

Chapter 2

Gamma-Spaces and S-Algebras

In this chapter we will introduce the so-called Γ -spaces. The reader can think of these as (very slight) generalizations of (simplicial) abelian groups. The surprising fact is that this minor generalization is big enough to encompass a wide and exotic variety of new examples.

The use of Γ -spaces also fixes another disparity. Quillen defined algebraic K-theory to be a functor from things with abelian group structure (such as rings or exact categories) to abelian groups. We have taken the view that K-theory takes values in spectra, and although spectra are almost as good as abelian groups, this is somehow unsatisfactory. The introduction of Γ -spaces evens this out, in that K-theory now takes things with a Γ -space structure (such as **S**-algebras, or the Γ -space analog of exact categories) to Γ -spaces.

Furthermore, this generalization enables us to include new fields of study, such as the K-theory of spaces, into serious consideration. It is also an aid—almost a prerequisite—when trying to understand the theories to be introduced in later chapters.

To be quite honest, Γ -spaces should not be thought of as a generalization of simplicial abelian groups, but rather of simplicial abelian (symmetric) monoids, since there need not be anything resembling inverses in the setting we use (as opposed to Segal's original approach). On the other hand, it is very easy to “group complete”: it is a stabilization process.

The reader should be aware that Γ -spaces give us just one of several solutions to an old and important problem in stable homotopy theory. After having been put on sound foundations by Boardman in the mid 1960's (see [295] or [4, III]), the smash product played a central rôle in stable homotopy theory for decades, but until the 1990's one only knew the construction in the “stable homotopy category”, and did not know how to realize the smash products in any category of spectra without inverting the stable equivalences.

Several solutions to this problem came more or less at the same time. In the summer of 1993 Elmendorf, Kriz and May's built the categorical foundations for a point

set level smash product, influenced by an observation of Hopkins [138] on coequalizers, making their original intended approach using “monadic bar constructions” more or less obsolete. The team was later joined by Mandell and the construction underwent some changes (for instance, some problems with the unit were solved) before it appeared in the book [80]. Around the same time, Jeff Smith gave talks where he offered another solution which he called “symmetric spectra”. Together with Hovey and Shipley he documented this approach in [141] (see also the unpublished notes on symmetric spectra by Schwede [251]).

The Γ -space approach to the problem of having a point set level construction of the smash product appeared in 1999 [188] in a paper by Lydakis, and has the advantage of being by far the simplest, but the disadvantage of only giving connective spectra. The solution is simple, and the techniques were well known in the 1970’s, and the authors have come to understand that the construction of the smash product in Γ -spaces was known, but dismissed as “much too simple” to have the right homotopy properties, see also Sect. 2.1.2.3 below.

Since then many variants have been introduced (most notably, orthogonal spectra) and there has been some reconciliation between the different setups (see in particular [196]). In retrospect, it turns out that Bökstedt in his investigations [30] in the 1980’s on topological Hochschild homology had struck upon the smash product for simplicial functors [187] in the sense that he gave the correct definition for what it means for a simplicial functor to be an algebra over the sphere spectrum, see also Gunnarsson’s preprint [117]. Bökstedt called what was to become S-algebras in simplicial functors (with some connectivity hypotheses) “FSP”, short for “functors with smash products”. Also, orthogonal ring spectra had appeared in [200], although not recognized as monoids in a monoidal structure.

The Γ -spaces have one serious shortcoming, and that is that they do not model strictly commutative ring spectra in the same manner as their competitors (see Lawson [170]). Although this mars the otherwise beautiful structure, it will not affect anything of what we will be doing, and we use Γ -spaces because of their superior concreteness and simplicity.

2.1 Algebraic Structure

2.1.1 Γ -Objects

A Γ -object is a functor from the category of finite sets. We need to be quite precise about this, and the details follow.

2.1.1.1 The Category Γ^o

Roughly, Γ^o is the category of pointed finite sets—a fundamental building block for much of mathematics. To be more precise, we choose a skeleton, and let Γ^o be the

category with one object $k_+ = \{0, 1, \dots, k\}$ for every natural number k , and with morphism sets $\Gamma^o(m_+, n_+)$ the set of functions $f: \{0, 1, \dots, m\} \rightarrow \{0, 1, \dots, n\}$ such that $f(0) = 0$. The notation k_+ is chosen to distinguish it from the *ordered* set $[k] = \{0 < 1 < \dots < k\}$.

In [257] Segal considered the opposite category and called it Γ , and this accounts for the awkward situation where we call the most fundamental object in mathematics the opposite of something. Some people object to this so strongly that they write Γ when Segal writes Γ^o . We follow Segal's convention.

2.1.1.2 Motivation

A *symmetric monoid* is a set M together with a multiplication and a unit element so that any two maps $M^{\times j} \rightarrow M$ obtained by composing maps in the diagram

$$\begin{array}{ccccc}
 * & \xrightarrow{\text{unit}} & M & \begin{array}{c} \xrightarrow{m \mapsto (1, m)} \\ \text{multiplication} \\ \xleftarrow{m \mapsto (m, 1)} \end{array} & M \times M & \begin{array}{c} \text{twist} \\ \curvearrowright \\ \xleftarrow{(m_1 m_2, m_3) \leftarrow (m_1, m_2, m_3)} \\ \xleftarrow{(m_1, m_2 m_3) \leftarrow (m_1, m_2, m_3)} \end{array} & M \times M \times M
 \end{array} \quad (2.1)$$

are equal. Thinking of multiplication as “two things coming together” as in the map $2_+ \rightarrow 1_+$ given by

$$\begin{array}{ccccc}
 2_+ = \{ & 0 & 1 & 2 \} \\
 & \downarrow & \downarrow & \swarrow \\
 1_+ = \{ & 0 & 1 & \}
 \end{array}$$

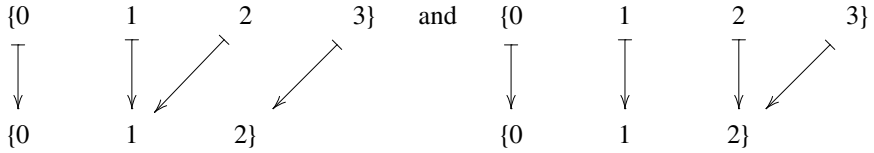
we see that the diagram (2.1) is mirrored by the diagram

$$0_+ \longrightarrow 1_+ \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} 2_+ \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} 3_+$$

in Γ^o , where the two arrows $1_+ \rightarrow 2_+$ are given by

$$\begin{array}{ccc}
 \begin{array}{cc} \{0 & 1\} \\ \downarrow & \downarrow \\ \{0 & 1 & 2\} \end{array} & \text{and} & \begin{array}{ccc} \{0 & 1 & \\ \downarrow & & \searrow \\ \{0 & 1 & 2\} \end{array}
 \end{array}$$

and the maps $3_+ \rightarrow 2_+$ are



(there are more maps in Γ^o , but these suffice for the moment). So we could say that a symmetric monoid is a functor from this part of Γ^o to pointed sets sending 0_+ to the one-point set and sending wedge sum to product (e.g., $3_+ = 2_+ \vee 1_+$ must be sent to the product of the values at 2_+ and 1_+ , i.e., the triple product of the value at 1_+).

This doesn't seem very helpful until one notices that this extends to all of Γ^o , and the requirement of sending 0_+ to the one-point set and wedge sum to product fixes the behavior in the sense that there is a one-to-one correspondence between such functors from Γ^o to sets and symmetric monoids; see Example 2.1.2.1(1) below for more details.

The reason for introducing this new perspective is that we can model multiplicative structures functorially, and relaxing the requirement that the functor sends wedge to product is just the trick needed to study more general multiplicative structures. For instance, one could imagine situations where the multiplication is not naturally defined on $M \times M$, but on some bigger space like $M \times M \times X$, giving an entire family of multiplications varying over the space X . This is exactly what we need when we are going to study objects that are, say, commutative only up to homotopy. Variants of this idea are Lawvere's algebraic theories, operads and multicategories.

2.1.1.3 Γ -Objects

If $\mathcal{C}$ is a pointed category (i.e., it has a chosen object which is both initial and final) one may consider pointed functors $\Gamma^o \rightarrow \mathcal{C}$ (often called a Γ -object in $\mathcal{C}$) and natural transformations between such functors. This defines a category we call $\Gamma\mathcal{C}$. Most notably we have the category

$$\Gamma S_*$$

of Γ -spaces, that is pointed functors from Γ^o to pointed simplicial sets, or equivalently, of simplicial Γ -objects in the category of pointed sets. If $\mathcal{A} = sAb$ is the category of simplicial abelian groups, we may define

$$\Gamma\mathcal{A},$$

the category of simplicial Γ -objects in abelian groups. Likewise for other module categories. Another example is the category of Γ -categories, i.e., pointed functors from Γ^o to categories. These must not be confused with the notion of ΓS_* -categories (see Sect. 2.1.6).

2.1.2 The Category $\Gamma\mathcal{S}_*$ of Γ -Spaces

We start with some examples of Γ -spaces.

Example 2.1.2.1

1. Let M be an abelian group. If we consider M as a mere pointed set, we can not reconstruct the abelian group structure. However, if we consider M as a Γ -pointed set, HM , as follows, there is no loss of structure. Send k_+ to the set

$$HM(k_+) = M \otimes \tilde{\mathbf{Z}}[k_+] \cong M^{\times k},$$

where $\tilde{\mathbf{Z}}[k_+]$ is the free abelian group on the pointed set k_+ (and so is the sum of k copies of $\mathbf{Z}$). A function $f \in \Gamma^o(k_+, n_+)$ gives rise to the homomorphism $f_*: HM(k_+) \rightarrow HM(n_+)$ sending the k -tuple $(m_1, \dots, m_k) \in M^{\times k}$ to the n -tuple

$$\left(\left(\sum_{j \in f^{-1}(1)} m_j \right), \dots, \left(\sum_{j \in f^{-1}(n)} m_j \right) \right)$$

(where $m_0 = 0$).

Alternative description: $HM(X) = \text{Ens}_*(X, M)$, and if $f: X \rightarrow Y \in \Gamma^o$, then $f_*: HM(X) \rightarrow HM(Y)$ sends ϕ to $y \mapsto f_*\phi(y) = \sum_{x \in f^{-1}(y)} \phi(x)$.

In effect, this defines a functor

$$\bar{H}: sAb = \mathcal{A} \rightarrow \Gamma\mathcal{A},$$

and we follow by the forgetful functor $U: \Gamma\mathcal{A} \rightarrow \Gamma\mathcal{S}_*$, so that

$$H = U\bar{H}.$$

Both HM and $\bar{H}M$ will be referred to as the *Eilenberg–Mac Lane* objects associated with M . The reason is that, through the functor from Γ -spaces to spectra defined in Definition 2.2.1.8, these Γ -objects naturally give rise to the so-called Eilenberg–Mac Lane spectra Sect. A.3.2.

2. The inclusion $\Gamma^o \subset \text{Ens}_* \subset \mathcal{S}_*$ is called in varying sources, **S** (for “sphere spectrum”), *Id* (for “identity”), etc. We will call it **S**.

Curiously, the Barratt–Priddy–Quillen theorem (see e.g., [257] or [12]) states that **S** is “stably equivalent” (defined in Definition 2.2.1.5) to the K-theory of the category Γ^o (in the interpretation of Sect. 2.3).

3. If X is a pointed simplicial set and M is a Γ -space, then $M \wedge X$ is the Γ -space sending $Y \in \text{ob}\Gamma^o$ to $M(Y) \wedge X$. Dually, we let $\underline{\mathcal{S}}_*(X, M)$ be the Γ -space

$$Y \mapsto \underline{\mathcal{S}}_*(X, M(Y)) = \{[q] \mapsto \mathcal{S}_*(X \wedge \Delta[q]_+, M(Y))\}$$

(see Sect. A.2.3 for facts on function spaces). Note that $\Gamma\mathcal{S}_*(M \wedge X, N)$ is naturally isomorphic to $\Gamma\mathcal{S}_*(M, \underline{\mathcal{S}}_*(X, N))$: the former set consists of natural

maps $M(Y) \wedge X \rightarrow N(Y)$, whereas the latter consists of natural maps $M(Y) \rightarrow \underline{\mathcal{S}}_*(X, N(Y))$. The natural isomorphism is then induced by the adjunction between $\wedge$ and the internal mapping space in $\mathcal{S}_*$.

For any simplicial set X , we let $\mathbf{S}[X] = \mathbf{S} \wedge X_+$, and we see that this is a left adjoint to the functor $R: \Gamma\mathcal{S}_* \rightarrow \mathcal{S}_*$ evaluating at 1_+ .

4. For $X \in ob\Gamma^o$, let $\Gamma^X \in ob\Gamma\mathcal{S}_*$ be given by

$$\Gamma^X(Y) = \Gamma^o(X, Y)$$

Note that $\mathbf{S} = \Gamma^{1_+}$.

The notion of Γ -spaces we are working with is slightly more general than Segal's [257]. It is usual to call Segal's Γ -spaces *special*:

Definition 2.1.2.2 A Γ -space M is said to be *special* if the canonical maps

$$M(k_+) \rightarrow \prod_k M(1_+)$$

(induced by the k maps $k_+ \rightarrow 1_+$ with support a single element) are weak equivalences for all $k_+ \in ob\Gamma$. This induces a symmetric monoid structure on $\pi_0 M(1_+)$ via

$$\pi_0 M(1_+) \times \pi_0 M(1_+) \xleftarrow{\cong} \pi_0 M(2_+) \longrightarrow \pi_0 M(1_+),$$

induced by the function $\phi: 2_+ \rightarrow 1_+$ with $\phi^{-1}(1) = \{1, 2\}$, and we say that M is *very special* if this is an abelian group structure.

The difference between Γ -spaces and very special Γ -spaces is not really important. Any Γ -space M gives rise to a very special Γ -space, say QM , in one of many functorial ways, such that there is a “stable equivalence” $M \xrightarrow{\sim} QM$ (see Definition 2.2.1.5). However, the larger category of all Γ -spaces is nicer for formal reasons, and the very special Γ -spaces are just nice representatives in each stable homotopy class.

2.1.2.1 The Smash Product

There is a close connection between Γ -spaces and spectra (there is a functor defined in Definition 2.2.1.8 that induces an equivalence on homotopy categories), and so the question of what the smash product of two Γ -spaces should be could be expected to be a complicated issue. M. Lydakis [187, 188] realized that this was not the case: the simplest candidate works just beautifully.

If we have two Γ -spaces M and N , we may consider the “external smash”, i.e., the functor $\Gamma^o \times \Gamma^o \rightarrow \mathcal{S}_*$ which sends (X, Y) to $M(X) \wedge N(Y)$. The category Γ^o

has its own smash product, and we want some “universal filler” in

$$\begin{array}{ccc} \Gamma^o \times \Gamma^o & \xrightarrow{(X,Y) \mapsto M(X) \wedge N(Y)} & \mathcal{S}_* \\ \wedge \downarrow & & \\ \Gamma^o & & \end{array},$$

where $\wedge: \Gamma^o \times \Gamma^o \rightarrow \Gamma^o$ is the smash (sending (X, Y) to $X \wedge Y$). The solutions to these kinds of questions are called “left Kan extensions” [191], and in our case it takes the following form:

Let $Z \in \Gamma^o$ and let $\wedge/Z$ be the over category (cf. Sect. A.2.4.2), i.e., the category whose objects are tuples (X, Y, v) where $(X, Y) \in \Gamma^o \times \Gamma^o$ and $v: X \wedge Y \rightarrow Z \in \Gamma^o$, and where a morphism $(X, Y, v) \rightarrow (X', Y', v')$ is a pair of functions $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ in Γ^o such that $v = v' \circ (f \wedge g)$.

Then the smash product $(M \wedge N)(Z)$ is defined as the colimit of the composite

$$\wedge/Z \xrightarrow{(X,Y,v) \mapsto (X,Y)} \Gamma^o \times \Gamma^o \xrightarrow{(X,Y) \mapsto M(X) \wedge N(Y)} \mathcal{S}_*,$$

that is

$$(M \wedge N)(Z) = \lim_{(X,Y,v) \in \wedge/Z} M(X) \wedge N(Y).$$

In the language of coends, this becomes particularly perceptive:

$$(M \wedge N)(Z) = \int^{(X,Y)} (M(X) \wedge N(Y)) \wedge \Gamma^o(X \wedge Y, Z)$$

the “weighted average of all the handcrafted smash products $M(X) \wedge N(Y)$ ”; the weight being the number of functions $X \wedge Y \rightarrow Z$.

Remark 2.1.2.3 Note that a map from a smash product $M \wedge M' \rightarrow N \in \Gamma\mathcal{S}_*$ is uniquely described by giving a map $M(X) \wedge M'(Y) \rightarrow N(X \wedge Y)$ which is natural in $X, Y \in \text{ob}\Gamma^o$.

2.1.2.2 The Closed Structure

Theorem 2.1.2.4 below states that the smash product endows the category of Γ -spaces with a structure of a *closed* category (which is short for closed symmetric monoidal category). For a thorough discussion see Definition A.10.1.1, but for now recall that *symmetric monoidal* means that the functor $\wedge: \Gamma\mathcal{S}_* \times \Gamma\mathcal{S}_* \rightarrow \Gamma\mathcal{S}_*$ is associative, symmetric and unital ($\mathbf{S}$ is the unit) up to coherent isomorphisms, and that it is *closed* means that in addition there is an “internal morphism object” with reasonable behavior.

The Γ -space of morphisms from M to N is defined by setting

$$\underline{\Gamma\mathcal{S}_*}(M, N) = \left\{ (k_+, [q]) \mapsto \underline{\Gamma\mathcal{S}_*}(M, N)(k_+)_q = \Gamma\mathcal{S}_*(M \wedge \Delta[q]_+, N(k_+ \wedge -)) \right\}.$$

Theorem 2.1.2.4 (Lydakis) *With these definitions of smash and morphism object, there are choices of coherency isomorphisms such that $(\Gamma\mathcal{S}_*, \wedge, \mathbf{S})$ becomes a closed category (cf. Definition A.10.1.1).*

Proof (See [188] for further details) First one uses the definitions to show that there is a natural isomorphism $\Gamma\mathcal{S}_*(M \wedge N, P) \cong \Gamma\mathcal{S}_*(N, \Gamma\mathcal{S}_*(M, P))$ (using the (co)end-descriptions of smash and internal morphism objects this can be written as follows

$$\begin{aligned} \Gamma\mathcal{S}_*(M \wedge N, P)(V) &= \int_Z \underline{\mathcal{S}}_* \left(\int_Z^{X,Y} M(X) \wedge N(Y) \wedge \Gamma^o(X \wedge Y, Z), P(V \wedge Z) \right) \\ &\cong \int_{X,Y} \underline{\mathcal{S}}_* \left(M(X) \wedge N(Y), \int_Z \underline{\mathcal{S}}_* (\Gamma^o(X \wedge Y, Z), P(V \wedge Z)) \right) \\ &\cong \int_{X,Y} \underline{\mathcal{S}}_* (M(X) \wedge N(Y), P(V \wedge (X \wedge Y))) \\ &\cong \int_X \underline{\mathcal{S}}_* \left(N(Y), \int_Y \underline{\mathcal{S}}_* (M(X), P((V \wedge X) \wedge Y)) \right) \\ &= \Gamma\mathcal{S}_*(N, \Gamma\mathcal{S}_*(M, P))(V), \end{aligned}$$

with $X, Y, Z, V \in \text{ob}\Gamma^o$ and where the isomorphisms are induced by the (dual) Yoneda lemma, associativity and the closed structure of $\underline{\mathcal{S}}_*$.

The symmetry $M \wedge N \cong N \wedge M$ follows from the construction of the smash product, and associativity follows by comparing with

$$M \wedge N \wedge P = \left\{ V \mapsto \lim_{X \wedge Y \wedge Z \rightarrow V} M(X) \wedge N(Y) \wedge P(Z) \right\}.$$

Recall from Example 2.1.2.1(4) that $\Gamma^X(Y) = \Gamma^o(X, Y)$ and note that $\mathbf{S} = \Gamma^{1+}$, $\Gamma\mathcal{S}_*(\Gamma^X, M) \cong M(X \wedge -)$ and $\Gamma^X \wedge \Gamma^Y \cong \Gamma^{X \wedge Y}$. We get that $M \wedge \mathbf{S} = M \wedge \Gamma^{1+} \cong M$ since $\Gamma\mathcal{S}_*(M \wedge \mathbf{S}, N) \cong \Gamma\mathcal{S}_*(M, \Gamma\mathcal{S}_*(\mathbf{S}, N)) \cong \Gamma\mathcal{S}_*(M, N)$ for any N .

That all diagrams that must commute actually do so follows from the crucial observation Lemma 2.1.2.5 below (with the obvious definition of the multiple smash product). $\square$

Lemma 2.1.2.5 *Any natural automorphism ϕ of expressions of the form*

$$M_1 \wedge M_2 \wedge \cdots \wedge M_n$$

must be the identity (i.e., $\text{Aut}(\bigwedge^n: \Gamma\mathcal{S}_^{\times n} \rightarrow \Gamma\mathcal{S}_*)$ is the trivial group).*

Proof The analogous statement is true in Γ^o , since any element in $X_1 \wedge X_2 \wedge \cdots \wedge X_n$ is in the image of a map from $1_+ \wedge 1_+ \wedge \cdots \wedge 1_+$, and so any natural automorphism must fix this element.

Fixing a dimension, we may assume that the M_i are discrete, and we must show that $\phi(z) = z$ for any $z \in \bigwedge M_i(Z)$. By construction, z is an equivalence class represented, say, by an element $(x_1, \dots, x_m) \in \bigwedge M_i(X_i)$ in the $f: \bigwedge X_i \rightarrow Z$ summand of the colimit. Represent each $x_i \in M_i(X_i)$ by a map $f_i: \Gamma^{X_i} \rightarrow M_i$ (so that $f_i(X_i = X_i) = x_i$). Then z is the image of $\wedge id_{X_i}$ in the f -summand of the composite

$$(\bigwedge \Gamma^{X_i})(Z) \xrightarrow{\wedge f_i} (\bigwedge M_i)(Z).$$

Hence it is enough to prove the lemma for $M_i = \Gamma^{X_i}$ for varying X_i . But $\bigwedge \Gamma^{X_i} \cong \Gamma \wedge^{X_i}$ and

$$\Gamma S_*(\Gamma \wedge^{X_i}, \Gamma \wedge^{X_i}) \cong \Gamma^o\left(\bigwedge X_i, \bigwedge X_i\right),$$

and we are done. $\square$

The crucial word in Lemma 2.1.2.5 is “natural”. There is just one automorphism of the **functor** $\bigwedge^n: \Gamma S_*^{\times n} \rightarrow \Gamma S_*$ whereas there are, of course, nontrivial actions on individual expressions $M_1 \wedge \dots \wedge M_n$. One should note that the functor in **one** variable $M \mapsto M \wedge \dots \wedge M$ has full Σ_n -symmetry.

2.1.2.3 Day’s Product

Theorem 2.1.2.4 also follows from a much more general theorem of Day [59], not relying on the special situation in Lemma 2.1.2.5.

In hindsight it may appear as a mystery that the smash product took so long to appear on the stage, given that the problem was well publicized and the technical construction had been known since 1970. Rainer Vogt had considered this briefly, and commented in an email in 2009: “I did not know of Day’s product but discovered it myself (later than Day in the 80’s). Then Roland [Schwänzl] and I thought a little about it. Since we considered special Γ -spaces only and the product did not preserve those we lost interest, in particular after we realised that we would get an associative and commutative smash product for connective spectra which we did not believe exists. When many years later Lydakis exploited this construction we could have kicked ourselves.”

2.1.3 Variants

The proof that ΓS_* is a closed category works if S_* is exchanged for other suitable closed categories with colimits. In particular $\Gamma \mathcal{A}$, the category of Γ -objects in the category $\mathcal{A} = sAb$ of simplicial abelian groups, is a closed category. The unit is $\bar{H}\mathbf{Z} = \{X \mapsto \tilde{\mathbf{Z}}[X]\}$ (it is $H\mathbf{Z}$ as a set, but we remember the group structure, see

Example 2.1.2.1(1)), the tensor is given by

$$(M \otimes N)(Z) = \lim_{X \wedge Y \rightarrow Z} M(X) \otimes N(Y)$$

and the internal function object is given by

$$\underline{\Gamma\mathcal{A}}(M, N) = \{X, [q] \mapsto \Gamma\mathcal{A}(M \otimes \mathbf{Z}[\Delta[q]], N(- \wedge X))\}.$$

2.1.3.1 $\Gamma\mathcal{S}_*$ vs. $\Gamma\mathcal{A}$

The adjoint functor pair between abelian groups and pointed sets

$$Ens_* \underset{U}{\overset{\tilde{\mathbf{Z}}}{\rightleftarrows}} Ab,$$

where U is the forgetful functor, induces an adjoint functor pair

$$\Gamma\mathcal{S}_* \underset{U}{\overset{\tilde{\mathbf{Z}}}{\rightleftarrows}} \Gamma\mathcal{A}.$$

The homomorphisms $\mathbf{Z} \rightarrow \tilde{\mathbf{Z}}(1_+)$ (sending n to $n \cdot 1$) and $\tilde{\mathbf{Z}}(X) \wedge \tilde{\mathbf{Z}}(Y) \rightarrow \tilde{\mathbf{Z}}(X \wedge Y)$ (sending the generator $x \otimes y$ to the generator $x \wedge y$) are isomorphisms and $\tilde{\mathbf{Z}}: (Ens_*, \wedge, 1_+) \rightarrow (Ab, \otimes, \mathbf{Z})$ is a *strong symmetric monoidal functor* (see Definition A.10.1.3 for details, but briefly a strong symmetric monoidal functor is a symmetric monoidal functor F such that the structure maps $F(a) \otimes F(b) \rightarrow F(a \otimes b)$ and $1 \rightarrow F(1)$ are isomorphisms). It follows immediately that $\tilde{\mathbf{Z}}: (\Gamma\mathcal{S}_*, \wedge, \mathbf{S}) \rightarrow (\Gamma\mathcal{A}, \otimes, \tilde{H}\mathbf{Z})$ is strong symmetric monoidal. In particular $\tilde{\mathbf{Z}}\mathbf{S} \cong \tilde{H}\mathbf{Z}$,

$$\tilde{\mathbf{Z}}(M \wedge N) \cong \tilde{\mathbf{Z}}M \otimes \tilde{\mathbf{Z}}N$$

and

$$\Gamma\mathcal{S}_*(M, UP) \cong U\underline{\Gamma\mathcal{A}}(\tilde{\mathbf{Z}}M, P)$$

satisfying the necessary associativity, commutativity and unit conditions.

Later, we will see that the category $\Gamma\mathcal{A}$, for all practical (homotopical) purposes can be exchanged for $sAb = \mathcal{A}$. The comparison functors come from the adjoint pair

$$\mathcal{A} \underset{R}{\overset{\tilde{H}}{\rightleftarrows}} \Gamma\mathcal{A}$$

where $\tilde{H}P(X) = P \otimes \tilde{\mathbf{Z}}[X]$ and $RM = M(1_+)$. We see that $R\tilde{H} = id_{\mathcal{A}}$. The other adjunction, $\tilde{H}R \rightarrow id_{\Gamma\mathcal{A}}$, is discussed in Lemma 2.1.3.1 below. Both $\tilde{H}$ and R are symmetric monoidal functors.

2.1.3.2 Special Objects

We say that $M \in \text{ob}\Gamma\mathcal{A}$ is *special* if its underlying Γ -space $UM \in \text{ob}\Gamma\mathcal{S}_*$ is special, i.e., if for all finite pointed sets X and Y the canonical map

$$UM(X \vee Y) \xrightarrow{\sim} UM(X) \times UM(Y)$$

is a weak equivalence in $\mathcal{S}_*$. The following lemma has the consequence that all special objects in $\Gamma\mathcal{A}$ can be considered to be in the image of $\bar{H}: sAb = \mathcal{A} \rightarrow \Gamma\mathcal{A}$:

Lemma 2.1.3.1 *Let $M \in \text{ob}\Gamma\mathcal{A}$ be special. Then the unit of adjunction $(\bar{H}RM)(k_+) \rightarrow M(k_+)$ is an equivalence.*

Proof Since M is special, we have that $M(k_+) \rightarrow \prod_k M(1_+)$ is an equivalence. On the other hand, if we precompose this map with the unit of adjunction

$$(\bar{H}RM)(k_+) = M(1_+) \otimes \tilde{\mathbf{Z}}[k_+] \rightarrow M(k_+)$$

we get an isomorphism. □

2.1.3.3 Additivization

There is also a Dold-Puppe-type construction: $L: \Gamma\mathcal{A} \rightarrow \mathcal{A}$ which is **left** adjoint to $\bar{H}$: Consider the three pointed functions $pr_1, pr_2, \nabla: 2_+ \rightarrow 1_+$ with nonzero value $pr_1(1) = \nabla(1) = \nabla(2) = pr_2(2) = 1$. Then L is given by

$$LM = \text{coker}\{M(pr_1) - M(\nabla) + M(pr_2): M(2_+) \rightarrow M(1_+)\}.$$

This functor is intimately connected with the subcategory of $\Gamma\mathcal{A}$ consisting of “additive”, or coproduct preserving functors $\Gamma^o \rightarrow \mathcal{A}$.

The additive objects are uniquely defined by their value at 1_+ , and we get an isomorphism $M \cong \bar{H}(M(1_+)) = \bar{H}RM$. Using this we may identify $\mathcal{A}$ with the full subcategory of additive objects in $\Gamma\mathcal{A}$, and the inclusion into $\Gamma\mathcal{A}$ has a left adjoint given by $\bar{H}L$.

Note that all the functors L, R and $\bar{H}$ between $\mathcal{A}$ and $\Gamma\mathcal{A}$ are strong symmetric monoidal.

Just the same considerations could be made with Ab exchanged for the category of k -modules for any commutative ring k .

2.1.4 S-Algebras

In any monoidal category there is a notion of a *monoid* (see Definition A.10.1.4). The reason for the name is that a monoid in the usual sense is a monoid in

$(Ens, \times, *)$. Furthermore, the axioms for a ring is nothing but the statement that it is a monoid in $(Ab, \otimes, \mathbf{Z})$. For a commutative ring k , a k -algebra is no more than a monoid in $(k\text{-mod}, \otimes_k, k)$, and so it is natural to define **S**-algebras the same way:

Definition 2.1.4.1 An **S**-algebra A is a monoid in $(\Gamma\mathbf{S}_*, \wedge, \mathbf{S})$.

This means that A is a Γ -space together with maps $\mu = \mu^A: A \wedge A \rightarrow A$ and $1: \mathbf{S} \rightarrow A$ such that the diagrams

$$\begin{array}{ccc}
 A \wedge (A \wedge A) & \xrightarrow{\cong} & (A \wedge A) \wedge A \xrightarrow{\mu \wedge id} A \wedge A \\
 \downarrow id \wedge \mu & & \downarrow \mu \\
 A \wedge A & \xrightarrow{\mu} & A
 \end{array}$$

(the isomorphism is the associativity isomorphism of the smash product) and

$$\begin{array}{ccccc}
 \mathbf{S} \wedge A & \xrightarrow{1 \wedge id} & A \wedge A & \xleftarrow{id \wedge 1} & A \wedge \mathbf{S} \\
 & \searrow & \downarrow \mu & \swarrow & \\
 & \cong & A & \cong &
 \end{array}$$

commute, where the diagonal maps are the natural unit isomorphisms.

We say that an **S**-algebra is *commutative* if $\mu = \mu \circ tw$ where

$$A \wedge A \xrightarrow[\cong]{tw} A \wedge A$$

is the twist isomorphism.

Remark 2.1.4.2 In the definition of an **S**-algebra, any knowledge of the symmetric monoidal category structure is actually never needed, since maps $M \wedge N \rightarrow P$ out of the smash products is uniquely characterized by a map $M(X) \wedge N(Y) \rightarrow P(X \wedge Y)$ natural in $X, Y \in ob \Gamma^o$. So, since the multiplication is a map from the smash $A \wedge A \rightarrow A$, it can alternatively be defined as a map $A(X) \wedge A(Y) \rightarrow A(X \wedge Y)$ natural in both X and Y .

This was the approach of Bökstedt [30] when he defined FSP's. These are simplicial functors from finite spaces to spaces with multiplication and unit, such that the natural diagrams commute, plus some stability conditions. These stability conditions are automatically satisfied if we start out with functors from Γ^o (and then apply degreewise and diagonalize if we want $X \in s\Gamma^o$ as input), see Lemma 2.2.1.3. On the other hand, we shall later see that there is no loss of generality to consider only **S**-algebras.

2.1.4.1 Variants

An $\bar{H}\mathbf{Z}$ -algebra is a monoid in $(\Gamma\mathcal{A}, \otimes, \bar{H}\mathbf{Z})$. (This is, for all practical purposes, equivalent to the more sophisticated notion of $H\mathbf{Z} = U\bar{H}\mathbf{Z}$ -algebras arising from the fact that there is a closed category $(H\mathbf{Z}\text{-mod}, \wedge_{H\mathbf{Z}}, H\mathbf{Z})$, see Example 2.1.5.6 below.) Since the functors

$$\Gamma\mathcal{S}_* \begin{array}{c} \xrightarrow{\tilde{Z}} \\ \xleftarrow{U} \end{array} \Gamma\mathcal{A} \begin{array}{c} \xrightarrow{\bar{H}} \\ \xleftarrow{R} \end{array} A$$

$\begin{array}{c} L \\ \hline \end{array}$

all are monoidal they send monoids to monoids. For instance, if A is a simplicial ring, then $\bar{H}A$ is an $\bar{H}\mathbf{Z}$ -algebra and HA is an $\mathbf{S}$ -algebra (it is even an $H\mathbf{Z}$ -algebra):

Example 2.1.4.3

1. Let A be a simplicial ring, then HA is an $\mathbf{S}$ -algebra with multiplication

$$HA \wedge HA \rightarrow H(A \otimes A) \rightarrow HA$$

and unit $\mathbf{S} \rightarrow \tilde{\mathbf{Z}}\mathbf{S} \cong H\mathbf{Z} \rightarrow HA$.

In particular, note the $\mathbf{S}$ -algebra $H\mathbf{Z}$. It is given by $X \mapsto \tilde{\mathbf{Z}}[X]$, the “integral homology”, and the unit map $X = \mathbf{S}(X) \rightarrow H\mathbf{Z}(X) = \tilde{\mathbf{Z}}[X]$ is the Hurewicz map of Appendix A.3.1.

2. Of course, $\mathbf{S}$ is the initial $\mathbf{S}$ -algebra. If M is a simplicial monoid, the *spherical monoid algebra* $\mathbf{S}[M]$ is given by

$$\mathbf{S}[M](X) = M_+ \wedge X$$

with obvious unit and with multiplication coming from the monoid structure. Note that $R\tilde{\mathbf{Z}}\mathbf{S}[M] \cong \mathbf{Z}[M]$.

3. If A is an $\mathbf{S}$ -algebra, then A^o , the *opposite* of A , is the $\mathbf{S}$ -algebra given by the Γ -space A with the same unit $\mathbf{S} \rightarrow A$, but with the twisted multiplication

$$A \wedge A \xrightarrow[\cong]{tw} A \wedge A \xrightarrow{\mu} A.$$

4. If A and B are $\mathbf{S}$ -algebras, their smash $A \wedge B$ is a new $\mathbf{S}$ -algebra with multiplication

$$(A \wedge B) \wedge (A \wedge B) \xrightarrow{id \wedge tw \wedge id} (A \wedge A) \wedge (B \wedge B) \rightarrow A \wedge B,$$

and unit $\mathbf{S} \cong \mathbf{S} \wedge \mathbf{S} \rightarrow A \wedge B$.

5. If A and B are two $\mathbf{S}$ -algebras, the product $A \times B$ is formed pointwise: $(A \times B)(X) = A(X) \times B(X)$ and with componentwise multiplication and diagonal unit. The coproduct also exists, but is more involved.

6. Matrices: If A is an $\mathbf{S}$ -algebra, we define the $\mathbf{S}$ -algebra of $n \times n$ matrices $Mat_n A$ by

$$Mat_n A(X) = \underline{\mathcal{S}}_*(n_+, n_+ \wedge A(X)) \cong \prod_n \bigvee_n A(X)$$

—the matrices with only “one entry in each column”. The unit is the diagonal, whereas the multiplication is determined by

$$\begin{aligned} Mat_n A(X) \wedge Mat_n A(Y) &= \underline{\mathcal{S}}_*(n_+, n_+ \wedge A(X)) \wedge \underline{\mathcal{S}}_*(n_+, n_+ \wedge A(Y)) \\ &\downarrow id \wedge (\text{smashing with } id_{A(X)}) \\ \underline{\mathcal{S}}_*(n_+, n_+ \wedge A(X)) \wedge \underline{\mathcal{S}}_*(n_+ \wedge A(X), n_+ \wedge A(X) \wedge A(Y)) \\ &\downarrow \text{composition} \\ \underline{\mathcal{S}}_*(n_+, n_+ \wedge A(X) \wedge A(Y)) \\ &\downarrow \text{multiplication} \\ \underline{\mathcal{S}}_*(n_+, n_+ \wedge A(X \wedge Y)) &= Mat_n A(X \wedge Y). \end{aligned}$$

We note that for a simplicial ring B , there is a natural map of $\mathbf{S}$ -algebras (sending some wedges to products, and rearranging the order)

$$Mat_n H B \rightarrow H M_n B$$

where $M_n B$ is the ordinary matrix ring. This map is a stable equivalence as defined in Definition 2.2.1.5. We also have a “Whitehead sum”

$$Mat_n(A) \times Mat_m(A) \xrightarrow{\vee} Mat_{n+m}(A)$$

which is the block sum listing the first matrix in the upper left hand corner and the second matrix in the lower right hand corner. This sum is sent to the ordinary Whitehead sum under the map $Mat_n H B \rightarrow H M_n B$.

2.1.5 A-Modules

If A is a ring, we define a left A -module to be an abelian group M together with a map $A \otimes M \rightarrow M$ satisfying certain properties. In other words, it is a “ $(A \otimes -)$ -algebra” where $(A \otimes -)$ is the triple on abelian groups sending P to $A \otimes P$. Likewise

Definition 2.1.5.1 Let A be an $\mathbf{S}$ -algebra. A (left) A -module is an $(A \wedge -)$ -algebra.

To be more explicit, a left A -module is a pair (M, μ^M) where $M \in ob \Gamma S_*$ and

$$A \wedge M \xrightarrow{\mu^M} M \in \Gamma S_*$$

such that

$$\begin{array}{ccc}
 A \wedge A \wedge M & \xrightarrow{id \wedge \mu^M} & A \wedge M \\
 \mu^A \wedge id \downarrow & & \mu^M \downarrow \\
 A \wedge M & \xrightarrow{\mu^M} & M
 \end{array}$$

commutes and such that the composite

$$M \cong \mathbf{S} \wedge M \xrightarrow{1 \wedge id} A \wedge M \xrightarrow{\mu^M} M$$

is the identity.

If M and N are A -modules, an A -module map $M \rightarrow N$ is a map of Γ -spaces compatible with the A -module structure (an “ $(A \wedge -)$ -algebra morphism”).

Remark 2.1.5.2

1. Note that, as remarked for $\mathbf{S}$ -algebras in Remark 2.1.4.2, the structure maps defining A -modules could again be defined directly without reference to the internal smash in $\Gamma\mathbf{S}_*$.
2. One defines *right A -modules* and *A -bimodules* as A^o -modules and $A^o \wedge A$ -modules.
3. Note that an $\mathbf{S}$ -module is no more than a Γ -space. In general, if A is a commutative $\mathbf{S}$ -algebra, then the concepts of left or right modules agree.
4. If A is a simplicial ring, then an HA -module does not need to be of the sort HP for an A -module P , but we shall see that the difference between A -modules and HA -modules is for most applications irrelevant.

Definition 2.1.5.3 Let A be an $\mathbf{S}$ -algebra. Let M be an A -module and M' an A^o -module. The smash product $M' \wedge_A M$ is the Γ -space given by the coequalizer

$$M' \wedge_A M = \lim_{\rightarrow} \{M' \wedge A \wedge M \rightrightarrows M' \wedge M\}$$

where the two maps represent the two actions.

Definition 2.1.5.4 Let A be an $\mathbf{S}$ -algebra and let M, N be A -modules. The Γ -space of A -module maps is defined as the equalizer

$$\underline{\mathcal{M}}_A(M, N) = \lim_{\leftarrow} \{ \underline{\Gamma\mathbf{S}}_*(M, N) \rightrightarrows \underline{\Gamma\mathbf{S}}_*(A \wedge M, N) \}$$

where the first map is induced by the action of A on M , and the second is

$$\underline{\Gamma\mathbf{S}}_*(M, N) \rightarrow \underline{\Gamma\mathbf{S}}_*(A \wedge M, A \wedge N) \rightarrow \underline{\Gamma\mathbf{S}}_*(A \wedge M, N)$$

induced by the action of A on N .

From these definitions, the following proposition is immediate.

Proposition 2.1.5.5 *Let k be a commutative $\mathbf{S}$ -algebra. Then the smash product and morphism object over k endow the category $\mathcal{M}_k$ of k -modules with the structure of a closed category.* ☺

Example 2.1.5.6 (k -Algebras) If k is a commutative $\mathbf{S}$ -algebra, the monoids in the closed monoidal category $(k\text{-mod}, \wedge_k, k)$ are called k -algebras. The most important example to us is the category of $H\mathbf{Z}$ -algebras. A crucial point we shall return to later is that the homotopy categories of $H\mathbf{Z}$ -algebras and simplicial rings are equivalent.

2.1.6 $\Gamma\mathbf{S}_*$ -Categories

Since $(\Gamma\mathbf{S}_*, \wedge, \mathbf{S})$ is a (symmetric monoidal) closed category it makes sense to talk of a $\Gamma\mathbf{S}_*$ -category, i.e., a collection of objects $ob\mathcal{C}$ and for each pair of objects $c, d \in ob\mathcal{C}$ a Γ -space $\underline{\mathcal{C}}(c, d)$ of “morphisms”, with multiplication

$$\underline{\mathcal{C}}(c, d) \wedge \underline{\mathcal{C}}(b, c) \longrightarrow \underline{\mathcal{C}}(b, d)$$

and unit

$$\mathbf{S} \longrightarrow \underline{\mathcal{C}}(c, c)$$

satisfying the usual identities analogous to the notion of an $\mathbf{S}$ -algebra (as a matter of fact: an $\mathbf{S}$ -algebra is precisely (the Γ -space of morphisms in) a $\Gamma\mathbf{S}_*$ -category with one object). See Sect. A.10.2 for more details on enriched category theory.

In particular, $\Gamma\mathbf{S}_*$ is itself a $\Gamma\mathbf{S}_*$ -category. As another example; from Definition 2.1.5.4 of the Γ -space of A -module morphisms, the following fact follows immediately.

Proposition 2.1.6.1 *Let A be an $\mathbf{S}$ -algebra. Then the category of A -modules is a $\Gamma\mathbf{S}_*$ -category.* ☺

Further examples of $\Gamma\mathbf{S}_*$ -categories:

Example 2.1.6.2

1. Any $\Gamma\mathbf{S}_*$ -category $\mathcal{C}$ has an underlying $\mathbf{S}_*$ -category $R\mathcal{C}$, or just $\mathcal{C}$ again for short, with function spaces $(R\mathcal{C})(c, d) = R(\underline{\mathcal{C}}(c, d)) = \underline{\mathcal{C}}(c, d)(1_+)$ (see Example 2.1.2.1(3)). The prime example being $\Gamma\mathbf{S}_*$ itself, where we always drop the R from the notation.

A $\Gamma\mathbf{S}_*$ -category with only one object is what we call an $\mathbf{S}$ -algebra (just as a k -mod-category with only one object is a k -algebra), and this is closely connected to Bökstedt’s notion of an FSP. In fact, a “ring functor” in the sense of [70] is the same as a $\Gamma\mathbf{S}_*$ -category when restricted to $\Gamma^o \subseteq \mathbf{S}_*$, and conversely, any $\Gamma\mathbf{S}_*$ -category is a ring functor when extended degreewise.

2. Just as the Eilenberg–Mac Lane construction takes rings to $\mathbf{S}$ -algebras Example 2.1.4.3(1), it takes Ab -categories to $\Gamma\mathcal{S}_*$ -categories. Let $\mathcal{E}$ be an Ab -category (i.e., enriched in abelian groups). Then using the Eilenberg–Mac Lane construction of Example 2.1.4.3(1) on the morphism groups gives a $\Gamma\mathcal{S}_*$ -category which we will call $\tilde{\mathcal{E}}$ (it could be argued that it ought to be called $H\mathcal{E}$, but somewhere there has got to be a conflict of notation, and we choose to sin here). To be explicit: if $c, d \in ob\mathcal{E}$, then $\tilde{\mathcal{E}}(c, d)$ is the Γ -space which sends $X \in ob\Gamma^o$ to $\mathcal{E}(c, d) \otimes \tilde{\mathbf{Z}}[X] = H(\mathcal{E}(c, d))(X)$.
3. Let $\mathcal{C}$ be a pointed $\mathcal{S}_*$ -category. The category $\Gamma\mathcal{C}$ of pointed functors $\Gamma^o \rightarrow \mathcal{C}$ is a $\Gamma\mathcal{S}_*$ -category by declaring that

$$\underline{\Gamma\mathcal{C}}(c, d)(X) = \Gamma\mathcal{C}(c, d(X \wedge -)) \in ob\mathcal{S}_*.$$

4. Let $(\mathcal{C}, \sqcup, e)$ be a symmetric monoidal category. An augmented symmetric monoid in $\mathcal{C}$ is an object c together with maps $c \sqcup c \rightarrow c$, $e \rightarrow c \rightarrow e$ satisfying the usual identities. A slick way of encoding all the identities of an augmented symmetric monoid c is to identify it with its bar complex (Eilenberg–Mac Lane object) $\bar{H}c: \Gamma^o \rightarrow \mathcal{C}$ where

$$\bar{H}c(k_+) = \sqcup^{k_+} c = \overbrace{c \sqcup \cdots \sqcup c}^{k \text{ times}} \quad (\sqcup^{0_+} c = e).$$

That is, an augmented symmetric monoid is a rigid kind of Γ -object in $\mathcal{C}$; it is an Eilenberg–Mac Lane object.

5. Adding (3) and (4) together we get a functor from symmetric monoidal categories to $\Gamma\mathcal{S}_*$ -categories, sending $(\mathcal{C}, \sqcup, e)$ to the $\Gamma\mathcal{S}_*$ -category with objects the augmented symmetric monoids, and with morphism objects

$$\Gamma\mathcal{C}(\bar{H}c, \bar{H}d(X \wedge -)).$$

6. Important special case: If $(\mathcal{C}, \vee, e)$ is a *category with sum* (i.e., e is both final and initial in $\mathcal{C}$, and $\vee$ is a coproduct), then **all** objects are augmented symmetric monoids and

$$\Gamma\mathcal{C}(\bar{H}c, \bar{H}d(k_+ \wedge -)) \cong \mathcal{C}\left(c, \bigvee^{k_+} d\right),$$

where $\bigvee^{k_+} d = d \vee \cdots \vee d$ (k -summands).

2.1.6.1 The $\Gamma\mathcal{S}_*$ -Category $\mathcal{C}^\vee$

The last Example 2.1.6.2(6) is so important that we introduce the following notation. Let $(\mathcal{C}, \vee, e)$ be a category with sum (i.e., e is both final and initial in $\mathcal{C}$, and $\vee$ is a

coproduct), then $\mathcal{C}^\vee$ is the $\Gamma\mathcal{S}_*$ -category with $ob\mathcal{C}^\vee = ob\mathcal{C}$ and

$$\mathcal{C}^\vee(c, d)(X) = \mathcal{C}\left(c, \bigvee^X d\right).$$

If $(\mathcal{E}, \oplus, 0)$ is an Ab -category with sum (what is often called an *additive category*), then the $\tilde{\mathcal{E}}$ of Example 2.1.6.2(2) and $\mathcal{E}^\oplus$ coincide:

$$\tilde{\mathcal{E}}(c, d)(n_+) \cong \mathcal{E}(c, d)^{\times n} \cong \mathcal{E}(c, d^{\oplus n}) = \mathcal{E}^\oplus(c, d)(n_+),$$

since finite sums and products coincide in an additive category, see [191, p. 194].

It is worth noting that the structure of Example 2.1.6.2(6) when applied to $(\Gamma\mathcal{S}_*, \vee, 0_+)$ is different from the $\Gamma\mathcal{S}_*$ -enrichment we have given to $\Gamma\mathcal{S}_*$ when declaring it to be a symmetric monoidal **closed** category under the smash product. Then $\underline{\Gamma\mathcal{S}_*}(M, N)(X) = \Gamma\mathcal{S}_*(M, N(X \wedge -))$. However, $\vee^X N \cong X \wedge N \rightarrow N(X \wedge -)$ is a stable equivalence (see Definition 2.2.1.5), and in some cases this is enough to ensure that

$$\underline{\Gamma\mathcal{S}_*}^\vee(M, N)(X) \cong \Gamma\mathcal{S}_*(M, X \wedge N) \rightarrow \Gamma\mathcal{S}_*(M, N(X \wedge -)) = \underline{\Gamma\mathcal{S}_*}(M, N)(X)$$

is a stable equivalence.

2.1.6.2 A Reformulation

When talking in the language of Ab -categories (linear categories), a ring is just (the morphism group in) an Ab -category with one object, and an A -module M corresponds to a functor from A to Ab : the ring homomorphism $A \rightarrow \text{End}(M)$ giving the abelian group M a structure of an A -module is exactly the data needed to give a functor from the category (corresponding to) A to Ab sending the single object to M .

In the setting of $\Gamma\mathcal{S}_*$ -categories, we can similarly reinterpret $\mathbf{S}$ -algebras and their modules. An $\mathbf{S}$ -algebra A is simply a $\Gamma\mathcal{S}_*$ -category with only one object, and an A -module is a $\Gamma\mathcal{S}_*$ -functor from A to $\Gamma\mathcal{S}_*$.

Thinking of A -modules as $\Gamma\mathcal{S}_*$ -functors $A \rightarrow \Gamma\mathcal{S}_*$ the definitions of smash and morphism objects can be elegantly expressed as

$$M' \wedge_A M = \int^A M' \wedge M$$

and

$$\underline{\text{Hom}}_A(M, N) = \int_A \underline{\Gamma\mathcal{S}_*}(M, N).$$

If B is another $\mathbf{S}$ -algebra, M' a $B \wedge A^o$ -module we get $\Gamma\mathcal{S}_*$ -adjoint functors

$$\mathcal{M}_A \begin{array}{c} \xrightarrow{M' \wedge_A -} \\ \xleftarrow{\quad} \end{array} \mathcal{M}_B \\ \mathcal{M}_B(M', -)$$

due to the canonical isomorphism

$$\begin{aligned}\underline{\mathcal{M}}_B(M' \wedge_A N, P) &= \int_B \underline{\Gamma\mathcal{S}}_* \left(\int^A M' \wedge N, P \right) \\ &\cong \int_A \underline{\Gamma\mathcal{S}}_* \left(M', \int_B \underline{\Gamma\mathcal{S}}_*(N, P) \right) = \underline{\mathcal{M}}_A(N, \underline{\mathcal{M}}_B(M', P))\end{aligned}$$

which follows from the definitions, the Fubini theorem for ends and the fact that $\Gamma\mathcal{S}_*$ is closed symmetric monoidal ($P \in \text{ob}\mathcal{M}_B$).

2.2 Stable Structures

In this section we will discuss the homotopical properties of Γ -spaces and $\mathbf{S}$ -algebras. Historically Γ -spaces are nice representations of connective spectra and the choice of equivalences reflects this. That is, in addition to the obvious pointwise equivalences, we have the so-called stable equivalences. The functors of $\mathbf{S}$ -algebras we will define, such as K-theory, should respect stable equivalences. Any $\mathbf{S}$ -algebra can, up to a canonical stable equivalence, be replaced by a very special one.

2.2.1 The Homotopy Theory of Γ -Spaces

To define the stable structure we need to take a different view to Γ -spaces.

2.2.1.1 Gamma-Spaces as Functors of Spaces

Let M be a Γ -space. It is a (pointed) functor $M: \Gamma^o \rightarrow \mathcal{S}_*$, and by extension by colimits and degreewise application followed by the diagonal we may think of it as a functor $\mathcal{S}_* \rightarrow \mathcal{S}_*$. To be explicit, we first extend from the skeletal category Γ^o to all finite pointed sets (in a chosen universe) by, for each finite pointed set S of cardinality $k + 1$, choosing a pointed isomorphism $\alpha_S: S \cong k_+$ (α_{k_+} is chosen to be the identity), setting $M(S) = M(k_+)$ and if $f: S \rightarrow T$ is a pointed function of finite sets we define $M(f)$ to be $M(\alpha_T f \alpha_S^{-1})$. If X is a pointed set, we define

$$M(X) = \lim_{\overline{Y} \subseteq \overline{X}} M(Y),$$

where the colimit varies over the finite pointed subsets $Y \subseteq X$, and so M is a (pointed) functor $\text{Ens}_* \rightarrow \mathcal{S}_*$. For this to be functorial, we—as always—assume that all colimits are actually chosen (and not something only defined up to unique isomorphism). Finally, if $X \in \text{ob}\mathcal{S}_*$, we set

$$M(X) = \text{diag}^* \{ [q] \mapsto M(X_q) \} = \{ [q] \mapsto M(X_q)_q \}.$$

Aside 2.2.1.1 For those familiar with the language of coends, the extensions of a Γ -space M to an endofunctor on spaces can be done all at once: if X is a space, then

$$M(X) = \int^{k_+} X^{\times k} \wedge M(k_+).$$

In yet other words, we do the left Kan extension

$$\begin{array}{ccc} \Gamma^o & \xrightarrow{M} & \mathcal{S}_* \\ \mathbf{s} \downarrow & \nearrow & \\ \mathcal{S}_* & & \end{array}$$

2.2.1.2 Gamma-Spaces as Simplicial Functors

The fact that these functors come from degreewise applications of a functor on (discrete) sets make them “simplicial” (more precisely: they are $\mathcal{S}_*$ -functors), i.e., they give rise to simplicial maps

$$\underline{\mathcal{S}}_*(X, Y) \rightarrow \underline{\mathcal{S}}_*(M(X), N(Y))$$

which results in natural maps

$$M(X) \wedge Y \rightarrow M(X \wedge Y) \tag{2.2}$$

coming from the identity on $X \wedge Y$ through the composite

$$\begin{aligned} \mathcal{S}_*(X \wedge Y, X \wedge Y) &\cong \mathcal{S}_*(Y, \underline{\mathcal{S}}_*(X, X \wedge Y)) \\ &\rightarrow \mathcal{S}_*(Y, \underline{\mathcal{S}}_*(M(X), M(X \wedge Y))) \cong \mathcal{S}_*(M(X) \wedge Y, M(X \wedge Y)) \end{aligned}$$

(where the isomorphisms are the adjunction isomorphisms of the smash/function space adjoint pair). In particular this means that Γ -spaces define spectra: the n th term is given by $M(S^n)$, and the structure map is $S^1 \wedge M(S^n) \rightarrow M(S^{n+1})$ where S^n is $S^1 = \Delta[1]/\partial\Delta[1]$ smashed with itself n times, see also Definition 2.2.1.8 below.

Definition 2.2.1.2 If $M \in ob\Gamma\mathcal{S}_*$, then the *homotopy groups* of M are defined as

$$\pi_q M = \varinjlim_k \pi_{k+q} M(S^k).$$

Note that $\pi_q M = 0$ for $q < 0$, by the following lemma.

Lemma 2.2.1.3 Let $M \in \Gamma\mathcal{S}_*$.

1. If $Y \xrightarrow{\sim} Y' \in \mathcal{S}_*$ is a weak equivalence then $M(Y) \xrightarrow{\sim} M(Y')$ is a weak equivalence also.

2. If X is an n -connected pointed space, then $M(X)$ is n -connected also.
3. If X is an n -connected pointed space, then the canonical map of Eq. (2.2) $M(X) \wedge Y \rightarrow M(X \wedge Y)$ is $2n$ -connected.

Proof Let $\mathbf{LM}$ be the simplicial Γ -space given by

$$\mathbf{LM}(X)_p = \bigvee_{Z_0, \dots, Z_p \in (\Gamma^o)^{\times p+1}} M(Z_0) \wedge \Gamma^o(Z_0, Z_1) \wedge \dots \wedge \Gamma^o(Z_{p-1}, Z_p) \wedge \Gamma^o(Z_p, X)$$

with operators determined by

$$d_i(f \wedge \alpha_1 \wedge \dots \wedge \alpha_p \wedge \beta) = \begin{cases} (M(\alpha_1)(f) \wedge \alpha_2 \wedge \dots \wedge \alpha_p \wedge \beta) & \text{if } i = 0, \\ (f \wedge \alpha_1 \wedge \dots \wedge \alpha_{i+1} \circ \alpha_i \wedge \dots \wedge \beta) & \text{if } 1 \leq i \leq p-1, \\ (f \wedge \alpha_1 \wedge \dots \wedge \alpha_{p-1} \wedge (\beta \circ \alpha_p)) & \text{if } i = p, \end{cases}$$

$$s_j(f \wedge \alpha_1 \wedge \dots \wedge \alpha_p \wedge \beta) = (f \wedge \dots \wedge \alpha_j \wedge \text{id} \wedge \alpha_{j+1} \wedge \dots \wedge \beta)$$

($\mathbf{LM}$ is an example of a “homotopy coend”, or a “bar construction”). Consider the natural transformation

$$\mathbf{LM} \xrightarrow{\eta} M$$

determined by

$$(f \wedge \alpha_1 \wedge \dots \wedge \beta) \mapsto M(\beta \circ \alpha_p \circ \dots \circ \alpha_1)(f).$$

For each $Z \in \text{ob} \Gamma^o$ we obtain a simplicial homotopy inverse to η_Z by sending $f \in M(Z)$ to $(f \wedge \text{id}_Z \wedge \dots \wedge \text{id}_Z)$. Since $\mathbf{LM}$ and M both commute with filtered colimits we see that η is an equivalence on all pointed sets and so by Theorem A.6.0.1, η is an equivalence for all pointed simplicial sets because $\mathbf{LM}$ and M are applied degreewise. Thus, for all pointed simplicial sets X the map η_X is a weak equivalence

$$\mathbf{LM}(X) \xrightarrow{\sim} M(X).$$

(1) If $Y \xrightarrow{\sim} Y'$ is a weak equivalence then $\mathcal{S}_*(k_+, Y) \cong Y^{\times k} \xrightarrow{\sim} (Y')^{\times k} \cong \mathcal{S}_*(k_+, Y')$ is a weak equivalence for all k . But this implies that $\mathbf{LM}(Y)_p \xrightarrow{\sim} \mathbf{LM}(Y')_p$ for all p and hence, by Theorem A.6.0.1, that $\mathbf{LM}(Y) \xrightarrow{\sim} \mathbf{LM}(Y')$.

(2) If X is n -connected for some $n \geq 0$, then $\mathcal{S}_*(k_+, X) \cong X^{\times k}$ is n -connected for all k and hence $\mathbf{LM}(X)_p$ is n -connected for all p . Thus, by Theorem A.6.0.5 we see that $\mathbf{LM}(X)$ is n -connected also.

(3) If X is n -connected and X' is m -connected then, by Corollary A.8.2.4, $X \vee X' \rightarrow X \times X'$ is $(m+n)$ -connected and so $Y \wedge (X \times X') \rightarrow (Y \wedge X') \times (Y \wedge X')$ is

$(m + n)$ -connected also by the commuting diagram

$$\begin{array}{ccc} Y \wedge (X \vee X') & \longrightarrow & Y \wedge (X \times X') \\ \cong \downarrow & & \downarrow \\ (Y \wedge X) \vee (Y \wedge X') & \longrightarrow & (Y \wedge X) \times (Y \wedge X') \end{array}$$

since both horizontal maps are $(m + n)$ -connected. By induction we see that

$$Y \wedge \mathcal{S}_*(k_+, X) \rightarrow \mathcal{S}_*(k_+, Y \wedge X)$$

is $2n$ -connected for all k and so $Y \wedge \mathbf{LM}(X)_p \rightarrow \mathbf{LM}(Y \wedge X)_p$ is $2n$ -connected for all p . Since a simplicial space which is $k > 0$ -connected in every degree has a k -connected diagonal (e.g., by Theorem A.6.0.5) we can conclude that $Y \wedge \mathbf{LM}(X) \rightarrow \mathbf{LM}(Y \wedge X)$ is $2n$ -connected. $\square$

Following Schwede [253] we now define two closed model category structures on $\Gamma\mathcal{S}_*$ (these differ very slightly from the structures considered by Bousfield and Friedlander [39] and Lydakis [188]). For basics on model categories see Appendix A.4. We will call these model structures the “pointwise” and the “stable” structures”:

Definition 2.2.1.4 Pointwise structure: A map $M \rightarrow N \in \Gamma\mathcal{S}_*$ is a *pointwise fibration* (resp. *pointwise equivalence*) if $M(X) \rightarrow N(X) \in \mathcal{S}_*$ is a fibration (resp. weak equivalence) for every $X \in \text{ob}\Gamma$. The map is a (pointwise) *cofibration* if it has the lifting property with respect to maps that are both pointwise fibrations and pointwise equivalences, i.e., $i : A \rightarrow X \in \Gamma\mathcal{S}_*$ is a cofibration if for every pointwise fibration $f : E \rightarrow B \in \Gamma\mathcal{S}_*$ that is a pointwise equivalence and for every solid commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & E \\ i \downarrow & \nearrow s & \downarrow f \\ X & \longrightarrow & B \end{array}$$

there exists a (dotted) map $s : X \rightarrow E$ making the resulting diagram commute.

From this one constructs the stable structure. Note that the cofibrations in the two structures are the same! Because of this we often omit the words “pointwise” and “stable” when referring to cofibrations.

Definition 2.2.1.5 Stable structure: A map of Γ -spaces is a *stable equivalence* if it induces an isomorphism on homotopy groups (defined in Definition 2.2.1.2). It is a (stable) *cofibration* if it is a (pointwise) cofibration, and it is a *stable fibration* if it has the lifting property with respect to all maps that are both stable equivalences and cofibrations.

As opposed to simplicial sets, not all Γ -spaces are cofibrant. Examples of cofibrant objects are the Γ -spaces Γ^X of Example 2.1.2.1(4) (and so the simplicial Γ -spaces $\mathbf{L}M$ defined in the proof of Lemma 2.2.1.3 are cofibrant in every degree, so that $\mathbf{L}M \rightarrow M$ can be thought of as a cofibrant resolution).

We shall see in Corollary 2.2.1.7 that the stably fibrant objects are the very special Γ -spaces which are pointwise fibrant.

2.2.1.3 Important Convention

The stable structure will by far be the most important to us, and so when we occasionally forget the qualification “stable”, and say that a map of Γ -spaces is a fibration, a cofibration or an equivalence this is short for it being a **stable** fibration, cofibration or equivalence. We will say “pointwise” when appropriate.

Theorem 2.2.1.6 *Both the pointwise and the stable structures define closed model category structures (see Sect. A.4.2) on $\Gamma\mathcal{S}_*$. Furthermore, these structures are compatible with the $\Gamma\mathcal{S}_*$ -category structure. More precisely: If $M \xrightarrow{i} N$ is a cofibration and $P \xrightarrow{p} Q$ is a pointwise (resp. stable) fibration, then the canonical map*

$$\underline{\Gamma\mathcal{S}}_*(N, P) \rightarrow \underline{\Gamma\mathcal{S}}_*(M, P) \prod_{\underline{\Gamma\mathcal{S}}_*(M, Q)} \underline{\Gamma\mathcal{S}}_*(N, Q) \quad (2.3)$$

is a pointwise (resp. stable) fibration, and if in addition i or p is a pointwise (resp. stable) equivalence, then (2.3) is a pointwise (resp. stable) equivalence.

Sketch proof (cf. Schwede [253]) That the pointwise structure is a closed simplicial model category is essentially an application of Quillen’s basic theorem [235, II4] to the category of Γ -sets. The rest of the pointwise claim follows from the definition of $\underline{\Gamma\mathcal{S}}_*(-, -)$.

As to the stable structure, all the axioms but one follow from the pointwise structure. If $f: M \rightarrow N \in \Gamma\mathcal{S}_*$, one must show that there is a factorization $M \xrightarrow{\sim} X \rightarrow N$ of f as a cofibration which is a stable equivalence, followed by a stable fibration. We refer the reader to [253]. We refer the reader to the same source for compatibility of the stable structure with the $\Gamma\mathcal{S}_*$ -enrichment. ☺

Note that, since the cofibrations are the same in the pointwise and the stable structure, a map is both a pointwise equivalence and a pointwise fibration if and only if it is both a stable equivalence and a stable fibration.

Corollary 2.2.1.7 *Let M be a Γ -space. Then M is stably fibrant (i.e., $M \rightarrow *$ is a stable fibration) if and only if it is very special and pointwise fibrant.*

Proof If M is stably fibrant, $M \rightarrow *$ has the lifting property with respect to all maps that are stable equivalences and cofibrations, and hence also to the maps that are

pointwise equivalences and cofibrations; that is, M is pointwise fibrant. Let $X, Y \in \text{ob } \Gamma^o$, then $\Gamma^X \vee \Gamma^Y \rightarrow \Gamma^{X \vee Y} \cong \Gamma^X \times \Gamma^Y$ is a cofibration and a (stable) equivalence. This means that if M is stably fibrant, then

$$\underline{\Gamma\mathcal{S}}_*(\Gamma^{X \vee Y}, M) \rightarrow \underline{\Gamma\mathcal{S}}_*(\Gamma^X \vee \Gamma^Y, M)$$

is a stable equivalence and a stable fibration, which is the same as saying that it is a pointwise equivalence and a pointwise fibration, which means that

$$M(X \vee Y) \cong \underline{\Gamma\mathcal{S}}_*(\Gamma^{X \vee Y}, M) \rightarrow \underline{\Gamma\mathcal{S}}_*(\Gamma^X \vee \Gamma^Y, M) \cong M(X) \times M(Y)$$

is an equivalence. Here, as elsewhere, we have written $\underline{\Gamma\mathcal{S}}_*(-, -)$ for the underlying morphism space $R\underline{\Gamma\mathcal{S}}_*(-, -)$. Similarly, the map

$$\mathbf{S} \vee \mathbf{S} \xrightarrow{in_1 + \Delta} \mathbf{S} \times \mathbf{S}$$

is a stable equivalence. When $\pi_0 \underline{\Gamma\mathcal{S}}_*(-, M)$ is applied to this map we get the assignment $(a, b) \mapsto (a, a + b): \pi_0 M(1_+)^{\times 2} \rightarrow \pi_0 M(1_+)^{\times 2}$.

If M is fibrant this must be an isomorphism, and so $\pi_0 M(1_+)$ has inverses.

Conversely, suppose that M is pointwise fibrant and very special. Let $M \xrightarrow{i} N \twoheadrightarrow *$ be a factorization into a map that is a stable equivalence and cofibration followed by a stable fibration. Since both M and N are very special i must be a pointwise equivalence, and so has a section (from the pointwise structure), which means that M is a retract of a stably fibrant object since we have a lifting in the diagram in the pointwise structure

$$\begin{array}{ccc} M & \xlongequal{\quad} & M \\ \downarrow \simeq & & \downarrow \\ N & \longrightarrow & * \end{array}$$

□

2.2.1.4 A Simple Fibrant Replacement Functor

In the approach we will follow, it is a strange fact that we will never need to replace a Γ -space with a cofibrant one, but we will constantly need to replace them by stably fibrant ones. There is a particularly easy way to do this: let M be any Γ -space, and set

$$QM(X) = \lim_{\overrightarrow{k}} \Omega^k M(S^k \wedge X),$$

cf. the analogous construction for spectra in Sect. A.3.2.1. Obviously the map $M \rightarrow QM$ is a stable equivalence, and QM is pointwise fibrant and very special (use e.g., Lemma 2.2.1.3). For various purposes, this replacement Q will not be good enough. Its main deficiency is that it will not take $\mathbf{S}$ -algebras to $\mathbf{S}$ -algebras.

2.2.1.5 Comparison with Spectra

We have already observed that Γ -spaces give rise to spectra:

Definition 2.2.1.8 Let M be a Γ -space. Then the *spectrum associated* with M is the sequence

$$\underline{M} = \{k \mapsto M(S^k)\}$$

where S^k is $S^1 = \Delta[1]/\partial\Delta[1]$ smashed with itself k times, together with the structure maps $S^1 \wedge M(S^k) \rightarrow M(S^1 \wedge S^k) = M(S^{k+1})$ of Eq. (2.2).

The assignment $M \mapsto \underline{M}$ is a simplicial functor

$$\Gamma S_* \xrightarrow{M \mapsto \underline{M}} \mathcal{Spt}$$

(where $\mathcal{Spt}$ is the category of spectra, see Appendix A.3.2 for details) and it follows from the considerations in [39] that it induces an equivalence between the stable homotopy categories of Γ -spaces and connective spectra.

Crucial for the general acceptance of Lydakis' definition of the smash product was the following (where $\text{conn}(X)$ is the connectivity of X):

Proposition 2.2.1.9 Let M and N be Γ -spaces and X and Y spaces. If M is cofibrant, then the canonical map

$$M(X) \wedge N(Y) \rightarrow (M \wedge N)(X \wedge Y)$$

is n -connected with $n = \text{conn}(X) + \text{conn}(Y) + \min(\text{conn}(X), \text{conn}(Y))$.

Sketch proof (see [188] for further details). The proof goes by induction, first treating the case $M = \Gamma^o(n_+, -)$, and observing that then $M(X) \wedge N(Y) \cong X^{\times n} \wedge N(Y)$ and $(M \wedge N)(X \wedge Y) \cong N((X \wedge Y)^{\times n})$. Hence, in this case the result follows from Lemma 2.2.1.3(3). ☺

Corollary 2.2.1.10 Let M and N be Γ -spaces with M cofibrant. Then $\underline{M \wedge N}$ is stably equivalent to a handcrafted smash product of spectra, e.g.,

$$n \mapsto \left\{ \lim_{\substack{\longrightarrow \\ k,l}} \Omega^{k+l}(S^n \wedge M(S^k) \wedge N(S^l)) \right\}. \quad \text{☺}$$

2.2.2 A Fibrant Replacement for S-Algebras

Note that if A is a simplicial ring, then the Eilenberg–Mac Lane object HA of Example 2.1.2.1(1) is a very special Γ -space, and so maps between simplicial rings

induce maps that are stable equivalences if and only if they are pointwise equivalences. Hence any functor respecting pointwise equivalences of $\mathbf{S}$ -algebras will have good homotopy properties when restricted to simplicial rings.

When we want to apply functors to all $\mathbf{S}$ -algebras A , we frequently need to replace our $\mathbf{S}$ -algebras by a very special $\mathbf{S}$ -algebras before feeding them to our functor in order to ensure that the functor will preserve stable equivalences. This is a potential problem since the fibrant replacement functor Q presented in Sect. 2.2.1.4 does not take $\mathbf{S}$ -algebras to $\mathbf{S}$ -algebras.

For this we need a gadget explored by Breen [41] and Bökstedt [30]. Breen noted the need for a refined stabilization of the Eilenberg-Mac Lane spaces for rings and Bökstedt noted that when he wanted to extend Hochschild homology to $\mathbf{S}$ -algebras or rather FSPs (see Chap. 4) in general, the face maps were problematic as they involved the multiplication, and this was not well behaved with respect to naïve stabilization. Both mention Illusie [145] as a source of inspiration.

2.2.2.1 The Category $\mathcal{I}$

Let $\mathcal{I} \subset \Gamma^o$ be the subcategory with all objects, but only the injective maps. This has much more structure than the natural numbers considered as the subcategory where we only allow the standard inclusion $\{0, 1, \dots, n-1\} \subset \{0, 1, \dots, n\}$. Most importantly, the wedge sum of two sets $x_0, x_1 \mapsto x_0 \vee x_1$ induces a natural transformation $\mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I}$. To be quite precise, the sum is given by $k_+ \vee l_+ = (k+l)_+$ with inclusion maps $k_+ \rightarrow (k+l)_+$ sending $i \in k_+$ to $i \in (k+l)_+$, and $l_+ \rightarrow (k+l)_+$ sending $j > 0 \in l_+$ to $k+j \in (k+l)_+$. Note that $\vee$ is strictly associative and unital: $(x \vee y) \vee z = x \vee (y \vee z)$ and $0_+ \vee x = x = x \vee 0_+$ (but symmetric only up to isomorphism).

This results in a simplicial category $\{[p] \mapsto \mathcal{I}^{p+1}\}$ with structure maps given by sending $\mathbf{x} = (x_0, \dots, x_q) \in \mathcal{I}^{q+1}$ to

$$d_i(\mathbf{x}) = \begin{cases} (x_0, \dots, x_i \vee x_{i+1}, \dots, x_q) & \text{for } 0 \leq i < q, \\ (x_q \vee x_0, x_1, \dots, x_{p-1}) & \text{for } i = q, \end{cases}$$

$$s_i(\mathbf{x}) = (x_0, \dots, x_i, 0_+, x_{i+1}, \dots, x_p) \quad \text{for } 0 \leq i \leq q.$$

Below, and many times later, we will use the symbol $Map_*(X, Y)$ to signify the homotopy-theoretically sensible mapping space $\underline{\mathcal{S}}_*(X, \sin|Y|)$ (which, in view of the geometric realization/singular complex adjunction is naturally isomorphic to the singular complex of the space $Top_*(|X|, |Y|)$ of pointed maps with the compact open topology) between pointed simplicial sets X and Y . If Y is fibrant, the map $\underline{\mathcal{S}}_*(X, Y) \rightarrow Map_*(X, Y)$ induced by the unit of adjunction $Y \rightarrow \sin|Y|$ is a weak equivalence. For more details on the geometric realization/singular complex adjoint pair, the reader may consult Appendix A.2.1. On several occasions we will need that smashing with a pointed space A induces a map $Map_*(X, Y) \rightarrow Map_*(A \wedge X, A \wedge Y)$, sending the q -simplex $\Delta[q]_+ \wedge X \rightarrow \sin|Y|$ to

$$\Delta[q]_+ \wedge (A \wedge X) \cong A \wedge (\Delta[q]_+ \wedge X) \rightarrow A \wedge \sin|Y| \rightarrow \sin|A \wedge Y|,$$

where the first isomorphism is the symmetry structure isomorphism of $\wedge$, the middle map is induced by the map in question and the last map is adjoint to the composite $A \rightarrow \underline{S}_*(Y, A \wedge Y) \rightarrow \underline{S}_*(\sin |Y|, \sin |A \wedge Y|)$ (where the first map is adjoint to the identity and the last map induced by the S_* -functor $\sin | - |$).

Definition 2.2.2.1 If $x = k_+ \in \text{ob}\mathcal{I}$, we let $|x| = k$ —the number of non-base points. We will often not distinguish notationally between x and $|x|$. For instance, an expression like S^x will mean S^1 smashed with itself $|x|$ times: $S^{0_+} = S^0$, $S^{(k+1)_+} = S^1 \wedge S^{k_+}$. Likewise Ω^x will mean $\text{Map}_*(S^x, -)$. If $\phi: x \rightarrow y \in \mathcal{I}$, then $S^{(|y|-|x|)} \wedge S^x \rightarrow S^y$ is the isomorphism which inserts the j th factor of S^x as the $\phi(j)$ th factor of S^y and distributes the factors of $S^{(|y|-|x|)}$ over the remaining factors of S^y , keeping the order. If M is a Γ -space and X is a finite pointed set, the assignment $x \mapsto \Omega^x M(S^x \wedge X)$ is a functor, where $\phi: x \rightarrow y$ is sent to

$$\begin{aligned} \Omega^x M(S^x \wedge X) &\rightarrow \Omega^{(|y|-|x|)+|x|} (S^{(|y|-|x|)} \wedge M(S^x \wedge X)) \rightarrow \\ \Omega^{(|y|-|x|)+|x|} M(S^{(|y|-|x|)} \wedge S^x \wedge X) &\cong \Omega^y M(S^y \wedge X), \end{aligned}$$

where the first map is the suspension, the second is induced by the structure map of M and the last isomorphism is conjugation by the isomorphism $S^{(|y|-|x|)} \wedge S^x \rightarrow S^y$ described above. Let $T_0 M$ be the Γ -space

$$T_0 M = \left\{ X \mapsto \text{holim}_{x \in \mathcal{I}} \Omega^x M(S^x \wedge X) \right\}$$

The reason for the notation $T_0 M$ will become apparent in Chap. 4 (no, it is not because it is the tangent space of something).

We would like to know that this has the right homotopy properties, i.e., that $T_0 M$ is equivalent to

$$QM = \left\{ X \mapsto \varinjlim_k \Omega^k M(S^k \wedge X) \right\}.$$

One should note that, as opposed to $\mathbf{N}$, the category $\mathcal{I}$ is not filtering, so we must stick with the homotopy colimits. However, $\mathcal{I}$ possesses certain good properties which overcome this difficulty. Bökstedt attributes the idea behind the following very important stabilization lemma to Illusie [145]. Still, we attach Bökstedt's name to the result to signify the importance his insight at this point was to the development of the cyclotomic trace. See [30, 1.5], but also [192, 2.3.7] or [42, 2.5.1] and compare with [145, VI, 4.6.12] and [41].

Lemma 2.2.2.2 (Bökstedt's Approximation Lemma) *Let $G: \mathcal{I}^{q+1} \rightarrow S_*$ be a functor, $\mathbf{x} \in \text{ob}\mathcal{I}^{q+1}$, and consider the full subcategory $F_{\mathbf{x}} \subseteq \mathcal{I}^{q+1}$ of objects supporting maps from $\mathbf{x}$. Assume G sends maps in $F_{\mathbf{x}}$ to n -connected maps. Then the canonical map*

$$G(\mathbf{x}) \rightarrow \varinjlim_{\mathcal{I}^{q+1}} G$$

is n -connected.

Proof Since $\mathcal{I}^{q+1}$ has an initial object, Lemma A.7.4.1 tells us that we may work with unbased homotopy colimits. Consider the functor

$$\mu_{\mathbf{x}}: \mathcal{I}^{q+1} \xrightarrow{\mathbf{y} \mapsto \mu_{\mathbf{x}}(\mathbf{y}) = \mathbf{x} \vee \mathbf{y}} \mathcal{I}^{q+1},$$

factoring over the inclusion $F_{\mathbf{x}} \subseteq \mathcal{I}^{q+1}$. The second inclusion $\mathbf{y} \subseteq \mathbf{x} \vee \mathbf{y}$ defines a natural transformation $\eta_{\mathbf{x}}$ from the identity to $\mu_{\mathbf{x}}$. This natural transformation translates to a homotopy from the identity to the map

$$\operatorname{holim}_{\mathcal{I}^{q+1}} G \xrightarrow{G\eta_{\mathbf{x}}} \operatorname{holim}_{\mathcal{I}^{q+1}} G\mu_{\mathbf{x}} \xrightarrow{(\mu_{\mathbf{x}})_*} \operatorname{holim}_{\mathcal{I}^{q+1}} G,$$

showing that $\operatorname{holim}_{F_{\mathbf{x}}} G \rightarrow \operatorname{holim}_{\mathcal{I}^{q+1}} G$ is a split surjection in the homotopy category. Likewise, the same natural transformation restricted to $F_{\mathbf{x}}$ gives a homotopy from the identity to

$$\operatorname{holim}_{F_{\mathbf{x}}} G \xrightarrow{G\eta_{\mathbf{x}}} \operatorname{holim}_{F_{\mathbf{x}}} G\mu_{\mathbf{x}} \xrightarrow{(\mu_{\mathbf{x}})_*} \operatorname{holim}_{F_{\mathbf{x}}} G,$$

showing that $\operatorname{holim}_{F_{\mathbf{x}}} G \rightarrow \operatorname{holim}_{\mathcal{I}^{q+1}} G$ is a split injection in the homotopy category. Together this shows that the map $\operatorname{holim}_{F_{\mathbf{x}}} G \rightarrow \operatorname{holim}_{\mathcal{I}^{q+1}} G$ is a weak equivalence. Hence it is enough to show that $G(\mathbf{x}) \rightarrow \operatorname{holim}_{F_{\mathbf{x}}} G$ is n -connected.

Repeating the same argument as above with the constant functor $*$ instead of G , we see that $B(F_{\mathbf{x}}) \cong \operatorname{holim}_{F_{\mathbf{x}}} * \rightarrow \operatorname{holim}_{\mathcal{I}^{q+1}} * \cong B(\mathcal{I}^{q+1})$ is an equivalence, and the latter space is contractible since $\mathcal{I}^{q+1}$ has an initial object. Quillen's theorem B in the form of Lemma A.7.4.2 then states that $G(\mathbf{x}) \rightarrow \operatorname{holim}_{F_{\mathbf{x}}} G$ is n -connected. $\square$

Lemma 2.2.2.3 *Let M be a Γ -space. Then T_0M is very special and the natural transformation $M \rightarrow T_0M$ is a stable equivalence of Γ -spaces.*

Proof Let X and Y be pointed finite sets and $y = k_+ \in \operatorname{ob} \mathcal{I}$. Consider the diagram

$$\begin{array}{ccc} \Omega^y[M(S^y) \wedge (X \vee Y)] & \longrightarrow & \Omega^y(M(S^y) \wedge X) \times \Omega^y(M(S^y) \wedge Y) \\ \downarrow & & \downarrow \\ \Omega^y[M(S^y \wedge (X \vee Y))] & \longrightarrow & \Omega^y M(S^y \wedge X) \times \Omega^y M(S^y \wedge Y) \\ \downarrow & & \downarrow \\ \operatorname{holim}_{x \in \mathcal{I}} \Omega^x[M(S^x \wedge (X \vee Y))] & \longrightarrow & \operatorname{holim}_{x \in \mathcal{I}} \Omega^x M(S^x \wedge X) \times \operatorname{holim}_{x \in \mathcal{I}} M(S^x \wedge Y) \end{array}$$

where the horizontal maps are induced by the projections from $X \vee Y$ to X and to Y and are $k - 2$ -connected by the Freudenthal suspension Theorem A.8.2.3; the

top vertical maps are the maps defined in Eq. (2.2) and are $k - 2$ -connected by Lemma 2.2.1.3(3); and the bottom vertical maps are the canonical maps into the homotopy colimits and are k -connected by Bökstedt's approximation Lemma 2.2.2.2. The bottom map is the map $T_0M(X \vee Y) \rightarrow T_0M(X) \times T_0M(Y)$ induced by the projections. Since k can be chosen arbitrarily, this shows that the Γ -space T_0M is very special.

The same reasoning shows that the canonical map

$$\operatorname{holim}_{k \in \mathbf{N}} \Omega^k M(S^k \wedge X) \rightarrow \operatorname{holim}_{x \in \mathcal{I}} \Omega^x M(S^x \wedge X)$$

induced by the inclusion $\mathbf{N} \subseteq \mathcal{I}$ is a weak equivalence under $M(X)$. Since by Lemma A.7.3.2 the canonical map $\operatorname{holim}_{k \in \mathbf{N}} \Omega^k M(S^k \wedge X) \rightarrow \lim_{k \in \mathbf{N}} \Omega^k M(S^k \wedge X) = (QM)(X)$ is a weak equivalence under $M(X)$ we are done. $\square$

Note that $T_0M(X)$ is usually not a fibrant space, and so T_0M is not stably fibrant either, but the lemma shows that e.g., $\sin |T_0M|$ is stably fibrant.

A *stable equivalence* of $\mathbf{S}$ -algebras is a map of $\mathbf{S}$ -algebras that is a stable equivalence when considered as a map of Γ -spaces.

Lemma 2.2.2.4 *The functor T_0 maps $\mathbf{S}$ -algebras to $\mathbf{S}$ -algebras, and the natural transformation $\operatorname{id} \rightarrow T_0$ is a stable equivalence of $\mathbf{S}$ -algebras.*

Proof Given Lemma 2.2.2.3, we only need to establish the multiplicative properties. Let A be an $\mathbf{S}$ -algebra. We have to define the multiplication and the unit of T_0A . The unit is obvious: $\mathbf{S} \rightarrow T_0\mathbf{S} \rightarrow T_0A$.

If $F, G: J \rightarrow \mathbf{S}_*$ are functors, distributivity of $\wedge$ over $\vee$ gives a natural isomorphism $(\operatorname{holim}_{\rightarrow} F) \wedge (\operatorname{holim}_{\rightarrow} G) \cong \operatorname{holim}_{\rightarrow \times} F \wedge G$. Using the map $\Omega^x A(S^x \wedge X) \wedge \Omega^y A(S^y \wedge Y) \rightarrow \Omega^{x \vee y} A(S^x \wedge X \wedge S^y \wedge Y)$ that smashes maps together, the multiplication in A and the sum in $\mathcal{I}$, the multiplication in T_0A is given by the composite

$$\begin{aligned} T_0A(X) \wedge T_0A(Y) &\longrightarrow \frac{\operatorname{holim}}{(x,y) \in \mathcal{I}^2} \Omega^{x \vee y} (A(S^x \wedge X) \wedge A(S^y \wedge Y)) \\ &\xrightarrow{\text{mult. in } A} \frac{\operatorname{holim}}{(x,y) \in \mathcal{I}^2} \Omega^{x \vee y} A(S^{x \vee y} \wedge X \wedge Y) \\ &\xrightarrow{\vee \text{ in } \mathcal{I}} \operatorname{holim}_{z \in \mathcal{I}} \Omega^z A(S^z \wedge X \wedge Y) = T_0A(X \wedge Y). \end{aligned}$$

Checking that this gives a unital and associative structure on T_0A follows by using the same properties in $\mathcal{I}$ and A . That the map $A \rightarrow T_0A$ is a map of $\mathbf{S}$ -algebras is now immediate. $\square$

Remark 2.2.2.5 It is noteworthy that the fibrant replacement Q is **not** monoidal and will not take $\mathbf{S}$ -algebras to $\mathbf{S}$ -algebras; the presence of nontrivial automorphisms

in $\mathcal{I}$ is of vital importance. We discuss this further in Note 4.2.2.6 since it crucial to Bökstedt's definition of topological Hochschild homology. Notice that if A is an $\mathbf{S}$ -algebra, then the multiplication in A provides $T_0A(1_+)$ with the structure of a simplicial monoid. However, even if A is commutative, the automorphisms of $\mathcal{I}$ prevent $T_0M(1_+)$ from being commutative (unless $A = *$), thus saving us from Lewis' pitfalls [174].

Corollary 2.2.2.6 *Any $\tilde{H}\mathbf{Z}$ -algebra is functorially stably equivalent to $\tilde{H}$ of a simplicial ring. In particular, if A is an $\mathbf{S}$ -algebra, then $\tilde{\mathbf{Z}}A$ is functorially stably equivalent to H of a simplicial ring.*

Proof The T_0 construction can equally well be performed in $\tilde{H}\mathbf{Z}$ -modules: let $\Omega_{Ab}^1 M$ be $\underline{S}_*(S^1, M)$, which is an $\tilde{H}\mathbf{Z}$ -module if M is, and let the homotopy colimit be given by the usual formula except the wedges are replaced by sums (see Sect. A.7.4.1 for further details). Let $R_0A = \operatorname{holim}_{x \in \mathcal{I}} \Omega_{Ab}^x A(S^x)$. This is an $\tilde{H}\mathbf{Z}$ -algebra if A is. There is a natural equivalence $R_0A \rightarrow R_0(\sin|A|)$ and a natural transformation $T_0UA \rightarrow UR_0(\sin|A|)$ (U is the forgetful functor). By Lemma A.7.4.6 and Lemma 2.2.1.3(2) you get that $T_0UA(S^n) \rightarrow UR_0(\sin|A|)(S^n)$ is $(2n - 1)$ -connected. But since both sides are special Γ -spaces, this means that

$T_0UA \xrightarrow{\sim} UR_0 \sin|A| \xleftarrow{\sim} UR_0A$ is a natural chain of weak equivalences. (Alternatively, we could have adapted Bökstedt's approximation theorem to prove directly that $A \rightarrow R_0A$ is a stable equivalence.)

Consequently, if A is a $\tilde{H}\mathbf{Z}$ -algebra, there is a functorial stable equivalence $A \rightarrow R_0A$ of $\tilde{H}\mathbf{Z}$ -algebras. But R_0A is special and for such algebras the unit of adjunction $\tilde{H}R \rightarrow 1$ is an equivalence by Lemma 2.1.3.1. $\square$

2.2.3 Homotopical Algebra in the Category of A -Modules

Although it is not necessary for the subsequent development, we list a few facts pertaining to the homotopy structure on categories of modules over $\mathbf{S}$ -algebras. The stable structure on A -modules is inherited in the usual way from the stable structure on Γ -spaces.

Definition 2.2.3.1 Let A be an $\mathbf{S}$ -algebra. We say that an A -module map is an *equivalence* (resp. *fibration*) if it is a stable equivalence (resp. stable fibration) of Γ -spaces. The *cofibrations* are defined by the lifting property.

Theorem 2.2.3.2 *With these definitions, the category of A -modules is a closed model category compatibly enriched in $\Gamma\mathcal{S}_*$: if $M \xrightarrow{i} N$ is a cofibration and $P \xrightarrow{p} Q$ is a fibration, then the canonical map*

$$\underline{\operatorname{Hom}}_A(N, P) \xrightarrow{(i^*, p_*)} \underline{\operatorname{Hom}}_A(M, P) \prod_{\underline{\operatorname{Hom}}_A(M, Q)} \underline{\operatorname{Hom}}_A(N, Q)$$

is a stable fibration, and if in addition i or p is an equivalence, then (i^*, p_*) is a stable equivalence.

Sketch proof (For a full proof, consult [253]). For the proof of the closed model category structure, see [255, 3.1.1]. For the proof of the compatibility with the enrichment, see the proof of [255, 3.1.2] where the commutative case is treated. ☺

The smash product behaves as expected (see [188] and [253] for proofs):

Proposition 2.2.3.3 *Let A be an $\mathbf{S}$ -algebra, and let M be a cofibrant A^o -module. Then $M \wedge_A - : A\text{-mod} \rightarrow \Gamma\mathcal{S}_*$ sends stable equivalences to stable equivalences. If N is an A -module there are first quadrant spectral sequences*

$$\begin{aligned} \mathrm{Tor}_p^{\pi_* A}(\pi_* M, \pi_* N)_q &\Rightarrow \pi_{p+q}(M \wedge_A N) \\ \pi_p(M \wedge_A (H\pi_q N)) &\Rightarrow \pi_{p+q}(M \wedge_A N) \end{aligned}$$

If $A \rightarrow B$ is a stable equivalence of $\mathbf{S}$ -algebras, then the derived functor of $B \wedge_A -$ induces an equivalence between the homotopy categories of A and B -modules. ☺

2.2.3.1 k -Algebras

Let k be a commutative $\mathbf{S}$ -algebra. In the category of k -algebras, we call a map a fibration or a weak equivalence if it is a stable fibration or stable equivalence of Γ -spaces. The cofibrations are as usual the maps with the right (right meaning correct: in this case left is right) lifting property. With these definitions the category of k -algebras becomes a closed simplicial model category [253]. We will need the analogous result for $\Gamma\mathcal{S}_*$ -categories.

2.2.4 Homotopical Algebra in the Category of $\Gamma\mathcal{S}_*$ -Categories

Definition 2.2.4.1 A $\Gamma\mathcal{S}_*$ -functor of $\Gamma\mathcal{S}_*$ -categories $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *stable equivalence* if for all $c, c' \in \mathrm{ob}\mathcal{C}$ the map

$$\mathcal{C}(c, c') \rightarrow \mathcal{D}(Fc, Fc') \in \Gamma\mathcal{S}_*$$

is a stable equivalence, and for any $d \in \mathrm{ob}\mathcal{D}$ there is a $c \in \mathrm{ob}\mathcal{C}$ and an isomorphism $Fc \cong d$.

Likewise, an $\mathcal{S}$ -functor of $\mathcal{S}$ -categories $F : \mathcal{C} \rightarrow \mathcal{D}$ is a *weak equivalence* if for all $c, c' \in \mathrm{ob}\mathcal{C}$ the map $\mathcal{C}(c, c') \rightarrow \mathcal{D}(Fc, Fc') \in \mathcal{S}$ is a weak equivalence, and for any $d \in \mathrm{ob}\mathcal{D}$ there is a $c \in \mathrm{ob}\mathcal{C}$ and an isomorphism $Fc \cong d$.

A functor which is surjective on isomorphism classes is sometimes called “essentially surjective”.

Recall that a ΓS_* -equivalence is a ΓS_* -functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ for which there exists a ΓS_* -functor $\mathcal{C} \xleftarrow{G} \mathcal{D}$ and ΓS_* -natural isomorphisms $id_{\mathcal{C}} \cong GF$ and $id_{\mathcal{D}} \cong FG$.

Lemma 2.2.4.2 *Every stable equivalence of ΓS_* -categories can be written as a composite of a stable equivalence inducing the identity on the objects and a ΓS_* -equivalence.*

Proof Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a stable equivalence. Let $\underline{F}$ be the ΓS_* -category with the same objects as $\mathcal{C}$, but with morphisms given by $\underline{F}(c, c') = \mathcal{D}(Fc, Fc')$. Then F factors as $\mathcal{C} \rightarrow \underline{F} \rightarrow \mathcal{D}$ where the first map is the identity on objects and a stable equivalence on morphisms, and the second is induced by F on objects, and is the identity on morphisms. The latter map is a ΓS_* -equivalence: for every $d \in ob \mathcal{D}$ choose a $c_d \in ob \mathcal{C}$ and an isomorphism $d \cong Fc_d$. As one checks, the application $d \mapsto c_d$ defines the inverse ΓS_* -equivalence. $\square$

So stable equivalences are the more general, and may be characterized as composites of ΓS_* -equivalences and stable equivalences that induce the identity on the set of objects. Likewise for weak equivalences of $\mathcal{S}$ -categories.

2.3 Algebraic K-Theory

2.3.1 K-Theory of Symmetric Monoidal Categories

A symmetric monoid can be viewed as a symmetric monoidal category (an SMC) with just identity morphisms. A symmetric monoid M gives rise to a Γ -space HM via the formula $k_+ \mapsto M^{\times k}$ (see Example 2.1.2.1(1) and [257]), the Eilenberg–Mac Lane object of M . Algebraic K-theory, as developed in Segal’s paper [257], is an extension of this to symmetric monoidal categories (see also [260] or [282]), such that for every symmetric monoidal category $\mathcal{C}$ we have a Γ -category $\bar{H}\mathcal{C}$.

For a finite set X , let $\mathcal{P}X$ be the set of subsets of X . If S and T are two disjoint subsets of X , then $S \sqcup T$ is again a subset of X . If all the coherence isomorphisms symmetric monoidal category $(\mathcal{C}, \sqcup, e)$ were identities we could define the algebraic K-theory as the Γ -category which evaluated at $k_+ \in \Gamma^o$ was the category whose objects were all functions $\mathcal{P}\{1, \dots, k\} \rightarrow ob \mathcal{C}$ sending $\sqcup$ to $\sqcup$ and $\emptyset$ to e :

$$\left(\begin{array}{c} \mathcal{P}\{1, \dots, k\} \\ \sqcup \\ \emptyset \end{array} \right) \rightarrow \left(\begin{array}{c} ob \mathcal{C} \\ \sqcup \\ e \end{array} \right).$$

Such a function is uniquely given by declaring what its values are on all subsets $\{i\} \subset \{1, \dots, k\}$ and so this is nothing but $\mathcal{C}$ times itself k times.

In the non-strict case this loosens up only a bit. If $(\mathcal{C}, \sqcup, e)$ is a symmetric monoidal category, $\bar{H}\mathcal{C}(k_+)$ is the symmetric monoidal category whose objects are the pointed functors $\mathcal{P}\{1, \dots, k\} \rightarrow \mathcal{C}$ taking $\coprod$ to $\sqcup$ up to coherent isomorphisms. More precisely (remembering that displayed diagrams commute unless otherwise explicitly stated not to)

Definition 2.3.1.1 Let $(\mathcal{C}, \sqcup, e)$ be a symmetric monoidal category. Let $k_+ \in \text{ob}\Gamma^o$. An object of $\bar{H}\mathcal{C}(k_+)$ is a function $a: \mathcal{P}\{1, \dots, k\} \rightarrow \text{ob}\mathcal{C}$ together with a choice of isomorphisms

$$\alpha_{S,T}: a_S \sqcup a_T \rightarrow a_S \sqcup\!\!\sqcup T$$

for every pair $S, T \subseteq \{1, \dots, k\}$ such that $S \cap T = \emptyset$, satisfying the following conditions:

1. $a_\emptyset = e$,
2. the morphisms $a_{\emptyset,S}: e \sqcup a_S \rightarrow a_\emptyset \sqcup\!\!\sqcup S = a_S$ and $a_{S,\emptyset}: a_S \sqcup e \rightarrow a_S \sqcup\!\!\sqcup \emptyset = a_S$ are the structure isomorphisms in $\mathcal{C}$,
3. the diagram

$$\begin{array}{ccc} (a_S \sqcup a_T) \sqcup a_U & \xrightarrow{\text{associativity}} & a_S \sqcup (a_T \sqcup a_U) \\ \downarrow \alpha_{S,T} \sqcup \text{id} & & \downarrow \text{id} \sqcup \alpha_{T,U} \\ a_S \sqcup\!\!\sqcup T \sqcup a_U & \xrightarrow{\alpha_{S \sqcup\!\!\sqcup T, U}} a_S \sqcup\!\!\sqcup T \sqcup\!\!\sqcup U & \xleftarrow{\alpha_{S,T} \sqcup\!\!\sqcup U} a_S \sqcup a_T \sqcup\!\!\sqcup U \end{array}$$

commutes and

4. the diagram

$$\begin{array}{ccc} a_S \sqcup a_T & \xrightarrow{\quad} & a_T \sqcup a_S \\ & \searrow \alpha_{S,T} & \downarrow \alpha_{T,S} \\ & & a_S \sqcup\!\!\sqcup T = a_T \sqcup\!\!\sqcup S \end{array}$$

commutes, where the unlabelled arrow is the corresponding structure isomorphism in $\mathcal{C}$.

A morphism $f: (a, \alpha) \rightarrow (b, \beta) \in \bar{H}\mathcal{C}(X)$ is a collection of morphisms

$$f_S: a_S \rightarrow b_S \in \mathcal{C}$$

such that

1. $f_\emptyset = \text{id}_e$ and

2. the diagram

$$\begin{array}{ccc}
 a_S \sqcup a_T & \xrightarrow{f_S \sqcup f_T} & b_S \sqcup b_T \\
 \alpha_{S,T} \downarrow & & \beta_{S,T} \downarrow \\
 a_S \sqcup\!\!\sqcup T & \xrightarrow{f_S \sqcup\!\!\sqcup T} & b_S \sqcup\!\!\sqcup T
 \end{array}$$

commutes.

If $\phi: k_+ \rightarrow l_+ \in \Gamma^o$, then $\bar{H}\mathcal{C}(k_+) \rightarrow \bar{H}\mathcal{C}(l_+)$ is defined by sending $a: \mathcal{P}\{1, \dots, k\} \rightarrow \mathcal{C}$ to

$$\mathcal{P}\{1, \dots, l\} \xrightarrow{\phi^{-1}} \mathcal{P}\{1, \dots, k\} \xrightarrow{a} \mathcal{C}$$

(this makes sense as ϕ was pointed at 0), with corresponding isomorphism

$$a_{\phi^{-1}(S), \phi^{-1}(T)}: a_{\phi^{-1}S} \sqcup a_{\phi^{-1}T} \rightarrow a_{\phi^{-1}(S) \sqcup\!\!\sqcup \phi^{-1}(T)} = a_{\phi^{-1}(S \sqcup\!\!\sqcup T)}.$$

This defines the Γ -category $\bar{H}\mathcal{C}$, which again is obviously functorial in $\mathcal{C}$, giving the functor

$$\bar{H}: \text{symmetric monoidal categories} \rightarrow \Gamma\text{-categories}.$$

The classifying space $B\bar{H}\mathcal{C}$ forms a Γ -space which is often called the (direct sum) *algebraic K-theory of $\mathcal{C}$* .

If $\mathcal{C}$ is discrete, or in other words, $\mathcal{C} = ob\mathcal{C}$ is a symmetric monoid, then this is exactly the Eilenberg–Mac Lane spectrum of $ob\mathcal{C}$.

Note that $\bar{H}\mathcal{C}$ becomes a **special** Γ -category in the sense that

Lemma 2.3.1.2 *Let $(\mathcal{C}, \sqcup, e)$ be a symmetric monoidal category. The canonical map*

$$\bar{H}\mathcal{C}(k_+) \rightarrow \bar{H}\mathcal{C}(1_+) \times \dots \times \bar{H}\mathcal{C}(1_+)$$

is an equivalence of categories.

Proof We do this by producing an equivalence $E_k: \mathcal{C}^{\times k} \rightarrow \bar{H}\mathcal{C}(k_+)$ such that

$$\begin{array}{ccc}
 \mathcal{C}^{\times k} & \xlongequal{\quad} & \mathcal{C}^{\times k} \\
 E_k \downarrow & & E_1^{\times k} \downarrow \\
 \bar{H}\mathcal{C}(k_+) & \longrightarrow & \bar{H}\mathcal{C}(1_+)^{\times k}
 \end{array}$$

commutes. The equivalence E_k is given by sending $(c_1, \dots, c_k) \in ob\mathcal{C}^{\times k}$ to $E_k(c_1, \dots, c_k) = \{(a_S, \alpha_{S,T})\}$ where

$$a_{\{i_1, \dots, i_j\}} = c_{i_1} \sqcup (c_{i_2} \sqcup \dots \sqcup (c_{i_{k-1}} \sqcup c_{i_k}) \dots)$$

and $\alpha_{S,T}$ is the unique isomorphism we can write up using only the structure isomorphisms in $\mathcal{C}$. Likewise for morphisms. A quick check reveals that this is an equivalence (check the case $k = 1$ first), and that the diagram commutes. $\square$

2.3.1.1 Enrichment in $\Gamma\mathcal{S}_*$

The definitions above make perfect sense also in the $\Gamma\mathcal{S}_*$ -enriched world, and we may speak about symmetric monoidal $\Gamma\mathcal{S}_*$ -categories $\mathcal{C}$.

A bit more explicitly: a *symmetric monoidal $\Gamma\mathcal{S}_*$ -category* is a tuple $(\mathcal{C}, \sqcup, e, \alpha, \lambda, \rho, \gamma)$ such that $\mathcal{C}$ is a $\Gamma\mathcal{S}_*$ -category, $\sqcup: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a $\Gamma\mathcal{S}_*$ -functor, $e \in ob\mathcal{C}$ and α, λ, ρ and γ are $\Gamma\mathcal{S}_*$ -natural transformations satisfying the usual requirements listed in Definition A.10.1.1.

The definition of $\tilde{H}\mathcal{C}$ at this generality is as follows: the objects in $\tilde{H}\mathcal{C}(k_+)$ are the same as before (2.3.1.1), and the Γ -space $\tilde{H}\mathcal{C}((a, \alpha), (b, \beta))$ is defined as the equalizer

$$\tilde{H}\mathcal{C}((a, \alpha), (b, \beta))(k_+) \longrightarrow \prod_{\emptyset \neq S \subseteq \{1, \dots, k\}} \mathcal{C}(a_S, b_S) \rightrightarrows \prod_{\substack{\emptyset \neq S, T \subseteq \{1, \dots, k\} \\ S \cap T = \emptyset}} \mathcal{C}(a_S \sqcup a_T, b_S \sqcup b_T).$$

The (S, T) -components of the two maps in the equalizer are the two ways around

$$\begin{array}{ccccc} \prod_U \mathcal{C}(a_U, b_U) & \xrightarrow{\text{proj}_S \times \text{proj}_T} & \mathcal{C}(a_S, b_S) \times \mathcal{C}(a_T, b_T) & \xrightarrow{\sqcup} & \mathcal{C}(a_S \sqcup a_T, b_S \sqcup b_T) \\ \downarrow \text{proj}_S \sqcup \text{proj}_T & & & & \downarrow (\beta_{S,T})_* \\ \mathcal{C}(a_S \sqcup a_T, b_S \sqcup b_T) & \xrightarrow{(\alpha_{S,T})^*} & & & \mathcal{C}(a_S \sqcup a_T, b_S \sqcup b_T). \end{array}$$

2.3.1.2 Categories with Sum

The simplest example of symmetric monoidal $\Gamma\mathcal{S}_*$ -categories comes from categories with sum (i.e., $\mathcal{C}$ is pointed and has a coproduct $\vee$). If $\mathcal{C}$ is a category with sum we consider it as a $\Gamma\mathcal{S}_*$ -category via the enrichment

$$\mathcal{C}^\vee(c, d)(k_+) = \mathcal{C}\left(c, \bigvee^k d\right)$$

(see Sect. 2.1.6.1).

The sum structure survives to give $\mathcal{C}^\vee$ the structure of a symmetric $\Gamma\mathcal{S}_*$ -monoidal category:

$$\begin{aligned}
& (\mathcal{C}^\vee \times \mathcal{C}^\vee)((c_1, c_2), (d_1, d_2))(k_+) \\
&= \mathcal{C}\left(c_1, \bigvee^k d_1\right) \times \mathcal{C}\left(c_2, \bigvee^k d_2\right) \\
&\xrightarrow{\vee} \mathcal{C}\left(c_1 \vee c_2, \left(\bigvee^k d_1\right) \vee \left(\bigvee^k d_2\right)\right) \cong \mathcal{C}\left(c_1 \vee c_2, \bigvee^k (d_1 \vee d_2)\right) \\
&= \mathcal{C}^\vee(c_1 \vee c_2, d_1 \vee d_2)(k_+).
\end{aligned}$$

Categories with sum also have a particular transparent K-theory. The data for a symmetric monoidal category above simplifies in this case to $\bar{H}\mathcal{C}(k_+)$ having as objects functors from the pointed category of subsets and inclusions of $k_+ = \{0, 1, \dots, k\}$, sending 0_+ to 0 and pushout squares to pushout squares, see also Sect. 3.2.1.1.

2.3.2 Quite Special Γ -Objects

Let $\mathcal{C}$ be a Γ - $\Gamma\mathcal{S}_*$ -category, i.e., a pointed functor $\mathcal{C}: \Gamma^o \rightarrow \Gamma\mathcal{S}_*$ -categories. We say that $\mathcal{C}$ is *special* if for each pair of finite pointed sets X and Y the canonical $\Gamma\mathcal{S}_*$ -functor $\mathcal{C}(X \vee Y) \rightarrow \mathcal{C}(X) \times \mathcal{C}(Y)$ is a $\Gamma\mathcal{S}_*$ -equivalence of $\Gamma\mathcal{S}_*$ -categories. So, for instance, if $\mathcal{C}$ is a symmetric monoidal category, then $\bar{H}\mathcal{C}$ is special. We need a slightly weaker notion.

Definition 2.3.2.1 Let $\mathcal{D}$ be a Γ - $\Gamma\mathcal{S}_*$ -category. We say that $\mathcal{D}$ is *quite special* if for each pair of finite pointed sets $X, Y \in \text{ob}\Gamma^o$ the canonical map $\mathcal{D}(X \vee Y) \rightarrow \mathcal{D}(X) \times \mathcal{D}(Y)$ is a stable equivalence of $\Gamma\mathcal{S}_*$ -categories (see Definition 2.2.4.1 for definition).

Likewise, a functor $\mathcal{D}: \Gamma^o \rightarrow \mathcal{S}$ -categories is *quite special* if $\mathcal{D}(X \vee Y) \rightarrow \mathcal{D}(X) \times \mathcal{D}(Y)$ is a weak equivalence of $\mathcal{S}$ -categories Definition 2.2.4.1.

Typically, theorems about special $\mathcal{D}$ remain valid for quite special $\mathcal{D}$.

Lemma 2.3.2.2 *Let $\mathcal{D}: \Gamma^o \rightarrow \mathcal{S}$ -categories be quite special. Then $B\mathcal{D}$ is special.*

Proof This follows since the classifying space functor B preserves products and by [75] takes weak equivalences of $\mathcal{S}$ -categories to weak equivalences of simplicial sets. $\square$

Recall the fibrant replacement functor T_0 of Definition 2.2.2.1. The same proof as in Lemma 2.2.2.4 gives that if we use T_0 on all the morphism objects in a $\Gamma\mathcal{S}_*$ -category we get a new category where the morphism objects are stably fibrant.

Lemma 2.3.2.3 *Let $\mathcal{D}: \Gamma^o \rightarrow \Gamma\mathcal{S}_*$ -categories be quite special. Then $T_0\mathcal{D}$ is quite special.*

Proof This follows since T_0 preserves stable equivalences, and since

$$T_0(M \times N) \xrightarrow{\sim} T_0M \times T_0N$$

is a stable equivalence for any $M, N \in \text{ob} \Gamma S_*$. Both these facts follow from the definition of T_0 and Bökstedt's approximation Lemma 2.2.2.2. $\square$

2.3.3 A Uniform Choice of Weak Equivalences

When considering a discrete ring A , the algebraic K-theory can be recovered from knowing only the isomorphisms of finitely generated projective A -modules. We will show in Sect. 3.2.1 that the algebraic K-theory of A , as defined through Waldhausen's S -construction in Chap. 1, is equivalent to what you get if you apply Segal's construction $\bar{H}$ to the groupoid $i\mathcal{P}_A$ of finitely generated A -modules and isomorphisms between them. So, non-invertible homomorphisms are not seen by algebraic K-theory.

This is not special for the algebraic K-theory of discrete rings. Most situations where you would be interested in applying Segal's $\bar{H}$ to an (ordinary) symmetric monoidal category, it turns out that only the isomorphisms matter.

This changes when one's attention turns to symmetric monoidal categories where the morphisms form Γ -spaces. Then one focuses on “weak equivalences” within the category rather than on isomorphisms. Luckily, the “universal choice” of weak equivalences, is the most useful one. This choice is good enough for our applications, but has to be modified in more complex situations where we must be free to choose our weak equivalences. For a trivial example of this, see Note 2.3.3.3 below.

Consider the path components functor π_0 as a functor from $\mathcal{S}$ -categories (or ΓS_* -categories) to categories by letting $\text{ob}(\pi_0\mathcal{C}) = \text{ob}\mathcal{C}$, and $(\pi_0\mathcal{C})(c, d) = \pi_0(\mathcal{C}(c, d))$. This works by the monoidality of π_0 . Likewise, the evaluation at 1_+ (see Example 2.1.2.1(3)), $R: \Gamma S_* \rightarrow \mathcal{S}$ induces a functor R from ΓS_* -categories to $\mathcal{S}$ -categories. Note that $\pi_0\mathcal{C} \cong \pi_0 R T_0\mathcal{C}$.

Define the functor

$$\omega: \Gamma S_*\text{-categories} \rightarrow \mathcal{S}\text{-categories}$$

by means of the (categorical) pullback

$$\begin{array}{ccc} \omega\mathcal{C} & \xrightarrow{w_{\mathcal{C}}} & R T_0\mathcal{C} \\ \downarrow & & \downarrow \\ i\pi_0\mathcal{C} & \longrightarrow & \pi_0\mathcal{C} \end{array}$$

where $i\pi_0\mathcal{C}$ is the subcategory of isomorphisms in $\pi_0\mathcal{C}$.

Lemma 2.3.3.1 *Let $\mathcal{C}$ be a quite special Γ - $\Gamma\mathcal{S}_*$ -category. Then $\omega\mathcal{C}$ is a quite special Γ - $\mathcal{S}$ -category.*

Proof That $\mathcal{C}$ is quite special implies that $RT_0\mathcal{C}$ is quite special, since stable equivalences of stably fibrant Γ -spaces are pointwise equivalences, and hence taken to weak equivalences by R . The map $RT_0\mathcal{C} \rightarrow \pi_0 RT_0\mathcal{C} \cong \pi_0\mathcal{C}$ is a (pointwise) fibration since R takes fibrant Γ -spaces to fibrant spaces.

Furthermore, $\pi_0\mathcal{C}$ is special since π_0 takes stable equivalences of $\Gamma\mathcal{S}_*$ -spaces to isomorphisms. The subcategory of isomorphisms in a special Γ -category is always special (since the isomorphism category in a product category is the product of the isomorphism categories), so $i\pi_0\mathcal{C}$ is special too.

We have to know that the pullback behaves nicely with respect to this structure. The map $RT_0\mathcal{C}(X \vee Y) \rightarrow RT_0\mathcal{C}(X) \times RT_0\mathcal{C}(Y)$ is a weak equivalence. Hence it is enough to show that if $\mathcal{A} \rightarrow \mathcal{B}$ is a weak equivalence of $\mathcal{S}$ -categories with fibrant morphism spaces, then $i\pi_0\mathcal{A} \times_{\pi_0\mathcal{A}} \mathcal{A} \rightarrow i\pi_0\mathcal{B} \times_{\pi_0\mathcal{B}} \mathcal{B}$ is a weak equivalence. Notice that $ob\mathcal{A} \cong ob(i\pi_0\mathcal{A} \times_{\pi_0\mathcal{A}} \mathcal{A})$ and that two objects in $\mathcal{A}$ are isomorphic if and only if they are isomorphic as objects of $i\pi_0\mathcal{A} \times_{\pi_0\mathcal{A}} \mathcal{A}$ (and likewise for $\mathcal{B}$). Hence we only have to show that the map induces a weak equivalence on morphism spaces, which is clear since pullbacks along fibrations are equivalent to homotopy pullbacks. $\square$

Lemma 2.3.3.2 *Let $\mathcal{E}$ be an Ab-category with subcategory $i\mathcal{E}$ of isomorphisms, and let $\tilde{\mathcal{E}}$ be the associated $\Gamma\mathcal{S}_*$ -category (see Example 2.1.6.2(2)). Then the natural map $i\mathcal{E} \rightarrow \omega\tilde{\mathcal{E}}$ is a stable equivalence.*

Proof Since $\tilde{\mathcal{E}}$ has stably fibrant morphism objects $T_0\tilde{\mathcal{E}} \xleftarrow{\sim} \tilde{\mathcal{E}}$ and by construction $R\tilde{\mathcal{E}} = \mathcal{E}$ (considered as an $\mathcal{S}$ -category). This means also that $\pi_0\tilde{\mathcal{E}} \cong \mathcal{E}$, and the result follows. $\square$

Note 2.3.3.3 So, for Ab-categories our uniform choice of weak equivalences essentially just picks out the isomorphisms, which is fine since that is what we usually want. For modules over S-algebras they also give a choice which is suitable for K-theory (more about this later).

However, occasionally this construction will not pick out the weak equivalences you had in mind. As an example, consider the category Γ^o itself with its monoidal structure coming from the sum. It turns out that the category of isomorphisms $i\Gamma^o = \coprod_{n \geq 0} \Sigma_n$ is an extremely interesting category: its algebraic K-theory is equivalent to the sphere spectrum by the Barratt–Priddy–Quillen theorem (see e.g., [257, Proposition 3.5]).

However, since Γ^o is a category with sum, by Sect. 2.3.1.2 it comes with a natural enrichment $(\Gamma^o)^\vee$. We get that $(\Gamma^o)^\vee(m_+, n_+)(k_+) \cong \Gamma^o(m_+, k_+ \wedge n_+)$. But in the language of Example 2.1.4.3(6), this is nothing but the n by m matrices over the sphere spectrum. Hence $(\Gamma^o)^\vee$ is isomorphic to the $\Gamma\mathcal{S}_*$ -category whose objects are the natural numbers, and where the Γ -space of morphisms from n to m

is $Mat_{n,m}\mathbf{S} = \prod_m \vee_n \mathbf{S}$. The associated uniform choice of weak equivalences are exactly the “homotopy invertible matrices” $\widehat{GL}_n(\mathbf{S})$ of Definition 3.2.3.1, and the associated algebraic K-theory is the algebraic K-theory of $\mathbf{S}$ —also known as Waldhausen’s algebraic K-theory of a point $A(*)$, see Sect. 3.2.3.

That $\mathbf{S}$ and $A(*)$ are different can for instance be seen from the fact that the stable homotopy groups of spheres are finite in positive dimension, whereas $A(*)$ is rationally equivalent to the K-theory of the integers, which is infinite cyclic in degree 5. For a further discussion, giving partial calculations, see Sect. 7.3.

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