Chapter 2
Introduction to Matrix Notation

2.1 Introduction

In this chapter matrix notation is introduced as a tool for solving a pair of linear equations. This reveals three important features about matrices: The first is the existence of a scalar value associated with a matrix called the *determinant*; the second is *matrix multiplication*, and the third is *matrix inversion*. This prepares us for the next chapter where we investigate the nature of the determinant for larger matrices.

2.2 Solving a Pair of Linear Equations

There are two simple ways to solve a pair of linear equations such as

$$24 = 6x + 4y \tag{2.1}$$

$$10 = 2x + 2y. \tag{2.2}$$

The first technique is graphical and the second is algebraic.

2.2.1 Graphical Technique

The graphical technique represents the equations as two straight lines which may be coincident, parallel or intersect. The point of intersection could be located anywhere with respect to the Cartesian axes, and could make it extremely difficult to identify an accurate x–y position.

Figure 2.1 shows two lines representing (2.1) and (2.2), where the solution is the point of intersection $(2, 3)$. Although the solution is easy to identify for these equations, a more reliable and accurate technique is required. So let's consider an algebraic approach.

J. Vince, *Matrix Transforms for Computer Games and Animation*,
DOI 10.1007/978-1-4471-4321-5_2,

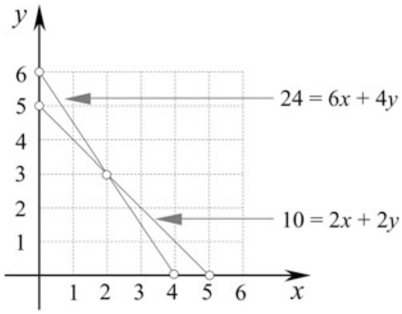

Fig. 2.1 Graphs of the simultaneous linear equations

2.2.2 Algebraic Technique

The algebraic strategy is to manipulate (2.1) and (2.2) such that when they are added or subtracted, the x or y coefficient disappears, which permits one variable to be identified. The second variable is revealed by substituting the first in one of the original equations. We begin by multiplying (2.2) by 2 to turn the $2y$ term into $4y$:

$$2 \times 10 = 2 \times 2x + 2 \times 2y$$

$$20 = 4x + 4y.$$

The pair of equations now become

$$24 = 6x + 4y \tag{2.3}$$

$$20 = 4x + 4y. \tag{2.4}$$

Subtracting (2.4) from (2.3) produces

$$4 = 2x,$$

which means that $x = 2$. To discover the corresponding value of y, we substitute $x = 2$ in (2.1) to discover that $y = 3$.

This algebraic approach always provides an accurate result, so long as the original equations are linearly independent. Now let's find a general solution for any pair of linear equations in two unknowns, with the proviso that they are linearly independent. We start with the following pair of equations:

$$r = ax + by \tag{2.5}$$

$$s = cx + dy. \tag{2.6}$$

To eliminate the y coefficient we multiply (2.5) by d and (2.6) by b:

$$dr = adx + bdy \tag{2.7}$$

$$bs = bcx + bdy. \tag{2.8}$$

Next, we subtract (2.8) from (2.7):

$$dr - bs = (ad - bc)x,$$

and

$$x = \frac{dr - bs}{ad - bc}. \tag{2.9}$$

To eliminate the x coefficient we multiply (2.5) by c and (2.6) by a:

$$cr = acx + bcy \tag{2.10}$$

$$as = acx + ady. \tag{2.11}$$

Next, we subtract (2.10) from (2.11):

$$as - cr = (ad - bc)y,$$

and

$$y = \frac{as - cr}{ad - bc}. \tag{2.12}$$

Note that (2.9) and (2.12) share the same denominator, $ad - bc$, where a, b, c and d are the coefficients of x and y in the original simultaneous equations. This denominator becomes zero if the equations are linearly dependent, and no solution is possible.

Let's test this general solution using the original equations (2.1) and (2.2) where:

$$a = 6, \qquad b = 4, \qquad c = 2, \qquad d = 2, \qquad r = 24, \qquad s = 10$$

$$x = \frac{dr - bs}{ad - bc} = \frac{48 - 40}{12 - 8} = 2$$

$$y = \frac{as - cr}{ad - bc} = \frac{60 - 48}{12 - 8} = 3.$$

The algebraic solution reveals some interesting patterns which emerge when we consider a third technique using matrix notation, which is covered next.

2.2.3 Matrix Technique

Given a pair of linearly independent equations such as (2.1) and (2.2):

$$24 = 6x + 4y$$

$$10 = 2x + 2y$$

their solution must depend on the constants and coefficients 24, 6, 4, 10, 2 and 2. Matrix notation describes equations such that the coefficients are isolated from the variables of x and y in a 2×2 *square matrix*:

$$\begin{bmatrix} 6 & 4 \\ 2 & 2 \end{bmatrix},$$

or in the general case:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $\mathbf{A}$ identifies the matrix. The denominator $ad - bc$ in (2.9) and (2.12) is the difference between the cross-multiplied terms ad and bc. This is called the *determinant* of the matrix, and is written.

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

To keep the notation consistent, the values 24 and 10, or the general values r and s, are represented as a *column matrix* or *column vector*:

$$\begin{bmatrix} 24 \\ 10 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} r \\ s \end{bmatrix}.$$

The variables x and y are also represented as a column vector:

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

Finally, we bring the above elements together as follows:

$$\begin{bmatrix} 24 \\ 10 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

or for the general case:

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In either case, the original equations are reconstructed using the following rules:

1. Select r followed by '=' and multiply the elements of the top row of the coefficient matrix by the elements of the x–y vector respectively:

$$r = ax + by.$$

2. Select s followed by '=' and multiply the elements of the bottom row of the coefficient matrix by the elements of the x–y vector respectively:

$$s = cx + dy.$$

For example, the following matrix equation

$$\begin{bmatrix} 30 \\ 20 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

represents the pair of linear equations

$$30 = 2x + 3y$$

$$20 = 4x + 5y.$$

The power of matrix notation is that there is no limit to the number of linear equations and variables one can manipulate. For example, the following matrix equation

$$\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

represents the three linear equations

$$10 = x + 2y + 3z$$

$$20 = 4x + 5y + 6z$$

$$30 = 7x + 8y + 9z.$$

Any pair of linear equations can be represented as a matrix equation, and permits us to write the solution in matrix form. For instance, the solution to the original equations requires the following matrix equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix},$$

which is another way of writing

$$x = er + fs$$

$$y = gr + hs.$$

But we have already computed two such equations:

$$x = \frac{dr - bs}{ad - bc}$$

$$y = \frac{as - cr}{ad - bc}$$

which can be expressed in matrix form. But before we do this, let's substitute $D = ad - bc$ to simplify the equations:

$$x = \frac{1}{D}(dr - bs) \tag{2.13}$$

$$y = \frac{1}{D}(as - cr). \tag{2.14}$$

Now let's express (2.13) and (2.14) as a matrix equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}. \tag{2.15}$$

So we now have a matrix solution for any pair of linear equations with two unknowns. Let's test (2.15).

Example Here are two linearly independent equations:

$$24 = 4x + 3y$$
$$11 = x + 2y$$

where

$$a = 4, \qquad b = 3, \qquad c = 1, \qquad d = 2, \qquad r = 24, \qquad s = 11, \qquad D = 5$$

therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 24 \\ 11 \end{bmatrix}$$
$$x = \frac{1}{5}(2 \times 24 - 3 \times 11) = 3$$
$$y = \frac{1}{5}(-1 \times 24 + 4 \times 11) = 4$$

which is correct.

Although matrix notation provides the same answers as algebra, so far, it does not appear to offer any advantages. However, these will become apparent as we discover more about matrix notation, such as matrix multiplication, which we cover next.

2.3 Matrix Multiplication

So far we have seen how to multiply a vector by a matrix. Now let's discover how to multiply one matrix by another. For example, given

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

what is

$$\mathbf{MN} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix}?$$

We can resolve this problem by solving the same problem using algebra. So let's start by declaring two linear equations of the form

$$x'' = ax' + by' \tag{2.16}$$
$$y'' = cx' + dy' \tag{2.17}$$

where

$$x' = ex + fy \tag{2.18}$$
$$y' = gx + hy. \tag{2.19}$$

Next, we substitute (2.18) and (2.19) in (2.16) and (2.17):

$$x'' = a(ex + fy) + b(gx + hy)$$
$$y'' = c(ex + fy) + d(gx + hy)$$
$$x'' = (ae + bg)x + (af + bh)y \tag{2.20}$$
$$y'' = (ce + dg)x + (cf + dh)y. \tag{2.21}$$

This algebraic answer *must* be the same as that given using matrix notation. So let's set up the same scenario using matrices. We begin with

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{2.22}$$
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2.23}$$

Next, we substitute (2.23) in (2.22)

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2.24}$$

The matrix form of (2.20) and (2.21) is

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{2.25}$$

which means that

$$\mathbf{MN} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}. \tag{2.26}$$

Equation (2.26) shows how the product **MN** must be evaluated.

The terms of the top row of the first matrix: a and b, multiply the terms of the first column of the second matrix e and g, giving the result $ae + bg$.

Table 2.1 Subscripts for matrix multiplication

$(mn)_{ij}$	$=$	$m_{ij} \times n_{ij}$	$+$	$m_{ij} \times n_{ij}$
$(mn)_{11}$	$=$	$m_{11} \times n_{11}$	$+$	$m_{12} \times n_{21}$
$(mn)_{12}$	$=$	$m_{11} \times n_{12}$	$+$	$m_{12} \times n_{22}$
$(mn)_{21}$	$=$	$m_{21} \times n_{11}$	$+$	$m_{22} \times n_{21}$
$(mn)_{22}$	$=$	$m_{21} \times n_{12}$	$+$	$m_{22} \times n_{22}$

The terms of the top row of the first matrix: a and b, multiply the terms of the second column of the second matrix f and h, giving the result $af + bh$.

The terms of the bottom row of the first matrix: c and d, multiply the terms of the first column of the second matrix e and g, giving the result $ce + dg$.

Finally, the terms of the bottom row of the first matrix: c and d, multiply the terms of the second column of the second matrix f and h, giving the result $cf + dh$.

Observe that the product result is placed in the common matrix element shared by the row in the first matrix and the column in the second matrix.

To formalise this operation, let's reference any matrix element using the subscripts (ij) where i is the row, and j the column.

Let m_{ij} be an element in **M**, n_{ij} an element in **N**, and $(mn)_{ij}$ be an element in the product **MN**. For example, $m_{11} = a$, $n_{22} = h$ and $(mn)_{12} = af + bh$.

Table 2.1 shows how the four elements of the matrix **MN** are formed from the individual elements of **M** and **N**, which can be generalised to

$$(mn)_{ij} = m_{i1} \times n_{1j} + m_{i2} \times n_{2j}.$$

For example, given

$$\mathbf{M} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

then

$$\begin{aligned} \mathbf{MN} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 2 + 2 \times 4 & 1 \times 3 + 2 \times 5 \\ 3 \times 2 + 4 \times 4 & 3 \times 3 + 4 \times 5 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix}. \end{aligned}$$

Although it may not be immediately obvious, matrix multiplication is non-commutative. i.e. in general, $\mathbf{MN} \neq \mathbf{NM}$. For example,

$$\begin{aligned} \mathbf{NM} &= \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{bmatrix} \end{aligned}$$

which does not equal **MN** (2.26). Consequently, one has to be careful whenever two or more matrices are multiplied together.

Now that we know how to form the product of two matrices, let's look at a special matrix, which when used to multiply another matrix results in the same matrix.

2.4 Identity Matrix

Algebra contains two important objects: the *additive identity* 0 and the *multiplicative identity* 1, which obey the following rules:

$$a+0=0+a=a$$
$$a\times 1=1\times a=a.$$

The matrix equivalent of 0 is **0** which contains only 0s:

$$\mathbf{0}=\begin{bmatrix}0&0\\0&0\end{bmatrix},$$

whereas the matrix equivalent of 1 is not a matrix of 1s, but 0s and 1s. It is easy to discover this matrix, simply by declaring the following pair of linear equations:

$$x=1x+0y$$
$$y=0x+1y$$

or in matrix form:

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}. \tag{2.27}$$

The matrix in (2.27) is called an *identity matrix* or a *unit matrix* **I**, and it is easy to see that it has no effect when it pre- or post-multiplies another matrix:

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}$$
$$\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

Now that we have an identity matrix, let's calculate the matrix equivalent of the *multiplicative inverse*.

2.5 Inverse Matrix

The *multiplicative inverse* in algebra is an object that obeys the following rule:

$$a\times a^{-1}=a^{-1}\times a=1.$$

This suggests that for any matrix $\mathbf{A}$, there is an *inverse matrix* $\mathbf{A}^{-1}$ such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. Unfortunately, an inverse matrix is not always possible, but for the moment let's assume that this is the case. Thus, if

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix},$$

then $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$:

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.28}$$

We can compute the inverse matrix by expanding (2.28):

$$\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.29}$$

Equating the matrix elements in (2.29) we have

$$ae + bg = 1,$$

where

$$g = \frac{1 - ae}{b}. \tag{2.30}$$

Similarly,

$$ce + dg = 0,$$

where

$$g = -\frac{ce}{d}. \tag{2.31}$$

Therefore equating (2.30) and (2.31) we have

$$\frac{1 - ae}{b} = -\frac{ce}{d}$$

$$d - ade = -bce$$

$$e = \frac{d}{ad - bc} = \frac{d}{D}$$

where $D = ad - bc$.

Substituting e in (2.31)

$$g = -\frac{ce}{d} = -\frac{c}{d}\frac{d}{D} = -\frac{c}{D}.$$

Equating the remaining matrix elements in (2.29) we have

$$af + bh = 0,$$

where

$$h = -\frac{af}{b}. \tag{2.32}$$

Similarly,

$$cf + dh = 1,$$

where

$$h = \frac{1 - cf}{d}. \tag{2.33}$$

Therefore, equating (2.32) and (2.33) we have

$$\begin{aligned}-\frac{af}{b} &= \frac{1 - cf}{d}\\ adf &= -b + bcf\\ f &= -\frac{b}{D}.\end{aligned}$$

Substituting f in (2.32)

$$h = -\frac{af}{b} = \frac{a}{b}\frac{b}{D} = \frac{a}{D}.$$

Substituting e, f, g and h in $\mathbf{A}^{-1}$ we have

$$\mathbf{A}^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

which is the same matrix we computed in (2.15). Therefore, given an invertible matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

its inverse $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where $D = ad - bc$. For example, given

$$\mathbf{A} = \begin{bmatrix} 6 & 4 \\ 3 & 3 \end{bmatrix},$$

then $D = 6$ and

$$\mathbf{A}^{-1} = \frac{1}{6}\begin{bmatrix} 3 & -4 \\ -3 & 6 \end{bmatrix}.$$

The product $\mathbf{AA}^{-1}$ must equal the identity matrix $\mathbf{I}$:

$$\begin{aligned}
\mathbf{AA}^{-1} &= \begin{bmatrix} 6 & 4 \\ 3 & 3 \end{bmatrix} \frac{1}{6}\begin{bmatrix} 3 & -4 \\ -3 & 6 \end{bmatrix} \\
&= \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

The inverse matrix is an extremely useful object and plays an important role in matrix transformations. It can also be used as an algebraic tool for rearranging matrix equations. For example, given the following matrix equation

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix},$$

if we let

$$\mathbf{v}' = \begin{bmatrix} r \\ s \end{bmatrix}, \qquad \mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

then we can write in shorthand

$$\mathbf{v}' = \mathbf{Mv}. \tag{2.34}$$

Multiplying both sides of (2.34) by $\mathbf{M}^{-1}$ we have

$$\begin{aligned}
\mathbf{M}^{-1}\mathbf{v}' &= \mathbf{M}^{-1}\mathbf{Mv} \\
&= \mathbf{Iv}
\end{aligned}$$

therefore,

$$\begin{aligned}
\mathbf{v} &= \mathbf{M}^{-1}\mathbf{v}' \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}.
\end{aligned}$$

We will employ this strategy later on when we investigate matrix algebra and transformations.

2.6 Worked Examples

Example 1 Solve the linearly independent equations:

$$9 = 3x + 2y$$
$$1 = 4x - y.$$

Using

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix},$$

where

$$r = 9, \qquad s = 1, \qquad a = 3, \qquad b = 2, \qquad c = 4, \qquad d = -1$$

then $D = -11$, and

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= -\frac{1}{11} \begin{bmatrix} -1 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \end{bmatrix} \\ &= -\frac{1}{11} \begin{bmatrix} -11 \\ -33 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix}. \end{aligned}$$

Example 2 Solve the linearly independent equations:

$$7 = 3x - y$$
$$0 = -2x - 4y.$$

Using

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix},$$

where

$$r = 7, \qquad s = 0, \qquad a = 3, \qquad b = -1, \qquad c = -2, \qquad d = -4,$$

then $D = -14$, and

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= -\frac{1}{14} \begin{bmatrix} -4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix} \\ &= -\frac{1}{14} \begin{bmatrix} -28 \\ 14 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Example 3 Solve the trivial, linearly independent equations:

$$1 = x$$
$$1 = y.$$

Using

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix},$$

where

$$r = 1, \qquad s = 1, \qquad a = 1, \qquad b = 0, \qquad c = 0, \qquad d = 1,$$

then $D = 1$, and

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Example 4 Solve the following equations:

$$4 = 6x - 4y$$
$$2 = 3x - 2y.$$

Using

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix},$$

where

$$r = 4, \qquad s = 2, \qquad a = 6, \qquad b = -4, \qquad c = 3, \qquad d = -2,$$

then $D = 0$, which confirms that the equations are not linearly independent. The second equation is half the first.

2.7 Summary

Hopefully, this chapter has provided a quick introduction to the ideas behind matrix notation, which is just another way of representing a collection of linear equations.

So far, we have only considered two simultaneous linear equations, but there is no limit—a matrix grows in size as the number of equations increases.

We employ special rules when multiplying matrices to ensure that the result agrees with that obtained using algebra. These rules apply to all of the matrices covered in this book.

As we tend to multiply matrices together, rather than add them, we are particularly interested in the identity matrix, which is equivalent to the number 1 in algebra. We are also interested in the inverse matrix, which when used to multiply the original matrix creates the identity matrix.

We have seen that very obvious patterns arise from the coefficients when solving pairs and triples of linear equations. One reoccurring pattern is given the name 'determinant', and is derived from the associated matrix. For a 2×2 matrix it is the difference of the cross products. Other rules are employed for 3×3 and 4×4 matrices.

Before developing a formal description of matrix algebra, we will explore how to compute the determinant for any size matrix.

http://www.springer.com/978-1-4471-4320-8

Matrix Transforms for Computer Games and Animation
Vince, J.
2012, XI, 166 p. 45 illus., Softcover
ISBN: 978-1-4471-4320-8