

Chapter 2

Apollonius's *Conics*

If there is some uncertainty about Halley's birth date, we are really in the dark about Apollonius's. The most one can say is that Apollonius of Perga was born around the middle of the third century B.C.E. and died sometime at the beginning of the second. Even this is only a matter of inference, based partly on a comment by Apollonius's late commentator and editor, Eutocius (c. 480–540 C.E.)—himself relying on yet another source, a certain Heraclius—that Apollonius was born "... in the time of Ptolemy Euergetes..." whose reign we know to have been from 246 to 221 B.C.E.¹⁹ What can be said with confidence, however, is that Apollonius was much admired as a mathematician and that his great work was the *Conics*. This, at least, was the view of Apollonius by the end of the 1st century B.C.E., for Eutocius tells us, according to Geminus (1st century, B.C.E.),²⁰ it was "... on account of the remarkable theorems he proved about conics [that Apollonius] was called the Great Geometer."²¹

The contents of the *Conics* are set out by Apollonius himself in the prefatory letter for Book I. Apollonius tells us there that the entire work comprises eight books. Of these eight, the first four, he says, should be considered a "course in the elements" (*agōgē stoixeiōdē*) while the remaining four are "in the manner of additions" (*periousiastikōtera*), "special topics," one might say. Apollonius's overview of the books in his "course in the elements" is as follows: Book I contains the generation of the three conic sections and the opposite sections²² and a "more complete and general" investigation of the characteristic properties, or *symptōmata*, of all these

¹⁹ The comment by Eutocius is from his *Commentary on the Conics* (Heiberg, *Apollonii Pergaei quae graece exstant cum commentariis antiquis*, II, p.168). More information may be found in Fried and Unguru (2001) and Toomer (1970).

²⁰ Little is known for sure about Geminus' life, and the time of his birth ranges from the 1st century B.C.E. to the 1st century C.E. In citing the earlier date, I am following Evans and Berggren (2006) here.

²¹ Eutocius, *Commentary*, p.170.

²² These are what we would call the two branches of the hyperbola. A hyperbola, for Apollonius, is only one of the branches. What this distinction says about Apollonius's general approach to the conics is discussed in Fried (2004).

sections²³; Book II looks at the properties of diameters and axes and asymptotes, as well as “other things necessary and useful for determining limits of possibility” (*diorismous*)²⁴; Book III includes theorems “useful for the synthesis of solid loci and for determining limits of possibility”; finally, Book IV concerns how many ways conic sections may meet one another and the circumference of a circle.

Except for the description of Book IV, which does give a fair picture of what that book is about, these descriptions are at best sketchy and give a far from complete account of Books I–III. Although Apollonius tells us that Book II looks at the properties of diameters and axes, for example, it is in Book I that many of the crucial theorems about diameters are proven—among others, that every conic section has an infinite number of diameters, that all diameters of the parabola are parallel, that all diameters of an ellipse or hyperbola are concurrent, that for every diameter of an ellipse or hyperbola there is a conjugate diameter and that the conjugate diameter of an ellipse is the mean proportional between the original diameter and its parameter, or *latus rectum* (a fact used over and over by Halley).

This is not to say that the descriptions of Books I–III are wholly uninformative as to their contents. Book I does contain much more than what Apollonius tells us, as I have just remarked; however, it truly does contain the generation of the sections and their *symptōmata*, which Apollonius chooses to emphasize in his prefatory letter. These things, moreover, are absolutely essential to Apollonius's treatment and conception of the conic sections, a conception that is a demonstrably geometric one, despite the ease with which key results in the *Conics* can be rewritten algebraically, at least by modern readers who are algebraically literate. To get a feeling for how Apollonius himself treats his subject, consider the statement and diagram for *Conics*, I.13, the proposition in which the ellipse is defined:

If a cone is cut by a plane through its axis and is also cut by another plane which on the one hand meets both sides of the axial triangle and which on the other hand, when extended, is neither parallel to the base nor subcontrariwise, and if the plane containing the base of the cone and the cutting plane meet in a straight line perpendicular either to the base of the axial triangle or to it produced, then any [straight] line which is drawn—parallel to the common section of the [base and cutting] planes—from the section of the cone to the diameter of the section will equal in square some area applied to a straight line [the parameter] (to which the diameter of the section has the same ratio as the square on the straight line drawn—parallel to the section's diameter—from the cone's vertex to the triangle's base has to the rectangle which is contained by the straight lines cut off [on the base] by this straight line in the direction of the sides of the [axial] triangle), an area which has as breadth the straight line on the diameter from the section's vertex to where the diameter is cut off by the straight line from the section to the diameter and which area is deficient (*elleipon*) by a figure similar and similarly situated to the rectangle contained by the diameter and parameter. And let such a section be called an ellipse (*elleipsis*).²⁵

²³ These *symptōmata* are given, in a somewhat simplified form, in appendix 1, as are some of the other basic Apollonian terms.

²⁴ The “limits of possibilities” are, roughly speaking, the conditions under which a problem has a solution at all or more than one solution. See below.

²⁵ I am taking advantage of the English translation by R. Catesby Taliaferro in Apollonius (1998).

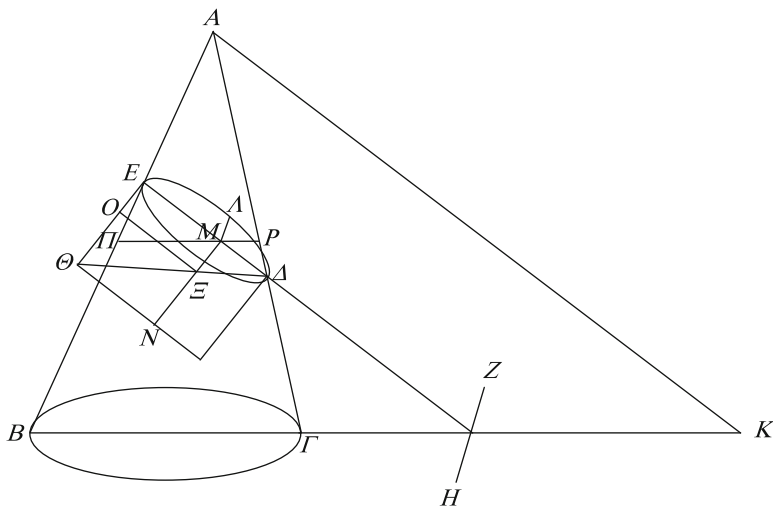


Fig. 2.1 *Conics* I.13

In fig. 2.1, then, a cone having base BF and vertex A is first cut by a plane $AB\Gamma$ through the axis (i.e., the line from A to the center of the circular base $B\Gamma$), producing the “axial triangle” $AB\Gamma$; it is cut, next, by a second plane, ΔEA , which meets two sides of the axial triangle and which, extended, intersects the plane of the base of the cone in a line HZ perpendicular to $B\Gamma$ (the line in which the plane $AB\Gamma$ meets the plane of the base of the cone). The section of the cone produced by the plane ΔEA is the ellipse. To define the *symptōma* of the ellipse we first need to determine the parameter $E\Theta$. It is the line that has the same ratio to ΔE as the square on AK has to the rectangle contained by BK and $K\Gamma$.²⁶ And with that, the *symptōma* is this: Let ΛM be drawn from the ellipse to the line ΔE , which is the diameter of the ellipse,²⁷ so that it is parallel to HZ . Then $\text{sq.}\Lambda M = \text{rect.}E\Theta, EM - \text{rect.}\Theta\Theta, O\Xi$, where the latter rectangle is similar and similarly situated to that contained by the parameter $E\Theta$ and the diameter ΔE .

The first thing one observes about this proposition is that it is long—impossibly long for most modern readers. One obvious reason for its length is that it is entirely stated in words: there are no symbols that formalize and condense statements and claims. After the enunciation, letters are introduced; however, they are not

²⁶ We shall use the shorthand, from now on, “sq. AK ” for “the square on AK ” and “rect. AK, KG ” for the “rectangle contained by the sides AK, KG .”

²⁷ That is, ΔE will bisect all chords of the ellipse drawn parallel to certain given line, in this case, line HZ . Earlier in Book I, in proposition I.7, Apollonius shows that any line such as ΔE , whether or not the plane ΔEA cuts sides AB and AG of the axial triangle, will be a diameter in this sense. Also note that while HZ is perpendicular to $B\Gamma$ extended, it is not necessarily perpendicular to the plane of the axial triangle $AB\Gamma$, and, therefore, lines such as AM are not necessarily perpendicular to the diameter ΔE . If HZ is perpendicular to the plane $AB\Gamma$, then lines AM will be perpendicular to the diameter and the diameter will be an axis.

introduced as symbols, but only as pointers to parts of the diagram. Greek mathematics is tied to diagrams. An inseparable part of proposition I.13, therefore, is its diagram.²⁸ And so the enunciation is long because, more than a statement, it is a *description* that appeals to our ability to visualize geometrical operations, relations, and objects with the aid of diagrams. Anyone who reads Apollonius's own text closely must be left ineluctably with a sense of this geometrical mode of presentation, however one interprets it and whether or not one sees it also as Apollonius's mode of thought.²⁹ This sense of a geometrical presentation must be assumed regarding Halley, for even as a young man he knew the first four books of the *Conics* intimately,³⁰ and later, when he edited and translated the entire extant work, that intimate knowledge was extended to the remaining books, Books V–VII. Later, we shall see that Halley not only appreciated the geometrical form in which Apollonius presented his work, but also largely recognized that this was representative of a genuinely geometrical approach to the subject.

Books V–VIII, as we noted above, were included “in the manner of additions,” according to Apollonius. It is not completely clear what Apollonius meant by this, but, minimally, he must have meant that these books were not conceived as general foundations on which a wide variety of investigations could be developed. It is in this sense that they can be said to differ from books of elements and reasonably referred to as books on “special topics.”³¹ Apollonius describes their contents in the letter introducing Book I as follows: Book V concerns minimum and maximum [lines], by which he means minimum and maximum of lines cut off between the axis of a section and the section itself³²; Book VI treats questions of similarity and equality of conic sections—it is here, for example, that Apollonius proves that all parabolas are similar; Book VII contains theorems on the determination of limits of possibility (*peri dioristikōn theōrēmatōn*); Book VIII, finally, Apollonius tells us, is a book of determinate conic problems (*problēmatōn kōnikōn diōrismenōn*).

²⁸ See Netz (1999).

²⁹ Naturally, I have in mind here particularly H. G. Zeuthen's (1886), *Die Lehre von den Kegelschnitten im Altertum*, where Apollonius's geometrical presentation is thought to hide an algebraic mode of thought. Indeed, Zeuthen's point of view dominated the historiography of Greek mathematics at least until the 1970s; it was wholly adopted, for example, by Thomas L. Heath and Bartel L. van der Waerden. But besides abundant material in the *Conics*, and elsewhere in Greek mathematics, which is awkward to explain by means of Zeuthen's thesis, to say the least, it is dubious that Apollonius should have one mode of presentation and another, conceptually distinct, mode of thought. That said, a divide between presentation and thought is perfectly possible in the case of Halley, when he is occupied in the reconstruction of an ancient text.

³⁰ Halley probably would have read Apollonius's *Conics*, I–IV from a text based on Federigo Commandino's famous edition of 1566. Commandino's edition included not only the first four books of the *Conics* but also Eutocius's commentary and Serenus's *On the Section of a Cylinder* and *On the Section of a Cone*. The latter were included with Halley's reconstruction of *Conics*, Book VIII.

³¹ Book IV also fits this description. In fact, its inclusion in the course of elements is not completely self-evident. The point is discussed at length in Fried and Unguru (2001), chapter III.

³² It has been often said that Book V concerns normals to conic section, but this characterization is moot.

More information about these later books of the *Conics* is found in the letters introducing each book individually. Particularly important for us is what is said in the letter introducing Book VII. For there, besides what is said in the introductory letter to Book I, Apollonius says the following:

In this book [Book VII] are many wonderful and beautiful things on the topic of diameters and the figures constructed on them, set out in detail. All of this is of great use in many types of problems, and there is much need for it in the kind of problems which occur in conic sections which we mentioned, among those which will be discussed and proven in the eighth book of this treatise (which is the last book in it).³³

So, here, unlike in the more general description in Book I, Apollonius explicitly connects the material of Book VII with the problems in Book VIII. As we have already seen, Apollonius's descriptions are not always completely adequate; however, the connection between Books VII and VIII is strengthened by the fact that, in Book VII of Pappus's (4th century C.E.) *Sunagōgē* or *Collection*, as it is usually known in English, lemmas to Books VII and VIII of the *Conics* are set out together, as if the two books were conceived as a pair. These hints about Book VIII in the *Conics* itself and in Pappus's *Collection* are important of course because Book VIII itself has been lost. We do not know for sure when it was lost, but by the 9th century, when great efforts were being made in the Islamic world to recover Greek mathematical and scientific texts, Book VIII was already missing even from the most complete manuscript of the *Conics* found in this period, the Banū Mūsā text.

³³ Toomer (1990), vol. I, p. 382. In general, all translations from the Arabic Books V–VII are from Toomer (1990).

Edmond Halley's Reconstruction of the Lost Book of
Apollonius's Conics

Translation and Commentary

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