z}\,dz$$

for each $s\in\mathbb{R}$, where the path $\gamma\colon [0,1]\to\mathbb{C}$ is given by $\gamma(t)=e^{2\pi it}$.

2.21. Compute the index $\operatorname{Ind}_\gamma(-1)$ for the path $\gamma\colon [0,2]\to\mathbb{C}$ given by

$$\gamma(t)=\left[1+t(2-t)\right]e^{2\pi it}$$

(see Figure 2.14).

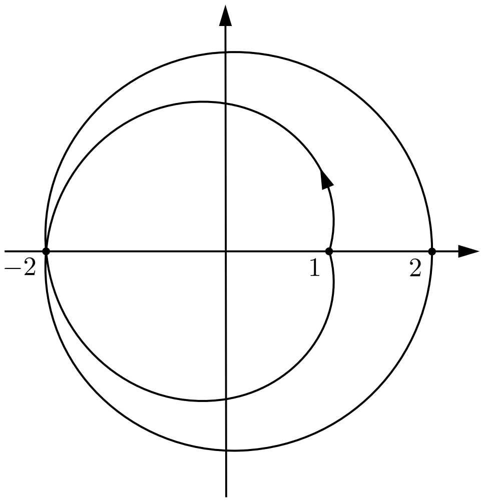

Figure 2.14 Path γ in Exercise 2.21

2.22. Find all functions $u\colon \mathbb{R}^2\to\mathbb{R}$ of class C^1 such that $f=u+iu^2$ is a holomorphic function in $\mathbb{C}$.

2.23. Find all holomorphic functions in $\mathbb{C}$ whose real part is twice the imaginary part.

2.24. For a function

$$f\left(re^{i\theta}\right) = a(r,\theta) + ib(r,\theta),$$

show that the Cauchy–Riemann equations are equivalent to

$$\frac{\partial a}{\partial r} = \frac{1}{r}\frac{\partial b}{\partial \theta} \quad \text{and} \quad \frac{\partial b}{\partial r} = -\frac{1}{r}\frac{\partial a}{\partial \theta}.$$

2.25. Show that the function $\log z$ is holomorphic in the open set $\mathbb{C} \setminus \mathbb{R}_0^-$. Hint: use Exercise 2.24.

2.26. Find all points where the function is differentiable:
(a) $\log(z-1)$;
(b) $(z-1)\log(z-1)$.

2.27. Show that

$$\left|\int_\gamma \frac{e^z}{z}\,dz\right| < \pi e$$

for the path $\gamma\colon [0,\pi] \to \mathbb{C}$ given by $\gamma(t) = e^{it}$.

2.28. For a path $\gamma\colon [0,1] \to \mathbb{C}$ satisfying $|\gamma(t)| < 1$ for every $t \in [0,1]$, show that

$$\sum_{n=1}^\infty \int_\gamma nz^{n-1}\,dz = \int_\gamma \frac{dz}{(1-z)^2}.$$

2.29. Given a function $f(x+iy) = u(x,y) + iv(x,y)$, since

$$x = \frac{z+\overline{z}}{2} \quad \text{and} \quad y = \frac{z-\overline{z}}{2i},$$

one can define

$$g(z,\overline{z}) = f\left(\frac{z+\overline{z}}{2} + i\frac{z-\overline{z}}{2i}\right) = f(x+iy). \tag{2.54}$$

Show that f satisfies the Cauchy–Riemann equations at (x_0,y_0) if and only if

$$\frac{\partial g}{\partial \overline{z}}(x_0+iy_0, x_0-iy_0) = 0.$$

2.30. For the function g in (2.54), show that if

$$\frac{\partial g}{\partial z} = \frac{\partial g}{\partial \overline{z}} = 0$$

in some connected open set $\Omega \subset \mathbb{C}$, then f is constant in Ω.

2.31. Show that the integral $\int_\gamma \overline{f(z)}f'(z)\,dz$ is purely imaginary for any closed piecewise regular path γ, and any function f of class C^1 in an open set containing the image of γ.

2.32. Show that if $f\colon \mathbb{C}\to\mathbb{C}$ is a bounded continuous function, then

$$\lim_{r\to\infty}\int_{\gamma_r}\frac{f(z)}{z^2}\,dz=0 \quad\text{and}\quad \lim_{r\to 0}\int_{\gamma_r}\frac{f(z)}{z}\,dz=2\pi i f(0),$$

where the path $\gamma_r\colon [0,2\pi]\to\mathbb{C}$ is given by $\gamma_r(t)=re^{it}$.

2.33. Let $f\colon \mathbb{C}\to\mathbb{C}$ be a holomorphic function in $\mathbb{C}$.

(a) For the path $\gamma\colon [0,2\pi]\to\mathbb{C}$ given by $\gamma(t)=z+re^{it}$, show that

$$|f(z)|\le\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(z+re^{it}\right)\right|dt.$$

(b) Show that if the function f has a maximum in the closed ball $\{w\in\mathbb{C}: |w-z|\le r\}$, then it occurs at the boundary.

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