

Chapter 2

Vorticity (Molecular Spin)

2.1 Introduction

The curl of the fluid velocity vector $\mathbf{v}$ is known as the vorticity or, physically, the angular velocity at a point in space (see [1, p. 68]). It has also been called *Rotationengeschwindigkeit* by Helmholtz, rotation by Kelvin, molecular rotation by Kelvin, and spin by Clifford.

While vorticity has an exact mathematical definition, its physical significance is still unclear. It is possible, indeed, to find examples of nonrotating flows with nonzero vorticity values. For example, in the shearing motion $u = y$, $v = 0$, and $w = 0$, where u , v , w are the x , y , z components, the particles move in straight lines, but the vorticity or rotation is nonzero.

While circulation is a large-scale measure of rotation, indicative of such features as the Hadley cell in atmospheric, vorticity is a measure of rotation that cannot be seen (microscopic).

Vorticity is the building block of circulation, and the individual locations of vorticity describe pure rotation. However, the sum of vorticity over an area (circulation) is not typically descriptive of pure rotation.

Vorticity, as defined, permits the smooth development of the mathematical study of fluid motion. Stresses within fluids depend on velocity *gradients* rather than on velocities, and vorticity is a combination of velocity gradients. It must be noted that a vorticity field is inherently a solenoidal field: if $\boldsymbol{\zeta}$ represents the vorticity vector, then

$$\text{div } \boldsymbol{\zeta} = 0, \quad (2.1)$$

because the divergence of the curl is identically zero. Because the divergence of vorticity is zero, the flux of vorticity out of any closed surface within the fluid is zero. If the closed surface is a vorticity tube, analogous to a stream tube, then the

vorticity tubes must either close upon themselves to form a ring or terminate on the boundaries. It appears that the vorticity vector has many of the properties that characterize the velocity vector of an incompressible fluid.

For a particle with zero vorticity, a scalar velocity can be defined such that

$$\mathbf{u} = \nabla\phi. \quad (2.2)$$

The continuity equation (fluid flow) then gives

$$\nabla \cdot \mathbf{u} = \nabla \cdot (\nabla\phi) = \nabla^2\phi = 0. \quad (2.3)$$

Thus, ϕ satisfies the Laplace equation.

If the fluid is assumed not to slip on a solid boundary, the relative velocity between the immediately adjacent fluid and the boundary must be zero. If the boundary is stationary, no vorticity tube can terminate on the boundary: so, at the solid stationary wall, there can be no component of vorticity perpendicular to the boundary. Similarly, there can be no component of vorticity parallel to a traction-free boundary.

2.2 Generation of Vorticity

How does a flow that is *initially* irrotational develop vorticity? Truesdell [1] discussed this problem at some length in connection with the Lagrange–Cauchy velocity potential theorem. This theorem deals with the permanence of irrotational motion – that is, a motion once irrotational stays that way. The extension of this theorem to *viscous* incompressible fluids increases the complexity of the problem greatly. Truesdell concluded that “in a motion of a homogenous viscous incompressible fluid subject to a conservative extraneous force and starting from rest, if there be a finite stationary boundary to which the liquid adheres without slipping, there must be some particles whose vorticity is not an analytic function of time at the initial instant.”

The measurement of vorticity is difficult but its presence in fluids is easily detected by the determination of circulation, Γ , which is defined as the line integral of the velocity field around any closed curve.

Kelvin’s theorem of circulation [2] leads to the following equation:

$$\frac{D\Gamma}{Dt} = \oint -\frac{\nabla p}{\rho} \cdot d\mathbf{l} + \oint \frac{1}{\rho} - \nabla \times (\mu\zeta) + \nabla \left[\left(\frac{4}{3} \right) \mu \nabla \cdot \mathbf{U} \right] \cdot d\mathbf{l} + \oint \mathbf{F} \cdot d\mathbf{l}. \quad (2.4)$$

This shows that the rate of change of circulation within a closed curve $\mathbf{l}$, always made up of the same fluid particles, is governed by the torques produced by pressure forces, body forces, and viscous forces. In (2.4), ρ is the fluid density, p is the fluid

pressure, $\boldsymbol{\zeta}$ is the vorticity vector, μ is the dynamic viscosity, $\mathbf{U}$ is the velocity vector, and $\mathbf{F}$ is the body force.

Irrotational body forces (conservative body forces with single-valued force potential), present in many fluid dynamic problems, do not produce circulation. Thus, the effects of the body force term can generally be omitted in the treatment of vorticity generation.

Under barotropic conditions ($p = p(\rho \text{ only})$), the density field is a function of the pressure field alone and the pressure forces do not produce circulation. In an inviscid incompressible fluid of uniform density and in an inviscid compressible, provided the flow field is homentropic, internal sources of vorticity do not exist. But, in a compressible, nonhomentropic fluid, pressure forces provide internal sources of vorticity. When all the torque-producing sources are absent, the dynamics of the fluid is governed by Helmholtz's vortex theorems. It is appropriate to recap these theorems at this point.

First theorem: this is general and valid for real flows. It states that the strength of vorticity in a vorticity tube is the same in all cross sections. In addition, a vorticity tube must be closed or must end at a boundary.

Second theorem: valid only for ideal flows of incompressible fluids. It states that vorticity in such a flow can be neither generated nor destroyed, since during the movement of vorticity, the fluid particles cannot leave the vorticity line on which they are positioned.

If the vorticity sources due to pressure are absent, vorticity flux or circulation cannot be created in the interior of the fluid [3]. In such cases, viscosity is generated by viscous forces at a solid boundary or at a free surface.

2.3 Generation by Shock Waves

There are many sources of vorticity in fluid motions and Hadamard [4] was the first to show that vortices are generated by shock waves and that the flow is no longer irrotational after shock waves. This is shown by the Crocco–Vazsonyi equation [5] for the steady flow of an inviscid fluid:

$$\nabla H = T \nabla S + \mathbf{U} \times \boldsymbol{\zeta}, \quad (2.5)$$

where H , T , and S are the total enthalpy, temperature, and entropy, respectively. The vorticity vector, therefore, is dependent on the rates of change of entropy and enthalpy normal to the streamlines. If all streamlines have the same H but different S , there is production of vorticity. This is the situation downstream of a curved shock wave because the entropy increase across a shock wave is determined by the local angle of the shock. Even if the upstream flow is irrotational, the flow downstream of a curved shock wave is rotational.

2.4 Generation by Free Convective Flow and Buoyancy

This is an important source of vorticity in atmospheric and oceanographic flows.

In fact, a free convection flow is produced by buoyancy forces. Temperature differences are introduced, for example, by boundaries maintained at different temperatures and the resulting density differences induce the motion – cold fluids tend to fall, while hot fluids tend to rise. The temperature changes cause variations in the fluid properties – for example, in the density and viscosity. A general analysis is extremely complex, and so some approximation is essential. The Boussinesq approximation is commonly used. In this approximation, variations of all fluid properties other than the density are ignored. Variations in density are considered only insofar as they result in a gravitational force. The continuity equation in its constant density form is

$$\nabla \cdot \mathbf{u} = 0. \quad (2.6)$$

The Navier–Stokes equation is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}, \quad (2.7)$$

where $\mathbf{F}$ represents the body force term (such forces act on the volume of a fluid particle) and is commonly the effect of gravity, so that $\mathbf{F} = \rho \mathbf{g}$. The gravitational acceleration is derivable from a potential and so $\mathbf{g} = -\nabla \phi$.

As density variations are important here, $\rho = \rho_0 + \Delta\rho$, and so

$$\mathbf{F} = -(\rho_0 + \Delta\rho)\nabla\phi = -\nabla(\rho_0\phi) + \Delta\rho\mathbf{g}. \quad (2.8)$$

Introducing, $P = p + \rho_0\phi$, then the Navier–Stokes equation becomes

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{u} + \Delta\rho\mathbf{g}. \quad (2.9)$$

If it is assumed that all accelerations in the flow are small compared to $|\mathbf{g}|$, then the dependence of ρ on T can be linearized:

$$\Delta\rho = -\alpha\rho_0\Delta T, \quad (2.10)$$

where α is the coefficient of expansion of the fluid. The Boussinesq dynamical equation then is

$$\rho \frac{D\mathbf{u}}{Dt} = \left(\frac{-1}{\rho}\right) \nabla p + \nu \nabla^2 \mathbf{u} - \mathbf{g}\alpha\Delta T, \quad (2.11)$$

where the normal ρ and p have been reverted to.

An equation for the temperature is also required.

To be consistent with the Boussinesq approximation, it is postulated that the fluid has a constant heat capacity per unit volume, ρC_P . $\rho C_P DT/Dt$ is the rate of heating per unit volume of a fluid particle. The heating is caused by transfer of heat from nearby fluid particles by thermal conduction and can also be due to internal heat generation. The corresponding terms in the thermal equation are analogous to the viscous term and the body force term in the dynamical equation, respectively. The conductive heat flux is

$$\mathbf{H} = -k \text{grad } T, \quad (2.12)$$

where k is the thermal conductivity of the fluid. Thus,

$$\rho C_P \frac{DT}{Dt} = -\text{div } \mathbf{H} + J, \quad (2.13)$$

where J is the rate of internal heat generation per unit volume. Taking k to be a constant, (2.13) can be modified to

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T + \frac{J}{\rho C_P}, \quad (2.14)$$

where $\kappa = k/\rho C_P$ is the thermal diffusivity.

Equations (2.6), (2.11), and (2.14) are the basic equations of convection in the Boussinesq approximation.

The additional term in the dynamical equation, $-\mathbf{g}\alpha\Delta T$, is known as the buoyancy force. The two terms on the right-hand side of (2.14) are the conduction term and heat generation term, respectively. $\mathbf{u} \cdot \nabla T$ is known as the advection term (transport of heat by the motion).

Equation (2.14) requires boundary conditions for the temperature field.

The most common type specifies the wall temperature, which is the temperature of the fluid in contact with the wall.

It must be noted that thermal conduction plays an integral role in convection.

A wide range of fluid dynamical behavior is to be expected, depending on the importance of the buoyancy force with respect to the other terms in (2.11). When the buoyancy force is negligible, one has forced convection – when it is the only cause of motion, there is free convection.

Free convective flows are normally rotational. Buoyancy forces directly generate vorticity. Applying the curl operation to (2.11):

$$\frac{D\zeta}{Dt} = \zeta \cdot \nabla \mathbf{u} + \nu \nabla^2 \zeta + \alpha \mathbf{g} \times \nabla(\Delta T). \quad (2.15)$$

This is just the vorticity equation, which will be dealt with in the next section, with the addition of the buoyancy force.

Horizontal components of the temperature gradient $\nabla(\Delta T)$ contribute to the last term; the vorticity so generated is also horizontal but perpendicular to the temperature gradient.

2.5 Generation by Baroclinic Effects

Kelvin's theorem for a potential body force and inviscid flow has the simple form:

$$\frac{D\Gamma}{Dt} = - \oint \frac{dp}{\rho}. \quad (2.16)$$

If the flow is baroclinic (with lines of constant ρ not parallel to lines of constant p), the baroclinicity will be a source of vorticity.

Note: in general, a barotropic situation is one in which surfaces of constant pressure and surfaces of constant density coincide; a baroclinic situation is one in which they intersect.

The sea-breeze problem [6] illustrates the baroclinic generation of circulation. A temperature difference between the air over the land and that over the sea generates density differences between the land air mass and the sea air mass. Thus, the flow isobars and the flow isopycnals are not coincident. For the typical situation shown in Fig. 2.1, Green [7] has shown that for a 15°C land–sea temperature difference ($\rho_2 = 1.22 \text{ kg/m}^3$ and $\rho_1 = 1.18 \text{ kg/m}^3$) and for an elevation change of 1 km ($p_0 - p_1 = 12 \text{ kPa}$), $D\Gamma/Dt = (p_0 - p_1)(1/\rho_2 - 1/\rho_1) = -333 \text{ m}^2/\text{s}$. The mean tangential velocity is evaluated from $\Gamma = v_{\text{mean}} \times (h + L)$ and it is found that

$$\frac{dv_{\text{mean}}}{dt} \approx \frac{-333 \text{ (m}^2/\text{s}^2\text{)}}{62,000 \text{ (m)}} = 20 \text{ m/s/h}.$$

Thus, this simple model predicts that in 1 h baroclinicity will induce a sea breeze of 20 m/s (40 knots) from the sea to land at low elevation and vice versa at high elevation. In atmospheric flows, then, baroclinicity can be an important generator of vorticity.

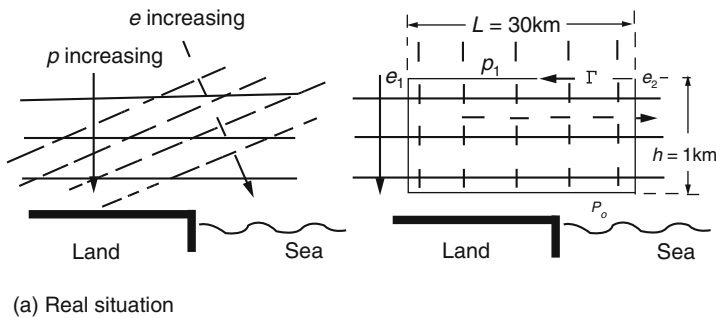


Fig. 2.1 Flow dynamics due to temperature differences between land and sea air masses

2.6 The General Vorticity Equation

The first step is to use the momentum equation written in an inertial frame of reference:

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{-\nabla p}{\rho} + \frac{\nabla \cdot \mathbf{T}}{\rho} + \mathbf{g}, \quad (2.17)$$

where $\mathbf{T}$ is the stress tensor and $\mathbf{g}$ is the body force per unit mass.

Now, $(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla((1/2)\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times \boldsymbol{\omega}$ and so (2.17) becomes

$$\frac{\partial\mathbf{u}}{\partial t} + \nabla\left(\frac{\mathbf{u} \cdot \mathbf{u}}{2}\right) - \mathbf{u} \times \boldsymbol{\omega} = \frac{-\nabla p}{\rho} + \frac{\nabla \cdot \mathbf{T}}{\rho} + \mathbf{g}. \quad (2.18)$$

The curl of this equation, after some simplification, is

$$\begin{aligned} \frac{D\boldsymbol{\omega}}{Dt} = & -\boldsymbol{\omega}(\nabla \cdot \mathbf{u}) + (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \left(\frac{1}{\rho^2}\right)\nabla \times \nabla p - \left(\frac{1}{\rho^2}\right)\nabla \rho \times (\nabla \cdot \mathbf{T}) \\ & + \left(\frac{1}{\rho}\right)\nabla \times (\nabla \cdot \mathbf{T}) + \nabla \times \mathbf{g}. \end{aligned} \quad (2.19)$$

Using the momentum equation to eliminate ∇p , another form of the general form of the vorticity equation is

$$\frac{D\boldsymbol{\omega}}{Dt} = -\boldsymbol{\omega}(\nabla \cdot \mathbf{u}) + (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \left(\frac{1}{\rho}\right)\nabla \rho \times \left(\mathbf{g} - \frac{D\mathbf{u}}{Dt}\right) + \left(\frac{1}{\rho}\right)\nabla \times (\nabla \cdot \mathbf{T}) + \nabla \times \mathbf{g}. \quad (2.20)$$

It must be noted here that the angular velocity vector $\boldsymbol{\omega}$ is being used synonymously with the vorticity vector symbol $\boldsymbol{\zeta}$ used above. This can be confusing at times.

The physical significance of the terms in (2.19) is very important. The left-hand side is the time rate of change of the vorticity following a specific fluid element – the convective transport of vorticity.

The first term on the right-hand side represents the reduction in vorticity due to fluid expansion. Vorticity is enhanced by fluid compression and this will be discussed in some more detail later. The next term represents a stretching of vortex lines that intensifies the vorticity. By Kelvin's theorem, the total circulation of the vortex lines must be constant and so the axial stretching of vorticity lines increases their vorticity. In terms of angular momentum, it is known that a thick solid rod spinning about its axes on frictionless bearings spins faster when stretched in order to preserve angular momentum – see Fig. 2.2. Tornadoes are highly stretched vortices resulting in powerful winds and are a good example of such vorticity intensification.

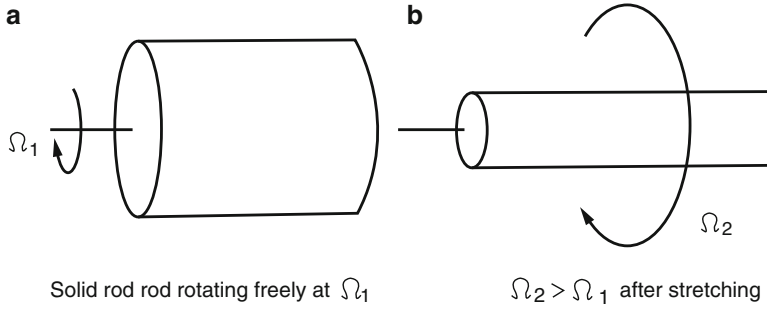


Fig. 2.2

The third term on the right-hand side is the “baroclinic torque” caused by the noncollinearity of the density and pressure gradients. This will be discussed further below. The fourth term is due to shear stress variations in a density gradient field resulting in a torque. In engineering flows it is neglected, being much smaller than the other terms; but it cannot be neglected in meteorological flows.

The fifth term represents the diffusion of vorticity due to viscosity and also will be treated in more detail later.

In engineering, one typically deals with body forces, such as gravity, which are potential, and so the last term on the right-hand side would be zero.

As defined previously, the vorticity, $\boldsymbol{\omega}$, of a flow field with velocity distribution $\mathbf{u}$ is

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}. \quad (2.21)$$

Equations (2.19) and (2.20) and the continuity equation

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0, \quad (2.22)$$

should be sufficient to determine the vorticity field everywhere. But (2.20) does not contain the pressure field. The equation necessary to find p is generated by taking the divergence of (2.18) above:

$$\begin{aligned} & \frac{\partial(\nabla \cdot \mathbf{u})}{\partial t} + 2(\mathbf{u} \cdot \nabla)(\nabla \cdot \mathbf{u}) + \nabla^2 \left(\frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) - \mathbf{u} \cdot (\nabla^2 \mathbf{u}) - \boldsymbol{\omega} \cdot \boldsymbol{\omega} \\ & = \left(\frac{-1}{\rho} \right) \nabla^2 p + \left(\frac{1}{\rho} \right) \nabla \cdot (\nabla \cdot \mathbf{T}) + (\nabla \cdot \mathbf{T} - \nabla p) \cdot \nabla \left(\frac{1}{\rho} \right) + \nabla \cdot \mathbf{g}. \end{aligned} \quad (2.23)$$

In the case of a constant density fluid ($\nabla \cdot \mathbf{u} = 0$ by continuity) and constant viscosity (then $\nabla \cdot \mathbf{T} = \mu \nabla^2 \mathbf{u}$) this very complex equation reduces to

$$\left(\frac{1}{\rho}\right) \nabla^2 p = \mathbf{u} \cdot (\nabla^2 \mathbf{u}) + \boldsymbol{\omega} \cdot \boldsymbol{\omega} - \nabla^2 \left(\frac{\mathbf{u} \cdot \mathbf{u}}{2}\right) + \nabla \cdot \mathbf{g}. \quad (2.24)$$

Once the vorticity and velocity fields are determined using (2.20), the pressure field is given by (2.24).

2.7 Viscous Diffusion of Vorticity

Here, the flow of a Newtonian fluid of fixed density and viscosity with only potential body forces will be considered.

Equation (2.19) reduces to

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \quad (2.25)$$

This has the form of a convection–diffusion equation, similar to the equations of thermal convection and substance diffusion. It explicitly implies the diffusion of vorticity due to the action of viscosity.

The “Oseen” or “Lamb” vortex is a classic example of this diffusion phenomenon. Oseen [8] has analyzed the case of a two-dimensional axisymmetric line vortex in an initially inviscid, infinite fluid. From time $t = 0$, the viscosity acts and the resulting diffusion of vorticity is analyzed. To facilitate interpretation, (2.25) is expanded as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla) \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \quad (2.26)$$

Following Truesdell’s terminology, the terms in this equation are given specific physical identities. The term $\partial \boldsymbol{\omega} / \partial t$ is called the diffusion of vorticity. $\mathbf{u} \cdot \text{grad} \boldsymbol{\omega}$ is called the convection of vorticity. $\boldsymbol{\omega} \cdot \text{grad} \mathbf{u}$ is the convective rate of change of vorticity, and $D\boldsymbol{\omega}/Dt$ is the diffusive rate of change of vorticity. Finally, the term $\nu \nabla^2 \boldsymbol{\omega}$ is called the dissipative rate of change of vorticity.

The Oseen vortex is two dimensional and so the third term is zero ($\boldsymbol{\omega}$ is normal to the plane of $\nabla \mathbf{u}$). The second term is zero for similar reasons and so

$$\frac{\partial \omega_z(r, t)}{\partial t} = \nu \nabla^2 \omega_z(r, t).$$

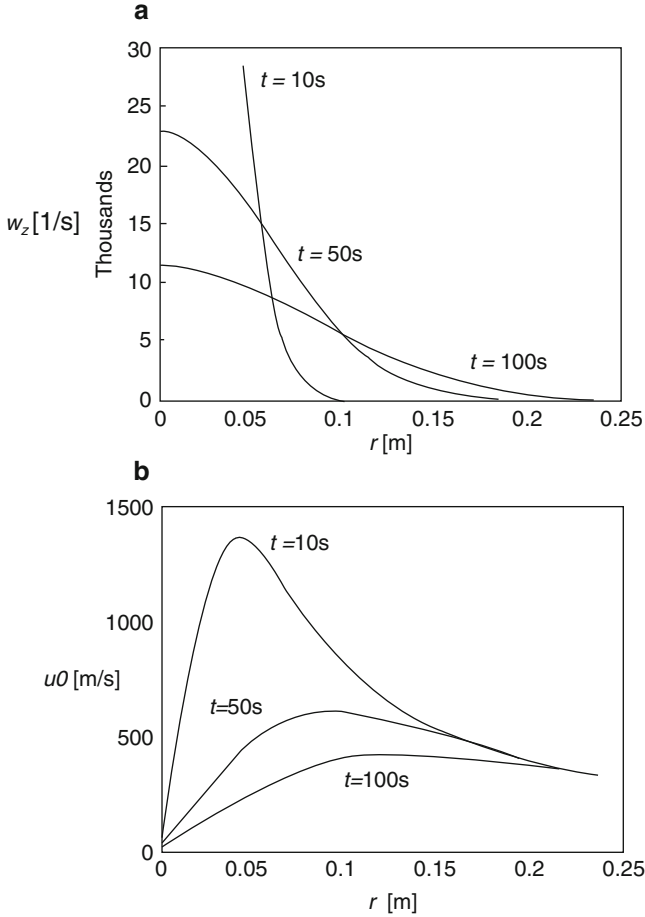


Fig. 2.3 (a) Vorticity distribution. (b) Resulting tangential velocity

This Poisson equation is readily solved and the solution is

$$\omega_z(r, t) = \left(\frac{\Gamma_0}{4\pi\nu t} \right) \exp\left(\frac{-r^2}{4\nu t} \right). \quad (2.27)$$

Figure 2.3a [7] shows the vorticity distribution for a circulation $\Gamma_0 = 500 \text{ m}^2/\text{s}$ and kinematic viscosity $\nu = 3.5 \times 10^{-5} \text{ m}^2/\text{s}^2$. These values are typical for the tip vortex generated by a large aircraft at cruising altitudes.

Figure 2.3b shows the resulting tangential velocity, $u_\theta = (\Gamma_0/2\pi r)[1 - \exp(-r^2/4\nu t)]$. There are three features of Fig. 2.3 which should be specially noted:

1. The circulation distant from the centerline is time independent as per Kelvin's theorem.

2. Distant from the vortex centerline the vorticity and velocity distributions are unchanged. It takes considerable time for viscosity to alter the vorticity over long distances.
3. Near the center of the vortex, where the velocity gradient is largest, the motion becomes a solid-body rotation quickly. Viscosity always acts to bring to convert vortex cores into solid-body rotation.

2.7.1 Viscous Diffusion at a Wall

Diffusion of vorticity is an exchange of momentum. Transport of vorticity can also occur by convection with the property that vorticity is preserved on a particle path. Thus, vorticity can be transferred to neighboring paths only by diffusion – that is, by the effect of viscosity. Immediately at a wall, to which the fluid particles adhere, vorticity can be transferred to the fluid only by diffusion. Boundary layers at surfaces have large velocity gradients and result in large viscous forces. Vorticity generation and diffusion at walls will be next considered.

Assuming constant density and constant Newtonian viscosity, and with some manipulation, (2.17) may be written as

$$\frac{D\mathbf{u}}{Dt} = \frac{-\nabla p}{\rho} + \mathbf{g} - \nu \nabla \times \boldsymbol{\omega}. \quad (2.28)$$

For two-dimensional flow ($u = u(x,y)$, $v = v(x,y)$, $w = 0$) near a wall ($\boldsymbol{\omega} = \omega_z$ only and $u(\text{wall}, y = 0) = 0$), and neglecting body forces, then the x -component of the momentum equation at the wall is

$$\frac{v\partial u}{\partial y} + \left(\frac{1}{\rho}\right) \frac{\partial p}{\partial z} = \frac{-v\partial \omega_z}{\partial y}. \quad (2.29)$$

There are two main implications of (2.29):

1. Lighthill [9] has identified the term $v(\partial \omega_z / \partial y)$ as the flux of vorticity away from a surface, with positive fluxes representing the flux of positive vorticity and negative fluxes representing the flux of vorticity of opposite sign [10]. Thus, only solid surfaces with pressure gradients, and porous walls with fluid blowing out, are sources of vorticity. In the Blasius boundary layer (formed by fluid flowing over a thin flat plate), in fact, all the vorticity in the boundary layer is generated in the small leading edge region where $\partial p / \partial x$ is negative (see Fig. 2.4). Vorticity in other regions of the Blasius boundary layer is due to convection from the leading edge.
2. Without a pressure gradient at the wall ($\partial p / \partial x$), or flow through it ($v = 0$), then $(\partial \omega_z / \partial y)_{y=0} = 0$, which implies that vorticity generated at the wall has diffused far into the flow (Fig. 2.4a). A porous wall with suction has $v(y = 0) < 0$

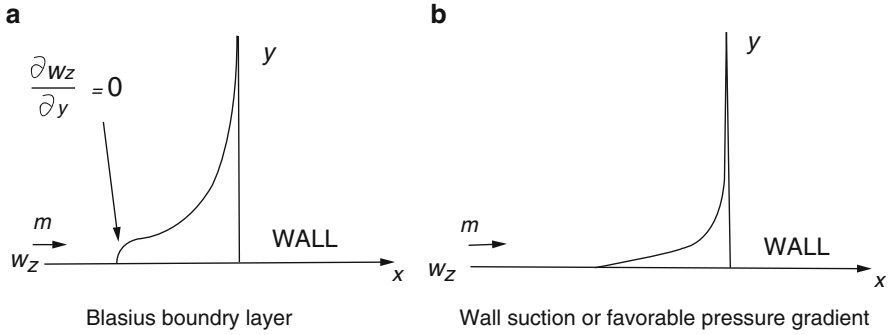


Fig. 2.4

(Fig. 2.4b). Equation (2.29) then implies $\partial \omega_z / \partial y_{y=0} > 0$. Thus, diffusion of wall vorticity into the flow has been inhibited by the wall suction. Contrariwise, blowing at a wall, or an adverse pressure gradient along a wall, will result in vorticity transport away from the wall. Batchelor [3] has solved the problem of suction at a wall quantitatively. Steady incompressible two-dimensional flow over a flat plate with suction velocity V , with no pressure gradients or body forces, must satisfy the following vorticity equation:

$$\frac{-V d\omega_z}{dy} = \frac{v d^2 \omega_z}{dy^2}. \quad (2.30)$$

The solution that satisfies the no slip ($u(y=0) = 0$) and freestream ($u(y \rightarrow \infty) = U_\infty$) conditions is

$$\omega_z = \left(\frac{-U_\infty V}{v} \right) \varepsilon^{-Vy/v} \quad \text{and} \quad u = U_\infty (1 - \varepsilon^{-Vy/v}). \quad (2.31)$$

This indicates that as the suction velocity increases, the thickness of the region (of the order v/V) with vorticity decreases. In the example described above, the convection of vorticity through the wall exactly compensates for diffusion into the free stream, resulting in a streamwise invariant flow. If $V = 0$, then a streamwise invariant boundary-layer flow would only be possible with a favorable pressure gradient.

A favorable pressure gradient will also inhibit vorticity diffusion into the free-stream. Boundary-layer flow near a stagnation point has a highly favorable pressure gradient and this problem has been solved numerically [11]. It was determined

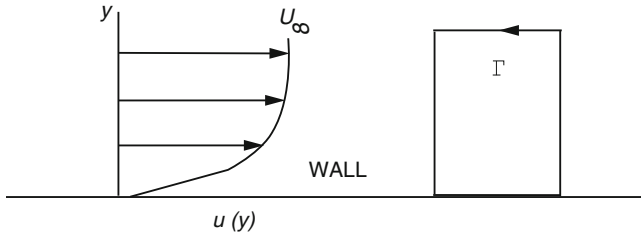


Fig. 2.5 Wall-generated vorticity

that the boundary-layer thickness, δ , in which ω is significant, is proportional to $(\nu/U_\infty)^{1/2}$, where U_∞ is the far field fluid velocity toward the stagnation point. The “vorticity layer” thickness does not vary with distance away from the stagnation point along the plate. Thus, the large favorable pressure gradient in effect halts the diffusion of vorticity from the wall, in spite of producing a surface vorticity flux.

2.7.2 Subsequent Motion of Wall-Generated Vorticity

Kelvin’s theorem implies that the flow region away from viscous effects remains irrotational for all time and may be computed by potential flow methods. Thus, determination of the complete flowfield around an object will require knowing the amount and motion of the wall-generated vorticity. As an approximate model of a boundary layer, consider a one-dimensional flow over a surface – see Fig. 2.5. Assuming that convection is the only mechanism for vorticity (valid at reasonably high Reynolds numbers because convection is a much faster process than diffusion), then the amount of vorticity convected past a fixed vertical line in time dt is

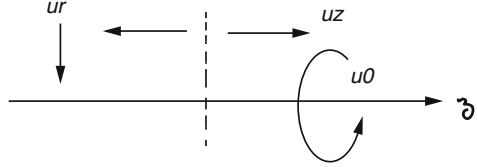
$$\Gamma = \int_0^\infty dt \cdot u(y) \omega_z(y) dy. \quad (2.32)$$

For one-dimensional flow, $\omega_z(y) = -du/dy$ and, therefore,

$$\Gamma = -dt \int_0^\infty u \left(\frac{du}{dy} \right) dy = -dt \int_{u(0)}^{u(\infty)} u du = -dt \left(\frac{u^2}{2} \right) \Big|_0^{U_\infty} = -U_\infty \left(\frac{U_\infty}{2} \right) dt. \quad (2.33)$$

This result is important as it indicates how much circulation must be injected into the flow at boundary-layer separation points. This is required for vortex method computations.

Fig. 2.6



2.7.3 Vorticity Increase by Vortex Stretching

This has already been referred to as another consequence of Kelvin's theorem. Burgers' vortex is an example of this phenomenon and will be analyzed next. For an inviscid barotropic flow with only potential body forces, (2.19) reduces to

$$\frac{D\omega}{Dt} = \omega \cdot \nabla \mathbf{u} - \omega(\nabla \cdot \mathbf{u}). \quad (2.34)$$

Using the continuity equation, $\nabla \cdot \mathbf{u} = (-1/\rho)(D\rho/Dt)$ and the product rule

$$D\left(\frac{\omega}{\rho}\right) = \left(\frac{1}{\rho}\right)\left(\frac{D\omega}{Dt}\right) - \left(\frac{\omega}{\rho^2}\right)\left[\frac{D\rho}{Dt}\right]; \quad \frac{D(\omega/\rho)}{Dt} = \left(\frac{\omega}{\rho}\right) \cdot \nabla \mathbf{u}. \quad (2.35)$$

Now, the length of an infinitesimal segment, $\mathbf{l}$, of a fluid line [7, p. 12] is given by

$$\frac{D\mathbf{l}}{Dt} = \mathbf{l} \cdot \nabla \mathbf{u}. \quad (2.36)$$

From (2.35) and (2.36) it is seen that for an arbitrary flow field

$$\mathbf{l} = \frac{C\omega}{\rho}, \quad (2.37)$$

where C is a constant. See [3] for a more rigorous derivation.

Equation (2.37) shows that by stretching a segment of fluid with vorticity, so that $\mathbf{l}$ increases, the vorticity magnitude of the segment will also increase. It is also obvious that fluid compression (increase in ρ) will also lead to vorticity augmentation.

Burgers [12] has solved a problem (the Burger vortex) that clearly shows the increase in vorticity by stretching. The vorticity equation in an incompressible, Newtonian zero body force fluid is [see (2.26)]

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla) \mathbf{u} + \nu \nabla^2 \omega. \quad (2.38)$$

For an axisymmetric vortex, aligned along the z -axis and placed in a uniaxial straining field along its length (see Fig. 2.6) $u_z = 2Cz$ (where C is a constant), then continuity demands the presence of a radial influx of fluid $u_r = -Cr$.

Equation (2.38) has a Lamb–Oseen vortex type of solution:

$$\begin{aligned} u_z &= 2C_z \\ u_r &= -Cr \\ u_\theta &= \left(\frac{\Gamma}{2\pi r}\right) \left[1 - \exp\left(\frac{-r^2}{4\delta^2}\right)\right], \end{aligned} \quad (2.39)$$

where

$$\delta^2 = \left(\frac{v}{C}\right) + \left(\delta_0^2 - \frac{v}{C}\right) \exp(-Ct). \quad (2.40)$$

δ may be defined as the vortex radius and δ_0 is then the initial vortex radius. This velocity field has a vorticity distribution given by

$$\omega_z = \left(\frac{\Gamma}{\pi\delta^2}\right) \exp\left(\frac{-r^2}{4\delta^2}\right). \quad (2.41)$$

The vorticity on the axis is given by $\omega_z(r=0) = \Gamma/\pi\delta^2$.

From (2.40) $\delta^2 \rightarrow v/C$ as t increases (for C positive). If the vortex is being compressed, C negative, δ increases continuously. If δ_0 is greater than $(v/C)^{1/2}$, the asymptotic value of δ , δ will decrease with time. For this situation $\omega_z(r=0)$ will rapidly increase with time. This is a good example of the increase in vorticity that results from vortex stretching.

2.8 Hill's Spherical Vortex [13, 14]

This is another good example in which stretching of vortex lines occurs. The spherical vortex is a model of the internal flow in a gas bubble moving in a liquid. The motion outside the bubble sets up an internal circulation. Figure 2.7 shows the spherical vortex, with a cylindrical coordinate system moving with the bubble.

The vortex lines are circular loops around the z -axis. As the flow moves the vortex lines to larger radial positions, the loops increase in length proportional to the radius. As a result of the vortex-line stretching effect, the vorticity is proportional to the radius (see [14] for the value of C as $5U/R^2$):

$$\omega_\theta = Cr = \left(\frac{5U}{R}\right) \frac{r}{R}. \quad (2.42)$$

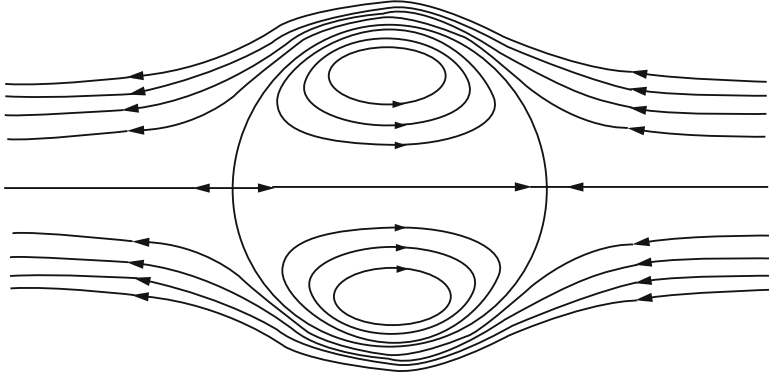


Fig. 2.7 Hill's spherical vortex

Consider the vorticity equation (2.38) as it applies to Hill's vortex and recall that $D\boldsymbol{\omega}/Dt$ is the rate of change of particle vorticity, $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ is the rate of deforming of vortex lines, and $\nu \nabla^2 \boldsymbol{\omega}$ is the net rate of viscous diffusion of $\boldsymbol{\omega}$.

Only the θ component has nonzero vorticity. The physical meaning of the terms in (2.38) is then

Convection: $D\omega_\theta/Dt = u_r \partial \omega_\theta / \partial r = Cu_r$

Stretching: $[\boldsymbol{\omega} \cdot \nabla \mathbf{u}] = \omega_\theta (u_r/r) = Cu_r$

Diffusion: $[\nu \nabla^2 \boldsymbol{\omega}]_\theta = \nu \partial / \partial r [(1/r) \partial / \partial r (r \omega_\theta)] = 0$

The vorticity balance then is between convection and stretching without any net viscous diffusion. There is no net diffusion of vorticity and the increase in vorticity is wholly due to vortex-line stretching. The vorticity is proportional to the circumference of the loop and does not depend on the movement of the circular vortex lines.

2.9 Vorticity in Rotating Frames of Reference

Kelvin's theorem for a potential force and inviscid flow is given in (2.16). This equation and other results developed so far are valid only in an inertial frame of reference. In studying atmospheric and oceanic flows, one deals with the noninertial, rotating frame of the Earth. It is appropriate to reformulate (2.16) in such a rotating frame.

Two additional forces occur in a rotating frame of reference (with rotational velocity $\boldsymbol{\Omega}$) – the centrifugal force and the Coriolis force.

The centrifugal force, $\mathbf{F}_{\text{cen}} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{x})$ per unit mass, where $\mathbf{x}$ is the vector displacement from the axis of rotation, generates no circulation because $\mathbf{F}_{\text{cen}} =$

$\nabla(1/2\Omega \times \mathbf{x})^2$ is curl free. Coriolis forces, $\mathbf{F}_{\text{cor}} = -2\Omega \times \mathbf{u}$ per unit mass, are not curl free and do generate circulation. This circulation can be calculated as follows.

An area of fluid A normal to the axis of rotation, in the rotating frame, has a circulation (about an axis parallel to the rotation) in the stationary frame of reference given by the rigid body rotation: $\Gamma = 2|\Omega|A$. If the area A has a normal at an inclination $\pi/2 - \phi$ to the axis of rotation (at the North Pole on Earth $\phi = \pi/2$ and at the South Pole $\phi = -\pi/2$), the area that determines the rotating frame circulation is the projected area of A on the equatorial plane, $A_e = A \sin\phi$.

Thus, the circulation in a rotating frame of reference is given by

$$\Gamma_{\text{rot}} = 2|\Omega|A\sin\phi. \quad (2.43)$$

This circulation is added to the circulation calculated relative to the rotating frame of reference, Γ_{rel} , to yield the circulation in the absolute frame of reference:

$$\Gamma_{\text{ab}} = \Gamma_{\text{rot}} + \Gamma_{\text{rel}} = 2|\Omega|A\sin\phi + \Gamma_{\text{rel}}. \quad (2.44)$$

Substituting (2.44) into (2.16), one obtains Bjerknes [15] theorem:

$$\frac{D\Gamma_{\text{rel}}}{Dt} = - \int \frac{dp}{\rho} - 2|\Omega| \frac{d(A\sin\phi)}{dt}. \quad (2.45)$$

Besides the baroclinic torque term discussed previously, there is a new source of vorticity convection of fluid from high latitudes to low latitudes. Consider a part of the atmosphere on the Earth at a latitude ϕ_1 , about which there is no relative circulation. Suppose the air is now brought barotropically to latitude ϕ_2 with no change in area, then the mean vorticity of the air mass will alter as

$$\frac{AD\omega}{Dt} = \frac{-2|\Omega|Ad \sin\phi}{dt} \quad \text{or} \quad \omega_2 - \omega_1 = -2|\Omega|(\sin\phi_2 - \sin\phi_1). \quad (2.46)$$

Counterclockwise fluid rotation ($+\omega$) is therefore enhanced in the Northern Hemisphere as fluid moves south and the reverse occurs in the Southern Hemisphere. Winds generated by this motion of the air equatorward across lines of latitude are called “cyclonic” winds, and are associated with low-pressure regions in the atmosphere. Coriolis force-generated circulation also plays an important part in oceanographic currents and in turbomachinery.

2.10 Atmospheric Fluid Motion and Vorticity

Analogous to absolute circulation, there is the absolute vorticity – the sum of the vorticity due to the rotation of the fluid itself (ζ) and that due to the Earth’s rotation (f). f is known as the Coriolis parameter and varies only with latitude. Under the conditions of nondivergent, frictionless flow, absolute vorticity is conserved and

$$\frac{d(\zeta + f)}{dt} = 0. \quad (2.47)$$

For two-dimensional flow, in an atmosphere of uniform density, and conservation of absolute vorticity, the vorticity equation becomes

$$\begin{aligned} \frac{d}{dt}(\zeta + f) = & -(\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \left(\left(\frac{\partial w}{\partial x} \right) \left(\frac{\partial v}{\partial z} \right) - \left(\frac{\partial w}{\partial y} \right) \left(\frac{\partial u}{\partial z} \right) \right) \\ & + \left(\frac{1}{\rho^2} \right) \left(\left(\frac{\partial r}{\partial x} \right) \left(\frac{\partial p}{\partial y} \right) - \left(\frac{\partial r}{\partial y} \right) \left(\frac{\partial p}{\partial x} \right) \right). \end{aligned} \quad (2.48)$$

The first term on the right-hand side arises because of the horizontal divergence. If there is a positive horizontal divergence, air flows out of the region in question and the vorticity will decrease. This is equivalent to the situation with a rotating body whose angular velocity decreases when its moment of inertia increases (angular momentum conservation). For synoptic scale (systems of 1,000 km or more in horizontal dimension) motions, the last two terms are much smaller than the others and to a first approximation

$$\frac{d_h}{dt}(\zeta + f) = -(\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \quad (2.49)$$

where d_h/dt denotes $\partial/\partial t + u\partial/\partial x + v\partial/\partial y$.

Applying (2.49) to a constant density and temperature atmosphere using the continuity equation for incompressible fluids, it becomes

$$\frac{d_h}{dt}(\zeta + f) = (\zeta + f) \frac{\partial w}{\partial z}. \quad (2.50)$$

Because of the constant temperature, the geostrophic (small friction, small curvature, and steady flow) wind is independent of the height z . The vorticity will not vary with height either, because to a first approximation the vorticity is equal to the vorticity of the geostrophic wind. Then, integrating (2.50) between levels z_1 and z_2 where $z_2 - z_1 = h$,

$$\frac{(1/(\zeta + f))d_h}{dt(\zeta + f)} = \frac{w(z_2) - w(z_1)}{h}. \quad (2.51)$$

Considering the fluid which at one instant is confined between the levels distance h apart, then $dh/dt = w(z_2) - w(z_1)$ and (2.50) may be written as

$$\frac{d_h}{dt(\zeta + f/h)} = 0. \quad (2.52)$$

This equation is a simplified statement of the conservation of *potential vorticity*. It has important consequences for atmospheric flow. For example, consider

adiabatic flow over a mountain barrier. As a column of air flows over the mountain, its vertical extent is decreased and so ζ must also decrease. A westward-moving wind will therefore move in the direction of the equator as it flows over the mountain.

2.11 Dissipation Function, Vorticity Function, and Curvature Function (Eddy or Vortex Motion)

Following Lamb [16], the rate of dissipation is defined as the energy expended in deforming a small fluid element and is mathematically defined by the dissipation function, Φ . For an incompressible fluid, in rectangular Cartesian coordinates,

$$\Phi = \mu \left[2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial y} \right)^2 + 2 \left(\frac{\partial w}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right]. \quad (2.53)$$

A vorticity function, Ω , may be defined by taking the scalar product of the vorticity vector with itself. This function is used as a measure of vorticity:

$$\Omega = \omega \cdot \omega = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial w} - \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)^2, \quad (2.54)$$

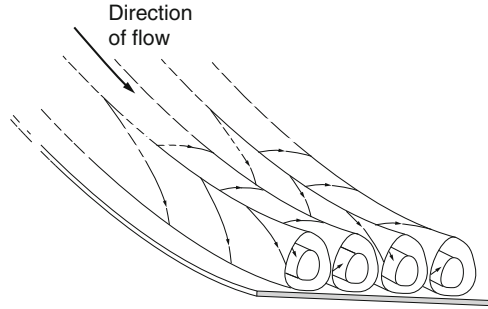
where ω is the vorticity vector. It can be shown [17] that Φ and Ω are mathematically independent functions; that is, dissipation is independent of vorticity.

A function called the K -function has been proposed as a measure of vortex motion [17]. It is not clear whether it is, indeed, a local measure of the physical phenomenon known as an eddy or vortex motion. It has been shown that it is a measure of curvature and that it does have significance in fluid dynamics. Requirements laid down for the function were that

1. It is zero for any straight translational motion, but is nonzero for any motion with rotation.
2. It is zero for an irrotational vortex, for rigid rotation, and for the Hagen–Poiseuille flow in a straight conduit.

This arbitrarily defined function is expressed in rectangular Cartesian coordinates as

$$K = \left[\left(\frac{\partial w}{\partial z} \right) \left(\frac{\partial v}{\partial y} \right) - \left(\frac{\partial w}{\partial y} \right) \left(\frac{\partial v}{\partial z} \right) \right] + \left[\left(\frac{\partial u}{\partial x} \right) \left(\frac{\partial w}{\partial z} \right) - \left(\frac{\partial u}{\partial z} \right) \left(\frac{\partial w}{\partial x} \right) \right] + \left[\left(\frac{\partial v}{\partial y} \right) \left(\frac{\partial u}{\partial x} \right) - \left(\frac{\partial v}{\partial x} \right) \left(\frac{\partial u}{\partial y} \right) \right]. \quad (2.55)$$

Fig. 2.8 Goertler vortices

It has been shown that the rate of dissipation is a linear combination of vorticity and the K -function:

$$\Phi = \mu[\Omega - 4K]. \quad (2.56)$$

where μ is the fluid viscosity.

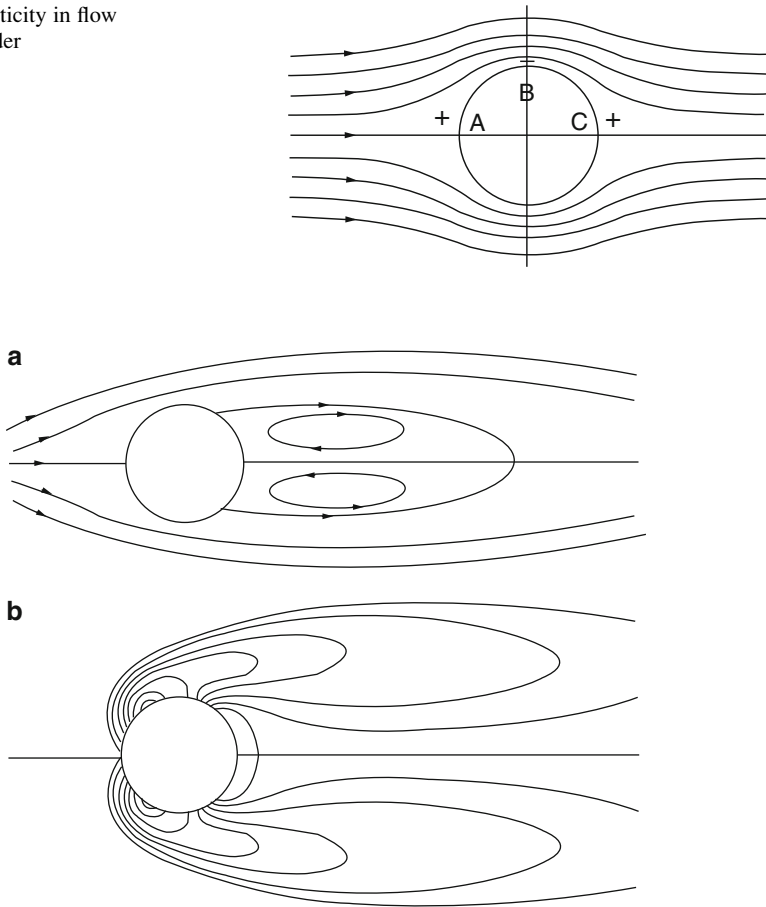
This equation indicates that a flow for which K is zero (the curvature is zero) has dissipation proportional to its vorticity. It also shows that a real fluid in motion may be dissipating energy in spite of the absence of vorticity. More importantly, it implies that if a flow has vorticity and has no dissipation, it must also have curvature.

Goertler [18–20] attached considerable significance to the curved flow condition shown in Fig. 2.8. He concluded that the concavity of the wall stabilizes the flow and convexity of the wall destabilizes the flow, and that the critical condition is that the streamlines be concave in the direction of increasing velocity. When such a condition exists, he predicted that longitudinal vortices would form [21]. By using the x - y coordinate system shown in Fig. 2.8 and assuming $\partial v / \partial y$ is zero, it is seen that the K -function is another way of expressing this condition.

Goertler in effect requires that the K -function be negative. If $\partial u / \partial y$ is zero, the K -function is zero; if the streamlines are not curved, the K -function is zero. If the streamlines are convex in the direction of increasing velocity gradient, the K -function is positive. The K -function appears to provide a mathematical measure of the Goertler criterion.

2.12 Generation of Vorticity in a Viscous Boundary Layer: Precursor to Turbulence

A boundary flow will now be considered with the object of elucidating the way in which the vorticity associated with a velocity gradient can be changed into distinct vortices (eddies), as an initial step in the transition to turbulence. The steady two-dimensional flow of an incompressible fluid over a cylinder, neglecting gravity, will serve as the specific flow. A qualitative approach will be adopted. It will be initially

Fig.2.9 Vorticity in flow over a cylinder**Fig. 2.10** Eddies shed from the cylinder

assumed that the Reynolds number, defined by $Re = \rho RU/\mu$, where R is the cylinder radius and U is the free stream velocity, is small. The no-slip condition results in a boundary layer at the cylinder surface (Fig. 2.9). Flow outside this boundary layer is effectively inviscid since velocity gradients vanish. If U is small, so that velocity gradients within the boundary layer are small, the flow here is also approximately inviscid.

Streamlines around the cylinder are compressed. Generally, they converge between A and B and diverge between B and C . As the fluid is approximately inviscid, the fluid pressure along a streamline decreases between A and B and increases between B and C . The negative pressure gradient between A and B is transformed to kinetic energy and the flow accelerates; with a positive pressure gradient between B and C , the flow decelerates. A constant total pressure (static plus dynamic) along the streamline is maintained. As Re increases, streamlines are increasingly compressed and velocity gradients in the boundary layer become larger. Viscous resistance to shear within the layer becomes significant. Energy is

thereby extracted from the as heat. As a consequence, the fluid in the boundary layer arriving at B does not have sufficient kinetic energy to overcome the adverse pressure gradient between B and C , so that fluid close to the cylinder stalls at some point between B and C . The adverse pressure gradient at this point can induce reverse flow close to the cylinder and provide the onset of concentrated vorticity in the lee of the cylinder (Fig. 2.10). The inviscid part of the flow is displaced outward. Thus, the flow is said to “separate” around the site of concentrated vorticity. The shear associated with this separation increases the rotational motion of the fluid next to the cylinder, and this region may become a distinct entity of rotating fluid – an eddy. The eddy, through shear with the surrounding fluid, will further extract energy from the main flow.

The eddies are shed from the cylinder and move downstream with the main flow. Shedding occurs with a regular frequency at low-to-moderate Re values. Eddies enlarge and then are shed alternately from either side of the cylinder. This pattern of regularly spaced eddies moving downstream is referred to as a Von Karman vortex street. This pattern of flow is a precursor to turbulence insofar that with further increase in Re , shedding becomes irregular, the eddy wake becomes disorganized, and a complex velocity field emerges.

2.13 Typical Vorticity Distributions

The first example will be the external flow over an airfoil (see [14]). It will be assumed that the Reynolds number is large and the flow is two dimensional (so that the vorticity vector is always perpendicular to the velocity) – see Fig. 2.11.

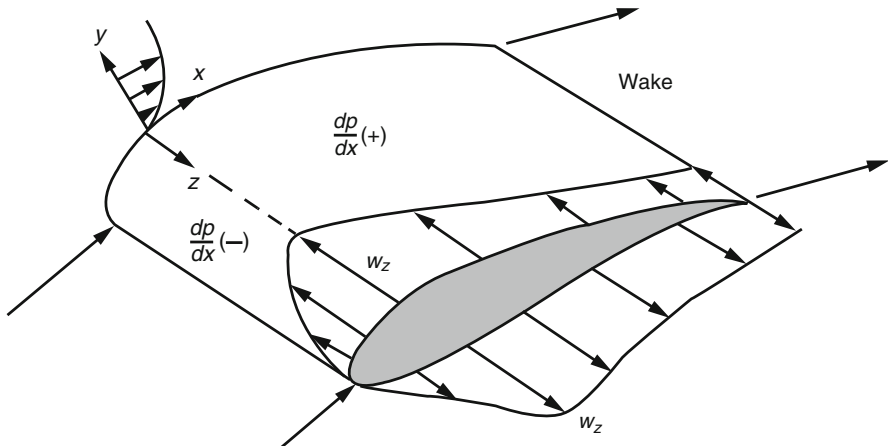


Fig. 2.11 Vorticity distribution in flow over an airfoil

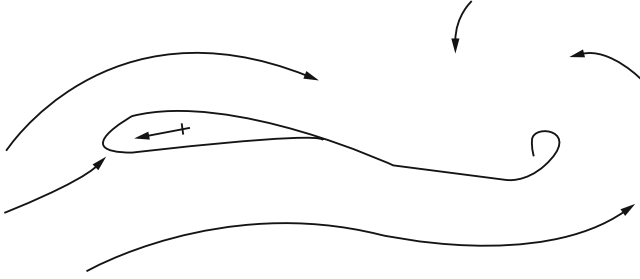


Fig. 2.12 Starting vortex

A local coordinate system with $y = 0$ on the surface of the airfoil and x in the flow direction is chosen. Vorticity diffusion will be primarily normal to the wall. At the stagnation point so that the positive x -axis is on the upper surface. Curvature will be ignored in this qualitative treatment. The stagnation point is a point of zero shear and hence zero vorticity ($F_{x \text{ viscous}} = \mu \partial u / \partial y = -\mu \omega_z$, see [14], where F is the shear stress). As the flow accelerates away from the stagnation point on the upper surface, the shear stress becomes positive and the vorticity becomes negative. In this region, the pressure drops and there is a flux of negative vorticity away from the wall: $\mu \sigma_z = -\mu \partial \omega_z / \partial y = \partial p / \partial x < 0$ where σ_z is the vorticity flux in the z direction. The surface is a source of negative vorticity. Near the front of the airfoil, the pressure reaches a minimum and then slowly increases as the trailing edge is approached. In this region, $\partial p / \partial x$ is positive and the wall in effect absorbs negative vorticity from the flow. The wall flux is positive—negative vorticity diffuses toward the wall. The maximum vorticity now occurs within the flow, as the sign of $\partial \omega_z / \partial y$ is negative at the wall. The process continues until the trailing edge is reached.

On the underside of the airfoil, similar processes occur, but the x coordinate now decreases in the flow direction and the signs of the events change. The pressure gradient that accelerates the flow generates positive vorticity, while the decelerating pressure gradient absorbs positive vorticity. At the trailing edge, the upper and lower streams merge. There is a discontinuity at this point that is washed out as the flow proceeds downstream. The negative vorticity from the upper surface and the positive from the lower merge into the wake. These regions in merging destroy the wake. Assuming vorticity has not diffused very far from the surface at the trailing edge, one can show that the net vorticity across the wake is zero:

$$\text{mean } \omega_z \text{ at trailing edge} = \int_{\delta_l}^{\delta_u} \omega_z dy = \int_{\delta_l}^{\delta_u} \frac{\partial u}{\partial y} = (u)_{\delta_l}^{\delta_u} = 0.$$

It can also be shown that [14] the net flux of vorticity from the surface of the airfoil is zero. However, there is a net vorticity within the flow. Integrating the vorticity in the region outside the airfoil to a radius R and then letting $R \rightarrow \infty$, it is found that, $\int \omega_z dA = \Gamma$ —a finite number equal to the circulation. The net nonzero

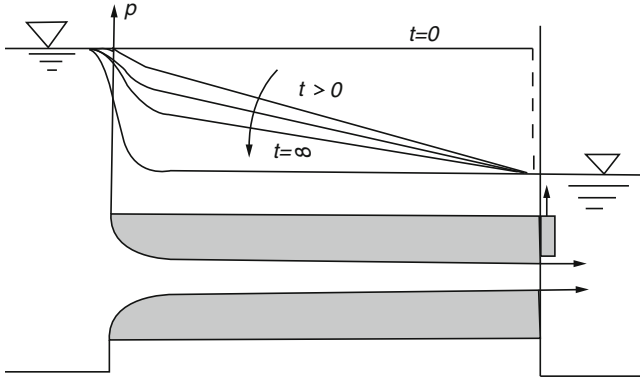


Fig. 2.13 Vorticity generation in channel flow

vorticity is inserted into the flow during the transient process by which the flow is established. In the transient process, the flow does not leave the trailing edge smoothly and the “starting vortex” is formed. Figure 2.12 illustrates a starting vortex formed by impulsively moving the airfoil.

The starting vortex contains the same net amount of vorticity as the airfoil but with opposite sign. A circulation loop around the airfoil and including the starting vortex has $\Gamma = 0$.

The second example will be that of flow through a channel connecting two reservoirs at different elevations – see Fig. 2.13. Consider that initially the fluid is at rest as the exit from the channel is sealed off.

The pressure in the channel is uniform and high. If the seal is rapidly removed, a pressure wave passes through the channel at the speed of sound – much higher than fluid velocities. Instantaneously, a linear pressure gradient is set up in the channel. The stationary fluid has $\omega = 0$ and the pressure forces do not create vorticity. The initial pressure gradient is constant and is required to accelerate the fluid. The momentum equation is

$$\rho \frac{\partial u_x}{\partial t} = -\frac{\partial p}{\partial x} - \mu \frac{\partial \omega_z}{\partial y} = -\frac{\partial p}{\partial x} - \mu \sigma_z. \quad (2.57)$$

The viscous force has been expressed in this equation as a flux of vorticity. The initial vorticity is zero, but there is a flux of vorticity at each wall:

$$-\frac{\partial p}{\partial x} = \mu \sigma_z|_0, \quad (2.58)$$

where $\sigma_z = \partial \omega_z / \partial y$.

The final state will depend on the competing pressure and viscous effects.

Viscous timescale, t_{vis} : this is the time taken for vorticity to diffuse halfway across the channel of width d . The Rayleigh analogy will be used, so that $t_{\text{vis}} = (2d)^2/(3.6\nu_A)$ where ν is the kinematic viscosity of the fluid (see [14], p. 347). For $d = 10$ cm and an airflow, t_{vis} is about 60 s (kinematic viscosity = $0.15 \text{ cm}^2/\text{s}$). For a viscous vegetable oil, $\nu = 1.1 \text{ cm}^2/\text{s}$ and $t_{\text{vis}} = 7$ s.

Suppose the flow is very fast, so that $Re \rightarrow \infty$ with d/L finite. In this situation, the vorticity flux term in (2.57) is zero except near the walls. Most fluid particles traverse the channel so fast that vorticity diffusion has no effect on them.

Particles start with no vorticity in the upstream reservoir and traverse the channel in irrotational flow. Vorticity is confined to a small region near each wall. After the pressure gradient generates new vorticity it diffuses only a short distance from the wall before convection moves it downstream and into the exit reservoir.

2.14 Vorticity in a Compressible Fluid

There are several unique features of vorticity in a compressible fluid. The first is that a vortex may develop a vacuum in the core if the strength is enough to centrifuge fluid away from the center of the vortex. Another is vorticity enhancement by fluid compression, represented by the second term in the vorticity equation (2.19), $\boldsymbol{\omega}(\nabla \cdot \mathbf{u})$.

A third unique feature was revealed by Crocco's work [22]. Starting with the momentum equation (2.18) for an inviscid fluid without body forces,

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{\mathbf{u} \cdot \mathbf{u}}{2} \right) - \mathbf{u} \times \boldsymbol{\omega} = \left(\frac{1}{\rho} \right) \nabla p. \quad (2.59)$$

Using the Gibbs equation [23] for the entropy, density and pressure may be eliminated from this equation:

$$Tds = dh - \left(\frac{1}{\rho} \right) dp, \quad (2.60)$$

where T is the temperature, h is the specific enthalpy, and s is the specific entropy. Equation (2.60) can be written as

$$T\nabla s = \nabla h - \left(\frac{1}{\rho} \right) \nabla p. \quad (2.61)$$

Substituting (2.61) in (2.59) gives Crocco's equation:

$$\mathbf{u} \times \boldsymbol{\omega} + T\nabla s = \nabla h_0 + \frac{\partial \mathbf{u}}{\partial t}, \quad (2.62)$$

where h_0 is the stagnation enthalpy, $h_0 = h + 0.5\mathbf{u} \cdot \mathbf{u}$.

It has been shown that [24] for steady adiabatic flow of an inviscid without body forces, h_0 is constant along streamlines. The vector ∇h_0 is, therefore, perpendicular to the streamlines, as is the vector $\mathbf{u} \times \boldsymbol{\omega}$ (first and third terms in (2.62)). The term $T\nabla s$ must therefore also be perpendicular to the streamlines. If it is further stipulated that all fluids originate from a constant stagnation enthalpy region, so that $\nabla h_0 = 0$, then $\mathbf{u} \times \boldsymbol{\omega}$ and $T\nabla s$ must be collinear vectors and (2.62) becomes

$$|\mathbf{u}||\boldsymbol{\omega}| + \frac{Tds}{dn} = 0, \quad (2.63)$$

where n is perpendicular to the streamlines, in the direction of $\mathbf{u} \times \boldsymbol{\omega}$. This shows that if s is constant, $|\boldsymbol{\omega}|$ must be zero. In other words, *isentropic flows are irrotational* (for steady inviscid flows without body forces and with constant h_0). Thus, if such a specific flow is also isentropic, the well-developed theory of irrotational flow may be used to determine the flow field. If the assumption of isentropicity cannot be made, generally closed form solutions cannot be found and numerical methods must be resorted to.

2.15 Vorticity and the Electromagnetic Analogy

There is a close analogy between [25] calculations involving vortices (in a nonviscous fluid, or superfluid) and calculations involving current filaments in a magnetic field.

This analogy is of particular importance in astrophysical hydrodynamics.

In electromagnetism (mks units),

$$\nabla \cdot \mathbf{B} = 0, \quad \oint \mathbf{B} \cdot d\mathbf{l} = \mu I, \quad W_B = \frac{B^2}{2\mu}, \quad (2.64)$$

where $\mathbf{B}$ is the magnetic field intensity, $d\mathbf{l}$ is an element of a filament, I is the current in the filament, μ is the permeability of the surrounding medium, and W_B is the magnetic field energy density. Now the hydrodynamic equations for a vortex field are as follows:

$$\nabla \cdot \mathbf{v} = 0, \quad \oint \mathbf{v} \cdot d\mathbf{l} = \Gamma, \quad W = \frac{\rho v^2}{2}, \quad (2.65)$$

where $\mathbf{v}$ is the velocity vector, Γ is the vortex strength(circulation), ρ is the fluid density, and W is the velocity field energy density. Equations (2.64) and (2.65) are equivalent with the substitutions:

$$\frac{\mathbf{B}}{\mu} \rightarrow \mathbf{v}, \quad I \rightarrow \Gamma, \quad \mu \rightarrow \rho. \quad (2.66)$$

Recapping, the evolution equation for vorticity was

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega + \nabla \times (\mathbf{v} \times \omega), \quad (2.67)$$

where $\omega = \nabla \times \mathbf{v}$ and ν is the kinematic viscosity.

Note that $\nabla \cdot \omega = 0$ analogous to the magnetic flux condition $-\nabla \cdot \mathbf{B} = 0$. Batchelor has argued that magnetic fields can be generated by vortex-dominated turbulence. There is then a “magnetic viscosity” η , analogous to the fluid viscosity, leading to a new dimensionless number – the *magnetic Reynolds number* – defined by

$$Re_M \equiv \frac{U l}{\eta}.$$

As vorticity leads to dissipative processes, in the case of turbulence one would expect that magnetic dissipation will be larger than the ohmic rate. This was first discussed by Spitzer [26] in connection with the magnetic fields of stellar interiors.

It turns out that a frequently encountered field in astrophysical plasmas is the force-free field. The solar corona is one such example. The magnetic helicity is defined as

$$H_M \equiv \int \mathbf{B} \cdot \mathbf{A} dV, \quad (2.68)$$

where $\mathbf{A}$ is the vector potential, and $\mathbf{B} = \nabla \times \mathbf{A}$ (cf. $\omega = \nabla \times \mathbf{v}$). The magnetic helicity is a very important entity in that it is minimal for a force-free field. Just like vorticity, it can be used to determine the field.

The equation for the field evolution with magnetic viscosity is given by

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \quad (2.69)$$

For magnetic fields, the last term is the same as $-\eta \nabla \times (\nabla \times \mathbf{B})$.

An equation for the vector potential $\mathbf{A}$ is obtained by removing the curl from this equation:

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B} - \nabla \Phi - \eta \nabla \times \mathbf{B}. \quad (2.70)$$

The scalar products of (2.69) with $\mathbf{A}$ and of (2.70) with $\mathbf{B}$ are taken, added, and the remaining terms integrated over volume. This leads to an evolution equation for the magnetic helicity:

$$\begin{aligned} \frac{\partial}{\partial t} \int \mathbf{A} \cdot \mathbf{B} d\mathbf{x} &= -\eta \int (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) d\mathbf{x} \\ &\quad - \eta \int (\nabla \times \mathbf{B}) \cdot \mathbf{B} d\mathbf{x} = -2\eta \left(\frac{4\pi}{c} \right) \int \mathbf{J} \cdot \mathbf{B} d\mathbf{x}, \end{aligned} \quad (2.71)$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ and $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}$.

This equation states that the evolution of magnetic helicity in a fluid is driven by dissipation and that if $\mathbf{B}$ is parallel to $\mathbf{J}$ the magnetic field decays; otherwise, it increases. Also, in a fluid with $\eta \rightarrow 0$, the magnetic field is a conserved quantity. Just as in the case of vorticity in a fluid, the magnetic helicity serves as a topological tool for understanding the dissipative mechanisms and the instabilities of the fluid. Magnetic field lines tangle, merge, split, and eventually decay.

2.16 Quantization of Circulation and Vorticity

The description of a superfluid in terms of a single wave function, $|\psi| \exp(iS)$, where S is the phase function, leads to the following superfluid current equations:

$$j_s = \left(\frac{h}{2\pi}\right) |\psi|^2 \nabla S \quad (\text{HeII, superfluid}), \quad (2.72)$$

$$\mathbf{J}_e = \left(\frac{eh}{2\pi m}\right) |\psi|^2 \nabla S - \left(\frac{2e^2}{m}\right) |\psi|^2 \mathbf{A}. \quad (2.73)$$

The mass current density j_s is used for He II and the electric current density $\mathbf{J}_e$ for the superconductor (valid for cubic superconductors only [27]).

Consider He II in an annular region between two concentric cylinders (a multiply connected region), as depicted in Fig. 2.14.

The temperature is absolute zero, so that the He II is a pure superfluid.

To determine the flow pattern, consider the circulation

$$\Gamma = \oint v_s \cdot d\mathbf{l}, \quad (2.74)$$

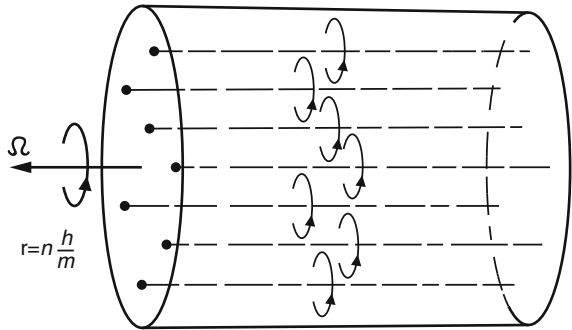


Fig. 2.14 Helium II between two concentric cylinders

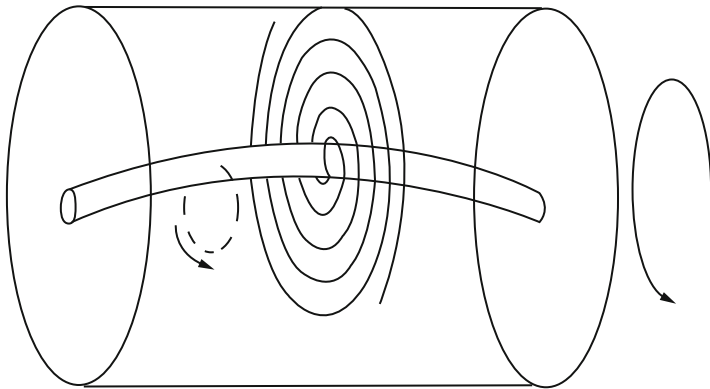


Fig. 2.15 Rotating drum with superfluid in it

where v_s is the velocity of the superfluid and the integral is taken around any contour wholly within the fluid. Equation (2.72) implies [27] and the superfluid equation can be written as

$$v_s = \left(\frac{h}{2\pi m_4} \right) \nabla S, \quad (2.75)$$

where m_4 is the atomic mass of the ^4He atom, and so it is possible to express the circulation in terms of the wave function phase S :

$$\Gamma = \left(\frac{h}{2\pi m_4} \right) \oint \nabla S \cdot d\mathbf{l}. \quad (2.76)$$

For the circle 1 in Fig. 2.14, the circulation is:

$$\Gamma = \left(\frac{h}{2\pi m_4} \right) (\Delta S)_1. \quad (2.77)$$

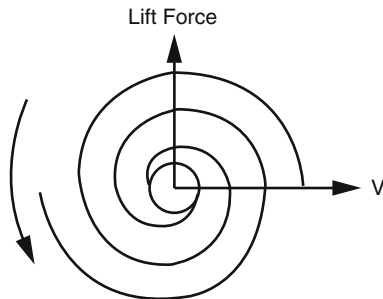
Since the superfluid is single valued, a traverse around a closed contour must leave it unaltered, so that the change in S can be only an integral multiple of 2π or zero. From (2.77) it can be seen that the circulation (and hence the vorticity) is quantized, with the values

$$\Gamma = n \left(\frac{h}{m_4} \right) \quad \text{where } n = 0, 1, 2, \quad (2.78)$$

h/m_4 is known as the quantum of circulation and has the value $9.98 \times 10^{-8} \text{ m}^2/\text{s}$.

Vinen [28] was the first to experimentally demonstrate that circulation is quantized in He II. The apparatus is sketched in Fig. 2.15 and consisted of a cylindrical

Fig. 2.16 The Magnus force is the perpendicular lift force on the wire when it and the vortex are dragged at velocity V



container with a fine conducting wire stretched along its axis. The He II filling the container thus occupied a multiply connected region.

When fluid is encircling the wire and the wire is vibrating, the Magnus force induces circular motion in the wire. The circulating fluid is dragged along with the wire

A solid cylinder, around which there is a fluid circulation, experiences a force – the Magnus force – when it moves through the fluid (see Fig. 2.16).

Treating the superfluid as an ideal Euler fluid, the Magnus force on the wire in Vinen's experiment is given by

$$f_M = \rho_s \Gamma \times \mathbf{V}, \quad (2.79)$$

where $\mathbf{V}$ is the wire velocity relative to the superfluid outside the circulation and Γ is the vector indicating the sense and strength of the circulation.

The wire was placed in a magnetic field and transverse vibrations excited by passage of an AC electric current through it. When the surrounding liquid is not rotating, the normal modes of the wire are two plane polarized waves at right angles to each other and with the same frequency. When the superfluid circulates smoothly around the wire, the latter is acted upon by the Magnus force, causing both planes of vibration to precess. The normal modes of the wire can now be viewed as circularly polarized in opposite directions, with frequencies differing by $\Delta\nu = \rho_s \Gamma / 2\pi W$ where W is the sum of the mass per unit length of the wire plus half the mass of the fluid displaced by this length. The difference $\Delta\nu$ appears as a beat frequency of the voltage induced in the wire and this provided a direct method of measuring the circulation Γ .

2.17 Quantized Vortices in He II

Rotation of the superfluid: when the two-fluid model of He II was first suggested, it was believed that it would be difficult to set the superfluid fraction into rotation because superfluid flow was characterized by the irrotationality condition introduced by Landau [29]:

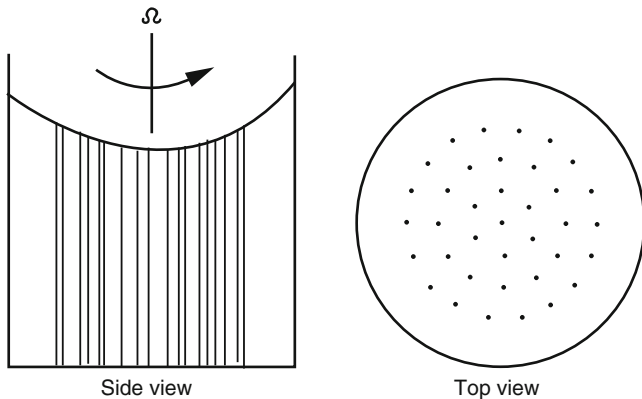


Fig. 2.17 Array of quantized vortices in a rotating bucket. Some vortices are missing near the outer edge

$$\nabla \times \mathbf{v}_s = 0. \quad (2.80)$$

Consider He II contained in a cylindrical bucket and the contour 2 – see Fig. 2.17.

By Stokes' theorem, the circulation around the contour 2 can be written as an integral over the surface $\mathbf{A}$ enclosed by the contour:

$$\Gamma = \oint_2 \mathbf{v}_s \cdot d\mathbf{l} = \int_A (\nabla \times \mathbf{v}_s) \cdot d\mathbf{A}. \quad (2.81)$$

Combining (2.80) and (2.81),

$$\Gamma = \oint_2 \mathbf{v}_s \cdot d\mathbf{l} = 0 \quad (2.82)$$

Thus, the circulation for any contour in the continuous fluid is zero. If (2.82) holds everywhere in the superfluid, rotation is not possible. But it only holds if $\mathbf{v}_s$ is zero at every point. This state of He II from which superfluid rotation is completely absent is called the Landau state. The rotation of the superfluid can be explained by assuming that it is threaded by parallel straight vortex lines – see Fig. 2.17. It has been explained above how one can have a circulation around a region from which the superfluid is excluded. Contours that enclose a solid obstacle or a vortex core (cf. Contour 1 in Fig. 2.14) yield a quantized circulation:

$$\Gamma = \oint_1 \mathbf{v}_s \cdot d\mathbf{l} = \oint_A (\nabla \times \mathbf{v}_s) \cdot d\mathbf{l} = \frac{nh}{m_4}. \quad (2.83)$$

Note: the core of the vortex is defined by that region in which $\text{curl } \mathbf{v}_s \neq 0$.

The earliest suggestion that vorticity should play an important role in superfluid hydrodynamics is credited to Onsager [30].

Vortex lines in rotating He II:

It can now be explained how, in Osborne's [31] rotating bucket experiment, the presence of a uniform array of vortex lines enables the superfluid to undergo solid-body rotation. Suppose there are n_v vortex lines per unit area of the bucket, all with their cores parallel to the axis of rotation (Fig. 2.17) and each with the same circulation Γ , represented by an axial vector of magnitude Γ pointing in the direction consistent with the sense of rotation. The strength of the array is specified by the vorticity ϖ , defined to be equal to the total circulation within unit area

$$\varpi = n_v \Gamma. \quad (2.84)$$

From (2.81) it can be seen that ϖ can be identified with the average value of $\text{curl } \mathbf{v}_s$. When $\text{curl } \mathbf{v}_s$ is nonzero it indicates the presence of vortices. In the bucket the total circulation enclosed by a contour of radius R centered on the axis is $\pi n_v R^2 \Gamma$. For the superfluid to appear to rotate with uniform angular velocity Ω , the total circulation must also be equal to $2\pi R(R\Omega)$. Thus, the condition for the simulation of solid-body rotation is that the vortex-line density is

$$n_v = \frac{2\Omega}{\Gamma}. \quad (2.85)$$

Alternatively, this can be used to determine the required vorticity,

$$\varpi = \nabla \times \mathbf{v}_s = 2\Omega. \quad (2.86)$$

Hall [32] has shown that the ground state of He II contains a regular array of vortex lines all having the smallest possible circulation h/m_4 and with a maximum total number of lines. Viewed from a frame rotating with the container, the equilibrium vortex array is a triangular lattice. It can be shown [27, p. 186] that the critical angular velocity for the formation of one vortex with minimum circulation is

$$\Omega_{c1} = \left(\frac{h}{2\pi m_4 R_0^2} \right) \ln \left(\frac{R_0}{a_0} \right) \quad (2.87)$$

For a typical value of R_0 of 1 cm, $\Omega_{c1} \approx 10^{-3}$ rad/s. Thus, it is easy for vortex lines to appear.

For sufficiently low angular velocities of the container ($\Omega \leq \Omega_{c1}$), the equilibrium state of the superfluid is the Landau state as described earlier. The Landau state was observed by Hess and Fairbank [33] in an experiment in which liquid 4He was cooled through the λ point while inside a rotating cylinder made from a closed capillary tube. Above T_λ , the He I was in solid-body rotation. On starting from rest, provided the cylinder was rotated slowly enough, it was found that the superfluid formed in a state of zero angular momentum relative to the laboratory. Packard and

Sanders [34] have developed a method of counting vortex lines that exploits the trapping of electrons on vortex cores. This was used to demonstrate that a small amount of vorticity could be detected for up to 30 min after the container was brought to rest. This highlighted an important aspect of vorticity: the *persistence of vorticity*.

A single electron self-trapped inside a cavity whose size is fixed by a balance between the outward pressure due to the zero-point motion of the electron and the inward pressure of the surrounding liquid is known as a negative ion, and can be trapped on the cores of vortex lines. The positive ion, which can also be trapped on vortex lines, is a “snowball” consisting of an α particle to which several neutral ^4He atoms are bound electrostatically. It has been shown that [35] the motion of ions in He II can create quantized vortex rings, which become coupled to the ions. The mechanism by which a bare ion nucleates a vortex ring in He II and then becomes trapped on it is unknown. The solution of this problem will probably cast light on the creation of vorticity on a microscopic scale. This knowledge is required in order to understand the breakdown of superfluidity through the agency of vorticity.

References

1. Truesdell, C.: The Kinematics of Vorticity. Indiana University Press, Bloomington (1954)
2. Thompson, W. (Lord Kelvin): On vortex motion. Math. Phys. 4, 49 (1869)
3. Batchelor, G.K.: An Introduction to Fluid Dynamics. Cambridge University Press, Cambridge (1967)
4. Hadamard, J.: Sur les Tourbillons Produit par les Ondes De Choc. Note III. Lecons sur la Propagation des Ondes. In: Herman, A. (ed.). Paris, 362 (1903)
5. Vazsonyi, A.: On rotational gas flows. Quart. J. Appl. Math. 3(1), 29–37 (1945)
6. Holton, J.R.: An Introduction to Dynamic Meteorology. Academic, New York, NY (1979)
7. Green, S.I. (ed.): Fluid Vortices. Kluwer Academic, Boston, Massachusetts (1995) (Chapter 11)
8. Oseen, C.W.: Über Wirbelbewegung in einer Reibenden Flussigkeit. Ark. f. Mat. Astron. Fyz 7, 14 (1912)
9. Lighthill, M.J.: Boundary layer theory. Part II. In: Rosenhead, L. (ed.) Laminar Boundary Layers, p. 389. Dover, Mineola (1963)
10. Morton, B.R.: The generation and decay of vorticity. Geophys. Astrophys. Fluid Dyn. 28, 277–308 (1984)
11. Heimenz, H.: On integrals of the hydrodynamical equations which express vortex motion. Dingler's Polytech. 326, 311 (1867)
12. Burgers, J.M.: A mathematical model illustrating the theory of turbulence. Adv. Appl. Mech. 17, 1 (1948)
13. Hill, M.J.M.: On a spherical vortex. Phil. Trans. Roy. Soc. Lond. A 185, 213 (1894)
14. Panton, R.I.: Incompressible Flow. Wiley, New York, NY (1984). p. 333
15. Bjerknes, V.: Überdie Bildung Circulations bewegung Skrifter (1898)
16. Lamb, H.: Hydrodynamics. Cambridge University Press, Cambridge (1932)
17. Mockros, L.: Ph.D. Thesis, University of California, Los Angeles (1962)
18. Goertler, H.: Über eine Dreidimensionale Instabilität Laminaren Grenzschichten am Koncaven Wander. Naschr Wiss Gas,Gottingen Math Phys Klasse, 2(1) (1940)
19. Witting, H.: Theorie der Sekundären Instabilität der Laminaren Grenzschichten. In: Goertler, H. (ed.) Boundary Layer Research. Springer, Berlin (1958)

20. Witting, H.: Einige Neuer Ergebnisse zur Hydrodynamischen Stabilitats Theorie. Zeits.fur Flugwissenschaften 8 Jahrgang, Heft 1 (1960)
21. McCormack, P., Welcher, H.: J. Heat Transfer (1967)
22. Crocco, L.: Zeits. Angew. Math. Mech. **17**, p. 1 (1937)
23. Black, W.Z., Hartley, J.G.: Thermodynamics. Harper Collins, New York, NY (1991)
24. Currie, I.G.: Fundamental Mechanics of Fluids. McGraw-Hill, New York (1974)
25. Fetter, A., Donnelly, R.: On the equivalence of vortices and current filaments. Phys. Fluids **9**, 619 (1966)
26. Spitzer Jr., L.: Influence of fluid motions in the decay of an external magnetic field. Astrophys. J. **125**, 525 (1957)
27. Tilley, D.R., Tilley, J.: Superfluidity and Super-Conductivity, 3rd edn. Graduate Student Series in Physics, Institute of Physics Publishing, Philadelphia (1990)
28. Vinen, W.F., et al.: Quantized Vortex Dynamics and Superfluid Turbulence. Springer, Berlin (2001)
29. Landau, L.D.: The theory of superfluidity of helium II. J. Phys. Moscow **5**, 71 (1941)
30. Onsager, L.: Nuevo Cimento. **6**(2), 249 (1949)
31. Osborne, L.: Proc. Phys. Soc. **A63**, 909 (1949)
32. Hall, H.E.: On the rotation of liquid helium II. Adv. Phys. **9**, 89 (1960)
33. Hess, G.B., Fairbank, W.M.: Measurements of angular momentum in superfluid helium. Phys. Rev. Letts. **19**, 216 (1967)
34. Packard, R.F., Sanders, T.M.: Observation of single vortex lines in rotating superfluid helium. Phys. Rev. **A6**, 799 (1972)
35. Rayfield, G.W., Reif, F.: Quantized vortex rings in superfluid helium. Phys. Rev. **136**, 1194 (1964)

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