

Preface

Algebraic K-theory draws its importance from its effective codification of a mathematical phenomenon which occurs in as separate parts of mathematics as number theory, geometric topology, operator algebras, homotopy theory and algebraic geometry. In reductionistic language the phenomenon can be phrased as

there is no canonical choice of coordinates,

or, as so elegantly expressed by Hermann Weyl [312, p. 49]:

The introduction of numbers as coordinates ... is an act of violence whose only practical vindication is the special calculatory manageability of the ordinary number continuum with its four basic operations.

As such, algebraic K-theory is a meta-theme for mathematics, but the successful codification of this phenomenon in homotopy-theoretic terms is what has made algebraic K-theory a valuable part of mathematics. For a further discussion of algebraic K-theory we refer the reader to Chap. 1 below.

Calculations of algebraic K-theory are very rare and hard to come by. So any device that allows you to obtain new results is exciting. These notes describe one way to produce such results.

Assume for the moment that we know what algebraic K-theory is; how does it vary with its input?

The idea is that algebraic K-theory is like an analytic function, and we have this other analytic function called *topological cyclic homology* (TC) invented by Bökstedt, Hsiang and Madsen [27], and

the difference between K and TC is locally constant.

This statement will be proven below, and in its integral form it has not appeared elsewhere before.

The good thing about this, is that TC is occasionally possible to calculate. So whenever you have a calculation of K-theory you have the possibility of calculating all the K-values of input “close” to your original calculation.

So, for instance, if somebody (please) can calculate K-theory of the integers, many “nearby” applications in geometric topology (simply connected spaces) are

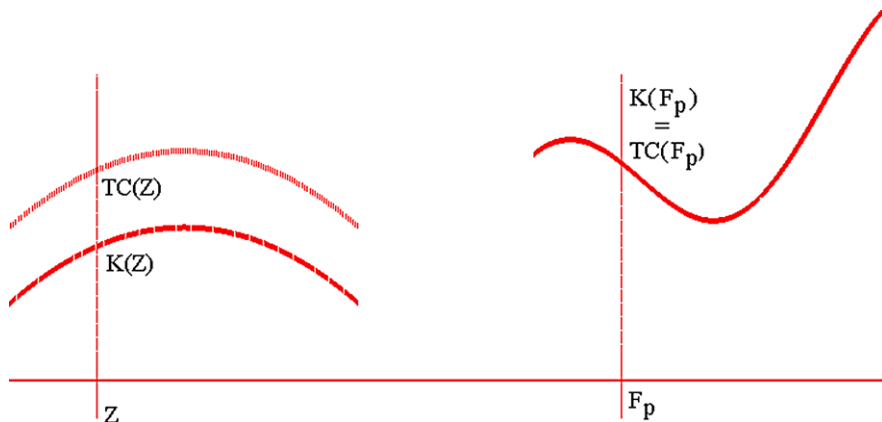


Fig. 0.1 The difference between K and TC is locally constant. The *left part of the figure* illustrates the difference between $K(\mathbf{Z})$ and $TC(\mathbf{Z})$ is quite substantial, but once you know this difference you know that it does not change in a “neighborhood” of $\mathbf{Z}$. In this neighborhood lies for instance all applications of algebraic K-theory of simply connected spaces, so here TC -calculations ultimately should lead to results in geometric topology as demonstrated by Rognes. On the *right hand side of the figure* you see that close to the finite field with p elements, K-theory and TC agree (this is a connective and p -adic statement: away from the characteristic there are other methods that are more convenient). In this neighborhood you find many interesting rings, ultimately resulting in Hesselholt and Madsen’s calculations of the K-theory of local fields

available through TC -calculations (see e.g., [242, 243]). This means that calculations in motivic cohomology (giving K-groups of e.g., the integers) will actually have bearing on our understanding of diffeomorphisms of manifolds!

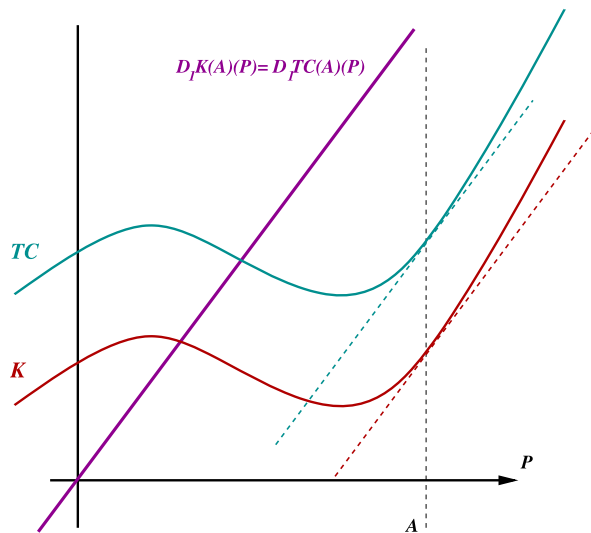
On a different end of the scale, Quillen’s calculation of the K-theory of finite fields gives us access to “nearby” rings, ultimately leading to calculations of the K-theory of local fields [131]. One should notice that the illustration offered by Fig. 0.1 is not totally misleading: the difference between $K(\mathbf{Z})$ and $TC(\mathbf{Z})$ is substantial (though locally constant), whereas around the field $\mathbf{F}_p$ with p elements it is negligible.

Taking K-theory for granted (we’ll spend quite some time developing it later), we should say some words about TC . Since K-theory and TC differ only by some locally constant term, they must have the same differential: $D_1 K = D_1 TC$ (see Fig. 0.2). For ordinary rings A this differential is quite easy to describe: it is the *homology* of the category $\mathcal{P}_A$ of finitely generated projective modules.

The homology of a category is like Hochschild homology, and as Connes observed, certain models of Hochschild homology carry a circle action which is useful when comparing with K-theory. Only, in the case of the homology of categories it turns out that the ground ring over which to take Hochschild homology is not an ordinary ring, but the so-called sphere spectrum. Taking this idea seriously, we end up with Bökstedt’s *topological Hochschild homology* THH .

One way to motivate the construction of TC from THH is as follows. There is a transformation $K \rightarrow THH$ which we will call the *Dennis trace map*, and there is

Fig. 0.2 The differentials “at an S -algebra A in the direction of the A -bimodule P ” of K and TC are equal. For discrete rings the differential is the homology of the category of finitely generated projective modules. In this illustration the differential is the *magenta straight line* through the origin, K -theory is the *red curve* and TC is the shifted curve in *cyan* (Colon figure online)



a model for THH for which the Dennis trace map is just the *inclusion of the fixed points under the circle action*. That is, the Dennis trace can be viewed as a composite

$$K \cong THH^{\mathbb{T}} \subseteq THH$$

where $\mathbb{T}$ is the circle group.

The unfortunate thing about this statement is that it is *model dependent* in that fixed points do not preserve weak equivalences: if $X \rightarrow Y$ is a map of $\mathbb{T}$ -spaces which is a weak equivalence of underlying spaces, normally the induced map $X^{\mathbb{T}} \rightarrow Y^{\mathbb{T}}$ will not be a weak equivalence. So, TC is an attempt to construct the $\mathbb{T}$ -fixed points through techniques that **do** preserve weak equivalences.

It turns out that there is more to the story than this: THH possesses something called an *epicyclic structure* (which is not the case for all $\mathbb{T}$ -spaces), and this allows us to approximate the $\mathbb{T}$ -fixed points even better.

So in the end, the *cyclotomic trace* is a factorization

$$K \rightarrow TC \rightarrow THH$$

of the Dennis trace map.

The cyclotomic trace is the theme for this book. There is another paper devoted to this transformation, namely Madsen’s eminent survey [192]. If you can get hold of a copy it is a great supplement to the current text.

It was originally an intention that readers who were only interested in discrete rings would have a path leading far into the material with minimal contact with ring spectra. This idea has to a great extent been abandoned since ring spectra and the techniques around them have become much more mainstream while these notes have matured. Some traces of this earlier approach can still be seen in that Chap. 1 does not depend at all on ring spectra, leading to the proof that stable K-theory

of rings corresponds to homology of the category of finitely generated projective modules. Topological Hochschild homology is, however, interpreted as a functor of ring spectra, so the statement that stable K-theory is *THH* requires some background on ring spectra.

General Plan The general plan of the book is as follows.

In Sect. 1.1 we give some general background on algebraic K-theory. The length of this introductory section is justified by the fact that this book is primarily concerned with algebraic K-theory; the theories that fill the last chapters are just there in order to shed light on K-theory, we are not really interested in them for any other reason. In Sect. 1.2 we give Waldhausen’s interpretation of algebraic K-theory and study in particular the case of radical extensions of rings. Finally, Sect. 1.3 compares stable K-theory and homology.

Chapter 2 aims at giving a crash course on ring spectra. In order to keep the presentation short we have limited our presentation only the simplest version: Segal’s Γ -spaces. This only gives us connective spectra and the behavior with respect to commutativity issues leaves something to be desired. However, for our purposes Γ -spaces suffice and also fit well with Segal’s version of algebraic K-theory, which we are using heavily later in the book.

Chapter 3 can (and perhaps should) be skipped on a first reading. It only asserts that various reductions are possible. In particular, K-theory of simplicial rings can be calculated degree-wise “locally” (i.e., in terms of the K-theory of the rings appearing in each degree), simplicial rings are “dense” in the category of (connective) ring spectra, and all definitions of algebraic K-theory we encounter give the same result.

In Chap. 4, topological Hochschild homology is at long last introduced, first for ring spectra, and then in a generality suitable for studying the correspondence with algebraic K-theory. The equivalence between the topological Hochschild homology of a ring and the homology of the category of finitely generated projective modules is established in Sect. 4.3, which together with the results in Sect. 1.3 settle the equivalence between stable K-theory and topological Hochschild homology of rings.

In order to push the theory further we need an effective comparison between K-theory and *THH*, and this is provided by the Dennis trace map $K \rightarrow THH$ in the following chapter. We have here chosen a model which “localizes at the weak equivalences”, and so conforms nicely with the algebraic case. For our purposes this works very well, but the reader should be aware that other models are more appropriate for proving structural theorems about the trace. The comparison between stable K-theory and topological Hochschild homology is finalized in Sect. 5.3, using the trace. As a more streamlined alternative, we also offer a new and more direct trace construction in Sect. 5.4.

In Chap. 6 topological cyclic homology is introduced. This is the most involved of the chapters in the book, since there are so many different aspects of the theory that have to be set in order. However, when the machinery is set up properly, and the trace has been lifted to topological cyclic homology, the local correspondence between K-theory and topological cyclic homology is proved in a couple of pages in Chap. 7.

Chapter 7 ends with a quick and inadequate review of the various calculations of algebraic K-theory that have resulted from trace methods. We first review the general framework set up by Bökstedt and Madsen for calculating topological cyclic homology, and follow this through for three important examples: the prime field $\mathbf{F}_p$, the (p -adic) integers $\mathbf{Z}_p$ and the Adams summand ℓ_p . These are all close enough to $\mathbf{F}_p$ so that the local correspondence between K-theory and topological cyclic homology make these calculations into actual calculations of algebraic K-theory. We also discuss very briefly the Lichtenbaum-Quillen conjecture as seen from a homotopy theoretical viewpoint, which is made especially attractive through the comparison with topological cyclic homology. The inner equivariant workings of topological Hochschild homology display a rich and beautiful algebraic structure, with deep intersections with log geometry through the de Rham-Witt complex. This is prominent in Hesselholt and Madsen's calculation of the K-theory of local fields, but facets are found in almost all the calculations discussed in Sect. 7.3. We also briefly touch upon the first problem tackled through trace methods: the algebraic K-theory Novikov conjecture.

The Appendix collects some material that is used freely throughout the notes. Much of the material is available elsewhere in the literature, but for the convenience of the reader we have given the precise formulations we actually need and set them in a common framework. The reason for pushing this material to an appendix, and not working it into the text, is that an integration would have produced a serious eddy in the flow of ideas when only the most diligent readers will need the extra details. In addition, some of the results are used at places that are meant to be fairly independent of each other.

The rather detailed index is meant as an aid through the plethora of symbols and complex terminology, and we have allowed ourselves to make the unorthodox twist of adding hopefully helpful hints in the index itself, where this has not taken too much space, so that in many cases a brief glance at the index makes checking up the item itself unnecessary.

Displayed diagrams commute, unless otherwise noted. The ending of proofs that are just sketched or referred away and of statements whose verification is embedded in the preceding text are marked with a $\odot$.

Acknowledgments This book owes a lot to many people. The first author especially wants to thank Marcel Bökstedt, Bjørn Jahren, Ib Madsen and Friedhelm Waldhausen for their early and decisive influence on his view on mathematics. The third author would like to thank Marcel Bökstedt, Dan Grayson, John Klein, Jean-Louis Loday and Friedhelm Waldhausen whose support has made all the difference.

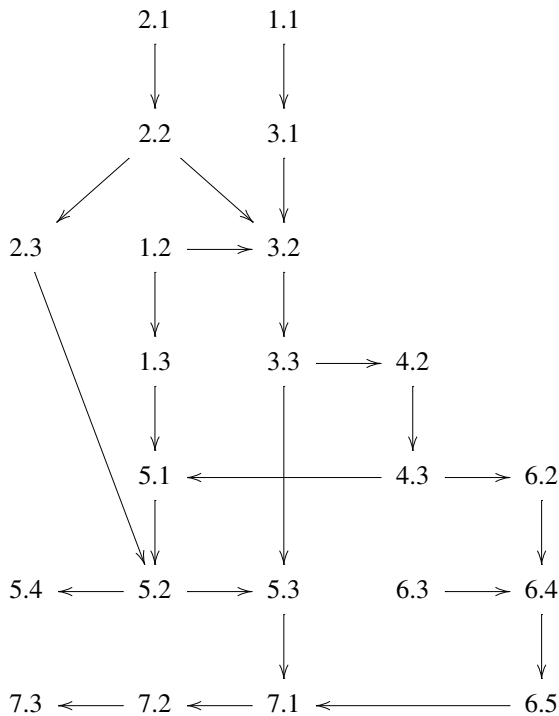
These notes have existed for quite a while on the net, and we are grateful for the helpful comments we have received from a number of people, in particular from Morten Brun, Lars Hesselholt, Harald Kittang, Birgit Richter, John Rognes, Stefan Schwede and Paul Arne Østvær.

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Leitfaden For the convenience of the reader we provide the following Leitfaden. It should not be taken too seriously, some minor dependencies are not shown, and many sections that are noted to depend on previous chapters are actually manageable if one is willing to retrace some cross referencing. In particular, Chap. 3 should be postponed upon a first reading.



Contents

1	Algebraic K-Theory	1
1.1	Introduction	1
1.1.1	Motivating Example from Geometry: Whitehead Torsion	2
1.1.2	K_1 of Other Rings	6
1.1.3	The Grothendieck Group K_0	7
1.1.4	The Mayer–Vietoris Sequence	13
1.1.5	Milnor’s $K_2(A)$	15
1.1.6	Higher K-Theory	16
1.1.7	Some Results and Calculations	20
1.1.8	Where to Read	24
1.2	The Algebraic K-Theory Spectrum	24
1.2.1	Categories with Cofibrations	24
1.2.2	Waldhausen’s S -Construction	29
1.2.3	The Equivalence $obSC \rightarrow BiSC$	33
1.2.4	The Spectrum	35
1.2.5	K-Theory of Split Radical Extensions	38
1.2.6	Categories with Cofibrations and Weak Equivalences	44
1.2.7	Other Important Facts About the K-Theory Spectrum	45
1.3	Stable K-Theory is Homology	47
1.3.1	Split Surjections with Square-Zero Kernels	47
1.3.2	The Homology of a Category	48
1.3.3	Incorporating the S -Construction	50
1.3.4	K-Theory as a Theory of Bimodules	53
1.3.5	Stable K-Theory	56
1.3.6	A Direct Proof of “ F is an Ω -Spectrum”	58
2	Gamma-Spaces and S-Algebras	63
2.1	Algebraic Structure	64
2.1.1	Γ -Objects	64
2.1.2	The Category $\Gamma\mathcal{S}_*$ of Γ -Spaces	67

2.1.3	Variants	71
2.1.4	S -Algebras	73
2.1.5	A -Modules	76
2.1.6	$\Gamma\mathbf{S}_*$ -Categories	78
2.2	Stable Structures	81
2.2.1	The Homotopy Theory of Γ -Spaces	81
2.2.2	A Fibrant Replacement for S -Algebras	87
2.2.3	Homotopical Algebra in the Category of A -Modules	92
2.2.4	Homotopical Algebra in the Category of $\Gamma\mathbf{S}_*$ -Categories	93
2.3	Algebraic K-Theory	94
2.3.1	K-Theory of Symmetric Monoidal Categories	94
2.3.2	Quite Special Γ -Objects	98
2.3.3	A Uniform Choice of Weak Equivalences	99
3	Reductions	103
3.1	Degree-wise K-Theory	103
3.1.1	The Plus Construction	104
3.1.2	K-Theory of Simplicial Rings	111
3.1.3	Degree-wise K-Theory	114
3.1.4	K-Theory of Simplicial Radical Extensions May Be Defined Degree-wise	118
3.2	Agreement of the Various K-Theories	122
3.2.1	The Agreement of Waldhausen's and Segal's Approaches	122
3.2.2	Segal's Machine and the Plus Construction	129
3.2.3	The Algebraic K-Theory Space of S -Algebras	132
3.2.4	Agreement of the K-Theory of S -Algebras Through Segal's Machine and the Definition Through the Plus Construction	135
3.3	Simplicial Rings Are Dense in S -Algebras	137
3.3.1	A Resolution of S -Algebras by Means of Simplicial Rings	138
3.3.2	K-Theory is Determined by Its Values on Simplicial Rings	141
4	Topological Hochschild Homology	143
4.1	Introduction	143
4.1.1	Where to Read	145
4.2	Topological Hochschild Homology of S -Algebras	146
4.2.1	Hochschild Homology of k -Algebras	146
4.2.2	Topological Hochschild Homology of S -Algebras	148
4.2.3	First Properties of Topological Hochschild Homology	155
4.2.4	THH is Determined by Its Values on Simplicial Rings	159
4.2.5	An Aside: A Definition of the Trace from the K-Theory Space to Topological Hochschild Homology for S -Algebras	162

4.3	Topological Hochschild Homology of $\Gamma\mathcal{S}_*$ -Categories	164
4.3.1	Functoriality	165
4.3.2	The Trace	167
4.3.3	Comparisons with the Ab -Cases	167
4.3.4	Topological Hochschild Homology Calculates the Homology of Additive Categories	168
4.3.5	General Results	170
5	The Trace $K \rightarrow THH$	179
5.1	THH and K-Theory: The Linear Case	179
5.1.1	The Dennis Trace with the S -Construction	183
5.1.2	Comparison with the Homology of an Additive Category and the S -Construction	186
5.1.3	More on the Trace Map $K \rightarrow THH$ for Rings	187
5.1.4	The Trace and the K-Theory of Endomorphisms	188
5.2	The General Construction of the Trace	189
5.2.1	Localizing at the Weak Equivalences	189
5.2.2	Comparison with Other Definitions of Algebraic K-Theory	193
5.2.3	The Trace	194
5.2.4	The Weak Trace	197
5.2.5	The Category of Finitely Generated Free A -Modules	199
5.3	Stable K-Theory and Topological Hochschild Homology	201
5.3.1	Stable K-Theory	201
5.3.2	THH of Split Square Zero Extensions	202
5.3.3	Free Cyclic Objects	204
5.3.4	Relations to the Trace $\tilde{K}(A \ltimes P) \rightarrow \tilde{T}(A \ltimes P)$	206
5.3.5	Stable K-Theory and THH for S -Algebras	208
5.4	The Normal Trace	209
5.4.1	Moore Singular Simplices	212
5.4.2	The Homotopy Nerve	222
6	Topological Cyclic Homology	227
6.1	Introduction	227
6.1.1	Connes' Cyclic Homology	227
6.1.2	Bökstedt, Hsiang, Madsen and $TC_{\widehat{p}}$	228
6.1.3	TC of the Integers	229
6.1.4	Other Calculations of TC	230
6.1.5	Where to Read	231
6.2	The Fixed Point Spectra of THH	231
6.2.1	Cyclic Spaces and the Edgewise Subdivision	232
6.2.2	The Edgewise Subdivision	234
6.2.3	The Restriction Map	237
6.2.4	Properties of the Fixed Point Spaces	240
6.2.5	Spherical Group Rings	245
6.3	(Naïve) G -Spectra	247

6.3.1	Circle and Finite Cyclic Actions	248
6.3.2	The Norm Map	249
6.4	Topological Cyclic Homology	253
6.4.1	The Definition and Properties of $TC(-; p)$	253
6.4.2	Some Structural Properties of $TC(-; p)$	255
6.4.3	The Definition and Properties of TC	261
6.5	The Connection to Cyclic Homology of Simplicial Rings	263
6.5.1	On the Spectral Sequences for the $\mathbb{T}$ -Homotopy Fixed Point and Orbit Spectra	264
6.5.2	Cyclic Homology and Its Relatives	266
6.5.3	Structural Properties for Integral TC	277
7	The Comparison of K-Theory and TC	281
7.1	Lifting the Trace and Square Zero Extensions	283
7.1.1	The Cyclotomic Trace	284
7.1.2	Split Square Zero Extensions and the Trace	285
7.2	The Difference Between K-Theory and TC is Locally Constant	290
7.2.1	The Split Algebraic Case	290
7.2.2	The General Case	296
7.3	Some Hard Calculations and Applications	299
7.3.1	General Framework for Calculating $TC(A; p)$	300
7.3.2	The Lichtenbaum–Quillen Conjecture, the Milnor Conjecture and the Redshift Conjecture	311
7.3.3	Topological Cyclic Homology of Local Number Fields	315
7.3.4	The de Rham–Witt Complex	318
7.3.5	Curves and Nil Terms	321
7.3.6	The Algebraic K-Theory Novikov Conjecture	322
7.3.7	Pointed Monoids and Truncated Polynomial Rings	324
7.3.8	Spherical Group Rings and Thom Spectra	327
7.3.9	Topological Cyclic Homology of Schemes and Excision	330
Appendix	Homotopical Foundations	333
A.1	Foundations	333
A.1.1	The Category Δ	333
A.1.2	Simplicial and Cosimplicial Objects	334
A.1.3	Resolutions from Adjoint Functors	335
A.2	Simplicial Sets	336
A.2.1	Simplicial Sets vs. Topological Spaces	337
A.2.2	The Standard Simplices, and Homotopies	338
A.2.3	Function Spaces	339
A.2.4	The Nerve of a Category	340
A.2.5	Filtered Colimits in $\mathcal{S}_*$	342
A.2.6	The Classifying Space of a Group	344
A.2.7	Path Objects	346
A.2.8	Cosimplicial Spaces	346

A.3	Spectra and Simplicial Abelian Groups	347
A.3.1	Simplicial Abelian Groups	347
A.3.2	Spectra	350
A.3.3	Cofibrant Spectra	351
A.4	Homotopical Algebra	352
A.4.1	Examples	353
A.4.2	The Axioms	355
A.4.3	The Homotopy Category	356
A.5	Fibrations in $\mathcal{S}_*$ and Actions on the Fiber	357
A.5.1	Actions on the Fiber	357
A.5.2	Actions for Maps of Grouplike Simplicial Monoids	359
A.6	Bisimplicial Sets	362
A.6.1	Linear Simplicial Spaces	366
A.7	Homotopy Limits and Colimits	368
A.7.1	Connection to Categorical Notions	369
A.7.2	Functoriality	370
A.7.3	(Co)Simplicial Replacements	372
A.7.4	Homotopy (Co)Limits in Other Categories	374
A.7.5	Enriched Homotopy (Co)Limits	378
A.7.6	Completions and Localizations	381
A.8	Cubical Diagrams	385
A.8.1	Cubes and (Co)Simplicial Spaces	386
A.8.2	The Blakers–Massey Theorem	387
A.8.3	Uniformly Cartesian Cubes	390
A.9	G -Spaces	392
A.9.1	The Orbit and Fixed Point Spaces	393
A.9.2	The Homotopy Orbit and Homotopy Fixed Point Spaces	394
A.10	A Quick Review on Enriched Categories	396
A.10.1	Closed Categories	396
A.10.2	Enriched Categories	400
A.10.3	Monoidal V -Categories	403
A.10.4	Modules	404
A.10.5	Ends and Coends	405
A.10.6	Functor Categories	407
	References	409
	Index	423

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