Chapter 2
Concepts of Probability and Statistics

2.1 Introduction

The analysis of communication systems involves the study of the effects of noise on the ability to transmit information faithfully from one point to the other using either wires or wireless systems (in free space), including microwave signals (Lindsey and Simon 1973; Schwartz 1980; Schwartz et al. 1996; Middleton 1996). Thus, it is important to understand the properties of noise, which requires the study of random variables and random processes. Additionally, the presence of fading and shadowing in the channel (mainly the wireless ones which are more susceptible to uncertainty) demands an understanding of the properties of random variables, distribution, density functions, and so on (Helstrom 1968; Rohatgi and Saleh 2001; Papoulis and Pillai 2002).

In this chapter, we will review, among others, the concepts of random variables, the properties of the distribution and density functions, and moments. We will also look at characterizing two or more random variables together so that we can exploit their properties when we examine techniques such as the diversity combining algorithms employed to mitigate fading and shadowing seen in wireless systems (Brennan 1959; Suzuki 1977; Hansen and Meno 1977; Shepherd 1977; Jakes 1994; Sklar 1997a, b; Vatalaro 1995; Winters 1998). In very broad terms, fading and shadowing represent the random fluctuations of signal power observed in wireless channels. Their properties are characterized in terms of their statistical nature, mainly through the density functions (Saleh and Valenzuela 1987; Vatalaro 1995; Yacoub 2000, 2007a, b; Cotton and Scanlon 2007; Nadarajah and Kotz 2007; da Costa and Yacoub 2008; Karadimas and Kotsopoulos 2008, 2010; Shankar 2010; Papazafeiropoulos and Kotsopoulos 2011) and we will review all the pertinent statistical aspects to facilitate a better understanding of signal degradation and mitigation techniques to improve the signal quality.

P.M. Shankar, *Fading and Shadowing in Wireless Systems*,
DOI 10.1007/978-1-4614-0367-8_2, © Springer Science+Business Media, LLC 2012

2.2 Random Variables, Probability Density Functions, and Cumulative Distribution Functions

A random variable is defined as a function that maps a set of outcomes in an experiment to a set of values (Rohatgi and Saleh 2001; Papoulis and Pillai 2002). For example, if one tosses a coin resulting in "heads" or "tails," a random variable can be created to map "heads" and "tails" into a set of numbers which will be discrete. Similarly, temperature measurements taken to provide a continuous set of outcomes can be mapped into a continuous random variable. Since the outcomes (coin toss, roll of a die, temperature measurements, signal strength measurements etc.) are random, we can characterize the random variable which maps these outcomes to a set of numbers in terms of it taking a specific value or taking values less than or greater than a specified value, and so forth. If we define X as the random variable, $\{X \leq x\}$ is an event. Note that x is a real number ranging from $-\infty$ to $+\infty$. The probability associated with this event is the distribution function, more commonly identified as the cumulative distribution function (CDF) of the random variable. The CDF, $F_X(x)$, is

$$F_X(x) = \text{Prob}\{X \leq x\} \ . \tag{2.1}$$

The probability density function (pdf) is defined as the derivative of the CDF as

$$f_X(x) = \frac{\mathrm{d}[F_X(x)]}{\mathrm{d}x}. \tag{2.2}$$

From the definition of the CDF in (2.1), it becomes obvious that CDF is a measure of the probability and, therefore, the CDF has the following properties:

$$\begin{aligned}
&F_X(-\infty) = 0, \\
&F_X(\infty) = 1, \\
&0 \leq F_X(x) \leq 1, \\
&F_X(x_1) \leq F_X(x_2), \quad x_1 \leq x_2.
\end{aligned} \tag{2.3}$$

Based on (2.2) and (2.3), the probability density function has the following properties:

$$\begin{aligned}
&0 \leq f_X(x), \\
&\int_{-\infty}^{\infty} f_X(x)\ \mathrm{d}x = 1, \\
&F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\ \mathrm{d}\alpha, \\
&\text{Prob}\{x_1 \leq X \leq x_2\} = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(\alpha)\ \mathrm{d}\alpha.
\end{aligned} \tag{2.4}$$

A few comments regarding discrete random variables and the associated CDFs and pdfs are in order. For the discrete case, the random variable X takes discrete values (1, 2, 3,. . ., n). The CDF can be expressed as (Helstrom 1991; Rohatgi and Saleh 2001; Papoulis and Pillai 2002)

$$F_X(x) = \sum_{m=1}^{n} \mathrm{Prob}\{X = m\} U(x - m). \tag{2.5}$$

In (2.5), $U(.)$ is the unit step function. The pdf of the discrete random variable becomes

$$f_X(x) = \sum_{m=1}^{n} \mathrm{Prob}\{X = m\} \delta(x - m). \tag{2.6}$$

In (2.6), $\delta(.)$ is the delta function (Abramowitz and Segun 1972).

It can be easily seen that for a continuous random variable there is no statistical difference between the probabilities of outcomes where the random variable takes a value less than ($<$) or less than or equal to ($\leq$) a specified outcome because the probability that a continuous random variable takes a specified value is zero. However, for the case of a discrete random variable, the probabilities of the two cases would be different. We will now examine a few properties of the random variables and density functions.

The moments of a random variable are defined as

$$\mu_k = \langle X^k \rangle = E(X^k) = \int_{-\infty}^{\infty} x^k f(x)\, \mathrm{d}x. \tag{2.7}$$

The first moment ($k = 1$) is the mean or the expected value of the random variable and $<\cdot>$ as well as $E(.)$ represents the statistical average. The variance (var) of the random variable is related to the second moment ($k = 2$) and it is defined as

$$\mathrm{var}(x) = \sigma^2 = \langle X^2 \rangle - \langle X \rangle^2. \tag{2.8}$$

The quantity σ is the standard deviation. Variance is a measure of the degree of uncertainty and

$$\sigma^2 \geq 0. \tag{2.9}$$

The equality in (2.9) means that the degree of uncertainty or randomness is absent; we have a variable that is deterministic. Existence of higher values of variance suggests higher level of randomness. For the case of a discrete random variable, the moments are expressed as

$$\langle X^k \rangle = \sum_{m=1}^{n} m^k \mathrm{Prob}\{X = m\}. \tag{2.10}$$

The survival function $S(x)$ of a random variable is defined as the probability that $\{X > x\}$. Thus,

$$S(x) = \int_{x}^{\infty} f_X(\alpha)\,\mathrm{d}\alpha = 1 - F_X(x). \tag{2.11}$$

The coefficient of skewness is defined as (Evans et al. 2000; Papoulis and Pillai 2002)

$$\eta_3 = \frac{\mu_3}{\sigma^3}. \tag{2.12}$$

The coefficient of kurtosis is defined as

$$\eta_4 = \frac{\mu_4}{\sigma^4}. \tag{2.13}$$

The entropy or information content of a random variable is defined as

$$I = -\int_{-\infty}^{\infty} f(x)\log_2[f(x)]\mathrm{d}x. \tag{2.14}$$

The mode of a pdf $f(x)$ is defined as the value x where the pdf has the maximum. It is possible that a pdf can have multiple modes. Some of the density functions will not have any modes. The median x_{m} of a random variable is defined as the point where

$$\int_{-\infty}^{x_{\mathrm{m}}} f(x)\mathrm{d}x = \int_{x_{\mathrm{m}}}^{\infty} f(x)\mathrm{d}x. \tag{2.15}$$

2.3 Characteristic Functions, Moment Generating Functions and Laplace Transforms

The characteristic function (CHF) of a random variable is a very valuable tool because of its use in the performance analysis of wireless communication systems (Nuttall 1969, 1970; Tellambura and Annamalai 1999; Papoulis and Pillai 2002; Tellambura et al. 2003; Goldsmith 2005; Withers and Nadarajah 2008). This will be demonstrated later in this chapter when we discuss the properties of the CHFs. The CHF, $\psi(x)$, of a random variable having a pdf $f(x)$ is given by

$$\psi_X(\omega) = \langle \exp(j\omega X)\rangle = \int_{-\infty}^{\infty} f(x)\exp(j\omega x)\mathrm{d}x. \tag{2.16}$$

Equation (2.16) shows that CHF is defined as the statistical average of exp $(j\omega x)$. Furthermore, it is also the Fourier transform of the pdf of the random variable.

Rewriting (2.16) by replacing ($j\omega$) by s, we get the moment generating function (MGF) of the random variable as (Alouini and Simon 2000; Papoulis and Pillai 2002)

$$\phi_X(s) = \langle \exp(sx) \rangle = \int_{-\infty}^{\infty} f(x) \exp(sx)\, \mathrm{d}x. \tag{2.17}$$

Defining (2.16) slightly differently, we can obtain the expression for the Laplace transform of pdf as (Beaulieu 1990; Tellambura and Annamalai 1999, 2000; Papoulis and Pillai 2002)

$$L_X(s) = \int_{-\infty}^{\infty} f(x) \exp(-sx)\, \mathrm{d}x. \tag{2.18}$$

Consequently, we have the relationship between the bilateral Laplace transform and the MGF of a random variable:

$$L_X(s) = \phi_X(-s). \tag{2.19}$$

Going back to (2.16), we can define the inverse Fourier transform of the CHF to obtain the pdf as

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(\omega) \exp(-j\omega x) \mathrm{d}\omega. \tag{2.20}$$

By virtue of the property (2.20), CHF will uniquely determine the probability density function. We will explore the use of CHFs in diversity analysis later in this chapter and we will use the Laplace transform, CHF, and MGF to estimate the error rates and outage probabilities to be seen in later chapters (Tellambura and Annamalai 1999).

2.4 Some Commonly Used Probability Density Functions

We will now look at several random variables which are commonly encountered in wireless communication systems analysis. We will examine their properties in terms of the pdfs, CDFs and CHFs. Within the context of wireless communications, we will explore the relationships among some of these random variables.

2.4.1 Beta Distribution

The beta distribution is not commonly seen in wireless communications. However, it arises in wireless system when we are studying issues related to signal-to-interference ratio (Jakes 1994; Winters 1984, 1987). The beta random variable is generated when the ratio of certain random variables is considered (Papoulis and Pillai 2002). The beta density function is

$$f(x) = \begin{cases} \dfrac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)}, & 0<x<1, \\ 0 & \text{elsewhere.} \end{cases} \tag{2.21}$$

In (2.21), $\beta(a,b)$ is the beta function given in integral form as

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\mathrm{d}x. \tag{2.22}$$

Equation (2.22) can also be written in terms of gamma functions (Abramowitz and Segun 1972):

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{2.23}$$

where $\Gamma(.)$ is the gamma function. The CDF of the beta random variable is

$$F(x) = \begin{cases} 0, & x \le 0, \\ \displaystyle\int_0^x \frac{y^{a-1}(1-y)^{b-1}}{\beta(a,b)}\mathrm{d}y, & 0<x<1, \\ 1, & x \ge 1. \end{cases} \tag{2.24}$$

The moments of the beta random variable are

$$\langle X^k \rangle = \frac{\Gamma(k+a)\Gamma(k+b)}{\Gamma(k+a+b)\Gamma(k)}. \tag{2.25}$$

The mean and variance of the beta variable are

$$\langle X \rangle = \frac{a}{a+b}, \tag{2.26}$$

$$\mathrm{var}(X) = \frac{ab}{(a+b+1)(a+b)^2}. \tag{2.27}$$

The beta density function is shown in Fig. 2.1 for three sets of values of (a,b). It can be seen that for the case of $a = b = 1$, the beta density becomes the uniform density function as seen later in this chapter. Note that X is a beta variate with parameters a and b; then $Y = 1-X$ is also a beta variate with parameters b and a.

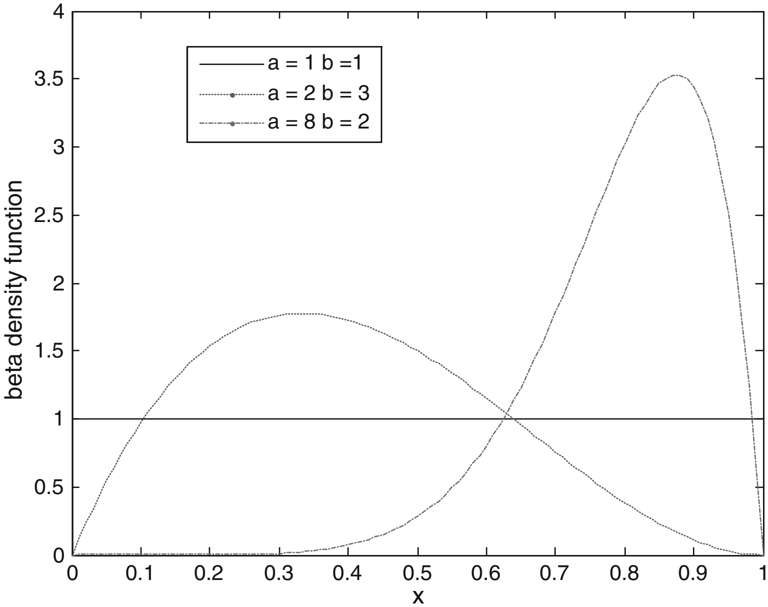

Fig. 2.1 The density function of the beta random variable

The beta density function allows significant flexibility over the standard uniform distribution in the range $\{0, 1\}$ as one can see from Fig. 2.1. As mentioned in the beginning if X_1 and X_2 are two gamma variables with parameters (a, A) and (b, A) respectively, the ratio $X_1/(X_1 + X_2)$ will be a beta variate with parameters (a,b). This aspect will be shown later in this chapter when we examine the density function of linear and nonlinear combination of two or more random variables.

2.4.2 Binomial Distribution

The binomial distribution is sometimes used in wireless systems when estimating the strength of the interfering signals at the receiver, with the number of interferers contributing to the interfering signal being modeled using a binomial random variable (Abu-Dayya and Beaulieu 1994; Shankar 2005). For example, if there are n interfering channels, it is possible that all of them might not be contributing to the interference. The number of actual interferers contributing to the interfering signal can be statistically described using the binomial distribution. This distribution is characterized in terms of two parameters, with the parameter n representing the number of Bernoulli trials and the parameter p representing the successes from the n trials. (A Bernoulli trial is an experiment with only two possible outcomes that have probabilities p and q such that $(p + q) = 1$). While n is an integer, the quantity

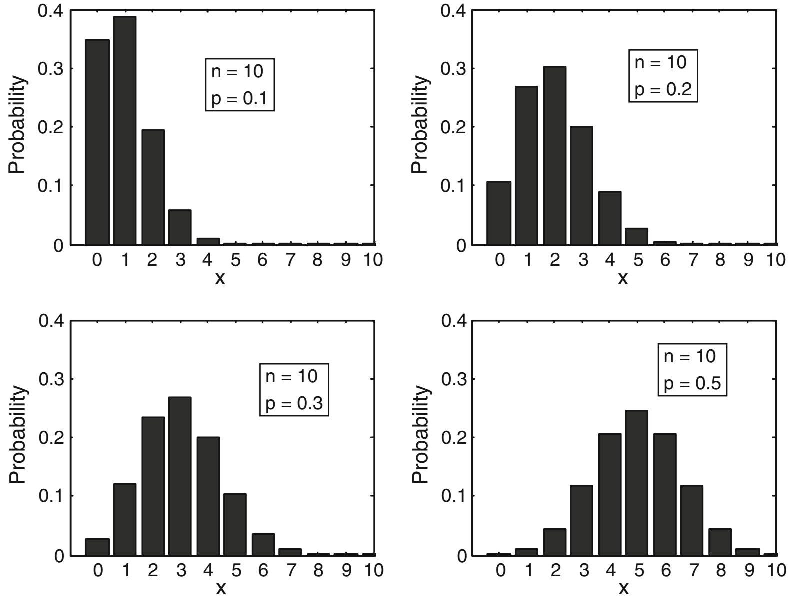

Fig. 2.2 Binomial probabilities

p is bounded as $0 < p < 1$. The binomial probability is given by (Papoulis and Pillai 2002)

$$\text{Prob}\{X = k\} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots. \tag{2.28}$$

In (2.28),

$$\binom{n}{k} = C_k^n = \frac{n!}{(n-k)!k!}. \tag{2.29}$$

The mean of the variable is given by (np) and the variance is given by (npq) where $q = (1 - p)$. The binomial probabilities are shown in Fig. 2.2.

The binomial variate can be approximated by a Poisson variate (discussed later in this section) if $p << 1$ and $n < 10$. Later we will describe how the binomial distribution approaches the normal distribution, when $npq > 5$ and $0.1 < p < 0.9$. The transition toward the normal behavior is seen in Fig. 2.2.

We alluded to Bernoulli trials which had only two outcomes. We can extend such experiments to have more than two outcomes resulting in the generalized Bernoulli trials. Instead of two outcomes, let us consider the case of r outcomes (mutually exclusive) such that the total probability

$$p_1 + p_2 + \cdots + p_r = 1. \tag{2.30}$$

Now, we repeat the experiment n times. Our interest is in finding out the probability that outcome #1 occurs k_1 times, #2 occurs k_2 times, and so on. In other words, we are interested in the Prob{#1 occurs k_1 times, #2 occurs k_2 times,.., #r occurs k_r times}. Noting that the number of ways in which these events can occur is $n!/(k_1!k_2!\ldots k_r!)$, the required probability becomes

$$p(k_1, k_2, \ldots, k_r) = \frac{n!}{k_1!k_2!\ldots k_r!} p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}, \quad \sum_{j=1}^{r} k_r = n. \tag{2.31}$$

We will use some of these results when we examine the order statistics later in this chapter.

2.4.3 Cauchy Distribution

The Cauchy distribution arises in wireless systems when we examine the pdfs of random variables which result from the ratio of two Gaussian random variables (Papoulis and Pillai 2002). What is unique in terms of its properties is that its moments do not exist. The Cauchy pdf is expressed as

$$f(x) = \frac{1}{\pi\beta[1 + ((x-\alpha)/\beta)^2]}, \quad -\infty < x < \infty. \tag{2.32}$$

The associated CDF is

$$F_X(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-\alpha}{\beta}\right). \tag{2.33}$$

The CHF is

$$\psi(\omega) = \exp(j\alpha\omega - |\omega|\beta). \tag{2.34}$$

As mentioned, its moments do not exist and its mode and median are (each) equal to α. Note that Cauchy distribution might appear similar to the normal (Gaussian) distribution. While both the Cauchy and Gaussian distributions are unimodal (only a single mode exists) and are symmetric (Gaussian around the mean and Cauchy around α), the Cauchy distribution has much heavier tails than the Gaussian pdf. The Cauchy pdf is shown in Fig. 2.3 for the case of $\alpha = 0$. The heavier tails are seen as β goes up

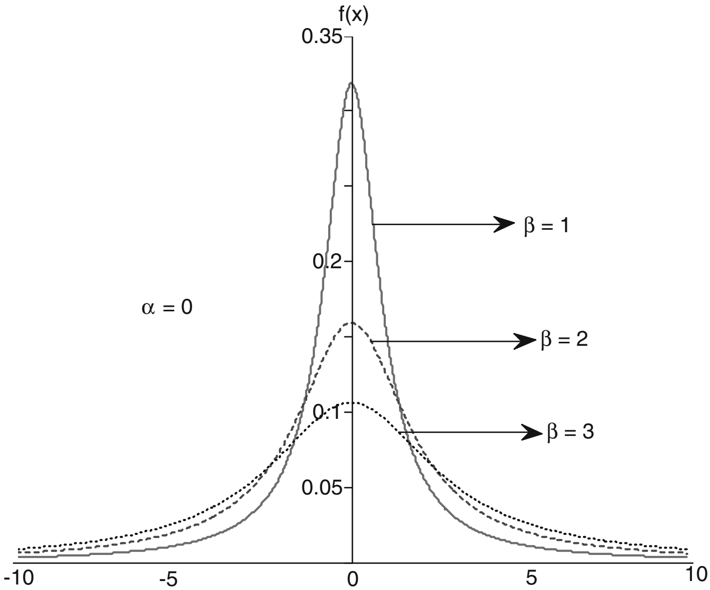

Fig. 2.3 Cauchy pdf is shown

2.4.4 Chi-Squared Distribution

This distribution arises in statistics (and in wireless systems) when we examine the pdf of the sum of the squares of several Gaussian random variables. It is also used in hypothesis testing such as the χ^2 goodness of fit test (Papoulis and Pillai 2002). The chi square (or chi squared) random variable has a pdf given by

$$f(x) = \frac{x^{(n-2)/2} \exp(-(x/2))}{2^{n/2}\Gamma(n/2)}, \quad 0 < x < \infty. \tag{2.35}$$

The shape parameter n is designated as the degrees of freedom associated with the distribution. This density function is also related to the Erlang distribution used in the analysis of modeling of the grade of service (GOS) in wireless systems and the gamma distribution used to model fading and shadowing seen in wireless systems. The gamma and Erlang densities are described later in this section. The distribution is often identified as $\chi^2(n)$. The density function also becomes the exponential pdf in the limiting case ($n = 2$).

There is no simple analytical expression for the CDF, but, it can be expressed in terms of incomplete gamma function as

$$F_X(x) = 1 - \frac{\Gamma\left(\frac{n}{2}, \frac{x}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \frac{\gamma\left(\frac{n}{2}, \frac{x}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}, \tag{2.36}$$

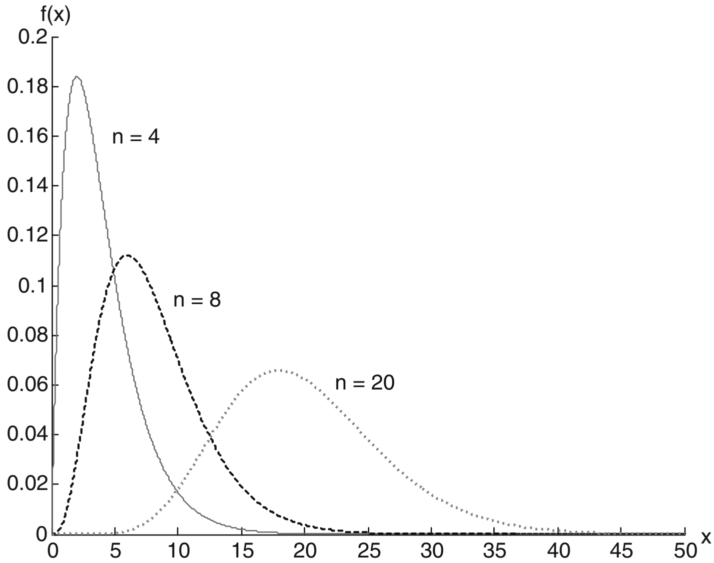

Fig. 2.4 The Chi squared pdf. The density function becomes more symmetric as the degree of freedom n goes up

where

$$\Gamma(a,b) = \int_b^{\infty} x^{a-1} \exp(-x)\, \mathrm{d}x, \quad \gamma(a,b) = \int_0^{b} x^{a-1} \exp(-x)\, \mathrm{d}x. \tag{2.37}$$

$\Gamma(.,.)$ is the (upper) incomplete gamma function, $\gamma(.,.)$ the (lower) incomplete gamma function and $\Gamma(.)$ is the gamma function (Abramowitz and Segun 1972; Gradshteyn and Ryzhik 2007). Note that the pdf in (2.35) is a form of gamma pdf of parameters (n/2) and 2 (as we will see later). The CHF is given by

$$\psi(\omega) = (1 - 2j\omega)^{-(n/2)}. \tag{2.38}$$

The mean of the random variable is n and the variance is $2n$. The mode is $(n{-}2)$, $n > 2$ and the median is $(n{-}2/3)$. The density function is plotted in Fig. 2.4. It can be seen that as the degrees of freedom (n) increases, the symmetry of the pdf increases and it will approach the Gaussian pdf.

There is a related distribution called the non-central chi squared distribution which is described later in this chapter.

An associated distribution is the *chi* distribution. The *chi* variable is the positive square root of a *chi* squared variable. The chi pdf is given by

$$f(x) = \frac{x^{n-1} \exp(-(x^2/2))}{2^{(n/2)-1}\Gamma(n/2)}, \quad 0 \le x \le \infty. \tag{2.39}$$

The cumulative distribution becomes

$$F_X(x) = 1 - \frac{\Gamma\left(\frac{n}{2}, \frac{x^2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \frac{\gamma\left(\frac{n}{2}, \frac{x^2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}. \tag{2.40}$$

When $n = 2$, the *chi* pdf becomes the Rayleigh pdf, described later in this section.

2.4.5 Erlang Distribution

When we examine the density of a sum of exponential random variables, we get the so-called Erlang pdf. The Erlang pdf is given by

$$f(x) = \frac{(x/\beta)^{c-1} \exp(-(x/\beta))}{\beta(c-1)!}, \quad 0 \leq x \leq \infty. \tag{2.41}$$

Note that the shape parameter c is an integer. Equation (2.41) becomes the gamma pdf if c is a non-integer. The CHF is given by

$$\psi(\omega) = (1 - j\beta\omega)^{-c}. \tag{2.42}$$

The mean of the random variable is (βc) and the variance is ($\beta^2 c$). The mode is $\beta(c-1)$, $c \geq 1$. The CDF can be expressed as

$$F_X(x) = 1 - \left[\exp\left(-\frac{x}{\beta}\right)\right]\left(\sum_{k=0}^{c-1} \frac{(x/\beta)^k}{k!}\right). \tag{2.43}$$

Note that the Erlang density function and chi-squared pdf in (2.35) have similar functional form with $c = n/2$ and $\beta = 2$.

2.4.6 Exponential Distribution

The exponential distribution (also known as the negative exponential distribution) arises in communication theory in the modeling of the time interval between events when the number of events in any time interval has a Poisson distribution. It also arises in the modeling of the signal-to-noise ratio (SNR) in wireless systems (Nakagami 1960; Saleh and Valenzuela 1987; Simon and Alouini 2005). Furthermore, the exponential pdf is a special case of the Erlang pdf when $c = 1$ and a special case of chi-squared pdf when $n = 2$. The exponential pdf is given by

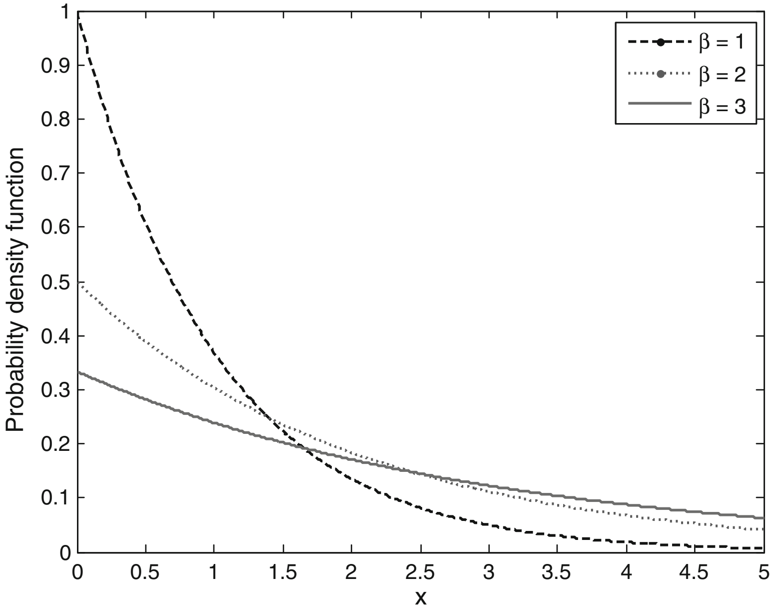

Fig. 2.5 The exponential densities are plotted for three values of the mean

$$f(x) = \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right), \quad 0 < x < \infty. \tag{2.44}$$

The associated CDF is given by

$$F(x) = 1 - \exp\left(-\frac{x}{\beta}\right). \tag{2.45}$$

The CHF is given by

$$\psi(\omega) = \frac{1}{1 - j\omega\beta}. \tag{2.46}$$

The exponential has no mode. The mean is β and the variance is β^2 and, thus, the exponential pdf is uniquely characterized in terms of the fact that the ratio of its mean to its standard deviation is unity. The exponential pdf is shown in Fig. 2.5 for three values of β.

The exponential CDF is a measure of the outage probability (probability that the signal power goes below a threshold) is shown in Fig. 2.6.

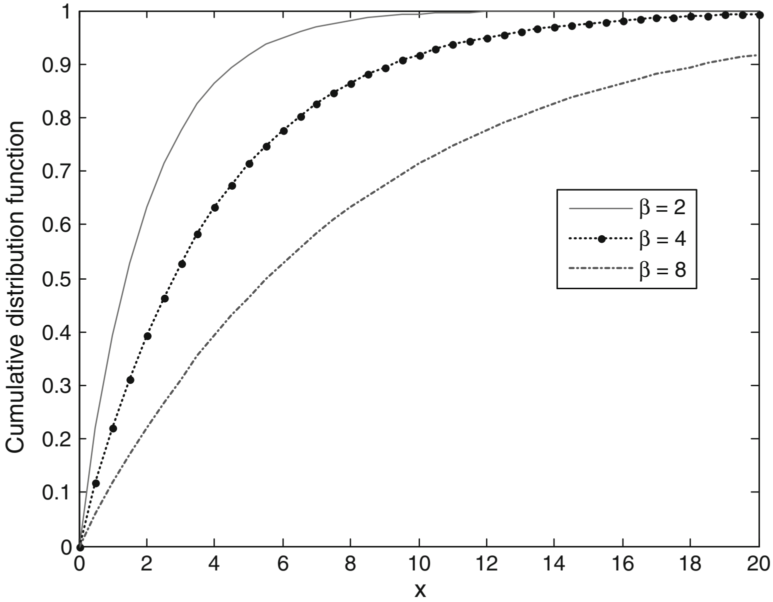

Fig. 2.6 (CDFs) of the exponential random variables are shown

2.4.7 F *(Fisher-Snedecor) Distribution*

The SNR in fading is often modeled using gamma distributions (Simon and Alouini 2005; Shankar 2004). The ratio of two such variables is of interest when examining the effects of interfering signals in wireless communications (Winters 1984). The *F* distribution arises when we examine the density function of the ratio of two chi-squared random variables of even degrees of freedom or two Erlang variables or two gamma variables of integer orders (Lee and Holland 1979; Nadarajah 2005; Nadarajah and Gupta 2005; Nadarajah and Kotz 2006a, b). The *F* density can be written as

$$f(x) = \frac{\Gamma((m+n)/2)m^{m/2}n^{n/2}}{\Gamma(m/2)\Gamma(n/2)} \frac{x^{m/2-1}}{(n+mx)^{[(m+n)/2]}} U(x). \tag{2.47}$$

In (2.47), m and n are integers. The mean is given by

$$\langle X \rangle = \frac{n}{n-2}, \quad n > 2. \tag{2.48}$$

The variance is

$$\operatorname{var}(x) = 2n^2 \frac{(n+m-2)}{m(n-4)(n-2)^2}, \quad n > 4. \tag{2.49}$$

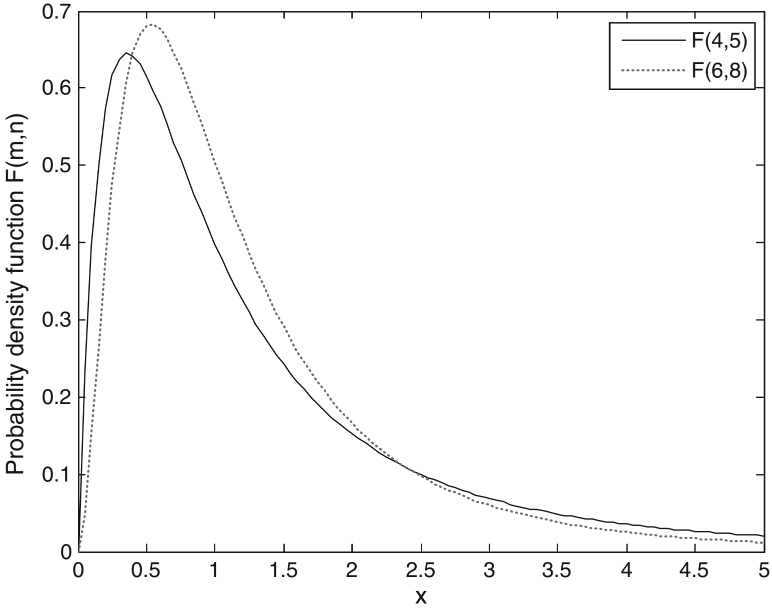

Fig. 2.7 The F distribution is plotted for two sets of values

The density function is identified as $F(m,n)$, with (m,n) degrees of freedom. The F distribution is shown in Fig. 2.7. The density function is sometimes referred to as Fisher's variance ratio distribution.

2.4.8 Gamma Distribution

The gamma distribution is used extensively in wireless communications to model the power in fading channels (Nakagami 1960; Abdi and Kaveh 1999; Atapattu et al. 2010). As mentioned earlier, the gamma pdf is a special case of the Erlang distribution in (2.41). The pdf is given by

$$f(x) = \frac{(x/\beta)^{c-1}\exp(-(x/\beta))}{\beta\Gamma(c)}, \quad 0 < x < \infty. \tag{2.50}$$

Comparing (2.41) and (2.50), we see that for the gamma pdf c can be any positive number while for the Erlang pdf, c must be an integer. The CDF can be expressed in terms of the incomplete gamma function as

$$F_X(x) = \frac{\gamma(c,(x/\beta))}{\Gamma(c)} = 1 - \frac{\Gamma(c,(x/\beta))}{\Gamma(c)}. \tag{2.51}$$

Table 2.1 Relationship of gamma pdf to other distributions

c	β	Name of the distribution	Probability density function
1	–	Exponential	$\frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right)$
Integer	–	Erlang	$\frac{(x/\beta)^{c-1}\exp(-(x/\beta))}{\beta(c-1)!}$
$c = 2n$	2	Chi-squared	$\frac{x^{(n-2)/2}\exp(-(x/2))}{2^{n/2}\Gamma(n/2)}$

The moments are given by

$$E(X^k) = b^k \frac{\Gamma(c+k)}{\Gamma(c)}. \tag{2.52}$$

The moments are identical to that of the Erlang distribution. The exponential distribution is a special case of the gamma pdf with $c = 1$. The received SNR or power is modeled using the gamma pdf. The CHF of the gamma distribution is

$$\psi(\omega) = (1 - j\beta\omega)^{-c}. \tag{2.53}$$

The gamma pdf in (2.50) becomes the chi-squared pdf in (2.35) when $n = 2c$ and $\beta = 2$. The relationship of the gamma pdf to other distributions is given in Table 2.1.

The gamma pdf is plotted in Fig. 2.8 for three values of c, all having identical mean of unity. One can see that as the order of the gamma pdf increases, the peaks of the densities move farther to the right. This aspect will be seen later in Chap. 5 when we examine the impact of diversity in fading channels . The corresponding CDFs are shown in Fig. 2.9.

2.4.9 Generalized Gamma Distribution

Instead of the two parameter distribution in (2.50), the three parameter gamma distribution known as the generalized gamma distribution can be expressed as (Stacy 1962; Stacy and Mihram 1965; Lienhard and Meyer 1967; Griffiths and McGeehan 1982; Coulson et al. 1998a, b; Gupta and Kundu 1999; Bithas et al. 2006)

$$f(x) = \frac{\lambda x^{\lambda c-1}}{\beta^{\lambda c}\Gamma(c)} \exp\left[-\left(\frac{x}{\beta}\right)^{\lambda}\right], \quad 0 < x < \infty. \tag{2.54}$$

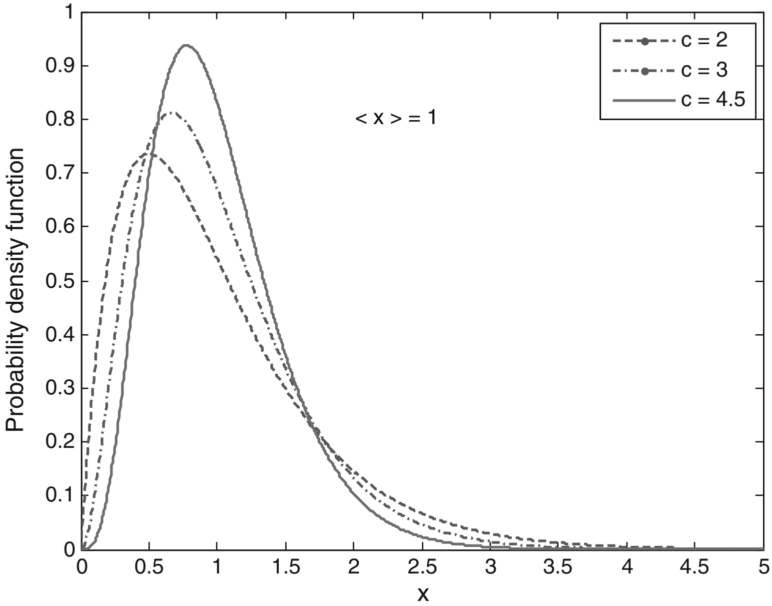

Fig. 2.8 The density functions of the gamma random variables for three values of the order. All have the identical means (unity)

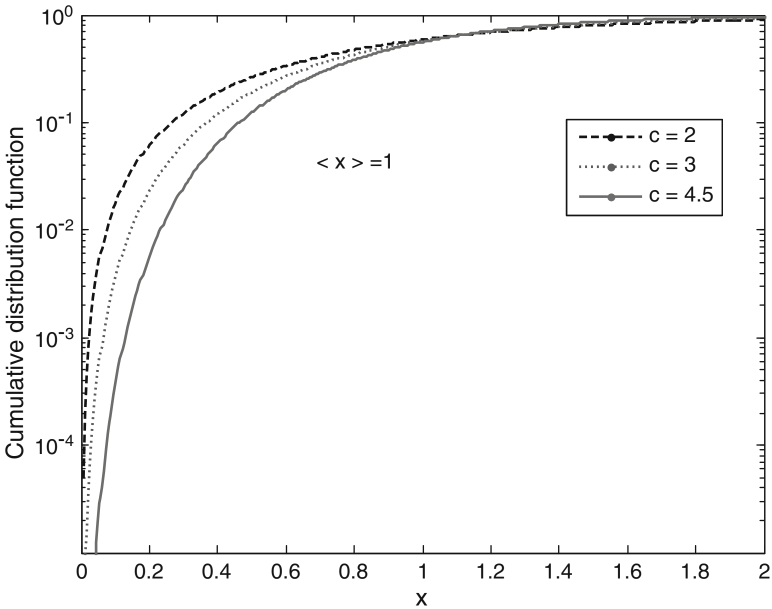

Fig. 2.9 The CDFs of the gamma random variables for three values of the order. All have the identical means (unity)

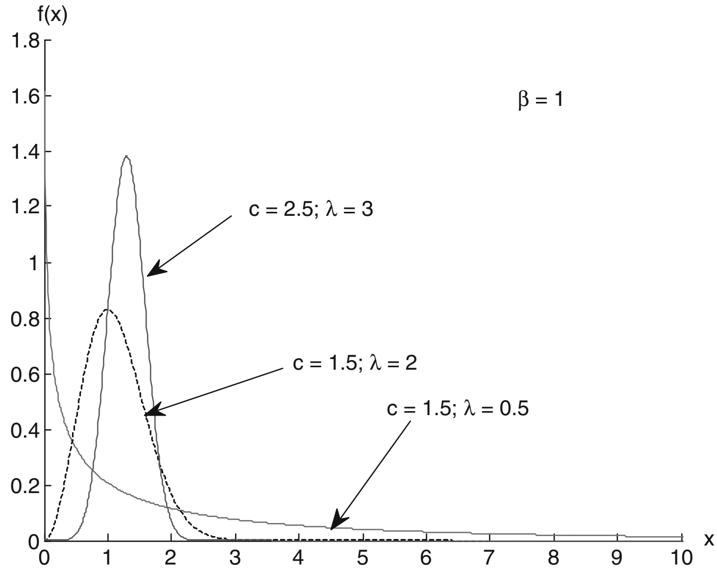

Fig. 2.10 The generalized gamma pdf in (2.56)

There is no simple expression for the CHF of the generalized gamma density function. The generalized gamma random variable can be obtained from a gamma random variable by scaling the random variable by $(1/\lambda)$. This distribution is also known as the Stacy distribution (Stacy 1962), which is expressed in a slightly different form as

$$f(x) = \left(\frac{p}{a^d}\right)x^{d-1}\frac{\exp\left(-\left(\frac{x}{a}\right)^p\right)}{\Gamma\left(\frac{d}{p}\right)}, \quad x \geq 0,\ a > 0,\ p > 0,\ d > 0. \tag{2.55}$$

Note that the generalized gamma pdf can also be expressed in yet another form as

$$f(x) = \frac{\lambda x^{\lambda c-1}}{\beta^c \Gamma(c)} \exp\left[-\frac{x^\lambda}{\beta}\right], \quad 0 < x < \infty. \tag{2.56}$$

The GG pdf in (2.56) is plotted in Fig. 2.10 for the case of $\beta = 1$. It can be seen that as λ increases the density function moves to the right, indicating that the density function will approach Gaussian as is possible with the case of Erlang pdf. The same effect will be present when c increases as well.

The generalized gamma (GG) distribution can be used to model power (or SNR) as well as the magnitude (or envelope) values in wireless communications (Coulson et al. 1998a, b). This can be accomplished by varying c and λ. Also, the GG distribution can morph into other distributions in limiting cases (Griffiths and

Table 2.2 The generalized gamma distribution and its special cases. In all the expressions in the last column, A is a normalization factor

m	n	b	Name of the distribution	Probability density function
1	–	–	Gamma	$Ax^n \exp\left(-\frac{x}{b}\right)$
1	>0 (integer)	–	Erlang	$Ax^n \exp\left(-\frac{x}{b}\right)$
1	>1 (integer)	2	Chi-squared	$Ax^n \exp\left(-\frac{x}{b}\right)$
2	–	–	Nakagami	$Ax^n \exp\left(-\frac{x^2}{b^2}\right)$
–	–	–	Stacy	$Ax^n \exp\left(-\frac{x^m}{b^m}\right)$
–	$n = m - 1$	–	Weibull	$Ax^{m-1} \exp\left(-\frac{x^m}{b^m}\right)$
1	0	–	Exponential	$A \exp\left(-\frac{x}{b}\right)$
2	0	–	One sided Gaussian	$A \exp\left(-\frac{x^2}{b^2}\right)$
	0	–	Generalized exponential	$A \exp\left(-\frac{x^m}{b^m}\right)$
2	1	–	Rayleigh	$Ax \exp\left(-\frac{x^2}{b^2}\right)$
–	1	–	Generalized Rayleigh	$Ax \exp\left(-\frac{x^m}{b^m}\right)$

McGeehan 1982). The CDF can once again be represented in terms of the incomplete gamma functions in (2.37) and (2.52). Using the representation in (2.56), the expression for the CDF becomes

$$F_X(x) = 1 - \frac{\Gamma(c, (x^\lambda/\beta))}{\Gamma(c)}. \tag{2.57}$$

If we express the generalized gamma pdf in (2.54) as

$$f(x) = Ax^n \exp\left(-\frac{x^m}{b^m}\right), \quad 0 \le x \le \infty, \tag{2.58}$$

where b is a scaling factor and A is a normalization factor such that

$$\int_0^\infty f(x)\,\mathrm{d}x = 1, \tag{2.59}$$

we can relate the pdf in (2.58) to several density functions as described in Table 2.2 (Griffiths and McGeehan 1982).

There are also other forms of gamma and generalized gamma distribution. Even though they are not generally used in wireless communications, we will still provide them so as to complete the information on the class of gamma densities. One such gamma pdf is (Evans et al. 2000)

$$f(x) = \frac{(x-\gamma)^{c-1}\exp(-(x-\gamma)/\beta)}{\beta^c\Gamma(c)}, \quad x>\gamma>0,\ c>0,\ \beta>0. \tag{2.60}$$

Note that when $\gamma = 0$, (2.60) becomes the standard two parameter gamma density defined in (2.50), another form of generalized gamma density function has four parameters and it is of the form

$$f(x) = \frac{\lambda(x-\gamma)^{\lambda c-1}}{\beta^{\lambda c}\Gamma(c)}\exp\left[-\left(\frac{x-\gamma}{\beta}\right)^{\lambda}\right], \quad x>\gamma>0,\ \lambda>0,\ c>0,\ \beta>0. \tag{2.61}$$

Note that (2.61) becomes the generalized gamma distribution (Stacy's pdf) in (2.54) when $\gamma = 0$.

While the gamma pdf and the generalized gamma densities mentioned so far are defined for positive values (single-sided), there also exists a double-sided generalized gamma density function which has been used to model noise in certain cases (Shin et al. 2005). The two-sided generalized gamma pdf is

$$f(x) = \frac{\lambda|x|^{c\lambda-1}}{2\beta^c\Gamma(c)}\exp\left(-\frac{|x|^{\lambda}}{\beta}\right), \quad -\infty<x<\infty. \tag{2.62}$$

The two-sided generalized gamma pdf is flexible enough that it can become Gaussian, Laplace, generalized gamma or gamma.

2.4.10 Inverse Gaussian (Wald) Distribution

This pdf is sometimes used to model shadowing in wireless systems because of the closeness of its shape to the lognormal density function (Karmeshu and Agrawal 2007; Laourine et al. 2009). The pdf is expressed as

$$f(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2}\exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right], \quad 0<x<\infty. \tag{2.63}$$

Note that both λ and μ are positive numbers. There is no simple analytical expression for the CDF. The CHF is given by

$$\psi(\omega) = \exp\left\{\frac{\lambda}{\mu}\left[1-\left(1-\frac{2j\mu^2\omega}{\lambda}\right)^{1/2}\right]\right\}. \tag{2.64}$$

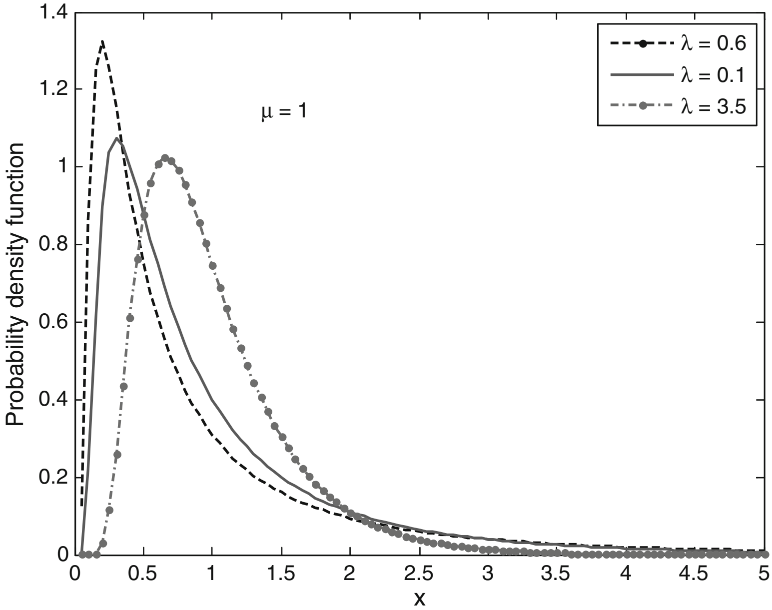

Fig. 2.11 Inverse Gaussian distribution

The mean is given by μ and the variance is given by μ^3/λ. The density function is shown in Fig. 2.11 for a few values of the parameters.

2.4.11 Laplace Distribution

This distribution is generally not used in wireless communication systems even though research exists into its use in communication systems (Sijing and Beaulieu 2010). It is related to the exponential distribution in that it extends the range of the variable down to $-\infty$. The pdf is expressed as

$$f(x) = \frac{1}{2\beta}\exp\left(-\frac{|x-\alpha|}{\beta}\right), \quad -\infty < x < \infty. \tag{2.65}$$

The associated CDF is given by

$$F(x) = \begin{cases} \frac{1}{2}\exp\left(-\frac{\alpha - x}{\beta}\right), & x < \alpha, \\ 1 - \frac{1}{2}\exp\left(-\frac{x-\alpha}{\beta}\right), & x \geq \alpha. \end{cases} \tag{2.66}$$

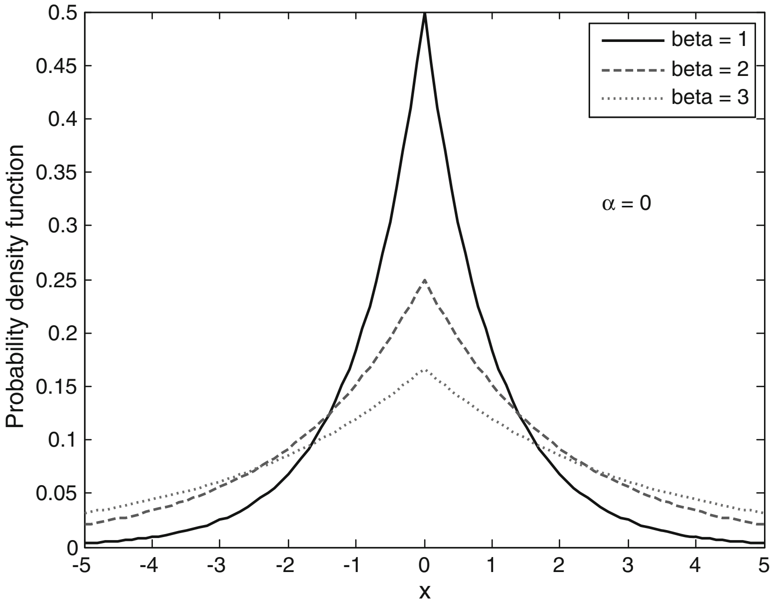

Fig. 2.12 The Laplace density functions

The CHF is given by

$$\psi(\omega) = \frac{\exp(j\omega\alpha)}{1 + \beta^2\omega^2}. \tag{2.67}$$

The mean, mode, and median are all equal to α, and the variance is equal to $2\beta^2$. The Laplace density function is shown in Fig. 2.12.

2.4.12 Lognormal Distribution

The lognormal pdf arises when central limit theorem for products of random variables is applied (Papoulis and Pillai 2002). It is used to model long-term fading or shadowing seen in wireless systems (Hudson 1996; Tjhung and Chai 1999; Coulson et al. 1998a, b; Patzold 2002; Kostic 2005; Stuber 2000; Cotton and Scanlon 2007). In certain cases, it finds applications in modeling short-term fading as well (Cotton and Scanlon 2007). The pdf of the lognormal random variable is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2 x}} \exp\left[-\frac{(\log_e(x) - \mu)^2}{2\sigma^2}\right], \quad 0 < x < \infty. \tag{2.68}$$

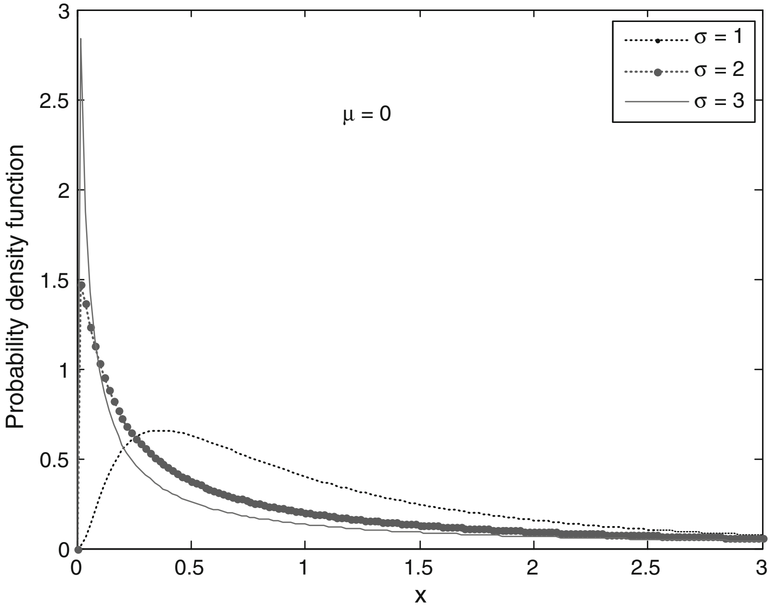

Fig. 2.13 Lognormal density function in (2.68) is shown

The CDF and CHF are not readily available in analytical form. The mean and variance can be expressed as

$$E(X) = \exp\left(\mu + \frac{1}{2}\sigma^2\right), \tag{2.69}$$

$$\mathrm{var}(X) = [\exp(\sigma^2 - 1)]\exp(2\mu + \sigma^2). \tag{2.70}$$

Equation (2.68) can also be expressed in a slightly different form using the logarithm with the base 10 as

$$f(x) = \frac{[10/\log_e(10)]}{\sqrt{2\pi\sigma^2 x^2}}\exp\left[-\frac{(10\log_{10}(x) - \mu)^2}{2\sigma^2}\right], \quad 0 < x < \infty. \tag{2.71}$$

In (2.71), μ and σ are in decibel units. Note that if the lognormal variable is converted to decibel units, the pdf of the variable in dB will be Gaussian. This density function (in terms of decibel units) is discussed in detail in Chap. 4.

Figure 2.13 shows the plots of the lognormal pdf in (2.68).

2.4.13 Nakagami Distribution

Even though there are several forms of the Nakagami distribution, the most commonly known one is the "Nakagami-*m* distribution" with a pdf given by (Nakagami 1960; Simon and Alouini 2005)

$$f(x) = 2\left(\frac{m}{\Omega}\right)^m \frac{x^{2m-1}}{\Gamma(m)} \exp\left(-m\frac{x^2}{\Omega}\right) U(x), \quad m \geq \frac{1}{2}. \tag{2.72}$$

In (2.72), m is the Nakagami parameter, limited to values greater than or equal to ½. The moments of the Nakagami distribution can be expressed as

$$\langle X^k \rangle = \frac{\Gamma(m + (k/2))}{\Gamma(m)} \left(\frac{\Omega}{m}\right)^{1/2}. \tag{2.73}$$

The mean is

$$E(X) = \frac{\Gamma\left(m + \dfrac{1}{2}\right)}{\Gamma(m)} \sqrt{\frac{\Omega}{m}}. \tag{2.74}$$

The variance is

$$\mathrm{var}(X) = \Omega\left[1 - \frac{1}{m}\left(\frac{\Gamma(m + (1/2))}{\Gamma(m)}\right)^2\right]. \tag{2.75}$$

Note that the Nakagami pdf becomes the Rayleigh density function when m is equal to unity. The square of the Nakagami random variable will have a gamma pdf. Under certain limiting conditions, the Nakagami density function can approximate the lognormal distribution (Nakagami 1960; Coulson et al. 1998a, b). The Nakagami pdf is plotted in Fig. 2.14.

The CDF associated with the Nakagami-*m* pdf can be expressed in terms of incomplete gamma functions. The Nakagami-*m* CDF is

$$F_X(x) = 1 - \frac{\Gamma(m, (m/\Omega)x^2)}{\Gamma(m)}. \tag{2.76}$$

There is no simple analytical expression for the CHF. The Nakagami pdf is used to model the magnitude or envelope of the signals in communication systems. The density function of the square of the Nakagami random variable,

$$Y = X^2 \tag{2.77}$$

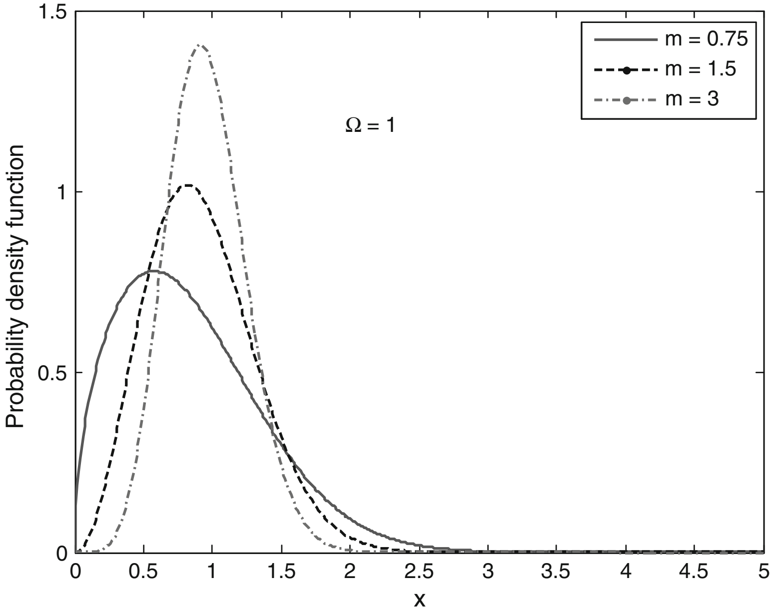

Fig. 2.14 The Nakagami density

with X having the Nakagami pdf in (2.72) is

$$f_Y(y) = \left(\frac{m}{\Omega}\right)^m \frac{y^{m-1}}{\Gamma(m)} \exp\left(-\frac{m}{\Omega}y\right), \quad y \geq 0,\ m \geq \frac{1}{2}. \tag{2.78}$$

It can be seen that there is no difference between the pdf of the power (square of the magnitude) expressed in (2.78) and the gamma pdf in (2.50) except that the order of the gamma pdf m must be larger than ½ for it be associated with the Nakagami pdf for the magnitude.

The CDF associated with the Nakagami-m distribution of power is

$$F_Y(y) = 1 - \frac{\Gamma(m, (m/\Omega)y)}{\Gamma(m)}. \tag{2.79}$$

In (2.79),

$$\Gamma\left(m, \frac{m}{\Omega}y\right) = \int_{(m/\Omega)y}^{\infty} x^{m-1} \exp(-x)\, \mathrm{d}x \tag{2.80}$$

is the incomplete gamma function (Gradshteyn and Ryzhik 2007). The other forms of the Nakagami distribution such as the Nakagami-Hoyt and Nakagami-Rice will be discussed in Chap. 4 where specific statistical models for fading will be presented (Okui 1981; Korn and Foneska 2001; Subadar and Sahu 2009).

It is also possible to define a generalized form of the Nakagami-m pdf called the generalized Nakagami density function. This density function is obtained by the exponential scaling of the Nakagami variable X as (Coulson et al. 1998a, b; Shankar 2002a, b)

$$Y = X^{(1/\lambda)}, \quad \lambda > 0. \tag{2.81}$$

The density function of Y is the generalized density function given as

$$f(y) = \frac{2\lambda m^m y^{2m\lambda-1}}{\Gamma(m)\Omega^m} \exp\left(-\frac{m}{\Omega} y^{2\lambda}\right) U(y). \tag{2.82}$$

The generalized Nakagami pdf in (2.82) becomes the Rayleigh density function in (2.94) when m and λ are each equal to unity. It can be easily observed that the generalized gamma random variable is obtained by squaring the generalized Nakagami random variable.

2.4.14 Non-Central Chi-Squared Distribution

While we looked at chi-squared distribution which arises from the sum of the squares of zero mean identical random variables, the non-central chi-squared distribution arises when the Gaussian variables have non-zero means. The density function can be expressed in several forms as (Evans et al. 2000; Papoulis and Pillai 2002; Wolfram 2011)

$$f(x) = \frac{\sqrt{\lambda}}{2(\lambda x)^{r/4}} \exp\left(-\frac{x+\lambda}{2}\right) x^{(r-1)/2} I_{(r/2)-1}\left(\sqrt{\lambda x}\right), \quad r > 0, \tag{2.83}$$

$$f(x) = \frac{1}{2^{(r/2)}} \exp\left(-\frac{x+\lambda}{2}\right) x^{(r-1)/2} \sum_{k=0}^{\infty} \frac{(\lambda x)^k}{2^{2k} k! \Gamma(k + (r/2))}, \quad r > 0, \tag{2.84}$$

$$f(x) = 2^{-(r/2)} \exp\left(-\frac{x+\lambda}{2}\right) x^{(r/2)-1} {}_0F_1\left([\,], \left[\frac{r}{2}\right], \frac{\lambda x}{4}\right), \quad r > 0. \tag{2.85}$$

When λ is zero, the non-central chi-squared distribution becomes the chi squared distribution. In (2.83), $I(.)$ is the modified Bessel function of the first kind and in (2.85) $F(.)$ is the hypergeometric function (Gradshteyn and Ryzhik 2007). The mean and variance of the pdf in (2.83) are

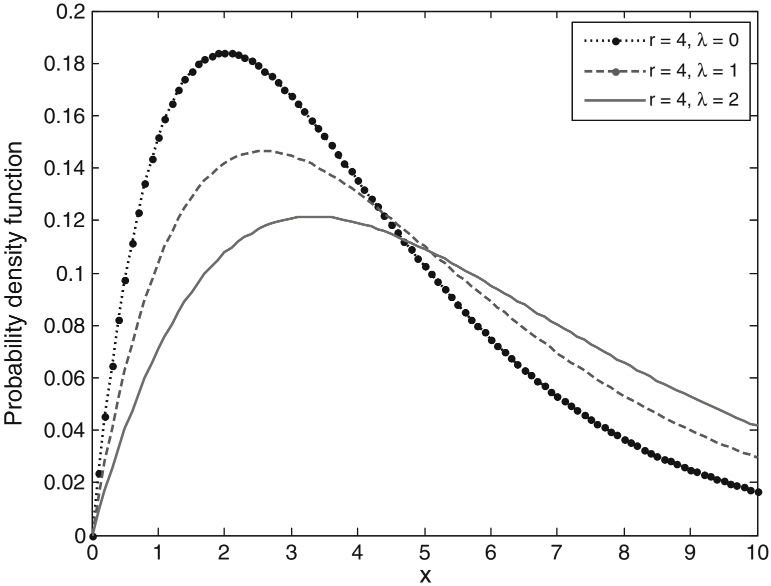

Fig. 2.15 Non-central chi-squared density function

$$\langle X \rangle = \lambda + r, \tag{2.86}$$

$$\operatorname{var}(x) = 2(2\lambda + r). \tag{2.87}$$

This distribution in its limiting form becomes the Rician distribution (Rice 1974; Papoulis and Pillai 2002), which is discussed later. This density function also occurs in clustering-based modeling of short-term fading in wireless systems. The CDF can be expressed in terms of Marcum Q functions (Helstrom 1968; Nuttall 1975; Helstrom 1992, 1998; Chiani 1999; Simon 2002; Gaur and Annamalai 2003; Simon and Alouini 2003).

The non-central chi-squared distribution is plotted in Fig. 2.15 for a few values of r and λ.

2.4.15 Normal (Gaussian) Distribution

The normal pdf is expressed as

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad -\infty < x < \infty. \tag{2.88}$$

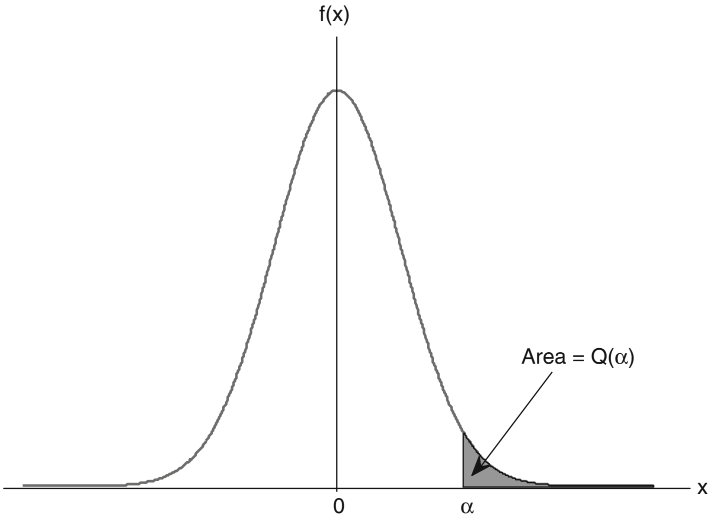

Fig. 2.16 The CDF of the Gaussian variable

The CDF can be expressed in terms of error functions or Q functions (Haykin 2001; Proakis 2001; Sklar 2001; Papoulis and Pillai 2002; Simon and Alouini 2005). The CDF is

$$F(x) = 1 - Q\left(\frac{x-\mu}{\sigma}\right). \tag{2.89}$$

In (2.89), the Q function is given by

$$Q(\alpha) = \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) \mathrm{d}w. \tag{2.90}$$

The shaded area in Fig. 2.16 represents the value of the $Q(.)$. The function and its properties are discussed in detail in Chap. 3.

The CHF of the normal random variable is

$$\psi(\omega) = \exp\left(j\mu\omega - \frac{1}{2}\sigma^2\omega^2\right). \tag{2.91}$$

The mean is μ and the standard deviation is σ. The mode and median are also equal to μ. Note that the normal pdf is used in the modeling of white noise (zero mean) in communication systems. The pdf of the sum of the squares of two independent identically distributed (zero mean) normal random variables leads to an exponentially distributed random variable with a pdf in (2.44). The sum of the squares of

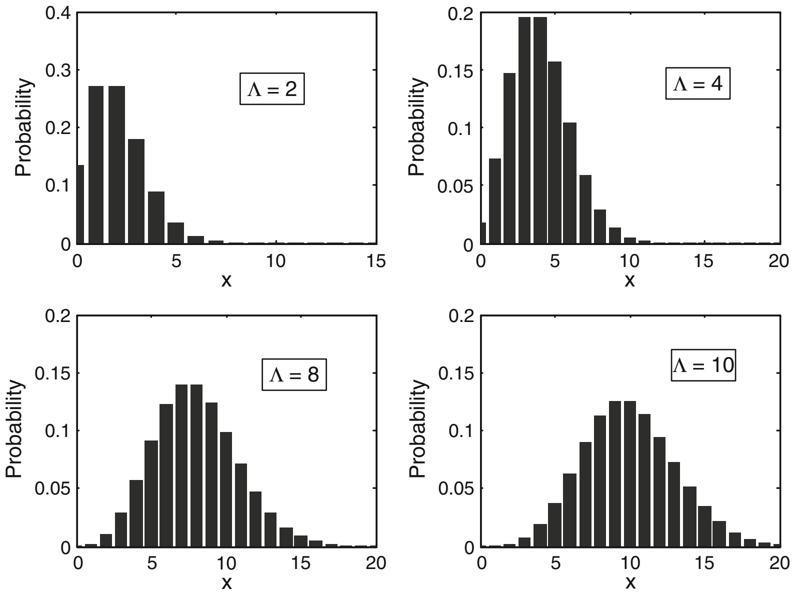

Fig. 2.17 Poisson probabilities

several Gaussian random variables (zero mean) also leads to a chi-squared distribution.

The normal distribution in (2.88) is identified in literature as $N(\mu,\sigma)$.

2.4.16 Poisson Distribution

The Poisson distribution is of the discrete type and is commonly used in communications to model the frequency of telephone calls being made (Stuber 2000; Papoulis and Pillai 2002; Molisch 2005; Gallager 2008). Since the random variable is of the discrete type, we need the probability that the number of outcomes equals a specific non-negative integer. This can be expressed as

$$\mathrm{Prob}\{X = k\} = \frac{\Lambda^k}{k!}\exp(-\Lambda), \quad k = 0, 1, 2, \ldots. \tag{2.92}$$

In (2.92), Λ is the average of the Poisson variable. It is also equal to the variance. When Λ increases, the density function approaches the normal or Gaussian pdf (Papoulis and Pillai 2002). This is illustrated in Fig. 2.17.

2.4.17 Rayleigh Distribution

The short-term fading observed in wireless channels is modeled by treating the magnitude of the signal as having the Rayleigh pdf (Jakes 1994; Schwartz et al. 1996; Sklar 1997a, b; Steele and Hanzó 1999; Patzold 2002; Rappaport 2002; Shankar 2002a, b). The density function results from the square root of the sum of two independent and identically distributed zero mean Gaussian random variables. In other words, if X_1 and X_2 are independent and identically distributed zero mean Gaussian variables, the Rayleigh variable X will be

$$X = \sqrt{X_1^2 + X_2^2}. \tag{2.93}$$

The Rayleigh pdf can be expressed as

$$f(x) = \frac{x}{\beta}\exp\left(-\frac{x^2}{2\beta}\right), \quad 0 < x < \infty. \tag{2.94}$$

The CDF can be expressed as

$$F(x) = 1 - \exp\left(-\frac{x^2}{2\beta}\right). \tag{2.95}$$

There is no simple expression for the CHF. The mean and variance can be expressed as

$$E(X) = \sqrt{\frac{\beta\pi}{2}}, \tag{2.96}$$

$$\mathrm{var}(X) = \left(2 - \frac{\pi}{2}\right)\beta. \tag{2.97}$$

Note that the Rayleigh random variable and exponential random variable are related, with the square of a Rayleigh random variable having an exponential distribution. The Rayleigh density function is also the special case of the Nakagami pdf in (2.72) when the Nakagami parameter $m = 1$. The Rayleigh pdf is shown in Fig. 2.18. It must be noted that the Rayleigh pdf is characterized by the fact that the ratio of its mean to its standard deviation is fixed at 1.91 and has no dependence on the parameter β. Thus, regardless of the fact that the peak of the density function moves to the right as β increases, such a shift would have no impact on the level of fading in Rayleigh fading channels (as we will see in Chap. 4).

There is also another density function closely related to the Rayleigh pdf. This is the so called Generalized Rayleigh distribution with a density function (Blumenson and Miller 1963; Kundu and Raqab 2005; V.Gh. Voda 2009) of

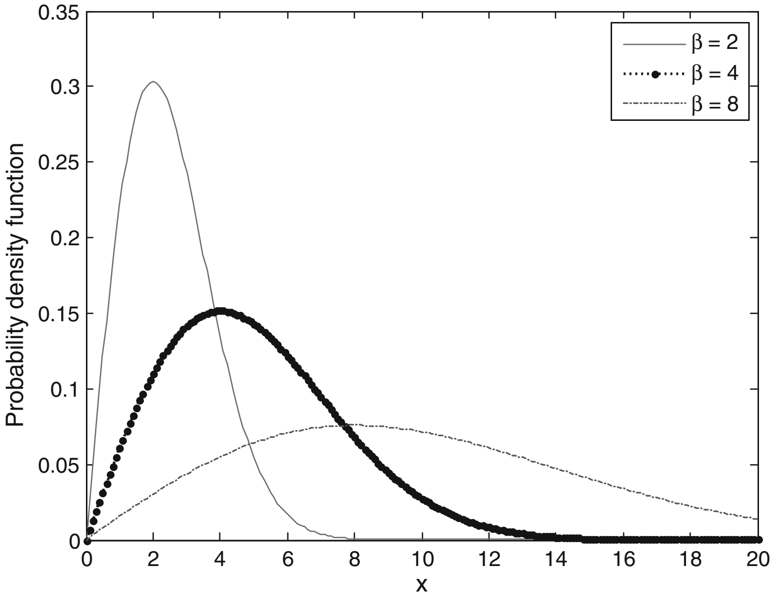

Fig. 2.18 Plot of Rayleigh pdf for three values of β

$$f(x) = 2\alpha\beta x \exp(-\beta x^2)[1 - \exp(-\beta x^2)]^{\alpha-1}, \quad 0 < x < \infty. \tag{2.98}$$

Note that (2.98) becomes the conventional Rayleigh density function when $\alpha = 1$. Another form of generalized Rayleigh is identified by the pdf

$$f(x) = \frac{m}{b^2\Gamma(2/m)} x \, \exp\left(-\frac{x^m}{b^m}\right). \tag{2.99}$$

Equation (2.99) is a special case of the generalized Gamma distribution in (2.54). It becomes the simple Rayleigh density when $m = 2$. There is no analytical expression for the CDF associated with the density function in (2.99). The density function in (2.99) is shown in Fig. 2.19.

There is another form of generalized Rayleigh distribution (Blumenson and Miller 1963; Kundu and Raqab 2005; V.Gh. Voda 2009) as follows:

$$f(x) = \frac{2\alpha^{k+1}}{\Gamma(k+1)} x^{2k+1} \exp(-\alpha x^2), \quad x \geq 0, \ \alpha > 0, \ k \geq 0. \tag{2.100}$$

Equation (2.100) becomes the Rayleigh density function when $k = 0$. Once again, there is no simple analytical expression for the CDF associated with the pdf in (2.100). It must also be noted that the generalized Rayleigh distribution in (2.100) is a more general form of the chi distribution in (2.39). Other aspects of the Rayleigh density function are discussed in Chap. 4.

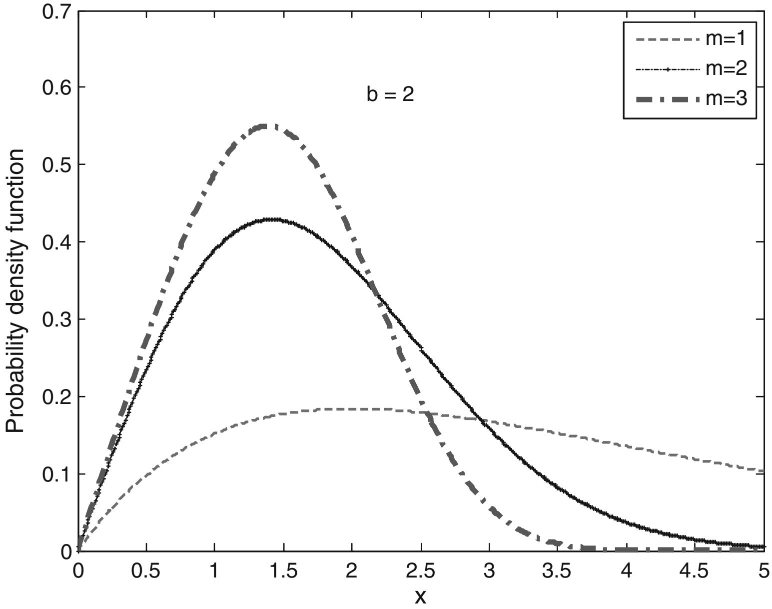

Fig. 2.19 The plot of the generalized Rayleigh pdf in (2.99) for three values of m and $b = 2$. The case of $m = 2$ corresponds to the Rayleigh pdf in (2.94)

2.4.18 Rectangular or Uniform Distribution

Uniform distribution is widely used in communication systems to model the statistics of phase (Stuber 2000; Papoulis and Pillai 2002; Shankar 2002a, b; Vaughn and Anderson 2003). If there are two independent normal random variables (X and Y) with zero means and identical variances, the pdf of the random variable

$$Z = \tan^{-}\left(\frac{Y}{X}\right) \tag{2.101}$$

will have a uniform distribution. The density function $f(z)$ can be expressed as

$$f(z) = \frac{1}{\beta - \alpha}, \quad \alpha < z < \beta. \tag{2.102}$$

The CDF is given by

$$F(z) = \frac{z - \alpha}{\beta - \alpha}. \tag{2.103}$$

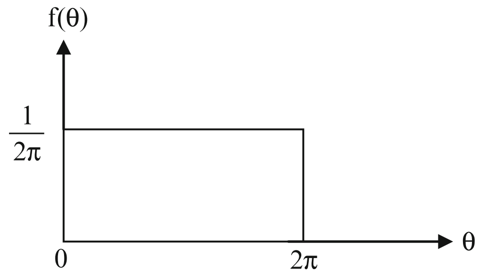

Fig. 2.20 Rectangular or uniform pdf

The CHF is given by

$$\psi(\omega) = \frac{[\exp(j\beta\omega) - \exp(j\alpha\omega)]}{j\omega(\beta - \alpha)}. \tag{2.104}$$

The mean and variance are given by

$$E(Z) = \frac{\alpha + \beta}{2}, \tag{2.105}$$

$$\mathrm{var}(Z) = \frac{(\beta - \alpha)^2}{12}. \tag{2.106}$$

In multipath fading, the phase is uniformly distributed in the range [0,2π]. The phase statistics can be displayed in two different ways. One is the conventional way of sketching the pdf as shown in Fig. 2.20. The other one is using polar plots, in which case uniformly distributed phase will appear as a circle. This is illustrated in Fig. 2.21 which was obtained by taking the histogram of several random numbers with the uniform pdf in the range [0,2π]. The histogram is seen as a circle.

The latter representation using the polar plot is convenient in understanding and interpreting the fading seen in Rician fading channels, as we will see in Chap. 4.

2.4.19 Student's t *Distribution*

The student *t* distribution in its shape looks very similar to the Gaussian and Cauchy distributions, and in the limiting cases it approaches either the Gaussian or the Cauchy distribution. Even though this density function is not directly used in wireless communications it can arise when we examine the ratio of a normal random variable to a normalized chi variable. This will be shown later in this chapter when we explore the random variables generated by mixing two or more random variables. This density function is also extensively used for testing to validate the statistical fits to densities.

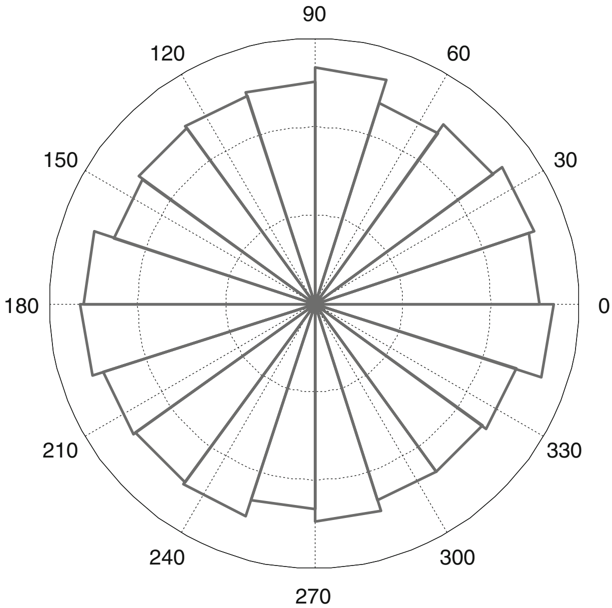

Fig. 2.21 Polar representation of the uniform random variable $[0, 2\pi]$

A random variable z has a Student t distribution with n degrees of freedom if the pdf of Z is

$$f_Z(z) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)} \frac{1}{\sqrt{(1+(z^2/n))^{n+1}}}, \quad -\infty < x < \infty. \tag{2.107}$$

When $n = 1$, the Student t pdf in (2.107) becomes the Cauchy pdf in (2.32). The Student t density is often identified as $t(n)$.

As n becomes large, the density function in (2.107) approaches a Gaussian pdf since

$$\left(1+\frac{z^2}{n}\right)^{-((n+1)/2)} \to \exp\left(-\frac{z^2}{2}\right) \quad \text{as } n \to \infty. \tag{2.108}$$

A simple expression for the CDF is not readily available. The moments of the Student t distribution become

$$\langle Z^k \rangle = \begin{cases} 0, \ k \text{ odd}, \\ \dfrac{1.3.5...(k-1)n^{k/2}}{(n-2)(n-4)\ldots(n-k)}, \quad k \text{ even}, \\ k, \ k \text{ even}, \ n > k. \end{cases} \tag{2.109}$$

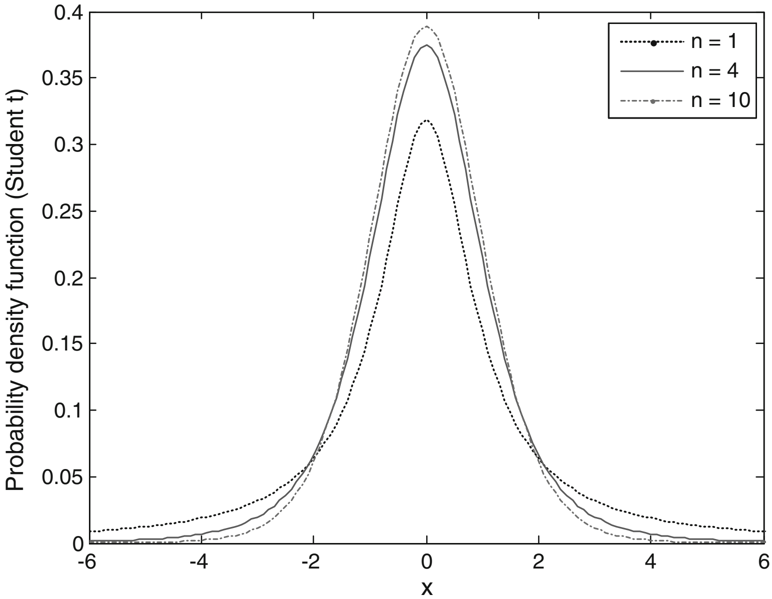

Fig. 2.22 Student *t* distribution

Thus, this random variable has a mean of zero and variance of $n/(n-2), n > 2$. The student *t* distribution is shown in Fig. 2.22 for three values of *n*. The heavier "tails" of the pdf are clearly seen at the lower values of *n*, indicating that the Student *t* distribution belongs to a class of density functions with "heavy tails" or heavy tailed distributions (Bryson 1974).

The transition to the Gaussian distribution is demonstrated in Fig. 2.23 where the CDFs of the Student *t* variable and the normal variable with identical variance are plotted. As *n* goes from 4 to 10, the CDFs almost match, indicating that with increasing values of *n*, the Student *t* distribution approaches the Gaussian distribution.

2.4.20 Weibull Distribution

In wireless communication systems, the Weibull pdf is also used to model the SNR in short-term fading (Shepherd 1977; Cheng et al. 2004; Sahu and Chaturvedi 2005; Alouini and Simon 2006; Ismail and Matalgah 2006). The Weibull density function can be expressed as

$$f(x) = \frac{\eta x^{\eta-1}}{\beta^{\eta}} \exp\left[-\left(\frac{x}{\beta}\right)^{\eta}\right], \quad 0 < x < \infty. \tag{2.110}$$

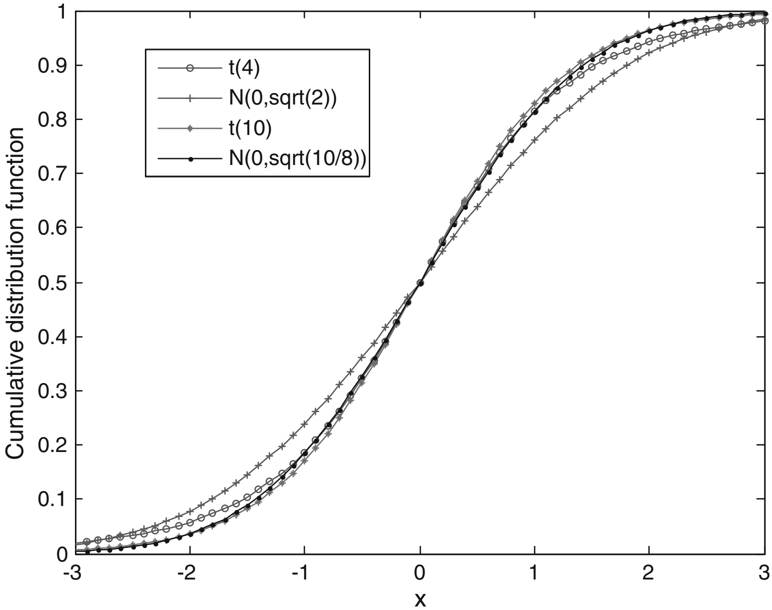

Fig. 2.23 The cumulative distributions functions of the Student *t* distribution and the corresponding Gaussian CDF with identical variances are shown

The Weibull pdf can also be written in a much simpler form as

$$f(x) = \alpha x^{\alpha-1}\exp(-x^{\alpha}), \quad x \geq 0,\ \alpha > 0. \tag{2.111}$$

The pdf in (2.111) is shown in Fig. 2.24. The CDF associated with the pdf in (2.110) is given by

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^{\eta}\right]. \tag{2.112}$$

There is no simple analytical expression for the CHF. The mean and variance can be expressed as

$$E(X) = \beta\Gamma\left(1 + \frac{1}{\eta}\right), \tag{2.113}$$

$$\mathrm{var}(X) = \beta^2\left[\Gamma\left(1 + \frac{2}{\eta}\right) - \Gamma^2\left(1 + \frac{1}{\eta}\right)\right]. \tag{2.114}$$

Note that the Weibull and generalized gamma (GG) random variables are related. If we put $c = 1$ and $\eta = \lambda$ in (2.56), the GG variable becomes a Weibull random variable.

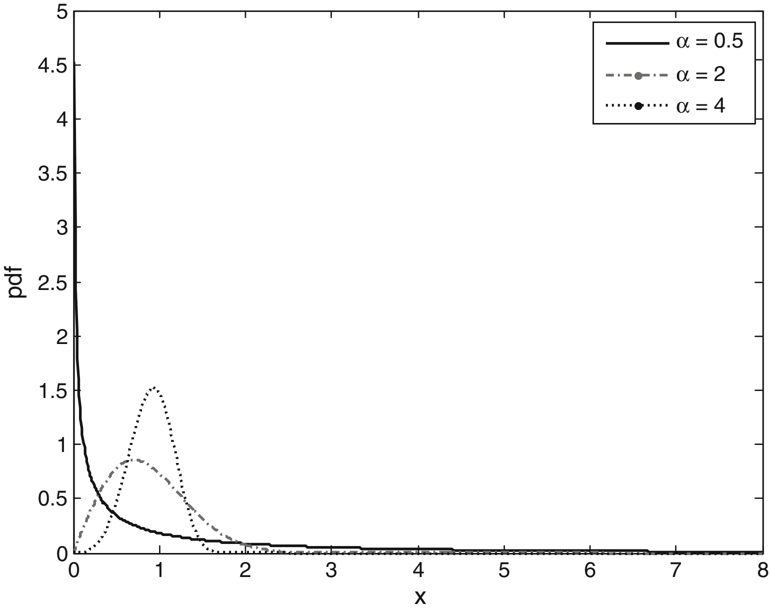

Fig. 2.24 The Weibull densities

2.5 Joint, Marginal and Conditional Densities

If $X_1,\ldots, X_k$ are a set of random variables, the joint CDF is defined as (Rohatgi and Saleh 2001; Papoulis and Pillai 2002)

$$F(x_1, \ldots, x_k) = P(X_1 \le x_1, \ldots, X_k \le x_k). \tag{2.115}$$

The joint pdf is obtained by differentiating (2.115) with respect to $x_1,\ldots, x_k$,. We have

$$f(x_1, \ldots, x_k) = \frac{\partial^k}{\partial x_1 \ldots \partial x_k}[F(x_1, \ldots, x_k)]. \tag{2.116}$$

Marginal densities can be obtained from the joint density. The marginal density of x_1 is given as

$$f(x_1) = \int_{x_2=-\infty}^{\infty}\int_{x_3=-\infty}^{\infty}\cdots\int_{x_k=-\infty}^{\infty} f(x_1, \ldots, x_k)\,\mathrm{d}x_2\,\mathrm{d}x_3 \ldots \mathrm{d}x_k. \tag{2.117}$$

Similarly,

$$f(x_1, x_2) = \int_{x_3=-\infty}^{\infty} \int_{x_4=-\infty}^{\infty} \cdots \int_{x_k=-\infty}^{\infty} f(x_1, \ldots, x_k)\ \mathrm{d}x_3 \ldots \mathrm{d}x_k. \tag{2.118}$$

If the random variables are independent, we have

$$f(x_1, \ldots, x_k) = f(x_1)f(x_2) \ldots f(x_k), \tag{2.119}$$

$$F(x_1, \ldots, x_k) = F(x_1)F(x_2) \ldots F(x_k). \tag{2.120}$$

We can also define conditional densities. If we have a joint pdf of two random variables $f(x,y)$, the conditional density function of Y, conditioned on $X = x$ is defined as

$$f(y|X = x) = f(y|x) = \frac{f(x, y)}{f(x)}. \tag{2.121}$$

It is obvious from (2.121), that if X and Y are independent, the conditional density of Y is unaffected by the presence of X. It is given by $f(y)$ itself. The conditioning expressed in (2.121) can also be extended to multiple random variables. On the other hand, if X and Y are dependent, the marginal density function of Y can be obtained as

$$f(y) = \int_{-\infty}^{\infty} f(y|x)f(x)\ \mathrm{d}x = \int_{-\infty}^{\infty} f(x, y)\ \mathrm{d}x. \tag{2.122}$$

Equation (2.122) can be interpreted as the Bayes Theorem for continuous random variables (Papoulis and Pillai 2002). Extending eqn. (2.121) to multiple random variables, we can express

$$f(x_n, \ldots, x_{k+1}|x_k, \ldots, x_1) = \frac{f(x_1, x_2, \ldots, x_k, \ldots, x_n)}{f(x_1, \ldots, x_k)}. \tag{2.123}$$

2.6 Expectation, Covariance, Correlation, Independence, and Orthogonality

The expected value of a function of random variables, $g(x_1, \ldots, x_k)$ is defined as (Papoulis and Pillai 2002)

$$\begin{aligned} E[g(x_1, \ldots, x_k)] &= \langle g(x_1, \ldots, x_k) \rangle \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_k)f(x_1, \ldots, x_k)\ \mathrm{d}x_1 \ldots \mathrm{d}x_k. \end{aligned} \tag{2.124}$$

Let us now look at the case of two random variables, X and Y. The joint expected value of the product of the two random variables,

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.125}$$

The *covariance* C_{xy} of two random variables is defined as

$$C_{xy} = E[(X-\eta_x)(Y-\eta_y)]. \tag{2.126}$$

In (2.126),

$$\eta_x = \int\int xf(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{2.127}$$

and

$$\eta_y = \int\int yf(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.128}$$

Expanding (2.126), we have

$$C_{xy} = E(XY) - \eta_x\eta_y. \tag{2.129}$$

Note that if X and Y are independent random variables, using (2.119), we have

$$C_{xy} = 0. \tag{2.130}$$

The *correlation coefficient* of two random variables, ρ_{xy} is defined as

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x\sigma_y}. \tag{2.131}$$

It can be shown that

$$|\rho_{xy}| \le 1. \tag{2.132}$$

Two random variables X and Y are said to be *uncorrelated* if

$$\rho_{xy} = 0 \quad \text{or} \quad C_{xy} = 0 \quad \text{or} \quad E(XY) = E(X)E(Y). \tag{2.133}$$

It can be easily seen that if the two random variables are independent, they will be uncorrelated. The converse is true only for Gaussian random variables. Two random variables are called *orthogonal* if

$$E(XY) = 0. \tag{2.134}$$

2.7 Central Limit Theorem

If we have n independent random variables X_i, $i = 1,2,\ldots,n$, the pdf $f(y)$ of their sum Y

$$Y = \sum_{i=1}^{n} X_i \tag{2.135}$$

approaches a normal distribution,

$$f_Y(y) \simeq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\eta)^2}{2\sigma^2}\right]. \tag{2.136}$$

In (2.136), η and σ are the mean and standard deviation of Y. If the random variables are identical, a lower value of n would be adequate for the pdf to become almost Gaussian while a higher value of n will be required if the random variables are not identical. In fact, for the case of independent identically distributed random variables with marginal pdfs that are smooth, a value of n of 5 or 6 would be enough to make the pdf of the sum approach the Gaussian distribution. Note that one of the requirements for the CLT to hold true is that none of the random variables can have variances of infinity. This would means that CLT is not applicable if the random variables have Cauchy densities. It can also be concluded that the chi-squared pdf in (2.35) will also approximate the Gaussian pdf when the number of degrees of freedom n is large.

Instead of the sum of n independent random variables, we can consider the product of n independent random variables, i.e.,

$$Z = \prod_{i=1}^{n} X_i. \tag{2.137}$$

Then the pdf of Z is approximately lognormal.

$$f_Z(z) \simeq \frac{1}{\sqrt{2\pi z\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(\ln y - \eta)^2\right]. \tag{2.138}$$

In (2.138),

$$\eta = \sum_{i=1}^{n} E(\ln X_i), \tag{2.139}$$

$$\sigma^2 = \sum_{i=1}^{n} \mathrm{var}(\ln X_i). \tag{2.140}$$

Note that in the equations above, var is the variance and ln is the natural logarithm. The central limit theorem for products can be restated by defining Y as

$$Y = \ln Z = \sum_{i=1}^{n} \ln(X_i). \tag{2.141}$$

Now, if we use the CLT for the sum, the pdf of Y will be Gaussian. One of the justifications for the lognormal pdf for modeling the shadowing component in wireless channels arises from the fact that the shadowing process resulting in the lognormal pdf is a consequence of multiple scattering/reflections and consequent applicability of the central limit theorem for products. This aspect will be covered in Chap. 4 when we discuss the various models for describing, fading and shadowing. It will be possible to determine the lower limit on the value of n such that the central limit theorem for the products would hold.

2.8 Transformation of Random Variables

Although we examined the properties of random variables, often in wireless communications, the signals pass through filters. We are interested in obtaining the statistical properties of the outputs of the filters. The input to the filters may be single variable or multiple variables. For example, in systems operating in the diversity mode, the output might be the sum of the inputs or the strongest of the inputs (Brennan 1959). The output might also be a scaled version of the input. The output of interest might be the ratio of two random variables such as in the case of the detection of desired signals in the presence of cochannel interference.

We will now look at techniques to obtain the density functions of the outputs, knowing the density functions of the input random variables.

2.8.1 *Derivation of the pdf and CDF of* $\mathbf{Y} = \mathbf{g}(\mathbf{X})$

As mentioned above, in wireless communications it is necessary to determine the statistics of the signal when it has passed through filters, linear and nonlinear. For example, if a signal passes through a square law device or an inverse law device, we need to find out the density function of the output from the density function of the input random variable. As a general case, we are interested in obtaining the pdf of Y which is the output of a filter as shown in Fig. 2.25.

If we consider the transformation that is monotonic, i.e., $\mathrm{d}y/\mathrm{d}x$ is either positive or negative, it is easy to determine the pdf of Y knowing the pdf of X. An example of monotonic transformation is shown in Fig. 2.26. Starting with the definition of the

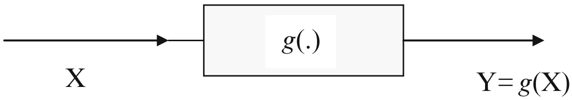

Fig. 2.25 Input–output relationship

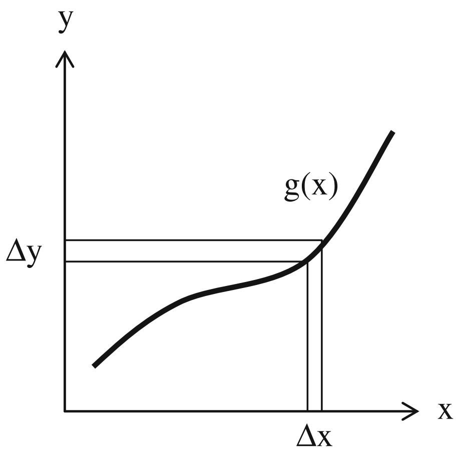

Fig. 2.26 Monotonic transformation of the random variable

random variable and CDF, the probability that the variable Y lies between y and $y + \Delta y$ is

$$P\{y < Y < y + \Delta y\} = P\{x < X < x + \Delta x\}. \tag{2.142}$$

Equation (2.142) is possible because of the monotonic nature of the transformation. Once again, using the definition of the CDF we can rewrite (2.142) as

$$f_Y(y)\Delta y = f_X(x)\Delta x. \tag{2.143}$$

By letting Δx and, hence, $\Delta y \to 0$, (2.143) becomes

$$f_Y(y) = \left.\frac{f_X(x)}{|\mathrm{d}y/\mathrm{d}x|}\right|_{x=g^{-1}(y)}. \tag{2.144}$$

The absolute sign in (2.144) merely reflects inclusion of both the monotonically increasing and decreasing nature of Y and shows that the pdf is always positive. The CDF can be found either from (2.144) or directly from the definition of the CDF as

$$F_Y(y) = P\{Y < y\} = P\{g(x) < y\} = F_X[g(x)]. \tag{2.145}$$

We can now consider the case of a non-monotonic transformation shown in Fig. 2.27.

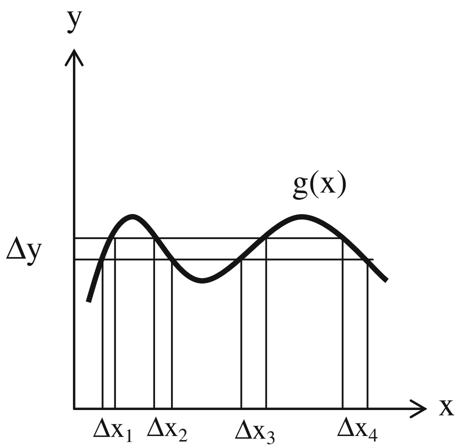

Fig. 2.27 Non-monotonic transformation. Multiple solutions are seen

If the transformation from X to Y is not monotonic as shown in Fig. 2.7, then, (2.144) can be modified to

$$f_Y(y) = \frac{f_X(x)}{|\mathrm{d}y/\mathrm{d}x|}\bigg|_{x_1=g^{-1}(y)} + \frac{f_X(x)}{|\mathrm{d}y/\mathrm{d}x|}\bigg|_{x_2=g^{-1}(y)} + \ldots \frac{f_X(x)}{|\mathrm{d}y/\mathrm{d}x|}\bigg|_{x_n=g^{-1}(y)}. \tag{2.146}$$

In (2.146), x_1, x_2,..., x_n are the n roots of the non-monotonic transformation between X and Y.

It is also possible that there may be instances where non-monotonic transformation might have infinite roots. Consider the case where X is a Gaussian random variable with a zero mean and standard deviation of σ. For example, let us consider the case of a half wave rectifier, i.e.,

$$Y = \begin{cases} X, & X \geq 0, \\ 0, & X < 0. \end{cases} \tag{2.147}$$

This is shown in Fig. 2.28.

From (2.147), we have

$$F_Y(y) = \begin{cases} \dfrac{1}{2}, & y = 0, \\ F_X(y), & y > 0. \end{cases} \tag{2.148}$$

The pdf is obtained by differentiating keeping in mind that $Y = 0$ is an event with a probability of 0.5, resulting in

$$f_Y(y) = \frac{1}{2}\delta(y) + f_X(y)U(y). \tag{2.149}$$

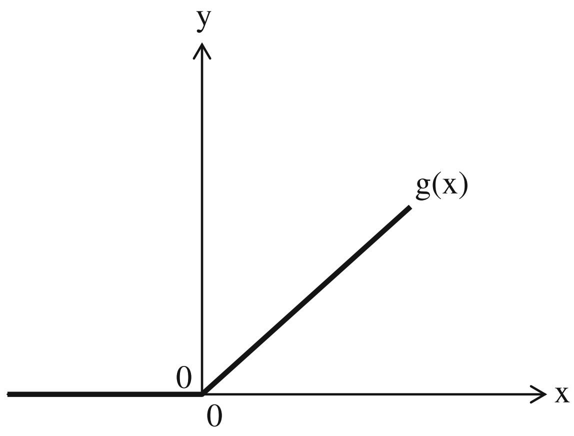

Fig. 2.28 Many-to-one transformation. For all negative values of X, there is only a single solution ($Y = 0$)

2.8.2 Probability Density Function of $Z = X + Y$

Let us find out the density function of the sum of two random variables

$$Z = X + Y. \tag{2.150}$$

Using the fundamental definition of the probability, we have

$$F_Z(z) = P\{Z < z\} = P\{X + Y < z\} = \iint\limits_{x+y<z} f(x, y)\, \mathrm{d}x\, \mathrm{d}y. \tag{2.151}$$

The region defined by (2.151) is shown in Fig. 2.29.
Rewriting (2.151),

$$F_Z(z) = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z-y} f(x, y)\, \mathrm{d}x\, \mathrm{d}y. \tag{2.152}$$

The pdf $f(z)$ is obtained by differentiating (2.152) with respect to z. Using the Leibnitz rule (Gradshteyn and Ryzhik 2007), we get

$$f_Z(z) = \int_{-\infty}^{\infty} f(z - y, y)\, \mathrm{d}y. \tag{2.153}$$

If X and Y are independent, (2.153) becomes

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y)\, \mathrm{d}y. \tag{2.154}$$

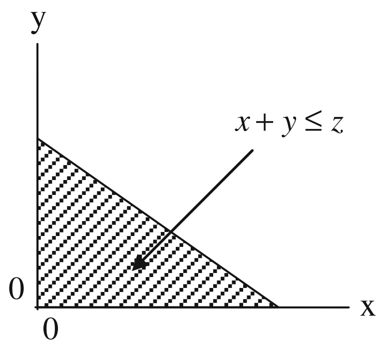

Fig. 2.29 The region defined by $x + y < z$

Equation (2.154) shows that the pdf of the sum of two independent random variables is the convolution of the marginal density functions. If X and Y exist only for positive values, (2.154) becomes

$$f_Z(z) = \int_0^z f_X(z - y) f_Y(y) \, \mathrm{d}y. \tag{2.155}$$

2.8.3 Joint pdf of Functions of Two or More Random Variables

We will now look at ways of obtaining the joint pdf of two variables which are functions of two random variables. For example, if U and V are two random variables given expressed as

$$\begin{aligned} U &= g(X, Y), \\ V &= h(X, Y). \end{aligned} \tag{2.156}$$

We are interested in obtaining $f(u,v)$ given the joint pdf of X and Y, namely $f(x,y)$. Extending the concept used in the derivation of (2.144) and (2.146), the joint pdf of U and V can be expressed as

$$f_{U,V}(u, v) = \left. \frac{f_{X,Y}(x, y)}{|J(x, y)|} \right|_{x=[u,v]^{-1}, y=[u,v]^{-1}}. \tag{2.157}$$

In (2.157), $J(x,y)$ is the Jacobian given by

$$J(x, y) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}. \tag{2.158}$$

If one is interested in the pdf of U or V, it can easily be obtained as

$$f_U(u) = \int f(u, v)\mathrm{d}v. \tag{2.159}$$

We can also use (1.16) to obtain the pdf of the sum of two random variables. Defining

$$\left.\begin{array}{l} Z = X + Y \\ V = Y \end{array}\right\}. \tag{2.160}$$

The Jacobian will be

$$J(x, y) = 1. \tag{2.161}$$

We have

$$f(z, v) = f_{x,y}(z - v, v). \tag{2.162}$$

Using (2.159), we have

$$f_Z(z) = \int_{-\infty}^{\infty} f_{z,v}(z - v, v)\,\mathrm{d}v. \tag{2.163}$$

Equation (2.163) is identical to (2.153) obtained earlier directly. We will now find the density function of the sum of two-scaled random variables such as

$$W = aX + bY. \tag{2.164}$$

In (2.164), a and b are real-valued scalars. Defining an auxiliary variable

$$V = Y, \tag{2.165}$$

we have the Jacobian of the transformation as

$$J(x, y) = a. \tag{2.166}$$

Using (2.157), we have the joint pdf

$$f(w, v) = \left.\frac{f_{X,Y}(x, y)}{|J(x, y)|}\right|_{y=v;\, x=\frac{w-by}{a}} = \frac{1}{|a|} f_{X,Y}\left(\frac{w - by}{a}, y\right). \tag{2.167}$$

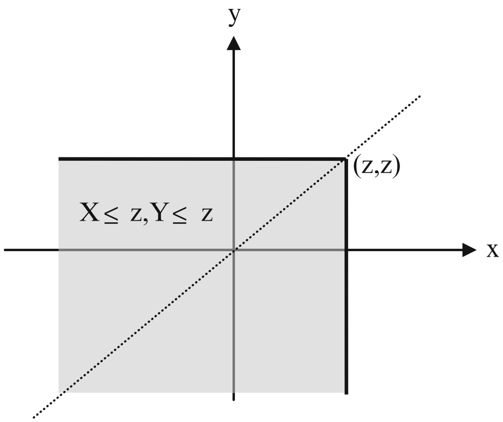

Fig. 2.30 The region of interest to obtain the pdf of the maximum of two random variables

The density function of the random variable in (2.164) now becomes

$$f(w) = \frac{1}{|a|}\int_{-\infty}^{\infty} f_{X,Y}\left(\frac{w-by}{a}, y\right)\,\mathrm{d}y. \tag{2.168}$$

We will look at a few more cases of interest in wireless systems. For example, one of the diversity combining algorithm uses the strongest signal from several branches. Let us determine the density function of

$$Z = \mathrm{Max}\{X, Y\}. \tag{2.169}$$

The CDF of Z becomes

$$F_Z(z) = P\{\mathrm{Max}(X,Y) < z\} = P\{X < z, Y < z\} = F_{X,Y}(z,z). \tag{2.170}$$

The CDF is the volume contained in the shaded area in Fig. 2.30.
If X and Y are independent, (2.170) becomes

$$F_Z(z) = F_X(z)F_Y(z). \tag{2.171}$$

The pdf of the maximum of two independent random variables is obtained by differentiating (2.171) w.r.t. z as

$$f_Z(z) = f_X(z)F_Y(z) + f_Y(z)F_X(z). \tag{2.172}$$

Furthermore, if X and Y are identical, (2.172) becomes

$$f_Z(z) = 2f_X(z)F_X(z) = 2f_Y(z)F_Y(z). \tag{2.173}$$

We can easily find out the density function of the minimum of two random variables. Let us define W as

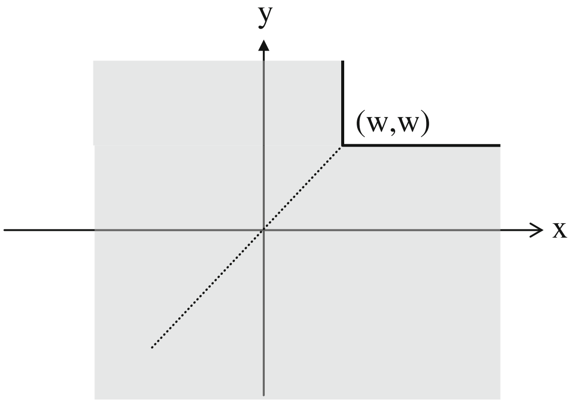

Fig. 2.31 The region of interest to obtain the CDF of the minimum of two variables

$$W = \mathrm{Min}\{X, Y\} = \begin{cases} Y, & X > Y, \\ X, & X \leq Y. \end{cases} \tag{2.174}$$

The CDF of W will be

$$F_W(w) = P\{\mathrm{Min}(X, Y) < w\} = P\{Y < w, X > Y\} + P\{X < w, X \leq Y\}. \tag{2.175}$$

The volume contained in the shaded area in Fig. 2.31 corresponds to this CDF. Assuming X and Y to be independent, (2.175) can result in

$$\begin{aligned} F_W(w) &= 1 - P\{W > w\} = 1 - P\{X > w, Y > w\} \\ &= 1 - P\{X > w\}P\{Y > w\}. \end{aligned} \tag{2.176}$$

Equation (2.176) simplifies to

$$F_W(w) = F_X(w) + F_Y(w) - F_X(w)F_Y(w). \tag{2.177}$$

We get the pdf by differentiating (2.177) w.r.t w, and we have

$$f(w) = f_X(w) + f_Y(w) - f_X(w)F_Y(w) - f_Y(w)F_X(w). \tag{2.178}$$

Another random parameter of interest is the product of two random variables. If U is the product of the two random variables

$$U = XY. \tag{2.179}$$

The density function of U might be obtained in a couple of different ways. First, let us define a dummy variable V as X,

$$V = Y. \tag{2.180}$$

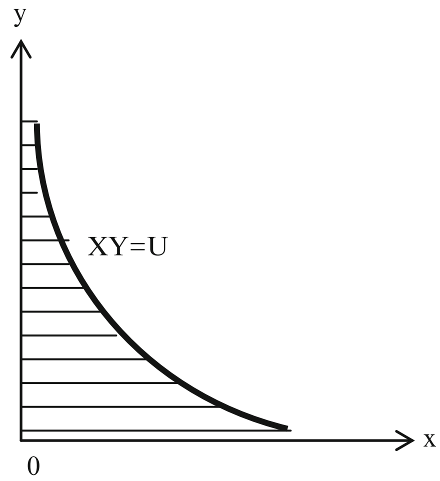

Fig. 2.32 The probability (volume) contained in the shaded region corresponds to the CDF

Thus, the joint pdf of U and V can be written using (2.157) as

$$f(u,v) = \frac{f_{X,Y}((u/v),v)}{|v|}. \tag{2.181}$$

The density function of the product of two random variables is obtained as

$$f_U(u) = \int_{-\infty}^{\infty} \frac{1}{|v|} f_{X,Y}\left(\frac{u}{v},v\right)\, \mathrm{d}v. \tag{2.182}$$

Assuming that both X and Y are independent and exist only for positive values, (2.182) becomes

$$f_U(u) = \int_0^{\infty} \frac{1}{y} f_{X,Y}\left(\frac{u}{y},y\right)\, \mathrm{d}y. \tag{2.183}$$

One can also obtain the pdf of U from the fundamental definition of the CDF as well. The CDF of U can be expressed as

$$F_U(u) = P\{XY < u\} = P\left\{X < \frac{u}{y}\right\} = \int_0^{\infty}\int_0^{u/y} f(x,y)\, \mathrm{d}x\, \mathrm{d}y. \tag{2.184}$$

The region with horizontal lines in Fig. 2.32 shows the region of interest for the calculation of the probability volume in (2.184).

The pdf is obtained by differentiating (2.184) w.r.t u and using the Leibniz's rule. Therefore, we have

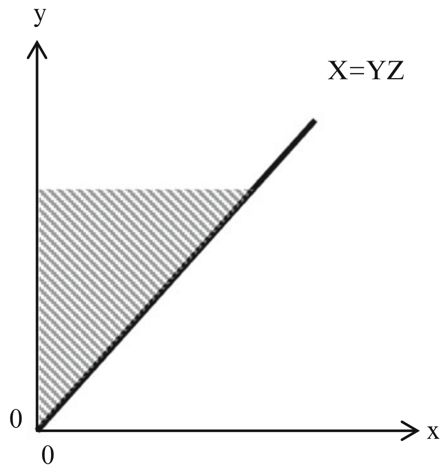

Fig. 2.33 The probability (volume) in the shaded region corresponds to the CDF $X = YZ$

$$f_U(u) = \int_0^\infty \frac{1}{y} f_{X,Y}\left(\frac{u}{y}, y\right) \, \mathrm{d}y. \tag{2.185}$$

Working similarly, if

$$Z = \frac{X}{Y}. \tag{2.186}$$

The pdf of the ratio of two random variables (X,Y $>$ 0) becomes,

$$f_Z(z) = \int_0^\infty y f_{X,Y}(zy, y) \, \mathrm{d}y. \tag{2.187}$$

It is also possible to obtain the CDF first. This was done for the case of the product of random variables in (2.184). Differentiating the CDF is given by the shaded region ($x > 0$, $y > 0$) in Fig. 2.33, we can also find the density function of

$$W = X^2 + Y^2. \tag{2.188}$$

Using the fundamental definition of CDF, we can write the CDF of the variable W as

$$F_W(w) = \text{Prob}\{X^2 + Y^2 < w\} \ . \tag{2.189}$$

Since $X^2 + Y^2 \leq w$ represents a circle of radius $\sqrt{w}$, the CDF becomes

$$F_W(w) = \int_{y=-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f(x, y) \, \mathrm{d}x \, \mathrm{d}y. \tag{2.190}$$

The pdf is obtained by differentiating the CDF in (2.190) resulting in

$$f(w) = \int_{-\sqrt{w}}^{\sqrt{w}} \frac{1}{2\sqrt{w-y^2}} \left[f_{x,y}\left(\sqrt{w-y^2}, y\right) + f_{x,y}\left(-\sqrt{w-y^2}, y\right)\right] \mathrm{d}y. \quad (2.191)$$

2.8.4 Use of CHF to Obtain pdf of Sum of Random Variables

One of the main uses of the CHF or the MGF is in diversity combining (Beaulieu 1990; Tellambura and Annamalai 2003; Annamalai et al. 2005). If the output of a diversity combining algorithm is given by the sum of several outputs such as

$$Y = X_1 + X_2 + \cdots + X_M, \quad (2.192)$$

and if we assume that the random variables $X_1, \ldots, X_M$ are independent, the pdf Y will be obtained by the M-fold convolution of the pdfs of those random variables. If the CHFs of X's are available, the CHF of Y can instead be expressed as

$$\psi_Y(\omega) = \langle \exp(j\omega X_1 + j\omega X_2 + \cdots + j\omega X_M) \rangle. \quad (2.193)$$

Using the Fourier transform properties, we have

$$\psi_Y(\omega) = \prod_{k=1}^{M} \psi_{X_k}(\omega). \quad (2.194)$$

If the random variables X's are identical, (2.194) becomes

$$\psi_Y(\omega) = [\psi_X(\omega)]^M. \quad (2.195)$$

The pdf of Y can now be obtained using the inverse Fourier transform property as

$$f_Y(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\psi_X(\omega)]^M \exp(-j\omega y)\, \mathrm{d}\omega. \quad (2.196)$$

Note that (2.196) is a single integral which replaces the M-fold convolution required if one were to use the marginal pdfs directly.

2.8.5 Some Transformations of Interest in Wireless Communications

We will now look at a few examples of transformations of random variables such as the sum or difference of two random variables, the sum of the squares of random variables, and the products and ratios of random variables.

Example #1 Let X and Y be two independent identically distributed random variables each with zero mean. The joint pdf is

$$f(x,y) = f(x)f(y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right). \tag{2.197}$$

We will find the joint pdf of

$$R = \sqrt{X^2+Y^2}, \tag{2.198}$$

$$\Theta = \tan^{-1}\left(\frac{Y}{X}\right). \tag{2.199}$$

Note that R is the envelope (or magnitude) and Θ is the phase. The Jacobian $J(x,y)$ defined in (2.158) for the set of these two variables

$$J(x,y) = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{r}. \tag{2.200}$$

The joint pdf now becomes

$$f(r,\theta) = \frac{f(x,y)}{|J(x,y)|} = \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right), \quad 0 \le r \le \infty,\ 0 \le \theta \le 2\pi. \tag{2.201}$$

The marginal density function of R is

$$f(r) = \int_{-\pi}^{\pi} f(r,\theta)\,\mathrm{d}\theta = \frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right), \quad 0 \le r \le \infty. \tag{2.202}$$

The marginal density function of the phase is

$$f(\theta) = \int_0^{\infty} f(r,\theta)\,\mathrm{d}r = \frac{1}{2\pi}, \quad 0 \le \theta \le 2\pi. \tag{2.203}$$

Note that R and Θ are independent with R having a Rayleigh distribution and Θ having a uniform distribution. Often, the range of the phase is also expressed as $-\pi < \theta < \pi$. A discussion on the phase statistics appears in connection with the next example.

Example #2 Another related case of interest arises when one of the Gaussian random variables in Example #1 has a non-zero mean. Let

$$f(x, y) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(x - A)^2}{2\sigma^2}\right] \exp\left(-\frac{y^2}{2\sigma^2}\right). \tag{2.204}$$

Our interest is still the joint and marginal pdfs of R and Θ in (2.198) and (2.199). The Jacobian for the transformation will be unaffected by the existence of the mean A of the random variable X. Thus, the joint pdf becomes

$$\begin{aligned} f(r, \theta) &= \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) \exp\left[\frac{rA\ \cos(\theta)}{\sigma^2}\right], \\ & 0 \le r \le \infty,\ 0 \le \theta \le 2\pi. \end{aligned} \tag{2.205}$$

The pdf of the magnitude R is

$$f(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) \int_0^{2\pi} \frac{1}{2\pi} \exp\left[\frac{rA\ \cos(\theta)}{\sigma^2}\right] \mathrm{d}\theta. \tag{2.206}$$

Noting the relationship between the integral in (2.206) and the modified Bessel function of the first kind $I_0(.)$ (Abramowitz and Segun 1972; Gradshteyn and Ryzhik 2007)

$$I_0(w) = \frac{1}{2\pi} \int_0^{2\pi} \exp[w\ \cos(\theta)]\ \mathrm{d}\theta. \tag{2.207}$$

We can write (2.206) as

$$f(r) = \frac{r}{\sigma^2} \exp\left[-\frac{r^2 + A^2}{2\sigma^2}\right] I_0\left(\frac{rA}{\sigma^2}\right), \quad 0 \le r \le \infty. \tag{2.208}$$

Equation (2.208) is known as the Rician distribution of the magnitude or envelope. This pdf arises in wireless systems when a direct path exists between the transmitter and receiver in addition to the multiple diffuse paths (Nakagami 1960; Rice 1974; Polydorou et al. 1999). When $A \to 0$, the Rician pdf in (2.208) becomes the Rayleigh pdf in (2.202). Another interesting observation on the Rician pdf is its characteristics when A becomes large. If

$$\frac{A}{\sigma^2} \gg 1 \tag{2.209}$$

we can use the approximation (Abramowitz and Segun 1972)

$$I_0(x) = \frac{\exp(x)}{\sqrt{2\pi x}}. \tag{2.210}$$

The Rician pdf now becomes (strong direct path and weak multipath component),

$$f(r) = \frac{1}{\sqrt{2\pi\sigma^2}}\left(\frac{r}{A}\right)^{1/2}\exp\left[-\frac{1}{2\sigma^2}(r-A)^2\right]. \tag{2.211}$$

Equation (2.211) has an approximate Gaussian shape but for the factor of $\sqrt{r/A}$. This situation also arises in electrical communication systems when we examine the sum of a strong sine wave signal and a weak narrow band additive Gaussian noise.

We can now look at the density function of the power Z,

$$Z = R^2. \tag{2.212}$$

The pdf can be obtained using the properties of the transformation of variables as

$$f(z) = \frac{f(r)}{|\mathrm{d}z/\mathrm{d}r|}. \tag{2.213}$$

Defining the Rician factor K_0 as

$$K_0 = \frac{A^2}{2\sigma^2} \tag{2.214}$$

and

$$Z_R = 2\sigma^2 + A^2 \tag{2.215}$$

(2.213) becomes

$$f(z) = \frac{(1+K_0)}{Z_R}\exp\left[-K_0 - (1+K_0)\frac{z}{Z_R}\right] I_0\left[2\sqrt{K_0(1+K_0)\frac{z}{Z_R}}\right], \quad 0 \le z \le \infty. \tag{2.216}$$

Equation (2.216) is the Rician distribution of the power or SNR. Another point to note is that the density function of the phase Θ will not be uniform (Goodman 1985; Papoulis and Pillai 2002). By observing the joint pdf of the envelope and phase in (2.205), we can also note that the variables R and Θ are not independent.

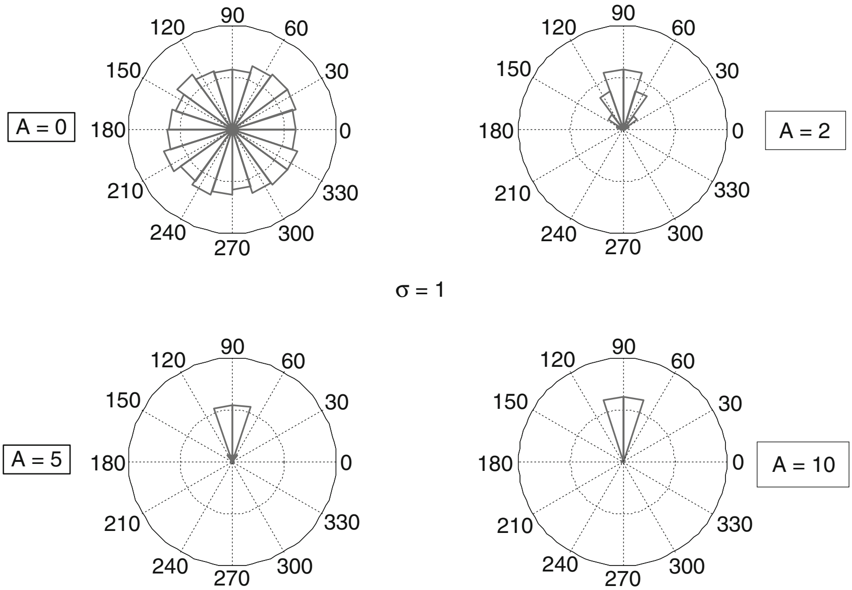

Fig. 2.34 The histogram of the phase associated with the Rician pdf

The density functions of the phase and the changes in the densities as the Rician factor changes can be observed in Fig. 2.34.

Example #3 Let X and Y be independent identically distributed exponential random variables. The joint density function becomes

$$f(x, y) = \frac{1}{a^2} \exp\left(-\frac{x+y}{a}\right), U(x)U(y). \tag{2.217}$$

We can find the pdf of

$$Z = X + Y. \tag{2.218}$$

Using (2.155), the pdf of Z becomes

$$f(z) = \int_0^z \frac{1}{a^2} \exp\left(-\frac{x}{a}\right) \exp\left[-\left(\frac{z-x}{a}\right)\right] \mathrm{d}x = \frac{z}{a^2} \exp\left(-\frac{z}{a}\right) U(z). \tag{2.219}$$

Note that the pdf of Z in (2.219) is the Erlang distribution in (2.41) for the case of $c = 2$. If we now write

$$W = \sum_{k=1}^{n} X_k \tag{2.220}$$

where each of the random variable on the right hand side is exponentially distributed with the parameter a, proceeding similarly we can get the pdf of W to be

$$f(w) = \frac{1}{a^N}\frac{w^{N-1}}{\Gamma(N)}\exp\left(-\frac{w}{a}\right)U(w). \tag{2.221}$$

Equation (2.221) is the Erlang pdf in (2.41).

Example #4 We will repeat Example #2 when the two random variables are gamma distributed with a joint pdf

$$\begin{aligned} f(x,y) = {} & \frac{(x/\beta)^{c-1}\exp(-(x/\beta))}{\beta\Gamma(c)}\frac{(y/\beta)^{c-1}\exp(-(y/\beta))}{\beta\Gamma(c)}, \\ & 0 < x < \infty,\ 0 < y < \infty. \end{aligned} \tag{2.222}$$

Let

$$R = X + Y. \tag{2.223}$$

Using (2.155), the pdf of R becomes

$$f(r) = \int_0^r \frac{(x/\beta)^{c-1}\exp(-(x/\beta))}{\beta\Gamma(c)}\frac{((r-x)/\beta)^{c-1}\exp(-(r-x)/\beta)}{\beta\Gamma(c)}\,\mathrm{d}x. \tag{2.224}$$

Using the table of integrals (Gradshteyn and Ryzhik 2007), the pdf of the sum of the two gamma random variable becomes

$$f(r) = \frac{r^{2c-1}}{\beta^{2c}\Gamma(2c)}\exp\left(-\frac{r}{\beta}\right), \quad 0 \le r \le \infty. \tag{2.225}$$

In arriving at (2.225), we have made use of the following identity of doubling formula for gamma functions (Gradshteyn and Ryzhik 2007; Wolfram 2011)

$$\Gamma(2c) = \frac{2^{2c-1}}{\sqrt{\pi}}\Gamma(c)\Gamma\left(c+\frac{1}{2}\right). \tag{2.226}$$

A comparison of (2.225) and (2.50) suggests that R is also a gamma random variable with order $2c$ and mean of $2\beta c$. In other words, R is a gamma random variable of parameters $2c$ and β.

Example #5 Let us continue with the case of two independent and identically distributed exponential random variables with a joint pdf in (2.217). We will now find out the pdf of the product of the two variables,

$$U = XY. \tag{2.227}$$

Using the approach given in (2.182), we have

$$f_U(u) = \int_0^{\infty} \frac{1}{x} f_{X,Y}\left(\frac{u}{x}, x\right) \, \mathrm{d}x = \int_0^{\infty} \frac{1}{a^2 x} \exp\left(-\frac{u}{ax}\right) \exp\left(-\frac{x}{a}\right) \, \mathrm{d}x. \tag{2.228}$$

Using tables of integrals (Gradshteyn and Ryzhik 2007), eqn. (2.228) becomes

$$f(u) = \frac{2}{a^2} K_0\left(\frac{2}{a}\sqrt{u}\right), \quad 0 \le u < \infty. \tag{2.229}$$

In (2.229), $K_0()$ is the modified Bessel function of the second kind of order zero (Gradshteyn and Ryzhik 2007). Note that the pdf in (2.229) arises when we examine the SNR in shadowed fading channels or cascaded channels (Shankar 2004; Bithas et al. 2006; Andersen 2002; Nadarajah and Gupta 2005; Salo et al. 2006; Nadarajah and Kotz 2006a, b).

If we define a new random variable as

$$W = \sqrt{U}, \tag{2.230}$$

we can identify the pdf of W as the double Rayleigh pdf (Salo et al. 2006). Using the property of the monotonic transformation of random variables in (2.144), we can write

$$f(w) = \frac{f(u)}{|\mathrm{d}u/\mathrm{d}w|} = \left(2\sqrt{u}\right) \frac{2}{a^2} K_0\left(\frac{2}{a}\sqrt{u}\right) = \frac{4w}{a^2} K_0\left(\frac{2}{a} w\right), \quad 0 \le w \le \infty. \tag{2.231}$$

Example #6 We will repeat Example #5 for the case of two gamma random variables, independent and identically distributed with a joint pdf in (2.222). Using (2.182) the pdf of the product becomes

$$f(u) = \int_0^{\infty} \frac{1}{x} \frac{(x/\beta)^{c-1} \exp(-(x/\beta))}{\beta \Gamma(c)} \frac{(u/\beta x)^{c-1} \exp(-(u/\beta x))}{\beta \Gamma(c)} \mathrm{d}x. \tag{2.232}$$

Using the table of integrals (Gradshteyn and Ryzhik 2007), we have

$$f(u) = \frac{2}{\beta^{2c} \Gamma^2(c)} u^{c-1} K_0\left(\frac{2}{\beta}\sqrt{u}\right), \quad 0 \le u < \infty. \tag{2.233}$$

Note that (2.233) becomes (2.229) when $c = 1$.

If we limit the value of $c > 1/2$, and the pdf of W in (2.230) is compared with the pdf of U in (2.233) the latter is identified as the double Nakagami pdf (Wongtrairat and Supnithi 2009; Shankar and Gentile 2010). In general, (2.233) is also known as the double gamma pdf and we will study its properties in Chap. 4 when we examine cascaded fading channels.

Using the procedure adopted in connection with (2.231), we get

$$(w) = \frac{f(u)}{|\mathrm{d}u/\mathrm{d}w|} = \left(2\sqrt{u}\right)\frac{2}{\beta^{2c}\Gamma^2(c)}u^{c-1}K_0\left(\frac{2}{\beta}\sqrt{u}\right) = \frac{4w^{2c-1}}{\beta^{2c}\Gamma^2(c)} \times K_0\left(\frac{2}{\beta}w\right), \quad 0 \le w \le \infty. \tag{2.234}$$

Equation (2.234) is the pdf associated with the product of two Nakagami variables.

Example #7 Another interesting case in wireless systems is variable created from the product of two nonidentical gamma random variables. Let X and Y be two gamma distributed variables with pdfs

$$f(x) = \frac{(x/\alpha)^{c-1}\exp(-(x/\alpha))}{\alpha\Gamma(c)}, \tag{2.235}$$

$$f(y) = \frac{(y/\beta)^{m-1}\exp(-(y/\beta))}{\beta\Gamma(m)}. \tag{2.236}$$

Let

$$S = XY. \tag{2.237}$$

Once again, pdf of S can be written using (2.183) as

$$f(s) = \int_0^\infty \frac{1}{x}\frac{(x/\alpha)^{c-1}\exp(-(x/\alpha))}{\alpha\Gamma(c)}\frac{(s/\beta x)^{m-1}\exp(-(s/\beta x))}{\beta\Gamma(m)}\mathrm{d}x. \tag{2.238}$$

Using the table of integrals (Gradshteyn and Ryzhik 2007), the pdf of S becomes

$$f(s) = \frac{2}{\left(\sqrt{\alpha\beta}\right)^{m+c}\Gamma(m)\Gamma(c)}s^{((m+c)/2)-1}K_{m-c}\left(2\sqrt{\frac{s}{\alpha\beta}}\right), \quad 0 \le s \le \infty. \tag{2.239}$$

The pdf in (2.239) is the gamma–gamma pdf or the generalized K distribution (Lewinsky 1983; McDaniel 1990; Abdi and Kaveh 1998; Anastassopoulos et al. 1999; Shankar 2004). Note that if $\alpha = \beta = b$ and $m = c$, (2.239) becomes (2.233). Furthermore, if $c = 1$, (2.233) is the so called K distribution or the K pdf (Jakeman and Tough 1987; Abdi and Kaveh 1998; Iskander et al. 1999).

Table 2.3 The relationship of GBK distribution to other pdfs

GBK pdf $f(x;m,n,\chi,\lambda)=\dfrac{2\lambda x^{[(\lambda/2)(m+n)]-1}}{(\chi)^{(\lambda/2)(m+n)}\Gamma(m)\Gamma(n)}K_{m-n}\left[2\left(\dfrac{x}{\chi}\right)^{(\lambda/2)}\right]$	Probability density functions (special case)
$f(x;1,\infty,\chi,1)$	Exponential
$f(x;1,\infty,\chi,2)$	Rayleigh
$f(x;m,n,\chi,1)$	GK distribution
$f(x;m,1,\chi,1)$	K distribution
$f(x;m,\infty,\chi,1)$	Gamma distribution
$f(x;m,\infty,\chi,\lambda)$	Generalized gamma distribution
$f\left(x;\frac{1}{2},\infty,\chi,2\right)$	Half Gaussian

Example #8 We will now extend the Example #7 to two generalized gamma random variables. Let

$$f(x)=\frac{\lambda x^{\lambda m-1}\exp[-(x/\alpha)^{\lambda}]}{\alpha^{\lambda m}\Gamma(m)},\quad 0\le x\le\infty, \tag{2.240}$$

$$f(y)=\frac{\lambda y^{\lambda n-1}\exp[-(y/\beta)^{\lambda}]}{\beta^{\lambda m}\Gamma(n)},\quad 0\le y\le\infty. \tag{2.241}$$

The pdf of $S=XY$ can be obtained using the integral in (2.183) as

$$f(s)=\frac{2\lambda s^{[(\lambda/2)(m+n)]-1}}{(\alpha\beta)^{(\lambda/2)(m+n)}\Gamma(m)\Gamma(n)}K_{m-n}\left[2\left(\frac{s}{\alpha\beta}\right)^{(\lambda/2)}\right],\quad 0\le s\le\infty. \tag{2.242}$$

Expressing

$$\chi=\alpha\beta, \tag{2.243}$$

(2.242) becomes a four parameter pdf given as

$$f(s)=\frac{2\lambda s^{[(\lambda/2)(m+n)]-1}}{(\chi)^{(\lambda/2)(m+n)}\Gamma(m)\Gamma(n)}K_{m-n}\left[2\left(\frac{s}{\chi}\right)^{(\lambda/2)}\right],\quad 0\le s\le\infty. \tag{2.244}$$

The pdf in (2.242) is known as the generalized Bessel K distribution (GBK) which becomes the GK distribution in (2.239) when $\lambda=1$ and becomes the K pdf when $\lambda=1$ and $n=1$ (Iskander and Zoubir 1996; Anastassopoulos et al. 1999; Frery et al. 2002). Note that the GBK pdf is a five parameter distribution as in (2.242) with shape parameters m, n, and λ and scaling factors α and β or a four parameter distribution in (2.244) with shape parameters m, n, and λ and scaling factor χ. Several density functions can be obtained from the GBK pdf by varying some of these shape parameters and following the details in Table 2.3.

Example #9 Let us look at another case of interest in wireless communications where the new random variable is the sum of several gamma random variables (Moschopoulos 1985; Kotz and Adams 1964; Provost 1989; Alouini et al. 2002; Karagiannidis et al. 2006). Let

$$Z = \sum_{k=1}^{N} X_k. \tag{2.245}$$

In (2.245), there are N independent and identically distributed random variables X's, each with a pdf of the form in (2.50). We will use the relationship between density functions and CHFs to obtain the pdf in this case. Since the pdf of Z will be a convolution of the pdfs of N identical density functions, the CHF of Z can be written as the product of N identical CHFs, each of them of the form given in (2.53). The CHF of Z is

$$\psi_Z(\omega) = [\psi_X(\omega)]^N = (1 - j\beta\omega)^{-cN}. \tag{2.246}$$

The pdf of Z is now obtained from the Fourier relationship between CHF and pdf.

$$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1 - j\beta\omega)^{cN}} \exp(-j\omega z)\, \mathrm{d}\omega. \tag{2.247}$$

Using the Fourier integral tables (Gradshteyn and Ryzhik 2007), eqn. (2.247) becomes

$$f(z) = \frac{z^{cN-1}}{\beta^{cN}\Gamma(cN)} \exp\left(-\frac{z}{\beta}\right), \quad z \geq 0. \tag{2.248}$$

From (2.248), it is seen that the sum of N identically distributed gamma random variables with parameters c and β is another gamma random variable with parameters cN and β.

Example #10 Shadowing in wireless systems is modeled using the lognormal pdf. In diversity systems, it might be necessary to obtain the density function of the sum of several lognormal random variables, each having a pdf of the form in (2.71). If

$$Z = \sum_{k=1}^{N} X_k \tag{2.249}$$

7a simple analytical expression for the density function of Z is not readily available. Several researchers have proposed approximate forms for the density function (Beaulieu et al. 1995; Slimane 2001; Beaulieu and Xie 2004; Filho et al. 2005;

Lam and Le-Ngoc 2006; Lam and Tho 2007; Liu et al. 2008). One such approximation results in the pdf of N independent identically distributed lognormal variables is expressed as a shifted gamma pdf (Lam and Le-Ngoc 2006)

$$f(z) = \begin{cases} \dfrac{[10\log_{10}(z)-\delta]^{\alpha-1}}{[10/\log_e(10)]\beta^{\alpha}\Gamma(\alpha)z}\exp\left[-\dfrac{10\log_{10}(z)-\delta}{\beta}\right], & z > 10^{\delta/10} \\ 0, \quad z \le 10^{\delta/10} \end{cases}. \tag{2.250}$$

The three parameters, namely α, β, δ (all in decibel units), can be obtained by matching the first three moments of the variable in (2.249) and the moments of the pdf in (2.250).

Example #11 We will explore another interesting case in wireless communications involving the product of several random variables (Karagiannidis et al. 2007; Shankar 2010). Let Z be the product of N random variables which forms a set of independent random variables with the same density functions but with different parameters.

$$Z = \prod_{k=1}^{N} X_k \tag{2.251}$$

Let the density of X_k be given by a gamma pdf as

$$f(x_k) = \frac{x_k^{m_k-1}}{b_k^{m_k}\Gamma(m_k)}\exp\left(-\frac{x_k}{b_k}\right), \quad k = 1, 2, \ldots, N. \tag{2.252}$$

Using MGFs and Laplace transforms, the density function of Z can be obtained as (Kabe 1958; Stuart 1962; Podolski 1972; Mathai and Saxena 1973; Carter and Springer 1977; Abu-Salih 1983; Nadarajah and Kotz 2006a, b; Karagiannidis et al. 2007; Mathai and Haubold 2008):

$$f(z) = \frac{1}{z\prod_{k=1}^{N}\Gamma(m_k)} G_{0,N}^{N,0}\left(\frac{z}{\prod_{k=1}^{N}\Gamma(b_k)}\middle| \begin{matrix} - \\ m_1, m_2, \ldots, m_N \end{matrix}\right) U(z). \tag{2.253}$$

In (2.253), G () is the Meijer's G function. The CDF can be obtained using the differential and integral properties of the Meijer's G function as (Springer and Thompson 1966, 1970; Mathai and Saxena 1973; Mathai 1993; Adamchik 1995):

$$F(z) = \frac{1}{\prod_{k=1}^{N}\Gamma(m_k)} G_{1,N+1}^{N,1}\left(\frac{z}{\prod_{k=1}^{N}\Gamma(b_k)}\middle| \begin{matrix} 1 \\ m_1, m_2, \ldots, m_N, 0 \end{matrix}\right) U(z). \tag{2.254}$$

If X's are identical and $m_k = m = 1$ and $b_k = b$, we have the pdf of the products of exponential random variables,

$$f(z) = \frac{1}{z} G_{0,N}^{N,0}\left(\frac{z}{b^N} \middle| \underbrace{\overset{-}{1, 1, \ldots, 1}}_{N-\text{terms}} \right) U(z). \tag{2.255}$$

Using the relationship between the Meijer's G function and modified Bessel functions (Mathai and Saxena 1973; Wolfram 2011)

$$G_{0,1}^{1,0}\left(\frac{z}{b} \middle| \begin{matrix} - \\ m \end{matrix} \right) = \left(\frac{z}{b} \right)^m \exp\left(-\frac{z}{b} \right) \tag{2.256}$$

(2.253) becomes the gamma pdf for $N = 1$ and

$$G_{0,2}^{2,0}\left(\frac{z}{b^2} \middle| \begin{matrix} - \\ m, n \end{matrix} \right) = 2\left(\frac{z}{b^2} \right)^{(1/2)(m+n)} K_{m-n}\left(\frac{2}{b}\sqrt{z} \right) \tag{2.257}$$

(2.253) becomes the GK pdf which is obtained for the pdf of the product of two gamma random variables (Shankar 2005; Laourine et al. 2008).

It is also interesting to find out the pdf of the cascaded channels when lognormal fading conditions exist in the channel. In this case, we take

$$f(x_k) = \frac{[10/\log_e(10)]}{\sqrt{2\pi\sigma_k^2 x_k^2}} \exp\left[-\frac{(10\log_{10}(x_k) - \mu_k)^2}{2\sigma_k^2} \right], \quad 0 < x < \infty, \ k = 1, 2, \ldots, N. \tag{2.258}$$

The cascaded output in (2.251) can be expressed in decibel form as

$$W = 10\log_{10}(Z) = \sum_{k=1}^{N} 10\log_{10}(X_k). \tag{2.259}$$

Since X's are lognormal and each term in the summation in (2.259) is therefore a Gaussian random variable, the density function of the random variable W will be Gaussian,

$$f(w) = \frac{1}{\sqrt{2\pi \sum_k^N \sigma_k^2}} \exp\left[-\frac{\left(w - \sum_{k=1}^N \mu_k \right)^2}{2\sum_k^N \sigma_k^2} \right]. \tag{2.260}$$

Converting back, the density function of Z will be lognormal given by

$$f(z) = \frac{[10/\log_e(10)]}{\sqrt{2\pi\left(\sum_{k=1}^N \sigma_k^2 \right) z^2}} \exp\left[-\frac{\left(10\log_{10}(z) - \sum_{k=1}^N \mu_k \right)^2}{2\sum_{k=1}^N \sigma_k^2} \right], \quad 0 < z < \infty. \tag{2.261}$$

Example #12 There is also interest in wireless communications to determine the density function of the ratio of two random variables. For example, the received signal generally has to be compared against an interference term which is also random (Winters 1984; Cardieri and Rappaport 2001; Shah et al. 2000). Thus, if X represents the signal (power) and Y represents the interference (power), we are interested in finding out the pdf of

$$Z = \frac{X}{Y}. \tag{2.262}$$

In practical applications, we would want Z to be a few dB so that the signal power will be stronger than the interference. Since we can assume that the signal and interference are independent, the pdf of Z can be written from (2.187). If both X and Y are gamma distributed (originating from Nakagami-m distributed envelope values), we have

$$f(x) = \left(\frac{m}{\alpha}\right)^m \frac{x^{m-1}}{\Gamma(m)} \exp\left(-m\frac{x}{\alpha}\right), \quad 0 < x < \infty, \tag{2.263}$$

$$f(y) = \left(\frac{n}{\beta}\right)^n \frac{x^{n-1}}{\Gamma(n)} \exp\left(-n\frac{y}{\beta}\right), \quad 0 < y < \infty. \tag{2.264}$$

The density function of Z is written using (2.187) as

$$f(z) = \int_0^\infty y\left(\frac{m}{\alpha}\right)^m \frac{(yz)^{m-1}}{\Gamma(m)} \exp\left(-m\frac{yz}{\alpha}\right)\left(\frac{n}{\beta}\right)^n \frac{y^{n-1}}{\Gamma(n)} \exp\left(-n\frac{y}{\beta}\right) \mathrm{d}y. \tag{2.265}$$

Equation (2.265) can be solved easily. We have the pdf for the ratio of two gamma distributed powers as

$$f(z) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} (\alpha n)^n (\beta m)^m \frac{z^{m-1}}{(mz\beta + n\alpha)^{m+n}}, \quad 0 < z < \infty. \tag{2.266}$$

For the special case of $m = n$ (corresponds to exponential distribution of power), we have the pdf of the ratio of the powers as

$$f(z) = \frac{\alpha\beta}{(\alpha + \beta z)^2}, \quad 0 < z < \infty. \tag{2.267}$$

Another interesting case in wireless communications arises when both the signal power and the interfering component power have lognormal distributions (Sowerby and Williamson 1987; Ligeti 2000). In this case, the density function of the ratio can be determined in a straight forward fashion. From (2.71) we have

$$f(x) = \frac{[10/\log_e(10)]}{\sqrt{2\pi\sigma_x^2 x^2}} \exp\left[-\frac{(10\log_{10}(x) - \mu_x)^2}{2\sigma_x^2}\right], \quad 0 < x < \infty, \tag{2.268}$$

$$f(x) = \frac{[10/\log_e(10)]}{\sqrt{2\pi\sigma_y^2 y^2}} \exp\left[-\frac{(10\log_{10}(y) - \mu_y)^2}{2\sigma_y^2}\right], \quad 0 < y < \infty. \tag{2.269}$$

Taking the logrithm and converting into decibel units, (2.262) becomes

$$10\log_{10}(Z) = 10\log_{10}(X) - 10\log_{10}(Y). \tag{2.270}$$

Since X and Y are lognormal random variables, the variables on the right-hand side of (2.270) will be Gaussian. Thus, the density function of

$$W = 10\log_{10}(Z) \tag{2.271}$$

will be Gaussian with a mean equal to the difference of the means of the two variables on the right-hand side of (2.270) and variance equal to the sum of the variances of the two variables on the right-hand side of (2.270). The pdf of W can now be expressed as

$$f(w) = \frac{1}{\sqrt{2\pi(\sigma_x^2 + \sigma_y^2)}} \exp\left\{-\frac{[w - (\mu_x - \mu_y)]^2}{2(\sigma_x^2 + \sigma_y^2)}\right\}. \tag{2.272}$$

Converting back from W to Z we can see that the density function of the ratio will also be lognormal. It can be expressed as

$$f(z) = \frac{[10/\log_e(10)]}{\sqrt{2\pi\sigma^2 z^2}} \exp\left[-\frac{(10\log_{10}(z) - \mu)^2}{2\sigma^2}\right], \quad 0 < z < \infty \tag{2.273}$$

with

$$\begin{aligned} \mu &= \mu_x - \mu_y, \\ \sigma^2 &= \sigma_x^2 + \sigma_y^2. \end{aligned} \tag{2.274}$$

Example #13 In wireless communications we must also address the situation that occurs when the receiver gets both the signal of interest (desired signal) and unwanted cochannels in addition to the noise (Winters 1984; Shah et al. 2000; Yacoub 2000; Aalo and Zhang 1999). The SNR at the receiver can be expressed as

$$Z = \frac{S_{\mathrm{i}}}{N + S_{\mathrm{c}}}. \tag{2.275}$$

In (2.275), S_i is the signal power and S_c is the cochannel power. Both are random, N is the noise power. Rewriting (2.275), we have

$$Z = \frac{X}{1+Y}. \tag{2.276}$$

In (2.276), X is the SNR of the desired signal and Y is the SNR of the cochannel. Defining an auxiliary variable W as

$$W = Y. \tag{2.277}$$

The Jacobian for the transformation involving Z and W becomes

$$J(x,y) = \begin{vmatrix} \frac{1}{1+y} & 0 \\ -\frac{x}{(1+y)^2} & 1 \end{vmatrix} = \frac{1}{1+y} = \frac{1}{1+w}. \tag{2.278}$$

The joint pdf of W and Y now becomes

$$f(w,z) = |1+w| f_{X,Y}(z(1+w), w). \tag{2.279}$$

The pdf of Z now becomes

$$f(z) = \int_{-\infty}^{\infty} |1+w| f_{X,Y}(z(1+w), w)\, \mathrm{d}w. \tag{2.280}$$

Since X and Y represent random variables which only take nonzero values, (2.280) can be rewritten as

$$f(z) = \int_{0}^{\infty} (1+w) f_{X,Y}(z(1+w), w)\, \mathrm{d}w. \tag{2.281}$$

Example #14 Instead of the SNR defined in (2.276), we might see cases where the random variable might be of the form

$$U = \frac{X}{X+Y}. \tag{2.282}$$

Note that the random variable U can be rewritten as

$$U = \frac{(X/Y)}{X/Y+1} = \frac{V}{1+V}. \tag{2.283}$$

Let us examine a specific case of interest in wireless systems where both X and Y are gamma distributed, as in (2.235) and (2.236) with the special case of $\alpha = \beta$. The CDF of the variable in (2.283) can be expressed as

$$F_U(u) = \text{Prob}\left[\frac{V}{1+V} \leq u\right] = \text{Prob}\left[V \leq \frac{u}{1-u}\right]. \tag{2.284}$$

Rewriting, we have

$$F_U(u) = F_V\left(\frac{u}{1-u}\right). \tag{2.285}$$

The density function becomes

$$f_U(u) = \frac{1}{(1-u)^2} f_V\left(\frac{u}{1-u}\right). \tag{2.286}$$

Using (2.266) for the density of the ratio of two gamma variables, we have

$$f(u) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} u^{m-1}(1-u)^{n-1}, \quad 0 < u < 1. \tag{2.287}$$

Note that (2.287) is the beta pdf described earlier.

Example #15 We will now establish the relationship among the Gaussian, Chi-squared and Student t distributions. Let

$$Z = \frac{X}{\sqrt{Y/n}}. \tag{2.288}$$

In (2.288), X is a Gaussian variable having a zero mean and unit standard deviation identified as $N(0,1)$ and Y is the chi squared variable identified by $\chi^2(n)$. Using the concept of an auxiliary or "dummy" variable, let us define

$$W = Y. \tag{2.289}$$

The Jacobian of the transformation becomes

$$J(x, y) = \sqrt{\frac{n}{w}}. \tag{2.290}$$

The joint density function of Z and W now becomes

$$\begin{aligned} f_{Z,W}(z,w) &= \sqrt{\frac{w}{n}} f(x,y) \\ &= \sqrt{\frac{w}{n}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w}{2n} z^2\right) \frac{w^{(n/2)-1}}{2^{(n/2)}\Gamma(n/2)} \exp\left(-\frac{w}{2}\right). \end{aligned} \tag{2.291}$$

The density function of Z is obtained as

$$f_Z(z) = \int_0^{\infty} f(z,w)\,\mathrm{d}w. \tag{2.292}$$

The limits of integration reflect the fact that $\chi^2(n)$ density function in (2.35) exists only in the range 0 to ∞. Carrying out the integration, we have

$$f(z) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)} \frac{1}{\sqrt{\left(1+(z^2/n)\right)^{n+1}}}, \quad -\infty < x < \infty. \tag{2.293}$$

Note that (2.293) is identical to the Student t-distribution seen in (2.107).

2.9 Some Bivariate Correlated Distributions of Interest in Wireless Communications

We will now look at a few joint distributions that are used in the analysis of wireless communication systems. While the joint pdf is the product of marginal pdfs, when the variables are independent (this property was used in some or most of the examples given above), their functional forms are unique when correlation exists between the two variables.

2.9.1 Bivariate Normal pdf

$$f(x,y) = A\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\eta_1)^2}{\sigma_1^2} - \frac{2\rho(x-\eta_1)(y-\eta_2)}{\sigma_1\sigma_2} + \frac{(x-\eta_2)^2}{\sigma_2^2}\right]\right\} \tag{2.294}$$

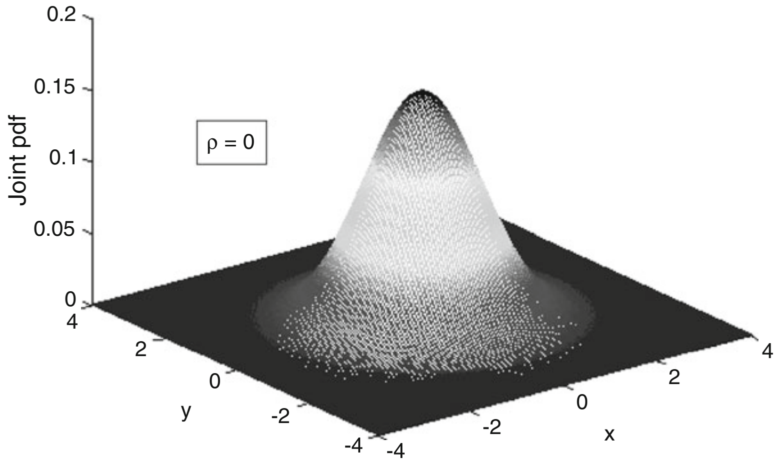

Fig. 2.35 Joint pdf of two Gaussian variables ($\rho = 0$)

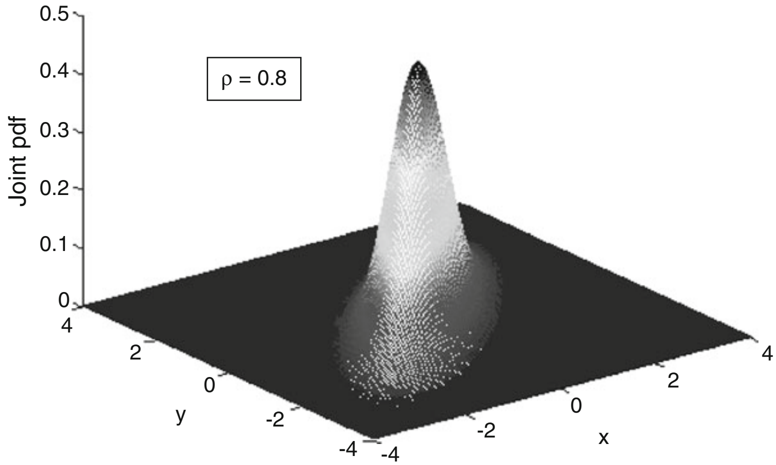

Fig. 2.36 Joint pdf of two Gaussian variables ($\rho = 0.8$)

In (2.294),

$$A = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}, \quad |\rho| \leq 1 \tag{2.295}$$

X and Y are Gaussian with means of η_1 and η_2 and standard deviations of σ_1 and σ_2 respectively, and ρ is the correlation coefficient defined earlier in (2.131). Note that when ρ is zero, (2.294) becomes the product of the marginal density functions

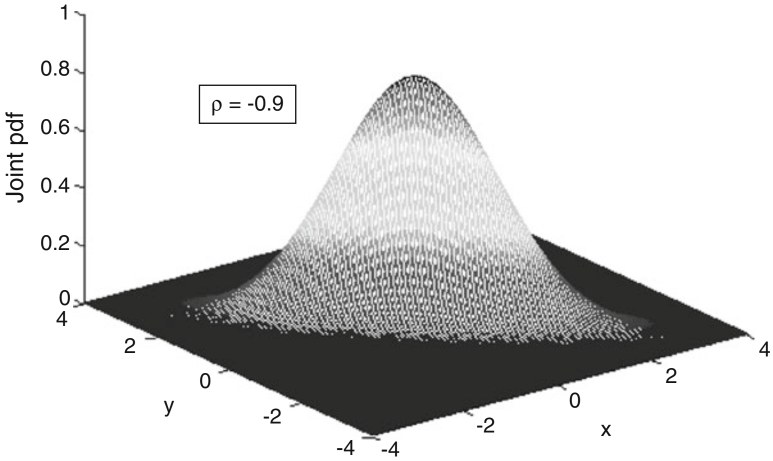

Fig. 2.37 Joint pdf of two Gaussian variables ($\rho = -0.9$)

of X and Y. Note that for the bivariate Gaussian, uncorrelatedness also implies independence. The joint Gaussian pdf is plotted in Figs. 2.35–2.37 for two zero mean variables each with unit variance for three values of the correlation coefficient ($\rho = 0$, 0.8 and -0.9). One can see that the joint pdf which is symmetric for the independent case ($\rho = 0$) takes on a ridge-like shape as the correlation increases (Papoulis and Pillai 2002).

2.9.2 Bivariate Nakagami pdf

The bivariate Nakagami pdf can be written as (Nakagami 1960; Tan and Beaulieu 1997; Karagiannidis et al. 2003a, b)

$$f(x,y) = B \exp\left[-\frac{m}{(1-\rho)}\left(\frac{x^2}{\Omega_x} + \frac{y^2}{\Omega_y}\right)\right] I_{m-1}\left[\frac{2mxy\sqrt{\rho}}{(1-\rho)\sqrt{\Omega_x \Omega_y}}\right] \tag{2.296}$$

with

$$B = \frac{4m^{m+1}(xy)^m}{\Gamma(m)\Omega_x\Omega_y(1-\rho)\left(\sqrt{\rho\Omega_x\Omega_y}\right)^{m-1}}. \tag{2.297}$$

In (2.296),

$$\Omega_x = E(X^2), \quad \Omega_y = E(Y^2). \tag{2.298}$$

The parameter ρ is the power correlation coefficient given by

$$\rho = \frac{\operatorname{cov}(X^2, Y^2)}{\sqrt{\operatorname{var}(X^2)\operatorname{var}(Y^2)}}. \tag{2.299}$$

In (2.299), cov(,) is the covariance defined in (2.126) and var is the variance. Note that m is the Nakagami parameter which has been considered to be identical for the two variables, X and Y. $I_{m-1}()$ is the modified Bessel function of the first kind of order $(m - 1)$.

The bivariate Rayleigh pdf is obtained from (2.299) by putting $m = 1$. We have

$$f(x,y) = \frac{4xy}{(1-\rho)\Omega_x\Omega_y} \exp\left[-\frac{1}{(1-\rho)}\left(\frac{x^2}{\Omega_x} + \frac{y^2}{\Omega_y}\right)\right] I_0\left[\frac{2xy\sqrt{\rho}}{(1-\rho)\sqrt{\Omega_x\Omega_y}}\right]. \tag{2.300}$$

2.9.3 Bivariate Gamma pdf

There are several forms of the bivariate gamma pdf. One of these representations is (Kotz and Adams 1964; Mathai and Moschopoulos 1991; Tan and Beaulieu 1997; Yue et al. 2001; Holm and Alouini 2004; Nadarajah and Gupta 2006; Nadarajah and Kotz 2006a, b)

$$f(x,y) = \frac{(xy/\rho)^{\left(\frac{m-1}{2}\right)}}{\alpha^{m+1}\Gamma(m)(1-\rho)} \exp\left[-\frac{x+y}{\alpha(1-\rho)}\right] I_{m-1}\left[\frac{2\sqrt{\rho xy}}{\alpha(1-\rho)}\right]. \tag{2.301}$$

Note that ρ is the correlation coefficient of X and Y (of identical order m and parameter α), and the density function in (2.301) is identified as the Kibble's bivariate gamma distribution. It must be noted that a pdf similar to (2.301) can also be obtained from (2.296) by converting to power values and replacing Ω/m by α. The plots of the bivariate correlated gamma density functions are shown in Figs. 2.38–2.40 for three values of the correlation ($m = 1$).

Another form of a bivariate gamma distribution is known as McKay's bivariate gamma distribution (Nadarajah and Gupta 2006) and the density function is given by

$$f(x,y) = \frac{a^{p+q}x^{p-1}}{\Gamma(p)\Gamma(q)}(y-x)^{q-1}\exp(-ay), \\ y > x > 0,\ a > 0,\ p > 0,\ q > 0. \tag{2.302}$$

An examination of the density function in (2.302) clearly shows that the two random variables X and Y are not independent.

A different form of bivariate gamma pdf is known as the Arnold and Strauss's bivariate gamma distribution. It has the joint pdf expressed as (Nadarajah 2005)

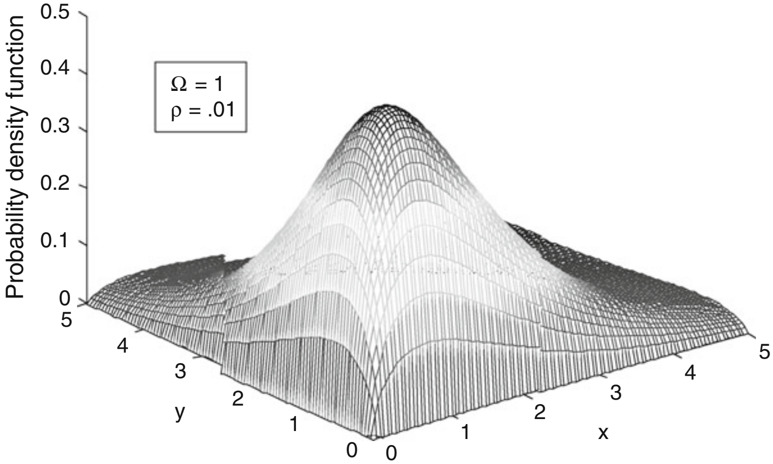

Fig. 2.38 Bivariate gamma pdf ($\rho = .01$ and $\alpha = \Omega = 1$)

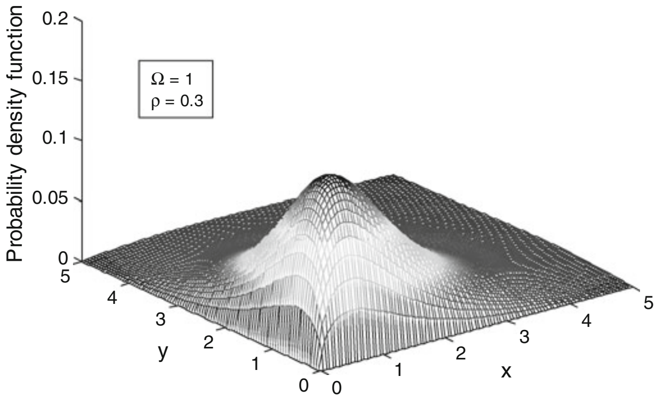

Fig. 2.39 Bivariate gamma pdf ($\rho = .3$ and $\alpha = \Omega = 1$)

$$f(x,y) = Kx^{\alpha-1}y^{\beta-1}\exp[-(ax+by+cxy)],\\ x>0,\ y>0,\ a>0,\ b>0,\ c>0,\ \alpha>0,\ \beta>0. \tag{2.303}$$

The parameter K is the normalization factor and it can be seen that the two random variables X and Y are not independent.

2.9.4 Bivariate Generalized Gamma pdf

The bivariate generalized gamma pdf can be obtained from the bivariate gamma pdf using the transformation of variables (Aalo and Piboongungon 2005). We have

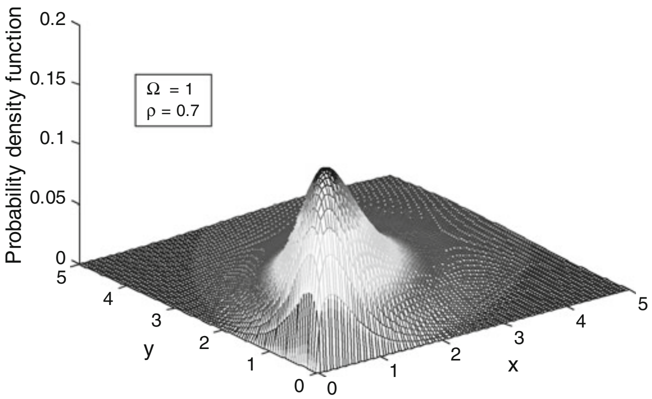

Fig. 2.40 Bivariate gamma pdf ($\rho = .7$ and $\alpha = \Omega = 1$)

$$f(z,w) = \frac{\lambda^2 m^{m+1} (zw)^{((\lambda(m+1))/2)-1} \rho^{((1-m)/2)}}{(\Omega_z \Omega_w)^{((1+m)/2)} (1-\rho)\Gamma(m)} \times \exp\left[-\frac{m}{(1-\rho)}\left(\frac{z^\lambda}{\Omega_z} + \frac{w^\lambda}{\Omega_w}\right)\right] I_{m-1}\left[\frac{2m}{1-\rho}\sqrt{\frac{\rho(zw)^\lambda}{\Omega_z\Omega_w}}\right]. \tag{2.304}$$

In (2.304),

$$\Omega_z = E(Z^\lambda), \quad \Omega_w = E(W^\lambda) \tag{2.305}$$

and ρ is the correlation coefficient between Z^2 and W^2. Note that by putting $m = 1$ in (2.305), we can obtain an expression for the bivariate Weibull pdf.

2.9.5 Bivariate Weibull pdf

The bivariate Weibull pdf can be obtained from the bivariate generalized gamma pdf in (2.304) by putting $m = 1$ as

$$f(z,w) = \frac{\lambda^2 (zw)^{\lambda-1}}{(\Omega_z\Omega_w)(1-\rho)} \exp\left[-\frac{1}{(1-\rho)}\left(\frac{z^\lambda}{\Omega_z} + \frac{w^\lambda}{\Omega_w}\right)\right] I_{m-1}\left[\frac{2}{1-\rho}\sqrt{\frac{\rho(zw)^\lambda}{\Omega_z\Omega_w}}\right] \tag{2.306}$$

with

$$\Omega_z = E(Z^\lambda), \quad \Omega_w = E(W^\lambda). \tag{2.307}$$

2.9.6 Bivariate Rician Distribution

The bivariate Rician distribution of the SNR values Z and W can be expressed as (Zogas and Karagiannidis 2005; Bithas et al. 2007; Panajotovic et al. 2009)

$$f(z,w) = \frac{(1+K_0)^2}{2\pi Z_R^2(1-\rho^2)} \exp\left[-\frac{2K_0}{1+\rho} - \frac{(1+K_0)(z+w)}{2Z_R(1-\rho^2)}\right] R. \tag{2.308}$$

In (2.308),

$$R = \int_0^{2\pi} \exp\left[\frac{2\rho(1+K_0)\sqrt{zw}\cos(\theta)}{(1-\rho^2)Z_R}\right] I_0\left[\sqrt{\frac{4K_0(1+K_0)(z+w+2\sqrt{zw}\cos(\theta))}{Z_R(1+\rho)^2}}\right] \mathrm{d}\theta. \tag{2.309}$$

The parameter ρ is correlation coefficient between the two envelope values A_1 and A_2 corresponding to the SNR values of Z and W respectively as

$$Z = A_1^2, \quad W = A_2^2. \tag{2.310}$$

The two variables (Z and W) are considered to be identical with equal average SNR values equal to $\mathrm{Z_R}$. The Rician factor is given by K_0 and was defined in (2.214).

2.10 Order Statistics

Another important and interesting statistical entity of interest in wireless systems is the order statistics (Rohatgi and Saleh 2001; Papoulis and Pillai 2002). For example, in selection combining (SC) algorithm, we are interested in finding the largest of a set of outputs and in generalized selection combining (GSC), we are interested in picking the M largest of a total of L outputs (Alouini and Simon 1999; Ma and Chai 2000; Alouini and Simon 2006; Annamalai et al. 2006). Both of these outcomes can be analyzed in terms of the order statistics. Let X_1, X_2,..., X_L correspond to the a set of L random variables. We will assume that the random variables are independent and identical. Our interest is in finding out the joint pdf of the largest M variables, i.e., in finding the joint pdf of

$$\{\mathrm{X}_1,\mathrm{X}_2,\ldots,\mathrm{X}_M\}, \mathrm{X}_1 > \mathrm{X}_2 > \mathrm{X}_3 > \cdots > \mathrm{X}_M, \quad M \le L. \tag{2.311}$$

Even though the random variables are the same, to make matters simple and eliminate confusion because the outputs and inputs of the selection process will contain X's, the output set will be identified as $\{Y_1,Y_2,\ldots,Y_M\}$. The joint CDF of $\{Y_1,Y_2,\ldots,Y_M\}$can be written as

$$F(\mathrm{y}_1,\mathrm{y}_2,\ldots,\mathrm{y}_M)=\sum_{i=1}^{L}F_{X_i}(y_1)\sum_{j=1,j\neq i}^{L}F_{X_j}(y_2)\ldots\prod_{n=1,n\neq i,j,\ldots}^{L}F_{Xn}(y_M),\quad y_1>y_2>y_3>\cdots>y_M\,. \tag{2.312}$$

Since the random variables have been considered to be identical, (2.312) becomes

$$F(\mathrm{y}_1,\mathrm{y}_2,\ldots,\mathrm{y}_M)=LF(y_1)(L-1)F(y_2)\ldots(L-M)F(y_{M-1})[F(y_M)]^{L-M+1}. \tag{2.313}$$

Equation (2.313) can be explained as follows. There are L different ways of choosing the largest. Thus, this probability will be $LF(y_1)$. There are $(L-1)$ ways of picking the second largest, and the probability of doing so will be $(L-1)F(y_2)$ and so on. There are (L–M) ways of picking the next to the last in that order, and the probability of doing so will be

$(L-M)F(y_{M-1})$. Since the rest of the variables $(L-M+1)$ will be either equal to or smaller than $\mathrm{y_M}$, the probability of this event will be the last term in (2.313). The joint pdf can be obtained by differentiating (2.313) with respect to $y_1,y_2,\ldots,y_M$, and we have

$$f(\mathrm{y}_1,\mathrm{y}_2,\ldots,\mathrm{y}_M)=Lf(y_1)(L-1)f(y_2)\ldots(L-M+1)f(y_m)[F(y_m)]^{L-M}. \tag{2.314}$$

Equation (2.314) can be easily expressed as

$$f(\mathrm{y}_1,\mathrm{y}_2,\ldots,\mathrm{y}_M)=\Gamma(M+1)\binom{L}{M}[F(y_m)]^{L-M}\prod_{k=1}^{M}f(y_k),\quad y_1>y_2>y_3>\cdots>y_M. \tag{2.315}$$

In (2.315),

$$\binom{L}{M}=\frac{L!}{M!(L-M)!}. \tag{2.316}$$

If we are choosing the largest output, $M=1$, and we have the expression for the pdf of the maximum of L independent and identically distributed variables as

$$f(\mathrm{y}_1)=L[F(y_1)]^{L-1}f(y_1). \tag{2.317}$$

If one puts $L=2$ in (2.317), we have the pdf of the largest of the two random variables, obtained earlier in (2.173).

We can now obtain the pdf of the kth largest variable. Let Y_k be the kth largest variable. If $f_M(y)$ is the pdf of the variable Y_k, we can write

$$f_k(y)\ \mathrm{d}y = \mathrm{Prob}\{y \le Y_k \le y + \mathrm{d}y\}. \tag{2.318}$$

The event $\{y \le Y_k \le y + \mathrm{d}y\}$ occurs *iff* exactly $(k - 1)$ variables less than y, one in the interval $\{y, y + \mathrm{d}y\}$, and $(L{-}k)$ variables greater than y. If we identify these events as I and II and III respectively,

$$I = (x \le y), \quad II = (y \le x \le y + \mathrm{d}y), \quad III = (x > y + \mathrm{d}y). \tag{2.319}$$

The probabilities of these events are

$$\mathrm{Prob}(I) = F_X(y), \quad \mathrm{Prob}(I) = f_X(y)\ \mathrm{d}y, \quad \mathrm{Prob}(III) = 1 - F_X(y). \tag{2.320}$$

Note that event I occurs $(k - 1)$ times, event II occurs once, and event III occurs $L{-}k$ times. Using the concept of generalized Bernoulli trial and (2.31), (2.318) becomes

$$f_k(y)\ \mathrm{d}y = \frac{L!}{(k-1)!1!(L-k)!}[F_X(y)]^{k-1} f_X(y)\ \mathrm{d}y[1 - F_X(y)]^{L-k}. \tag{2.321}$$

Equation (2.321) simplifies to

$$f_k(y) = \frac{L!}{(k-1)!(L-k)!}[F_X(y)]^{k-1}[1 - F_X(y)]^{L-k} f_X(y). \tag{2.322}$$

When k equals L, we get the pdf of the largest variable (maximum), and (2.322) becomes (2.317). If $k = 1$, we get the pdf of the smallest of the random variable (minimum) as described below.

We can also obtain the density function of the minimum of L random variables. Let us define

$$Z = \min\{x_1, x_2, \ldots, x_L\}. \tag{2.323}$$

Noting the probability that at least one of the variables is less than Z, we can express the CDF as

$$F_Z(z) = 1 - \mathrm{Prob}\{x_1 > z, x_2 > z, \ldots, x_L > z\}\ = 1 - [1 - F_X(z)]^L. \tag{2.324}$$

In (2.324), $F_X(.)$ and $f_X(.)$ are the marginal CDF and pdf of the X's which are treated to be identical and independent. Differentiating the CDF, we have the density function of the minimum of a set of random variables as

$$f_Z(z) = L[1 - F_X(z)]^{L-1} f_X(z). \tag{2.325}$$

Note that (2.322) can also be used to obtain the pdf of the minimum of the random variables by setting $k = 1$ and (2.322) reduces to (2.325).

Before we look at the specific cases of interest in wireless communications, let us look at the special case of the exponential pdf. If X's are independent and identically distributed with exponential pdf,

$$f(x_i) = \frac{1}{\alpha}\exp\left(-\frac{x_i}{\alpha}\right), \quad i = 1, 2, \ldots, L \tag{2.326}$$

the density function of the minimum of the set can be written using (2.325) as

$$f(z) = \frac{L}{\alpha}\exp\left(-\frac{L}{\alpha}z\right). \tag{2.327}$$

In other words, the minimum of a set of independent and identically distributed exponential random variables is another exponentially distributed random variable with a mean of (α/L), i.e., with a reduced mean.

2.10.1 A Few Special Cases of Order Statistics in Wireless Communications

We will now look at a few special cases of interest in wireless systems, specifically, the cases of bivariate Nakagami (or gamma) and bivariate lognormal distributions. We will start with the bivariate Nakagami distribution. The bivariate Nakagami pdf given in (2.296) is

$$\begin{aligned} f(a_1, a_2) = {} & \frac{4m^{m+1}(a_1 a_2)^m}{\Gamma(m) P_{01} P_{02} (1-\rho)\left(\sqrt{P_{01}P_{02}\rho}\right)^{m-1}} \\ & \times \exp\left[-\frac{m}{(1-\rho)}\left(\frac{a_1^2}{P_{01}} + \frac{a_2^2}{P_{02}}\right)\right] I_{m-1}\left(2m\frac{a_1 a_2\sqrt{\rho}}{(1-\rho)\sqrt{P_{01}P_{02}}}\right). \end{aligned} \tag{2.328}$$

In (2.328), m is the Nakagami parameter considered to be identical for the two variables, a_1 and a_2. The other parameters are

$$P_{01} = \langle a_1^2 \rangle, \quad P_{02} = \langle a_2^2 \rangle, \quad \rho = \frac{\operatorname{cov}(a_1^2, a_2^2)}{\operatorname{var}(a_1^2)\operatorname{var}(a_2^2)}. \tag{2.329}$$

In (2.329), cov is the covariance and var is the variance. Note that by putting $m = 1$ in (2.328), we get the bivariate Rayleigh pdf

$$f(a_1, a_2) = \frac{4(a_1 a_2)}{P_{01} P_{02}(1-\rho)} \exp\left[-\frac{1}{(1-\rho)}\left(\frac{a_1^2}{P_{01}} + \frac{a_2^2}{P_{02}}\right)\right] I_0\left(2\frac{a_1 a_2 \sqrt{\rho}}{(1-\rho)\sqrt{P_{01} P_{02}}}\right). \tag{2.330}$$

One of the interesting uses of the bivariate correlated pdf is in diversity combining. We will obtain the pdf of the selection diversity combiner in which the output of the diversity algorithm is the stronger of the two input signals. We will also simplify the analysis by assuming that the signals have identical powers, i.e., $P_{01} = P_{02} = P_0$. Since the comparison is made on the basis of power or SNR, we will rewrite (2.328) in terms of the powers

$$X = a_1^2, \quad Y = a_2^2 \tag{2.331}$$

as

$$f(x, y) = \frac{m^{m+1} (xy)^{((m-1)/2)} \rho^{((1-m)/2)}}{\Gamma(m) P_0^{m+1} (1-\rho) \left(\sqrt{\rho}\right)^{m-1}} \exp\left[-\frac{m(x+y)}{P_0(1-\rho)}\right] I_{m-1}\left(2m\frac{\sqrt{xy\rho}}{P_0(1-\rho)}\right). \tag{2.332}$$

Note that (2.332) is identical to the Kibble's bivariate gamma pdf given in (2.301) with an appropriately scaled average power. If we define

$$Z = \max(X, Y). \tag{2.333}$$

The pdf of the selection combining can be expressed as

$$f(z) = \frac{\mathrm{d}}{\mathrm{d}z}[\mathrm{Prob}(X < z, Y < z)]. \tag{2.334}$$

The expression for the pdf becomes

$$f(z) = 2\frac{m^m}{\Gamma(m) P_0}\left(\frac{z}{P_0}\right)^{m-1} \exp\left(-m\frac{z}{P_0}\right)\left[1 - Q_m\left(\sqrt{\frac{2m\rho}{1-\rho}\left(\frac{z}{P_0}\right)}, \sqrt{\frac{2m}{1-\rho}\left(\frac{z}{P_0}\right)}\right)\right]. \tag{2.335}$$

In (2.335), $Q_m(,)$ is the generalized Marcum Q function given by (Simon 2002)

$$Q_m(\alpha, \beta) = \frac{1}{\alpha^{m-1}} \int_\beta^\infty w^m \exp\left(-\frac{w^2 + \alpha^2}{2}\right) I_{m-1}(\alpha w)\, \mathrm{d}w. \tag{2.336}$$

We will also obtain the pdf of the maximum of two correlated lognormal random variables. As we will discuss in Chap. 3, lognormal density function is often used to model shadowing. It has been shown that short-term fading in some of the indoor wireless channels can be modeled using the lognormal density functions. Consequently, one can find the use of bivariate lognormal densities in the analysis of diversity algorithms (Ligeti 2000; Alouini and Simon 2003; Piboongungon and Aalo 2004).

Using the relationship between normal pdf and lognormal pdf, it is possible to write the expression for the joint pdf of two correlated lognormal random variables X and Y. The pdf is given by

$$f(x,y) = \frac{D}{xy}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{10\log_{10}(x)-\mu_x}{\sigma_x}\right)^2+\left(\frac{10\log_{10}(y)-\mu_y}{\sigma_y}\right)^2-2\rho g(x,y)\right]\right\}. \tag{2.337}$$

In (2.337),

$$D = \frac{A_0^2}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}, \tag{2.338}$$

$$g(x,y) = \left(\frac{10\log_{10}(x)-\mu_x}{\sigma_x}\right)\left(\frac{10\log_{10}(y)-\mu_y}{\sigma_y}\right). \tag{2.339}$$

Note that μ's are the means and σ's the standard deviations of the corresponding Gaussian random variables in decibel units. A_0 was defined earlier and it is given by

$$A_0 = \frac{10}{\log_e(10)}. \tag{2.340}$$

As in the case of the Nakagami random variables, ρ is the correlation coefficient (power). If Z represents the maximum of the two random variables, the pdf of the maximum can be written either as in (2.334) or as

$$f(z) = \int_0^z f_{X,Y}(z,y)\,\mathrm{d}y + \int_0^z f_{X,Y}(x,z)\,\mathrm{d}x. \tag{2.341}$$

In (2.341), $f(x,y)$ is the joint pdf of the dual lognormal variables expressed in (2.337). The integrations in (2.341) can be performed by transforming the variables to Gaussian forms by defining

$$\alpha = \frac{10\log_{10}(x)-\mu_x}{\sigma_x}, \tag{2.342}$$

$$\beta = \frac{10\log_{10}(y) - \mu_y}{\sigma_y}, \tag{2.343}$$

$$z_{\mathrm{dB}} = 10\log_{10}(z). \tag{2.344}$$

We get

$$f(z) = \exp\left[-\frac{1}{2}\left(\frac{z_{\mathrm{dB}} - \mu_x}{\sigma_x}\right)^2\right] g_1 + \exp\left[-\frac{1}{2}\left(\frac{z_{\mathrm{dB}} - \mu_y}{\sigma_y}\right)^2\right] g_2. \tag{2.345}$$

In (2.345), g_1 and g_2 are given by

$$g_1 = Q\left[-\frac{(z_{\mathrm{dB}} - \mu_y)/\sigma_y - \rho((z_{\mathrm{dB}} - \mu_x)/\sigma_x)}{\sqrt{1-\rho^2}}\right], \tag{2.346}$$

$$g_2 = Q\left[-\frac{(z_{\mathrm{dB}} - \mu_x)/\sigma_x - \rho((z_{\mathrm{dB}} - \mu_y)/\sigma_y)}{\sqrt{1-\rho^2}}\right]. \tag{2.347}$$

In (2.346) and (2.347),

$$Q(\lambda) = \int_{\lambda}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2}\right) \mathrm{d}\phi. \tag{2.348}$$

The CDF of the maximum of two lognormal random variables can now be expressed as

$$F(z) = 1 - Q\left(\frac{\alpha - \mu_x}{\sigma_x}\right) - Q\left(\frac{\beta - \mu_y}{\sigma_y}\right) + Q\left(\frac{\alpha - \mu_x}{\sigma_x}, \frac{\beta - \mu_y}{\sigma_y}; \rho\right). \tag{2.349}$$

In (2.349), the last function is given by (Simon and Alouini 2005)

$$Q(u, v; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_u^{\infty} \int_v^{\infty} \exp\left[-\frac{x^2 + y^2 - 2\rho xy}{2(1-\rho^2)}\right] \mathrm{d}x\, \mathrm{d}y. \tag{2.350}$$

2.11 Decision Theory and Error Rates

Digital communication mainly involves the transmission of signals with discrete values (0's and 1's or any other set of M values $M > 1$) and reception of those signals (Van Trees 1968; Helstrom 1968; Middleton 1996). The channel adds noise.

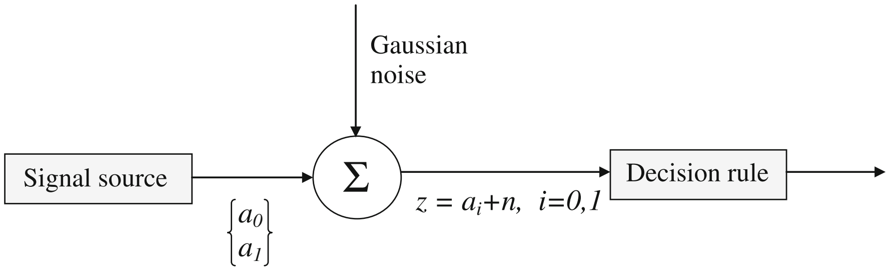

Fig. 2.41 Hypothesis testing problem

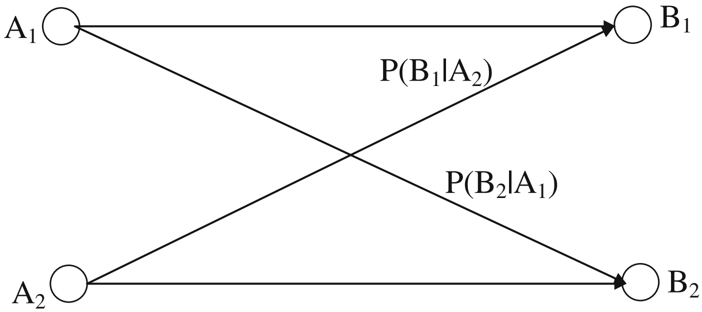

Fig. 2.42 The binary channel

Noise as stated earlier is typically modeled as a Gaussian random variable with a means of zero and a standard deviation. But, noise could also be non-Gaussian. We will examine the two cases separately.

2.11.1 Gaussian Case

As mentioned above, the received signal consists of the transmitted signal plus noise. Thus, the problem of identifying the received discrete values becomes one of hypothesis testing (Fig. 2.41). This problem is shown in Fig. 2.37. We will examine the case of a binary system where 0's and 1's are transmitted. Note that these two bits are represented by the values a_0 and a_1. (Benedetto and Biglieri 1999; Simon et al. 1995; Proakis 2001; Haykin 2001; Couch 2007.)

Let us first examine what happens when the signals move through the channel. Because of the noise, "0" might be detected as a "1" or "0." Similarly, "1" might be detected as a "0" or "1." This scenario points to two distinct ways of making an error at the output. These two are the detection of "0" as a "1" and detection of "1" as a "0." This is shown in Fig. 2.42. We can invoke Bayes theorem in probability theory to estimate this error (Papoulis and Pillai 2002). If we have two events, *A* and

B, each with probabilities of $P(A)$ and $P(B)$, respectively, the conditional probability can be expressed in terms of Bayes theorem as

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}. \tag{2.351}$$

Extending this notion to multiple events $A = \{A_1, A_2, \ldots, A_M\}$ and $B = \{B_1, B_2, \ldots, B_M\}$, the Bayes rule becomes

$$P(A_i|B_j) = \left.\frac{P(B_j|A_i)P(A_i)}{P(B_j)}\right\} \quad \begin{matrix} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, M \end{matrix}, \tag{2.352}$$

where

$$P(B_j) = \sum_{i=1}^{M} P(B_j|A_i)P(A_i). \tag{2.353}$$

In our description, A's represent the input and B's represent the received signals. We also identify $P(A_i)$ as the a priori probability, $P(B_j|A_i)$ as the conditional probability and $P(A_i|B_j)$ as the a posteriori probability. Equation (2.353) shows that each output will have contributions from the complete input set. We can now introduce the notion of errors by looking at a binary case ($M = 2$). In the case of binary signal transmission, we have two inputs $\{A_1, A_2\}$ and two corresponding outputs $\{B_1, B_2\}$. Since there is noise in the channel, B_1 could be read as A_2 and B_2 could be read as A_1. Thus, we have two ways in which error can occur. If the error is represented by e, the probability of error can be written using the Bayes rule as

$$p(e) = P(B_2|A_1)P(A_1) + P(B_1|A_2)P(A_2). \tag{2.354}$$

We will now expand this notion to the transmission of digital signals through a channel corrupted by noise. The received signal can be written as

$$z = a_i + n, \quad i = 1, 2. \tag{2.355}$$

Since the noise is Gaussian, we can represent the density function of noise as (Shanmugam 1979; Schwartz 1980; Schwartz et al. 1996; Sklar 2001)

$$f(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{n^2}{2\sigma^2}\right). \tag{2.356}$$

Note that even though a's only takes discrete values, z will be continuous as the noise is not discrete. The density function of the received signal can be represented as conditional density functions,

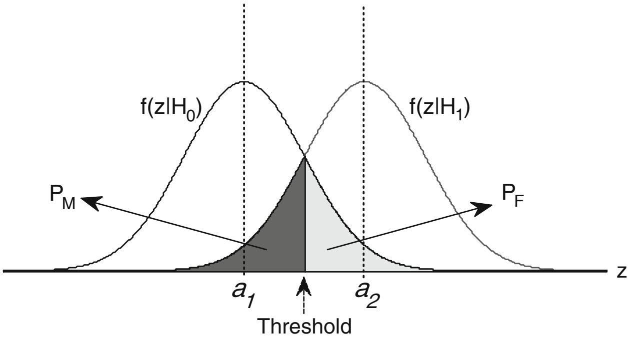

Fig. 2.43 Concept of hypothesis testing

$$f(z|a_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(z-a_1)^2}{2\sigma^2}\right], \tag{2.357}$$

$$f(z|a_2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(z-a_2)^2}{2\sigma^2}\right]. \tag{2.358}$$

We will now identify the two hypotheses we have as H_0 and H_1, the former corresponding to the transmission "0" (signal strength a_0) and the latter corresponding to the transmission of "1" (signal strength a_1). We will rewrite (2.357) and (2.358) as

$$f(z|a_0) \equiv f(z|H_0), \tag{2.359}$$

$$f(z|a_1) \equiv f(z|H_1). \tag{2.360}$$

These two density functions are plotted in Fig. 2.43.

Using simple logic, it would seem that we can decide on whether the received signal belongs to hypothesis H_0 or H_1 as (Van Trees 1968)

$$f(H_1|z) \underset{H_0}{\overset{H_1}{\gtrless}} f(H_0|z). \tag{2.361}$$

Using the Bayes rule in (2.351)

$$f(z|H_1)P(H_1) \underset{H_0}{\overset{H_1}{\gtrless}} f(z|H_0)P(H_0) \tag{2.362}$$

or

$$\left[\frac{f(z|H_1)}{f(z|H_0)}\right] \underset{H_0}{\overset{H_1}{\gtrless}} \left[\frac{P(H_0)}{P(H_1)}\right]. \tag{2.363}$$

The left-hand side of (2.363) is known as the likelihood ratio; the entire equation is identified as the likelihood ratio test. If both hypotheses are equally likely, the right-hand side of (2.363) will be unity and take logarithms. We arrive at the log likelihood ratio:

$$\log_e\left[\frac{f(z|H_1)}{f(z|H_0)}\right] \underset{H_0}{\overset{H_1}{\gtrless}} 0. \tag{2.364}$$

Substituting (2.357) and (2.358), we have the threshold for the decision as

$$z = z_{\text{thr}} = \frac{(a_1 + a_2)}{2} \tag{2.365}$$

Thus, (2.365) demonstrates that the threshold for making decision on whether H_1 is accepted for H_0 is accepted is the midpoint of the two amplitude values. The errors can occur in two ways. First, even when a "0" was transmitted, because of noise, it can be read/detected as a "1." This is called the probability of false alarm (P_{F}) and is given by the probability indicated by the shaded area in Fig. 2.12.

$$P_{\text{F}} = \int_{z_{\text{thr}}}^{\infty} f(z|H_0)\,\mathrm{d}z. \tag{2.366}$$

Second, when "1" was sent, it can be read/detected as "0." This is called the probability of miss (P_{M}) and it is given by the shaded area corresponding to the probability

$$P_{\text{M}} = \int_{-\infty}^{z_{\text{thr}}} f(z|H_1)\,\mathrm{d}z. \tag{2.367}$$

The average probability of error is given by

$$p(e) = p(e|H_1)P(H_1) + p(e|H_0)P(H_0) \tag{2.368}$$

or

$$p(e) = P_{\text{F}}P(H_1) + P_{\text{M}}P(H_0). \tag{2.369}$$

If both hypotheses are equally likely (0's and 1's being transmitted with equal probability), the average probability of error will be

$$p_{\mathrm{e}} = P_{\mathrm{F}} \quad \text{or} \quad P_{\mathrm{M}}. \tag{2.370}$$

It can be seen that the average probability of error can be expressed in terms of the CDF of the Gaussian random variable as well as the complimentary error functions (Sklar 2001).

2.11.2 Non-Gaussian Case

While the detection of signals in additive white Gaussian noise is generally encountered in communications, there are also instances when the receiver makes decisions based on the envelope of the signal. Examples of these include envelope detection of amplitude shift keying and frequency shift keying. It also includes distinguishing the signals at the wireless receiver from the signal of interest and the interference, both of which could be Rayleigh distributed when short-term fading is present. As discussed earlier the statistics of the envelope might also be Rician distribution. There is a need to minimize the error rate by determining the optimum value of the threshold.

We would like to examine three separate cases, the first where both hypotheses lead to Rayleigh densities, the second where one of the hypothesis leads to Rayleigh while the other hypothesis results in Rician, and the third where both hypothesis result in Rician densities. Let the two density functions be

$$f(z|H_0) = \frac{z}{\sigma_0^2} \exp\left[-\frac{z^2}{2\sigma_0^2}\right], \quad z > 0, \tag{2.371}$$

$$f(z|H_1) = \frac{z}{\sigma_1^2} \exp\left[-\frac{z^2}{2\sigma_1^2}\right], \quad z > 0 \tag{2.372}$$

with

$$\sigma_1 > \sigma_0. \tag{2.373}$$

From (2.364) the log likelihood ratio test leads to a threshold of

$$z_{\mathrm{thr}} = \sigma_0\sigma_1\sqrt{\frac{\log_e(\sigma_1/\sigma_0)}{2(\sigma_1^2 - \sigma_0^2)}}. \tag{2.374}$$

The decision region and the two regions for the computation of error are shown in Fig. 2.44.

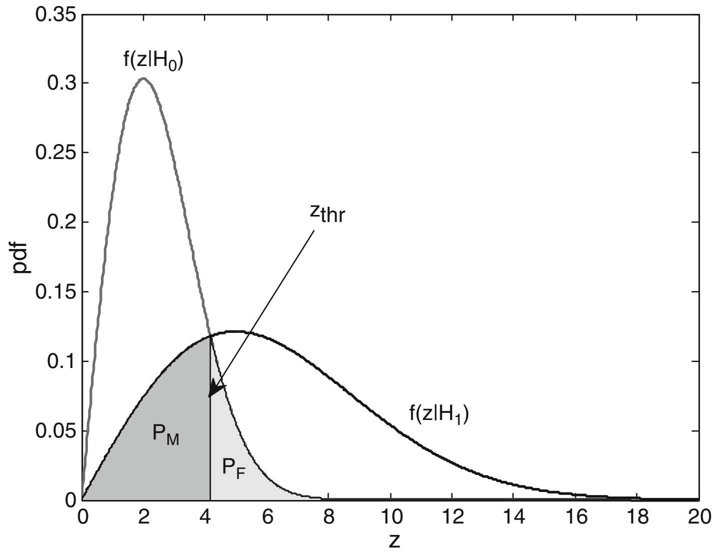

Fig. 2.44 Hypothesis testing (non-Gaussian case)

The values of σ_0 and σ_1 for the Fig. 2.44 are 2 and 5, respectively, and the threshold is at 4.177.

Next we consider the case where the density function of the envelope under the hypothesis H_0 is Rayleigh distributed, as previously given in (2.371), while the pdf of the envelope is Rician distributed for the second hypothesis, namely H_1. The pdf for the envelope under the hypothesis H_1 is as in (2.208)

$$f(z|H_1) = \frac{z}{\sigma_0^2} \exp\left[-\frac{z^2 + A^2}{2\sigma_0^2}\right] I_0\left(\frac{zA}{\sigma_0^2}\right), \quad z > 0. \tag{2.375}$$

Note that both the density functions in (2.371) and (2.375) come from Gaussian random variables of identical standard deviation σ_0 and A is >0. A direct solution of (2.364) is going to be difficult because of the existence of the Bessel function in (2.375). We will use an indirect approach for solving the threshold. Let us assume that η is the threshold. The probability of false alarm P_F will be given by (Van Trees 1968; Cooper and McGillem 1986)

$$P_\mathrm{F} = \int_\eta^\infty f(z|H_0)\,\mathrm{d}z = \exp\left(-\frac{\eta^2}{2\sigma_0^2}\right). \tag{2.376}$$

The probability of miss P_M will be

$$P_\mathrm{M} = \int_0^\eta f(z|H_1)\,\mathrm{d}z = \int_0^\eta \frac{z}{\sigma_0^2} \exp\left[-\frac{z^2 + A^2}{2\sigma_0^2}\right] I_0\left(\frac{zA}{\sigma_0^2}\right)\,\mathrm{d}z. \tag{2.377}$$

Since there is no analytical solution to (2.377), we will examine what happens if we assume that the direct component $A \gg \sigma_0$. Invoking the approximation to the modified Bessel function of the first kind mentioned in (2.210), and rewriting (2.304) as

$$P_{\mathrm{M}} = 1 - \int_{\eta}^{\infty} \frac{z}{\sigma_0^2} \exp\left[-\frac{z^2 + A^2}{2\sigma_0^2}\right] I_0\left(\frac{zA}{\sigma_0^2}\right) \mathrm{d}z \tag{2.378}$$

we get

$$P_{\mathrm{M}} = 1 - \int_{\eta}^{\infty} \frac{\sqrt{z/A}}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{(z-A)^2}{2\sigma_0^2}\right] \mathrm{d}z. \tag{2.379}$$

The integrand in (2.379) is sharply peaked at $z = A$, and the slowly varying factor $\sqrt{z/A}$ may be replaced with its value at the peak, i.e., unity, leading to a simpler expression for the probability of miss as

$$P_{\mathrm{M}} = 1 - \int_{\eta}^{\infty} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{(z-A)^2}{2\sigma_0^2}\right] \mathrm{d}z. \tag{2.380}$$

Using the Q function defined in (2.90), the probability of miss becomes

$$P_{\mathrm{M}} = 1 - Q\left[\frac{\eta - A}{\sigma_0}\right]. \tag{2.381}$$

The total error will be

$$p(e) = \frac{1}{2}[P_{\mathrm{F}} + P_{\mathrm{M}}]. \tag{2.382}$$

Taking the derivative of (2.382) and setting it equal to zero, we have

$$\frac{\mathrm{d}[P_{\mathrm{F}} + P_{\mathrm{M}}]}{\mathrm{d}\eta} = 0. \tag{2.383}$$

Using the derivative of the Q function as

$$\frac{\mathrm{d}[Q(x)]}{\mathrm{d}x} = -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.384}$$

(2.384) becomes

$$-\frac{\eta}{\sigma_0^2} \exp\left(-\frac{\eta^2}{2\sigma_0^2}\right) + \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{(\eta - A)^2}{2\sigma_0^2}\right] = 0. \tag{2.385}$$

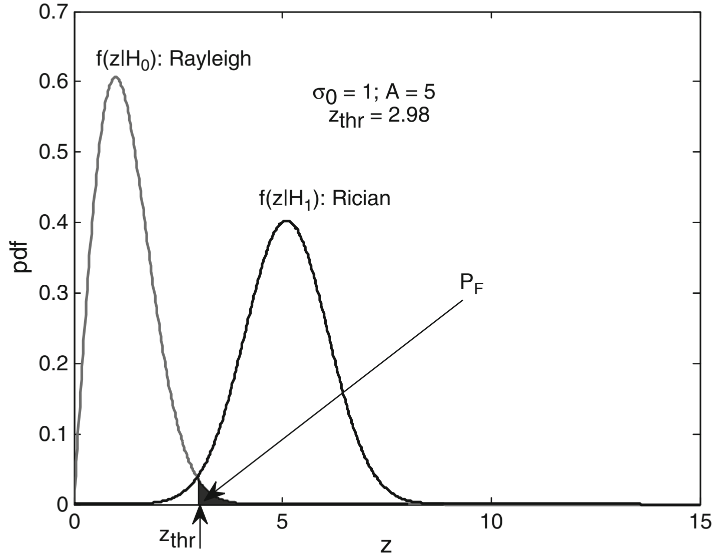

Fig. 2.45 Hypothesis testing (Rayleigh vs. Rician)

Once again, a solution to (2.385) is not straightforward. Instead, we import the concept from the Gaussian pdfs in (2.365). At high values of the SNR, as seen in (2.380), the envelope follows a Gaussian under the hypothesis H_1. We can argue that the threshold should be the midway point between the peaks of the two density functions. This is shown in Fig. 2.45. This would mean that the optimum threshold is

$$z_{\text{thr}} = \frac{1}{2}[\sigma_0 + A]. \tag{2.386}$$

Note that in (2.386), A is the mode of the pdf of the envelope under hypothesis H_1 and σ_0 the mode of the pdf under the hypothesis H_0. Since $A \gg \sigma_0$, the optimum threshold is

$$z_{\text{thr}} \approx \frac{A}{2}. \tag{2.387}$$

We can now look at the last case where the envelopes follow Rician pdf under both hypotheses, with direct components of A_0 and A_1 respectively. Using the Gaussian approximation to Rician, the optimum threshold will be

$$z_{\text{thr}} = \frac{1}{2}[A_1 + A_0]. \tag{2.388}$$

2.12 Upper Bounds on the Tail Probability

In wireless communications, it is often necessary to evaluate the probability that the SNR exceeds a certain value or that SNR fails to reach a certain threshold value. In Sect. 2.13, we saw two measures of error probabilities, a probability of false alarm, i.e., the area under the upper tail of the pdf curve beyond a certain value, and a probability of miss, i.e., the area under the lower tail below a certain value. Often, it might not be possible to evaluate such areas. It might be necessary to get an estimate of the upper bounds of such probabilities which correspond to areas under the tails of the pdf curves. We will look at two bounds, one based on the Chebyshev inequality and the other one called the Chernoff bound (Schwartz et al. 1996; Haykin 2001). The former one is a loose bound while the latter is much tighter.

2.12.1 Chebyshev Inequality

Let $f(x)$ be the pdf of a random variable X with a mean of m and standard deviation of σ. If δ is any positive number, the Chebyshev inequality is given by (Papoulis and Pillai 2002)

$$P(|X - m| \geq \delta) \leq \frac{\sigma^2}{\delta^2}. \tag{2.389}$$

In (2.389), $P(.)$ is the probability. Equation (2.389) can be established from the definition of variance as

$$\sigma^2 = \int_{-\infty}^{\infty} (x - m)^2 f(x)\, \mathrm{d}x. \tag{2.390}$$

Changing the lower limit in (2.390), we have

$$\sigma^2 \geq \int_{|x-m|\geq\delta}^{\infty} (x - m)^2 f(x)\, \mathrm{d}x = \delta^2 \int_{|x-m|\geq\delta}^{\infty} f(x)\, \mathrm{d}x. \tag{2.391}$$

Equation (2.391) simplifies to

$$\sigma^2 \geq \delta^2 P(|X - m| \geq \delta) \tag{2.392}$$

which is the Chebyshev inequality in (2.389). The Chebyshev inequality can also be obtained in a slightly different way that would provide more insight into its meaning. If we assume that the random variable has a mean of zero (i.e., define a new random variable $Y = X{-}m$), we can define a new function

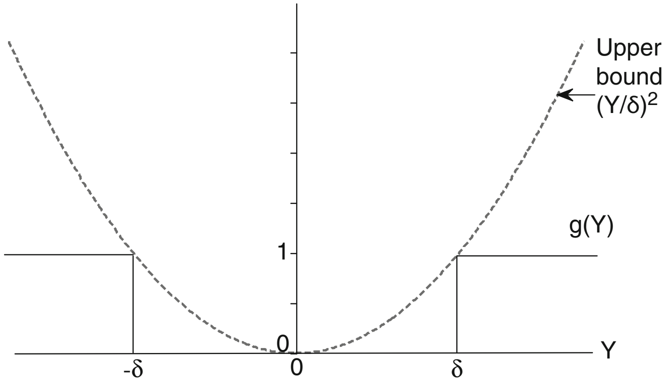

Fig. 2.46 Concept of Chebyshev inequality

$$g(Y) = \begin{cases} 1, & |Y| \geq \delta, \\ 0, & |Y| \leq \delta. \end{cases} \tag{2.393}$$

The left-hand side of the Chebyshev inequality in (2.389) is related to the mean of the new variable in (2.393)

$$\langle g(Y) \rangle = P(|Y| \geq \delta) = P(|X - m| \geq \delta). \tag{2.394}$$

The concept of this approach to Chebyshev inequality is shown in Fig. 2.46. From the figure, it is seen that the $g(Y)$ is bounded (upper) by the quadratic $(Y/\delta)^2$,

$$g(Y) \leq \left(\frac{Y}{\delta}\right)^2. \tag{2.395}$$

Note that Y is a zero mean random variable and hence

$$\langle Y^2 \rangle = \langle (X - m)^2 \rangle = \sigma^2. \tag{2.396}$$

Thus,

$$\langle g(Y) \rangle \leq \left\langle \frac{Y^2}{\delta^2} \right\rangle = \frac{\langle Y^2 \rangle}{\delta^2} = \frac{\sigma^2}{\delta^2}. \tag{2.397}$$

Since (2.394) is the tail probability, (2.397) provides the Chebyshev inequality in (2.389).

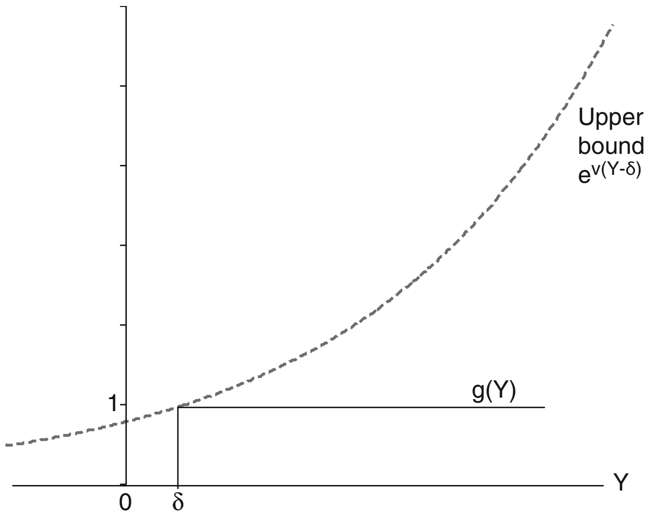

Fig. 2.47 Concept of the Chernoff bound

2.12.2 Chernoff Bound

In some of the cases we are interested in the area only under one tail. For most of the wireless communications, SNR takes positive values only. We are often interested in the probability that the SNR exceeds a certain value, i.e., the area under the tail from δ to ∞. Since only a single tail is involved, we could use an exponential function instead of the quadratic function in the Chebyshev inequality (Proakis 2001; Haykin 2001). Let $g(Y)$ be such that

$$g(Y) \leq \exp[v(Y-\delta)] \tag{2.398}$$

with

$$g(Y) = \begin{cases} 1, & Y \geq \delta, \\ 0, & Y < \delta. \end{cases} \tag{2.399}$$

In (2.398), the parameter v needs to be found and optimized. The exponential upper bound is shown in Fig. 2.47.

The expected value of $g(Y)$ is

$$\langle g(Y) \rangle = P(Y \geq \delta) \leq \langle \exp[v(Y-\delta)] \rangle. \tag{2.400}$$

Note that v must be positive. Its value can be obtained by minimizing the expected value of the exponential term in (2.400). A minimum occurs when

$$\frac{\mathrm{d}}{\mathrm{d}v}\langle \exp[v(Y-\delta)]\rangle = 0. \tag{2.401}$$

Changing the order of integration and differentiation in (2.401), we have

$$\exp(-v\delta)[\langle Y \exp(vY)\rangle - \delta\langle \exp(vY)\rangle] = 0. \tag{2.402}$$

The tightest bound is obtained from

$$\langle Y \exp(vY)\rangle - \delta\langle \exp(vY)\rangle = 0. \tag{2.403}$$

If $\hat{v}$ is the solution to (2.403), the upper bound on the one sided tail probability from (2.400) is

$$P(Y \geq \delta) \leq \exp(-\hat{v}\delta)\langle \exp(\hat{v}\delta)\rangle. \tag{2.404}$$

As it can be observed, the Chernoff bound is tighter than the tail probability obtained from the Chebyshev inequality because of the minimization in (2.401).

2.13 Stochastic Processes

We have so far examined the randomness of observed quantities, such as the signal amplitude or power and noise amplitude or power at a certain fixed time instant. While the signal and noise are spread over a long period of time, the random variable we studied constituted only a sample taken from the time domain function associated with the signal or noise (Taub and Schilling 1986; Gagliardi 1988; Papoulis and Pillai 2002). In other words, we need to characterize the temporal statistical behavior of the signals. This is accomplished through the concept of random processes. Consider a simple experiment. Take, for example, the measurement of a current or voltage through a resistor. Let us now set up several such experiments, all of them with identical characteristics. In each of these experiments, we measure the signal over a certain time period and we expect to see several time functions as illustrated in Fig. 2.48.

The voltages are plotted along the X-axis for each experiment while the temporal information is plotted along the Y-axis. For a fixed time (t_1), if we take the samples along the time line, we get different sample values which are random. In other words, if each of the temporal functions are represented by X, we can express it as a function of two variables, namely the "ensembles" ξ and "time" t. All such time functions together constitute the random process which describes the noise voltage. In other words, $X(\xi,t)$ represents the random process and for a fixed time instant t_k, $X(\xi,t_k)$ represents a random variable, simply represented by $X(t_k)$. If the sample value is fixed, i.e., for a fixed value of ξ, say ξ_n, we have a pure time function as $X(\xi_n,t)$. In other words, by fixing the time, we create a random variable from a

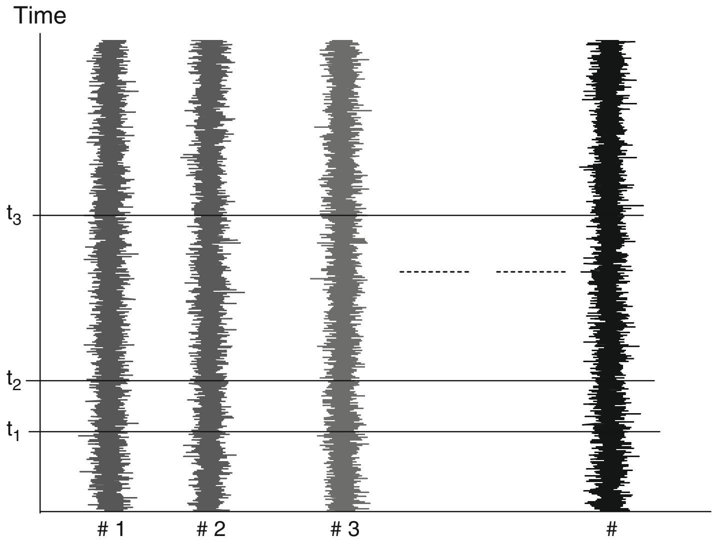

Fig. 2.48 Concept of a random process

process, and the random process becomes a pure time function when we fix the ensemble value. Thus, the random process provides both temporal and ensemble information on the noise or any other noise phenomenon. For example, we can create a random process by making the phase of a cosine wave random as

$$X(t) = \cos(2\pi f_0 t + \Theta). \tag{2.405}$$

In (2.405), Θ is a random variable. Since the random process contains both time and ensemble information, we can obtain two different types averages, autocorrelation values, and so on. Let us first define the density functions. For any value of t, $X(t)$ is a random variable, and hence, the first-order CDF is expressed as

$$F(x,t) = P[x(t) < x]. \tag{2.406}$$

The first order density function $f(x,t)$ is obtained by differentiating (2.406) w.r.t x as

$$f(x,t) = \frac{\partial F(x,t)}{\partial x}. \tag{2.407}$$

First, if we have time at two instants, t_1 and t_2, we can similarly obtain the second density function

$$f(x_1,x_2;t_1,t_2) = \frac{\partial^2 F(x_1,x_2;t_1,t_2)}{\partial x_1 \partial x_2}. \tag{2.408}$$

Similarly we can create higher-order density functions. Once we have the first-order density function, we have the so called first-order statistics and we get the nth order statistics of the random process by defining the density function for ensembles taken at the n unique time instants, t_1, t_2,. . .,t_n.

We define the ensemble average or statistical average $\eta(t)$ as

$$\eta(t) = \langle X(t) \rangle = \int x f(x, t)\, \mathrm{d}x. \tag{2.409}$$

The autocorrelation of $X(t)$ is the joint moment of $X(t_1)X(t_2)$ defined as

$$R(t_1, t_2) = \int\int x_1 x_2 f(x_1, x_2; t_1, t_2)\, \mathrm{d}x_1 \mathrm{d}x_2. \tag{2.410}$$

We can define the autocovariance as

$$C(t_1, t_2) = R(t_1, t_2) - \eta(t_1)\eta(t_2). \tag{2.411}$$

Equation (2.411) provides the variance of the random process when $t_1 - t_2 = 0$. The average values defined so far have been based on the use of density functions. Since the random process is a function of time as well, we can also define the averages in the time domain. Let

$$s = \int_a^b X(t)\, \mathrm{d}t \tag{2.412}$$

and

$$s^2 = \int_a^b \int_a^b X(t_1)X(t_2)\, \mathrm{d}t_1 \mathrm{d}t_2. \tag{2.413}$$

The averages in (2.412) and (2.413) are random variables. To remove the randomness and to get the true mean, we need to take a further expectation (statistical) of the quantities.

$$\eta(s) = \langle s \rangle = \int_a^b \langle X(t) \rangle\, \mathrm{d}t = \int_a^b \eta(t)\, \mathrm{d}t, \tag{2.414}$$

$$\langle s^2 \rangle = \int_a^b \int_a^b \langle X(t_1)X(t_2) \rangle\, \mathrm{d}t_1 \mathrm{d}t_2 = \int_a^b \int_a^b R(t_1, t_2)\, \mathrm{d}t_1 \mathrm{d}t_2. \tag{2.415}$$

In communications, it is easy to perform temporal averaging by observing the signal in the time domain as in (2.412) or (2.413). If the temporal averages and statistical averages are equal, it would be necessary for the integrals in (2.414) and

(2.415) to be equal. Such properties are associated with stationary processes. A stochastic process $X(t)$ is called strict sense stationary (SSS) if its statistical properties are invariant to a shift in the origin. Thus, a process is first-order stationary if

$$f(x,t) = f(x,t+c) = f(x), \quad c > 0. \tag{2.416}$$

This would mean that the temporal average and statistical average are identical. A process is second order stationary if the joint pdf depends only on the time difference and not on the actual values of t_1 and t_2 as

$$f(x_1,x_2;t_1,t_2) = f(x_1,x_2;\tau), \tag{2.417}$$

$$\tau = t_1 - t_2. \tag{2.418}$$

A process is called wide stationary sense (WSS) if

$$\langle X(t)\rangle = \eta, \tag{2.419}$$

$$\langle X(t_1)X(t_2)\rangle = R(\tau). \tag{2.420}$$

Since τ is midway point between t and $t + \tau$, we can write

$$R(\tau) = \left\langle X\left(t-\frac{\tau}{2}\right)X^*\left(t+\frac{\tau}{2}\right)\right\rangle. \tag{2.421}$$

In (2.421), we have treated the process as complex in the most general sense. Note that a complex process

$$X(t) = X_r(t) + jX_i(t) \tag{2.422}$$

is specified in terms of the joint statistics of the real processes $X_r(t)$ and $X_i(t)$. A process is called ergodic if the temporal averages are equal to the ensemble averages. For a random process to be ergodic, it must be SSS. Using this concept of ergodicity, we can see that a wide sense of stationary process is ergodic in the mean and ergodic in the autocorrelation. These two properties are sufficient for most of the analysis of communication systems.

The power spectral density (PSD) $S(f)$ of a wide sense stationary process $X(t)$ is the Fourier transform of its autocorrelation, $R(\tau)$. As defined in (2.421) the process may be real or complex. We have

$$S(f) = \int_{-\infty}^{\infty} R(\tau)\exp(-j2\pi f_0\tau)\,\mathrm{d}\tau. \tag{2.423}$$

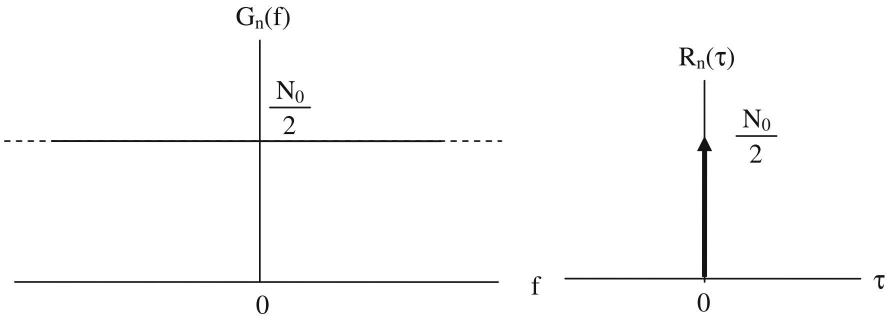

Fig. 2.49 Spectral density and the autocorrelation of noise

Note that $S(f)$ is real since $R(-\tau) = R^*(\tau)$. From the Fourier inversion property,

$$R(\tau) = \int_{-\infty}^{\infty} S(f) \exp(j2\pi f_0 \tau)\, \mathrm{d}f. \tag{2.424}$$

Furthermore, if $X(t)$ is real and $R(\tau)$ is real and even, then

$$S(f) = \int_{-\infty}^{\infty} R(\tau) \cos(j2\pi f_0 \tau)\, \mathrm{d}\tau = 2\int_{0}^{\infty} R(\tau) \cos(j2\pi f_0 \tau)\, \mathrm{d}\tau, \tag{2.425}$$

$$R(\tau) = 2\int_{0}^{\infty} S(f) \cos(j2\pi f_0 \tau)\, \mathrm{d}f. \tag{2.426}$$

As discussed earlier, the noise in communication systems is modeled as a random process. The primary characteristic of the noise in communication systems, referred to as thermal noise, is that it is zero mean Gaussian and that it is white (Taub and Schilling 1986). This means that the thermal noise at any given time instant has a Gaussian distribution with zero mean. The whiteness refers to the fact that its spectral density is constant (shown in Fig. 2.49).

The spectral density $G_n(f)$ of the noise $n(t)$ has a constant value of $(N_0/2)$ over all the frequencies. This means that its autocorrelation is a delta function as shown

$$R(\tau) = \frac{N_0}{2} \delta(\tau). \tag{2.427}$$

Equation (2.427) suggests that any two samples of the noise taken at any separation (however small this separation may be) will be uncorrelated. When the noise passes through a low pass filter of bandwidth B, the noise energy will be $((N_0/2)2B)$ or $N_0 B$.

2.14 Summary

In this chapter, we examined some theoretical aspects of probability density functions and distributions encountered in the study of fading and shadowing in wireless channels. We started with the basic definition of probability, then discussed density functions and properties relating to the analysis of fading and shadowing. We also examined the transformations of random variables in conjunction with relationships of different types of random variables. This is important in the study of diversity and modeling of specific statistical behavior of the wireless channels. The density functions of some of the functions of two or more random variables were derived. We examined order statistics, placing emphasis on the density functions of interest in diversity analysis. Concepts of stochastic processes and their properties were outlined. Similarly, the characteristics of noise were delineated within the context of signal detection. In addition, this study included an exploration of ways of expressing some of the densities in more compact forms using the hypergeometric functions and Meijer's *G* function.

References

Aalo, V. A. and T. Piboongungon (2005). On the multivariate generalized gamma distribution with exponential correlation. Global Telecommunications Conference, 2005. GLOBECOM '05. IEEE: 3–5.

Aalo, V. and J. Zhang (1999). "On the effect of cochannel interference on average error rates in Nakagami-fading channels." *Communications Letters, IEEE* **3**(5): 136–138.

Abdi, A. and M. Kaveh (1998). "K distribution: an appropriate substitute for Rayleigh-lognormal distribution in fading-shadowing wireless channels." *Electronics Letters* **34**(9): 851–852.

Abdi, A. and M. Kaveh (1999). *On the utility of gamma PDF in modeling shadow fading (slow fading)*. Vehicular Technology Conference, 1999 IEEE 49th: 2308–2312.

Abramowitz, M., Segun, I. A., eds. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, New York: Dover Publications.

Abu-Dayya, A. A. and N. C. Beaulieu (1994). "Outage probabilities in the presence of correlated lognormal interferers." *Vehicular Technology, IEEE Transactions on* **43**(1): 164–173.

Abu-Salih, M. (1983). "Distributions of the product and the quotient of power-function random variables." *Arab J. Math* **4**: 1–2.

Adamchik, V. (1995). "The evaluation of integrals of Bessel functions via G-function identities." *Journal of Computational and Applied Mathematics* **64**(3): 283–290.

Alouini, M., et al. (2002). "Sum of gamma variates and performance of wireless communication systems over Nakagami-fading channels." *Vehicular Technology, IEEE Transactions on* **50**(6): 1471–1480.

Alouini, M. S. and M. K. Simon (1999). "Performance of coherent receivers with hybrid SC/MRC over Nakagami-m fading channels." *Vehicular Technology, IEEE Transactions on* **48**(4): 1155–1164.

Alouini, M. S. and M. K. Simon (2000). "An MGF-based performance analysis of generalized selection combining over Rayleigh fading channels." *Communications, IEEE Transactions on* **48**(3): 401–415.

Alouini, M. and M. Simon (2003). "Dual diversity over correlated log-normal fading channels." *Communications, IEEE Transactions on* **50**(12): 1946–1959.

Alouini, M. S. and M. K. Simon (2006). “Performance of generalized selection combining over Weibull fading channels.” *Wireless Communications & Mobile Computing* **6**(8): 1077–1084.

Anastassopoulos, V. et al. (1999). “High resolution radar clutter statistics,” *Aerospace and Electronics Systems, IEEE Transactions on*, **35(1)**: 43–60.

Andersen, J. B. (2002). Statistical distributions in mobile communications using multiple scattering. presented at the *27th URSI General Assembly*, Maastricht, Netherlands.

Annamalai, A. et al. (2006). “Analysis of generalized selection diversity systems in wireless channels.” *Vehicular Technology, IEEE Transactions on* **55**(6): 1765–1775.

Annamalai, A. et al. (2005). “A general method for calculating error probabilities over fading channels.” *Communications, IEEE Transactions on* **53**(5): 841–852.

Atapattu, S. et al. (2010). *Representation of Composite Fading and Shadowing Distributions by Using Mixtures of Gamma Distributions*. Wireless Communications and Networking Conference (WCNC), 2010 IEEE. 1:5.

Beaulieu, N. C. (1990). “An infinite series for the computation of the complementary probability distribution function of a sum of independent random variables and its application to the sum of Rayleigh random variables.” *Communications, IEEE Transactions on* **38**(9): 1463–1474.

Beaulieu, N. et al. (1995). “Estimating the distribution of a sum of independent lognormal random variables.” *Communications, IEEE Transactions on* **43**(12): 2869.

Beaulieu, N. and Q. Xie (2004). “An optimal lognormal approximation to lognormal sum distributions.” *Vehicular Technology, IEEE Transactions on* **53**(2): 479–489.

Benedetto, S. and E. Biglieri (1999). *Principles of digital transmission: with Wireless Applications*. New York, Kluwer Academic/Plenum Press.

Bithas, P. et al. (2006). “On the performance analysis of digital communications over generalized-K fading channels.” *Communications Letters, IEEE* **10**(5): 353–355.

Bithas, P. S. et al. (2007). “Dual diversity over correlated Rician fading channels,” *J. Comm & Networks*. **9**(1) 1–8.

L. E. Blumenson and K. S. Miller (1963). “Properties of Generalized Rayleigh Distributions,” *The Annals of Mathematical Statistics* **34**(3) 903–910.

Brennan, D. G. (1959). “Linear diversity combining techniques”, *Proc. IRE* **47**(6): 1075–1102.

Bryson, B. C. (1974). “Heavy tailed distributions: Properties and tests,” *Technometrics* **16**(1) 61–68.

Cardieri, P. and T. Rappaport (2001). “Statistical analysis of co-channel interference in wireless communications systems.” *Wireless Communications and Mobile Computing* **1**(1): 111–121.

Carter, B. and M. Springer (1977). “The distribution of products, quotients and powers of independent H-function variates.” *SIAM Journal on Applied Mathematics* **33**(4): 542–558.

Cheng, J. et al. (2004). “Performance of digital linear modulations on Weibull slow-fading channels.” *Communications, IEEE Transactions on* **52**(8): 1265–1268.

Chiani, M. (1999). “Integral representation and bounds for Marcum Q-function.” *Electronics Letters* **35**(6): 445–446.

Cooper, G. R. and McGillem, C. D. (1986). *Modern communications and Spread Spectrum*. New York. McGraw-Hill.

Cotton, S. L. and W. G. Scanlon (2007). “Higher Order Statistics for Lognormal Small-Scale Fading in Mobile Radio Channels.” *Antennas and Wireless Propagation Letters, IEEE* **6**: 540–543.

Couch, L. W. (2007). *Digital and analog communication systems*. Upper Saddle River, N.J., Pearson/Prentice Hall.

Coulson, A. J. et al. (1998a). “Improved fading distribution for mobile radio,” *Communications, IEE Proceedings* **145**(3) 197–202.

Coulson, A. J., et al. (1998b). “A Statistical Basis for Lognormal Shadowing Effects in Multipath Fading Channels,” *Communications, IEEE Trans. on* **46**(4) 494–502.

da Costa, B. D. and M. D. Yacoub (2008). “Moment generating functions of generalized fading distributions and applications.” *Communications Letters, IEEE* **12**(2): 112–114.

Evans, M., N. A. J. Hastings, et al. (2000). *Statistical distributions*. New York, John Wiley.

Filho, J. C. S. S. et al. (2005). “Simple accurate lognormal approximation to lognormal sums.” *Electronics Letters* **41**(18): 1016–1017.

Frery, A. et al. (2002). “A model for extremely heterogeneous clutter.” *Geoscience and Remote Sensing, IEEE Transactions on* **35**(3): 648–659.

Gagliardi, R. M. (1988). *Introduction to Communications Engineering* New York, John Wiley.

Gallager, R. G. (2008). *Principles of digital communication*. Cambridge; New York, Cambridge University Press.

Gaur, S. and Annamalai A. (2003). "Some Integrals Involving the $Q_m(a\sqrt{x}, b\sqrt{x})$ with Application to Error Probability Analysis of Diversity Receivers," *Vehicular Technology, IEEE Transactions on* **52**(6) 1568–1575.

Goldsmith, A. (2005). *Wireless Communications*. New York, Cambridge University Press.

Goodman, J. W. (1985). *Statistical Optics*. New York. Wiley.

Gradshteyn, I. S., I. M. Ryzhik (2007). *Table of integrals, series and products*. Oxford, Academic.

Griffiths, J. and McGeehan, J. P. (1982). "Inter relationship between some statistical distributions used in radio-wave propagation", *Comm, Radar and Sig. Proc, IEE Proceedings on* **129**(6) 411–417.

Gupta, R. D. and Kundu, D. (1999). "Generalized exponential distributions," *Austral. & New Zealand J. Statistics* **41**(2), 173–188.

Hansen, F. and F. I. Meno (1977). "Mobile fading; Rayleigh and lognormal superimposed." *Vehicular Technology, IEEE Transactions on* **26**(4): 332–335.

Haykin, S. S. (2001). *Digital communications*. New York, Wiley.

Helstrom, C. W. (1968). *Statistical theory of signal detection*. Oxford, New York, Pergamon Press.

Helstrom, C. W. (1991). *Probability and stochastic processes for engineers*. New York, Macmillan.

Helstrom, C. W. (1992). "Computing the generalized Marcum Q function." *Information Theory, IEEE Transactions on* **38**(4): 1422–1428.

Helstrom, C. W. (1998). "Approximate inversion of Marcum's Q-function." *Aerospace and Electronic Systems, IEEE Transactions on* **34**(1): 317–319.

Holm, H. and M. S. Alouini (2004). "Sum and difference of two squared correlated Nakagami variates in connection with the McKay distribution." *Communications, IEEE Transactions on* **52**(8): 1367–1376.

Hudson, J. E. (1996). A lognormal fading model and cellular radio performance. Global Telecommunications Conference, 1996. GLOBECOM '96. 'Communications: The Key to Global Prosperity. **2**: 1187–1191.

Iskander, R. D. and Zoubir, A. M. (1996). "On Coherent Modeling of Non-Gaussian Radar Clutter," Proc. 8th IEEE Signal Processing Workshop on statistical Signal and Array Processing: 226–229.

Iskander, D. R. et al. (1999). "A method for estimating the parameters of the K distribution," *Signal Processing, IEEE Transactions on* **47**: 1147–1151.

Ismail, M. H. and M. M. Matalgah (2006). "On the use of Pade approximation for performance evaluation of maximal ratio combining diversity over Weibull fading channels." *Eurasip Journal on Wireless Communications and Networking* **6**: 62–6.

Jakeman, E. and R. Tough (1987). "Generalized K distribution: a statistical model for weak scattering." *Journal of the Optical Society of America A* **4**(9): 1764–1772.

Jakes, W. C. (1994). *Microwave mobile communications*. Piscataway, NJ, IEEE Press.

Kabe, D. G. (1958). "Some Applications of Meijer-G Functions to Distribution Problems in Statistics," *Biometrika*, **45**(3/4): 578–580.

Karadimas, P. and S. A. Kotsopoulos (2008). "A Generalized Modified Suzuki Model with Sectored and Inhomogeneous Diffuse Scattering Component." *Wireless Personal Communications* **47**(4): 449–469.

Karadimas, P. and S. A. Kotsopoulos (2010). "A Modified Loo Model with Partially Blocked and Three Dimensional Multipath Scattering: Analysis, Simulation and Validation." *Wireless Personal Communications* **53**(4): 503–528.

Karagiannidis, G., N. Sagias, et al. (2006). "Closed-form statistics for the sum of squared Nakagami-m variates and its applications." *Communications, IEEE Transactions on* **54**(8): 1353–1359.

Karagiannidis, G. K. et al. (2003a). "Performance analysis of triple selection diversity over exponentially correlated Nakagami- m fading channels," *Communications, IEEE Transactions on* **51**(8):1245–1248.

Karagiannidis, G. K. et al. (2003b). "On the multivariate Nakagami-m distribution with exponential correlation," *Communications, IEEE Transactions on* **51**(8):1240–1244.

Karagiannidis, G. K. et al. (2007). "N*Nakagami: A novel statistical model for cascaded fading channels," *Communications, IEEE Transactions on* **55**(8) 1453–145.

Karmeshu, J. and R. Agrawal (2007). "On efficacy of Rayleigh-inverse Gaussian distribution over K-distribution for wireless fading channels." *Wireless Communications & Mobile Computing* **7**(1): 1–7.

Korn, I. and Foneska, J. P. (2001). "M-CPM with MRC Diversity in Rician-, Hoyt-, and Nakagami-Fading Channels," *Vehicular Technology, IEEE Transactions on* **50**(4) 1182–118.

Kostic, I. M. (2005). "Analytical approach to performance analysis for channel subject to shadowing and fading." *Communications, IEE Proceedings-* **152**(6): 821–827.

Kotz, S. and J. Adams (1964). "Distribution of sum of identically distributed exponentially correlated gamma-variables." *The Annals of Mathematical Statistics*: 277–283.

Kundu, K. and Raqab, M. Z. (2005). "Generalized Rayleigh distribution: different methods of estimations," *Computational Statistics & Data Analysis* (**49**) 187–200.

Lam, C. L. J. and T. Le-Ngoc (2006). "Estimation of typical sum of lognormal random variables using log shifted gamma approximation." *Communications Letters, IEEE* **10**(4): 234–23.

Lam, C. L. J. and L.-N. Tho (2007). "Log-Shifted Gamma Approximation to Lognormal Sum Distributions." *Vehicular Technology, IEEE Transactions on* **56**(4): 2121–2129.

Laourine, A. et al. (2008). "On the capacity of generalized-K fading channels." *Wireless Communications, IEEE Transactions on* **7**(7): 2441–2445.

Laourine, A., et al. (2009). "On the Performance Analysis of Composite Multipath/Shadowing Channels Using the G-Distribution," *Communications, IEEE Transactions on* **57**(4): 1162–1170.

Lee, R. and B. Holland (1979). "Distribution of a ratio of correlated gamma random variables." *SIAM Journal on Applied Mathematics* **36**(2): 304–320.

D. J. Lewinsky (1983). "Nonstationary probabilistic target and cluttering scattering models," *Aerospace Electronics and Systems, IEEE Transactions on* **31**: 490–49.

Ligeti, A. (2000). "Outage probability in the presence of correlated lognormal useful and interfering components." *Communications Letters, IEEE* **4**(1): 15–17.

Lienhard, J. H. and Meyer, P. L. (1967). "A physical basis for the generalized gamma distribution," *Quarterly of Appl. Math.* **25**(3) 330–334.

Lindsey, W. C. and Simon M. K. (1973). *Telecommunication Systems Engineering*. Englewood Cliffs, N.J., Prentice-Hall.

Liu, Z. et al. (2008). "Approximating lognormal sum distributions with power lognormal distributions." *Vehicular Technology, IEEE Transactions on* **57**(4): 2611–2617.

Ma, Y. and C. C. Chai (2000). "Unified error probability analysis for generalized selection combining in Nakagami fading channels." *Selected Areas in Communications, IEEE Journal on* **18**(11): 2198–2210.

Mathai, A. and P. Moschopoulos (1991). "On a multivariate gamma." *Journal of Multivariate Analysis* **39**(1): 135–153.

Mathai, A. M. (1993). *A Handbook of Generalized Special Functions for Statistical and Physical Sciences*. Oxford University Press.

Mathai, A. M. and H. J. Haubold (2008). *Special functions for applied scientists*. New York, Springer Science+Business Media.

Mathai, A. M. and R. K. Saxena (1973). *Generalized hypergeometric functions with applications in statistics and physical sciences*. Berlin, New York, Springer.

McDaniel, S. T. (1990). "Seafloor reverberation fluctuations", *J. Acoust. Soc. Amer.* **88**(3): 1530–1535.

Middleton, D. (1996). *An Introduction to Statistical Communications Theory*. Piscataway, N. J, IEEE Press.

Molisch, A. F. (2005). *Wireless communications*. Chichester, U. K. John Wiley & Sons.

Moschopoulos, P. (1985). "The distribution of the sum of independent gamma random variables." *Annals of the Institute of Statistical Mathematics* **37**(1): 541–544.

Nadarajah, S. (2005). "Products, and ratios for a bivariate gamma distribution." *Applied Mathematics and Computation* **171**(1): 581–595.

Nadarajah, S. and A. Gupta (2005). "On the product and ratio of Bessel random variables." *International Journal of Mathematics and Mathematical Sciences* **2005**(18): 2977–2989.

Nadarajah, S. and A. Gupta (2006a). "Some bivariate gamma distributions." *Applied Mathematics Letters* **19**(8): 767–774.

Nadarajah, S. and S. Kotz (2006b). "Bivariate gamma distributions, sums and ratios." *Bulletin of the Brazilian Mathematical Society* **37**(2): 241–274.

Nadarajah, S. and S. Kotz (2006). "On the product and ratio of gamma and Weibull random variables." *Econometric Theory* **22**(02): 338–344.

Nadarajah, S. and S. Kotz (2007). "A class of generalized models for shadowed fading channels." *Wireless Personal Communications* **43**(4): 1113–1120.

Nakagami, M. (1960). The *m*-distribution—A general formula of intensity distribution of rapid fading, in *Statistical Methods in Radio Wave Propagation*, W. C. Hoffman, Ed. Elmsford, NY: Pergamon.

Nuttall, A. (1975). "Some integrals involving the Q_M function (Corresp.)." *Information Theory, IEEE Transactions on* **21**(1): 95–96.

Nuttall, A. H. (1970). "Alternate forms for numerical evaluation of cumulative probability distributions directly from characteristic functions." *Proceedings of the IEEE* **58**(11): 1872–1873.

Nuttall, A. H. (1969). "Numerical evaluation of cumulative probability distribution functions directly from characteristic functions." *Proceedings of the IEEE* **57**(11): 2071–2072.

Okui, S. (1981). "Probability Distributions for Ratios of Fading Signal Envelopes and Their Generalization," *Electronics and Communications in Japan* 64-B(3), 72–80.

Panajotovic, A. S. et al. (2009). "Effect of Microdiversity and Macrodiversity on Average Bit Error Probability in Shadowed Fading Channels in the Presence of Interference." *ETRI Journal* **31**(5): 500–505.

Papazafeiropoulos, A. K. and S. A. Kotsopoulos (2011). "The α-λ-μ and α-μ-μ small scale general fading distributions: A unified approach," *Wireless Personal Communications* **57**: 735–751.

Papoulis, A. and S. U. Pillai (2002). *Probability, random variables, and stochastic processes*. Boston, McGraw-Hill.

Patzold, M. (2002). *Mobile Fading Channels*. Chichester, U. K., John Wiley & Sons, Inc.

Piboongungon, T. and V. A. Aalo (2004). "Outage probability of L-branch selection combining in correlated lognormal fading channels." *Electronics Letters* **40**(14): 886–888.

Podolski, H. (1972). "The distribution of a product of n independent random variables with generalized gamma distribution." *Demonstratio Math* **4**(2): 119–123.

Polydorou, D. S. et al. (1999). "Statistical characterization of fading in LOS wireless channels with a finite number of dominant paths. Application in millimeter frequencies." *International Journal of Infrared and Millimeter Waves* **20**(3): 461–472.

Proakis, J. G. (2001). *Digital communications*. Boston, McGraw-Hill.

Provost, S. (1989). "On sums of independent gamma random variables." *STATISTICS*. **20**(4): 583–591.

Rappaport, T. S. (2002). *Wireless communications: principles and practice*. Upper Saddle River, N.J., Prentice Hall PTR.

Rice, S. (1974). "Probability Distributions for Noise Plus Several Sine Waves–The Problem of Computation." *Communications, IEEE Transactions on* **22**(6): 851–853.

Rohatgi, V. K. and A. K. M. E. Saleh (2001). *An introduction to probability and statistics*. New York, Wiley.

Saleh, A. & Valenzuela R. A. (1987) "A statistical model for indoor multipath propagation" *Selected Areas in Communications, IEEE Journal on* **5**: 128–137.

Sahu, P. R. & Chaturvedi, A. K. (2005), "Performance analysis of predetection EGC receiver in Weibull fading channel," *Electr. Lett.* 41(2) 85–86.

Salo, J. et al. (2006). "The distribution of the product of independent Rayleigh random variables." *Antennas and Propagation, IEEE Transactions on* **54**(2): 639–643.

Schwartz, M. (1980). *Information transmission, modulation, and noise: a unified approach to communication systems*. New York, McGraw-Hill.

Schwartz, M. et al. (1996). *Communication Systems and Techniques*. Piscataway, NJ., IEEE Press.

Shah, A. et al. (2000). “Exact bit-error probability for optimum combining with a Rayleigh fading Gaussian cochannel interferer.” *Communications, IEEE Transactions on* **48**(6): 908–912.

Shankar, P. (2002a). *Introduction to wireless systems*, Wiley New York.

Shankar, P. (2002b). “Ultrasonic tissue characterization using a generalized Nakagami model.” *Ultrasonics, Ferroelectrics and Frequency Control, IEEE Transactions on* **48**(6): 1716–1720.

Shankar, P. M. (2004). “Error rates in generalized shadowed fading channels.” *Wireless Personal Communications* **28**(3): 233–238.

Shankar, P. M. (2005). “Outage probabilities in shadowed fading channels using a compound statistical model.” *Communications, IEE Proceedings on* **152**(6): 828–832.

Shankar, P. M. and Gentile, C. (2010). Statistical Analysis of Short Term Fading and Shadowing in Ultra-Wideband Systems. Communications (ICC), 2010 IEEE International Conference on.

Shankar, P. M. (2010). “Statistical Models for Fading and Shadowed Fading Channels in Wireless Systems: A Pedagogical Perspective,” *Wireless Personal Communications*. DOI: 10.1007/s11277-010-9938-2

Shanmugam, K. S. (1979). *Digital and analog communication systems*. New York, Wiley.

Shepherd, N. H. (1977). “Radio wave loss deviation and shadow loss at 900 MHZ,” *Vehicular Technology, IEEE Transactions on* **26**(4) 309–313.

Shin, J. et al. (2005). “Statistical modeling of speech signals based on generalized gamma distribution.” *Signal Processing Letters, IEEE* **12**(3): 258–261.

Sijing, J. and N. C. Beaulieu (2010). BER of antipodal signaling in Laplace noise., 2010 25th Biennial Symposium on Communications (QBSC).

Simon, M. (2002). “The Nuttall Q function-its relation to the Marcum Q function and its application in digital communication performance evaluation.” *Communications, IEEE Transactions on* **50**(11): 1712–1715.

Simon, M. et al. (1995). *Digital communication techniques: signal design and detection*, Prentice Hall PTR.

Simon, M. K. and M. S. Alouini (2003). “Some new results for integrals involving the generalized Marcum Q function and their application to performance evaluation over fading channels.” Wireless *Communications, IEEE Transactions on* **2**(4): 611–615.

Simon, M. K. and M.-S. Alouini (2005). *Digital communication over fading channels*. Hoboken, N.J., Wiley-Interscience.

Sklar, B. (1997a). “Rayleigh fading channels in mobile digital communication systems. I. Characterization.” *Communications Magazine, IEEE* **35**(7): 90–100.

Sklar, B. (1997b). “Rayleigh fading channels in mobile digital communication systems. II. Mitigation.” *Communications Magazine, IEEE* **35**(7): 102–109.

Sklar, B. (2001). *Digital communications: fundamentals and applications*. Upper Saddle River, N.J., Prentice-Hall PTR.

Slimane, B. (2001). “Bounds on the distribution of a sum of independent lognormal random variables.” *Communications, IEEE Transactions on* **49**(6): 975–978.

Sowerby, K. W. and A. G. Williamson (1987). “Outage probability calculations for a mobile radio system having two log-normal interferers.” *Electronics Letters* **23**(25): 1345–1346.

Springer, M. and W. Thompson (1966). “The distribution of products of independent random variables.” *SIAM Journal on Applied Mathematics* **14**(3): 511–526.

Springer, M. and W. Thompson (1970). “The distribution of products of beta, gamma and Gaussian random variables.” *SIAM Journal on Applied Mathematics* **18**(4): 721–737.

Stacy, E. (1962). “A generalization of the gamma distribution.” *The Annals of Mathematical Statistics* **33**(3): 1187–1192.

Stacy, E. and G. Mihram (1965). “Parameter estimation for a generalized gamma distribution.” *Technometrics* **7**(3): 349–358.

Steele, R. and L. Hanzó (1999). *Mobile radio communications: second and third generation cellular and WATM systems*. Chichester, England; New York, John Wiley & Sons, Inc.

Stuart, A. (1962). “Gamma-distributed products of independent random variables.” *Biometrika* **49**(3–4): 564.

Stuber, G. L. (2000). *Principles of mobile communication*. New York, Kluwer Academic.

Subadar, R. & Sahu, P. R. (2009). “Performance Analysis of Dual MRC Receiver in Correlated Hoyt Fading Channels,” *Communications Letters, IEEE* **13**(6) 405–407.
Suzuki, H. (1977). “A statistical model for urban radio propagation”, *Communications, IEEE Transactions on* **25**: 673–680.
Tan, C. C. & Beaulieu, N. C. (1997). “Infinite Series Representations of the Bivariate Rayleigh and Nakagami- Distributions,” *Communications, IEEE Transactions on* **45**(10), 1159–1161.
Taub, H. & D. L. Schilling (1986). *Principles of communication systems*. New York, McGraw-Hill.
Tellambura, C. and A. Annamalai (1999). “An unified numerical approach for computing the outage probability for mobile radio systems.” *Communications Letters, IEEE* **3**(4): 97–99.
Tellambura, C. and A. Annamalai (2000). “Efficient computation of erfc(x) for large arguments.” *Communications, IEEE Transactions on* **48**(4): 529–532.
Tellambura, C. and A. Annamalai (2003). *Unified performance bounds for generalized selection diversity combining in fading channels*. Wireless Communications and Networking (WCNC), 2003 IEEE.
Tellambura, C. et al. (2003). “Closed form and infinite series solutions for the MGF of a dual-diversity selection combiner output in bivariate Nakagami fading.” *Communications, IEEE Transactions on* **51**(4): 539–542.
Tjhung, T. T. and C. C. Chai (1999). “Fade statistics in Nakagami-lognormal channels.” *Communications, IEEE Transactions on* **47**(12): 1769–1772.
Vaughn, R. A., J. B. Anderson, (2003). *Channels, propagation and Antennas for Mobile Communications*. Herts, U. K., IEE.
Van Trees, H. L. (1968) *Detection, estimation, and modulation theory*. Part I. New York: John Wiley and Sons.
V.Gh. Voda, B. (2009). “A method constructing density functions: the case for a Generalized Rayleigh variable,” *Applications of Mathematics* **54**(5) 417–43.
Vatalaro, F. (1995). “Generalised Rice-lognormal channel model for wireless communications.” *Electronics Letters* **31**(22): 1899–1900.
Winters, J. (1987). “Optimum Combining for Indoor Radio Systems with Multiple Users.” *Communications, IEEE Transactions on* **35**(11): 1222–1230.
Winters, J. H. (1984). “Optimum combining in digital mobile radio with cochannel interference.” *Vehicular Technology, IEEE Transactions on* **33**(3): 144–155.
Winters, J. H. (1998). “The diversity gain of transmit diversity in wireless systems with Rayleigh fading.” *Vehicular Technology, IEEE Transactions on* **47**(1): 119–123.
Withers, C. and S. Nadarajah (2008). “MGFs for Rayleigh Random Variables.” *Wireless Personal Communications* **46**(4): 463–468.
Wolfram (2011) http://functions.wolfram.com/, Wolfram Research, Inc.
Wongtrairat, W. and P. Supnithi (2009). “Performance of Digital Modulation in Double Nakagami-m Fading Channels with MRC Diversity.” *Communications, IEICE Transactions on* E92b(2): 559–566.
Yacoub, M. D. (2000). “Fading distributions and co-channel interference in wireless systems.” *Antennas and Propagation Magazine, IEEE* **42**(1): 150–160.
Yacoub, M. D. (2007a). “The α-μ Distribution: A Physical Fading Model for the Stacy Distribution.” *Vehicular Technology, IEEE Transactions on* **56**(1): 27–34.
Yacoub, M. D. (2007b). “The μ-μ distribution and the η-μ distribution.” *Antennas and Propagation Magazine, IEEE* **49**(1): 68–81.
Yue, S. et al. (2001). “A review of bivariate gamma distributions for hydrological applications,” *J. Hydrology* **246**: 1–18.
Zogas, D. and Karagiannidis, G. K. (2005). “Infinite-Series Representations Associated With the Bivariate Rician Distribution and Their Applications,” *Communications, IEEE Transactions on* **53**(11): 1790–179.

Springer
http://www.springer.com/978-1-4614-0366-1

Fading and Shadowing in Wireless Systems
Shankar, P.M.
2012, XIV, 466 p., Hardcover
ISBN: 978-1-4614-0366-1