Chapter 2
Note on Superstability of Mikusiński's Functional Equation

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Dedicated to the memory of S.M. Ulam
on the 100th anniversary of his birth

Abstract We show superstability of Mikusiński's functional equation

$$f(x+y)\,(f(x+y)-f(x)-f(y))=0.$$

Keywords Stability • Superstability • Conditional Cauchy equation • Mikusiński's equation

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2.1 Introduction

Certain geometrical considerations have led J. Mikusiński to the functional equation

$$f(x+y)((f(x+y)-f(x)-f(y))=0, \tag{2.1}$$

for the continuous real-valued function f of the real variable. Equation (2.1) is usually written in the conditional form

$$f(x+y)\neq 0 \Longrightarrow f(x+y)=f(x)+f(y), \tag{2.2}$$

which enables us to deal with it more generally – in structures endowed only with one binary operation.

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The general sulution of Mikusiński's equation (2.2) is described in [4]. Stability of Mikusiński's equation in both forms (2.1) and (2.2) is proved in [1].

We show that the method for proving superstability of conditional Cauchy equations, proposed in [6] (see also [2, 3]), is applicable to Mikusiński's equation.

2.2 Superstability

We use [1, Theorem 2] in order to prove superstability of Mikusiński's equation (2.1).

Theorem 2.1. *Let $(G,+)$ be an Abelian group and a function $f : G \to \mathbb{C}$ satisfy*

$$|f(x+y)(f(x+y)-f(x)-f(y))| \leq \varepsilon \quad \textit{for } x,y \in G, \tag{2.3}$$

with some $\varepsilon \geq 0$. Then f is additive, or bounded with $|f(x)| \leq 2\sqrt{6\varepsilon}$ for $x \in G$.

Proof. By [1, Theorem 2] there exists an additive function $a : G \to \mathbb{C}$ with

$$|\ f(x) - a(x)\ | \leq 2\sqrt{6\varepsilon} \qquad \text{for } x \in G. \tag{2.4}$$

If f is bounded, then $a = 0$ and $|\ f(x)\ | \leq 2\sqrt{6\varepsilon}$. Thus, let us consider f unbounded. According to (2.4) a is nontrivial and there is a bounded function b such that $f = a + b$. Taking into account this representation and (2.3) one can easily see that the function

$$G \ni y \longmapsto a(y)(b(x+y) - b(x) - b(y))$$

is bounded for an arbitrary $x \in G$. This implies that

$$b(x) = \lim_{n\to+\infty}(b(x+y_n) - b(y_n)) \qquad \text{for } x \in G,$$

where $(y_n)_{n\in\mathbb{N}}$ is an arbitrary sequence in G with $|\ a(y_n)\ | \to +\infty$ (such a sequence exists since a is nontrivial). Now, using the same argumentation as in [3] one can check that b is additive, and consequently has to be trivial, which yields $f = a$. □

Remark 2.1. It is to be noticed that Moszner [5] proposed another general method for proving superstability of some functional equations which is also applicable to Mikusiński's equation.

References

1. Batko, B.: On the stability of Mikusiński's equation. Publ. Math. Debrecen **66**, 17–24 (2005)
2. Batko, B.: On the stability of an alternative functional equation. Math. Inequal. Appl. **8**, 685–691 (2005)

3. Batko, B.: Stability of Dhombres' equation. Bull. Austral. Math. Soc. **70**, 499–505 (2004)
4. Dubikajtis, L., Ferens, C., Ger, R., Kuczma, M.: On Mikusiński's functional equation. Ann. Polon. Math. **28**, 39–47 (1973)
5. Moszner, Z.: On stability of some functional equations and topology of their target spaces. Result. Math. (submitted)
6. Schwaiger, J.: Remark 13. In: Report of Meeting, The 41st International Symposium on Functional Equations, p. 309. Aequationes Math. **67** (2004)

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