Energy and Nonlinear Dynamics of Hybrid Systems

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Abstract The transmission of energy between subsystems coupled in hybrid system is very important for different applications. For first as an introduction, by using the author's previously published references and that of her students, a short survey of an analytical study of the energy transfer between coupled subsystems is presented as a basis of this chapter. An analytical study of the mechanical energy transfer between two coupled subsystems, as well as, between two or more coupled rotation motions is presented. For starting, an analytical analysis of the mechanical energy transfer between a linear and a nonlinear oscillators of a hybrid system (see Refs. by Hedrih (Stevanović) 2002 [10, 11, 15–18, 20, 24]) in the free, as well as forced, vibrations of a different types of interconnections between subsystems is presented. Coupling element between subsystems of the considered hybrid systems are standard light elements with elastic, viscoelastic, hereditary, or creeping properties as well as dynamical constrain element realized by rolling element with inertia properties. Using Krilov–Bogolyubov–Mitropolskiy's asymptotic method, both the solutions in the first approximation and the system of nonlinear-coupled differential equations for the corresponding number of excited amplitudes and phases of multifrequency free as well as forced regimes are derived. By means of this asymptotic approximation of differential equations for the amplitudes and phases for forced vibrations of the coupled oscillators, the mutual influence of the nonlinear harmonics and energy transient were analyzed. The Lyapunov exponents corresponding to the each of two eigen like nonlinear modes are expressed by using energy of the corresponding eigen time components. A generalization of an analytical analysis of the transfer energy between linear and nonlinear oscillators for forced vibrations with different type constraints as a

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couple between two subsystems, each of them with one degree of freedom, is done. A mathematical analogy between discrete and complex discrete-continual hybrid systems is pointed out.

In the second part, an analytical analysis is extended to the transfer energy between plates for free and forced transversal vibrations of a visco and nonlinear elastically connected double plate system. The analysis showed that the visco- and nonlinear elastic connection between plates caused the appearance of two-frequency like regime of time function, which corresponds to one eigen amplitude function of one mode, and also that time functions of different vibration modes are coupled, as well as energy transfer between plates in one eigen mode appear.

Next, as an author's new research result, an analytical study of the energy transfer between two coupled-like string belts interconnected by light pure elastic layer in the axially moving sandwich double belt system in the free vibrations is presented.

1 Introduction

1.1 Importance

The study of the transfer of energy between subsystems coupled into hybrid system is very important for different kinds of applications.

For first as an introduction, by use author's previously published references as well as by her students, a short review of an analytical study of the energy transfer between coupled subsystems is presented as a basis of this lecture. An analytical study of the mechanical energy transfer between two coupled subsystems, as well as between two or more coupled rotation motions, also, is presented. For starting, an analytical analysis of the mechanical energy transfer between a linear and a nonlinear oscillators of a hybrid system (see Refs. by Hedrih (Stevanović) [10, 11, 15–18, 20, 24, 31, 34–38, 43]), in the free, as well as forced vibrations of a different type of interconnections (see Ref. by Goroško and Hedrih (Stevanović) [1–4]) between subsystems is presented. Coupling elements between subsystems of the considered hybrid systems are standard light elements with elastic, viscoelastic, hereditary, or creeping properties, as well as no light dynamical constraint element realized by rolling element with inertia properties.

Using well known Krilov–Bogolyubov–Mitropolskiy's asymptotic method, [73–80] both, the solutions in the first approximation and the system of nonlinear coupled differential equations for the corresponding number of excited amplitudes and phases of multifrequency free as well as forced regimes are derived. By means of this first asymptotic approximation of ordinary differential equations for the amplitudes and phases for forced vibrations of the coupled oscillators in resonant frequency intervals, the mutual influence of the nonlinear harmonics and energy transient were analyzed. The Lyapunov exponents corresponding to the each of two eigen like nonlinear modes are expressed by using energy of the corresponding eigen time components.

A generalization of an analytical analysis of the transfer energy between linear and nonlinear oscillators for forced vibrations with different type constraints as a couple between two subsystems, each of them with one degree of freedom is done. A mathematical analogy between discrete and complex discrete-continual hybrid systems (see Ref. by Hedrih (Stevanović) [42]) is pointed out. Mathematical analogy and phenomenological mapping (see Refs. by Hedrih (Stevanović) [34,46]) between different mechanical systems on the basic of the discretizations by subdynamics or subcomponents of dynamics are used also for transfer energy analysis.

For second, the study of the transfer energy between subsystems containing deformable body coupled in hybrid system is very important for different applications. *An* analytical analysis of the transfer energy between plates for free and forced transversal vibrations of a viscoelastically connected double plate system is pointed out. The analytical analysis showed that the viscoelastic connection between plates caused the appearance of two-frequency-like regime of time functions, which corresponds to one eigen amplitude function of one mode, and also that time functions of different vibration modes, in linear system, are uncoupled, but energy transfer between plates in one eigen mode appears. It was shown for each shape of vibrations. Series of the two Lyapunov exponents corresponding to the one eigen amplitude mode are expressed by using energy of the corresponding eigen amplitude time component.

In the same second part, an analytical analysis is extended to the transfer energy between plates for free and forced transversal vibrations of a visco- and nonlinear elastically connected double plate system. The analysis showed that the visco- and nonlinear elastic connection between plates caused the appearance of two-frequency-like regime of time function, which corresponds to one eigen amplitude function of one mode, and also that time functions of different vibration modes are coupled, as well as energy transfer between plates in one eigen mode appears. More than two resonant jumps in the amplitude-frequency as well as in phase-frequency curves appeared and caused more than two resonant jumps of the energy and corresponding influence between nonlinear modes, as nonlinear phenomena interactions. Using the analytical asymptotic approximation of the amplitudes and phases of multifrequency particular solutions of such a dynamics, it is possible to analyze transfer energy between nonlinear modes in stationary and nonstationary regimes passing through resonant frequency intervals.

Next, as an author's new research result, an analytical study of the energy transfer between two coupled-like string belts interconnected by light pure elastic layer in the axially moving sandwich double belt system, in the free vibrations is presented. On the basis of the obtained analytical expressions for the kinetic and potential energy of the belts and potential energy of the light pure elastic distributed layer numerous conclusions are derived. For the pure linear elastic double belt system no transfer energy between different eigen modes of transversal vibrations of the axially moving double belt system appears, but in each of the set of the infinite numbers eigen modes, there are transfer energy between belts. The corresponding free transversal vibrations are like two frequency, when changes of the potential energy of the booth belts are four frequency, and potential energy interaction is one frequency in the each eigen mode. Changes of the kinetic energy of the both belts of the sandwich double axially moving belt system are two frequency-like oscillatory regimes with two time multiplicities of the eigen frequencies of the corresponding eigen amplitude mode.

1.2 Literature Survey

1. The study of the transfer of energy between subsystems coupled in hybrid system (see Refs. Heedrih (Stevanović) [7, 8, 19–23, 25–27, 32, 33, 40, 47, 48], Hedrih (Stevanović) and Simonović [58, 60, 61] and Hedrih (Stevanović) and Hedrih A. [51, 52]) is very important for different applications. Two papers by the author (see Refs. Heedrih (Stevanović) [8, 25, 26] presents analytical analysis of the transfer of energy between plates for free and forced transversal vibrations of an elastically connected double-plate system. Energy analysis of vibro-impact system dynamics with curvilinear trajectories and no ideal constraints was done by Jović in 2009 and in 2011 in his two theses [115, 120], for Magistar of science as well for doctors of sciences degrees. Potential energy and stress state in material with crack was study by Jovanović and presented in his Doctor's Degree Thesis [109, 118] in 2009. Energy analysis of the nonlinear oscillatory motions of elastic and deformable bodies was done by Hedrih (Stevanović) in her doctor's degree thesis [100, 101] in 1975. The energy analysis of longitudinal oscillations of rods with changeable cross-sections was original research results in 1995 presented by Filipovski in his magistar of sciences degree thesis [11] (for all see References from list in *Appendix – References*),
2. When, at an international conference ICNO in Kiev in 1969, my professor of mechanics and mathematics, D. P. Rašković ([88, 89]) (see Refs. Rašković (1965, 1985) presented me to academician Yuri Alekseevich Mitropolskiy (1917–2008) (see Refs. Mitropolskiy ([73–80]) and when I started really to understand the differences between linear and nonlinear phenomena in dynamics of mechanical real systems, I knew I was on the right path of research which enchanted me ever more by understanding new phenomena and their variety in nonlinear dynamics of realistic engineering and other dynamical systems. (First, my knowledge about properties of nonlinearity and the nonlinear function I obtained in gymnasium from my excellent professor of mathematics Draginja Nikolić and during my research Matura work on the subject of *Nonlinear elementary functions and their graphics* as a final high-school examination.)

For beginning of this chapter, a review survey of original results of the author and of researchers from Faculty of Mechanical Engineering University of Niš (see References [97–122] from list in *Appendix – References*), inspired and/or obtained by the asymptotic method of Krilov–Bogolyubov–Mitropolyskiy, and as a direct influence of professor Rašković scientific instruction and also by published Mitropolskiy's papers and monographs [73–80], as well as publications by Kiev Mathematical institute scientists in area of nonlinear and stochastic dynamics. These results have been obtained during realizations of the series of the research projects supported by Ministry of Sciences of Serbia, Faculty of mechanical engineering University of Niš and Mathematical Institute SANU Belgrade (see List of Projects (period 1967–2011) in Appendix – References -List of Projects [123–134]). These results have been published in scientific journals and were presented on the scientific conferences and in the bachelor degree works (see Stevanović, (1967)), Magister of sciences theses (see [99,102,104,106,110,111,113,114]), and doctoral dissertations (see (Stevanović) [101,103,105,107,116–119,121,122]) supervised by Mitropolskiy (in period from 1972 to 1975) or by Rašković (in period from 1964 to 1974), and by Hedrih in period from 1976 to 2001 year as well. In area of stochastic stability, a scientific support by series of consultation to researchers was given by S.T. Ariaratnam (Canada) and A. Tylikowski (Polad) papers.

The original results contain asymptotic analysis of the nonlinear oscillatory motions of elastic bodies: beams, plates, shells, and shafts (see References by (Stevanović) [5, 6, 11–15, 24, 29–35, 38–40, 62–65, 93–95]). Also, late a series of new research results are obtained by Janevski in 2003 and by Simonovic [53–61] in 2008 an in 2011. The multifrequency oscillatory motion of elastic bodies was studied. Corresponding systems of partial differential equations of system dynamics, as well as system of first approximation of ordinary differential equations for corresponding numbers of amplitudes and phases of multifrequency regimes of elastic bodies nonlinear oscillations were composed. The characteristic properties of nonlinear systems passing through coupled multifrequency resonant state and mutual influences between excited modes were discovered.

In the same cited papers, amplitude-frequency and phase frequency curves for stationary and nonstationary coupled multifrequency resonant kinetic states based on the numerical experiment on the system of ordinary differential equations in first approximation are presented. Resonant jumps are pointed out in the both series of graphical presentation: amplitude-frequency and phase-frequency curves for the case of the resonant interactions between modes in the same frequency resonant intervals.

Using ideas of averaging and asymptotic methods Krilov–Bogoliyubov–Mitropolyskiy in the Doctoral dissertation and in References (see Refs. Hedrih (Stevanović) [5–50]), the author gives the first asymptotic approximations of the solutions for one-, two-, three- and four-frequency vibrations of nonlinear elastic beams, shaft, and thin elastic plates, as well as of the thin elastic shells with positive constant Gauss's curvatures and finite deformations, and system of the ordinary differential equations in first asymptotic approximation for corresponding numbers of amplitudes and phases for stationary and nonstationary vibration regimes.

Some results of an investigation of multifrequency vibrations in single-frequency regime in nonlinear systems with many degrees of freedom and with slow-changing parameters are presented by Stevanović and Rašković article (1974). Application of the Krilov–Bogolyubov–Mitropolskiy asymptotic method for study of elastic bodies nonlinear oscillations and energetic analysis of the elastic bodies oscillatory motions give new results in theses by Stevanović in 1975. One-frequency transversal oscillations of thin rectangular plate with nonlinear constitutive material stress-strain relations and nonlinear transversal vibrations of a plate with special analysis of influence of weak nonlinear boundary conditions are contents of the articles by Hedrih (1979, 1981).

First approximation of an asymptotic particular solution of the nonlinear equations of a thin elastic shell with positive Gauss' curvature in two-frequency regime is pointed out in the article by Hedrih (1983). Two-frequency oscillations of the thin elastic shells with finite deformations and interactions between harmonics have been studied by Hedrih and Mitić (1983), and multifrequency forced vibrations of thin elastic shells with a positive Gauss's curvature and finite displacements by Hedrih (1984). Also, on the mutual influence between modes in nonlinear systems with small parameter applied to the multifrequencies plate oscillations are studied [54, 62, 63, 65].

Multifrequency-forced vibrations of thin elastic shells with a positive Gauss' curvature and finite deformations and initial deformations influence of the shell middle surface to the phase-frequency characteristics of the nonlinear stationary forced shell's vibrations and numerical analysis of the four-frequency vibrations of thin elastic shells with Gauss' positive curvature and finite deformations are content of reference by Hedrih and Mitić (1985). Also, initial displacement deformation influence of the thin elastic shell middle surface to the resonant jumps appearance was investigated by same authors Hedrih and Mitić (1987). By means of the graphical presentations from the cited References, analysis was made and some conclusions about nonlinear phenomenon in multifrequency vibrations regimes were pointed out. Some of these conclusions are quoted here: Nonlinearities are the reason for the appearance of interaction between modes in multifrequency regimes; in the coupled resonant state, one or several resonant jumps appear on the amplitude-frequency and phase-frequency curves; these resonant jumps are from smaller to greater amplitudes and vice versa.

Unique trigger of coupled singularities (see Refs. [28, 30, 50, 96]) with one unstable homoclinic saddle type point, and with two singular stable center type points appear in one frequency stationary-resonant kinetic state. It is visible on the phase-frequency as well as on the amplitude-frequency graphs for stationary-resonant state.

In the case of the multifrequency-coupled resonant state and in the appearance of the more resonant-coupled modes in resonant range of corresponding frequencies, unique trigger of coupled singularities and multiplied triggers of coupled singularities (see Refs. by Hedrih, 2004, 2005) appear. Maximum number of triggers of coupled singularities is adequate to number of coupled modes and resonant frequencies of external excitations. Multiplied triggers contain multiple unstable saddle homoclinic points in the mapped phase plane as the number of resonant frequencies of external excitations. For example, if a four-frequency-coupled resonant process in u-v plane is in question, four homoclinic saddle-type points appear. The appearance of these unstable homoclinic saddle points requires further study, since it induces instability in a stationary nonlinear multifrequency kinetic process.

By use a double circular plate system, presented in the Refs. [53–61], the multifrequency analysis of the nonlinear dynamics with different approaches and by use different kinetic parameters of multifrequency regimes is pointed out. Series of the amplitude-frequency and phase-frequency graphs as well as eigen-time functions–frequency graphs are obtained for stationary resonant states and analyzed according to present singularities and triggers of coupled singularities, as well as resonant jumps.

An analogy between nonlinear phenomena in particular multifrequency stationary-resonant regimes of multi-circular plate system nonlinear dynamics, multibeam system nonlinear dynamics, and corresponding regimes in chain system nonlinear dynamics is identified (see References by Hedrih (Stevanović) listed in the reference list from period 1972–2010).

Using differential equations systems of the first approximation of multifrequency regime of stationary and no stationary-resonant kinetic states, we analyzed the energy of excited modes and transfer of energy from one to other modes. On the basis of this analysis, the question of excitation of lower frequency modes by higher frequency mode in the nonlinear multifrequency vibration regimes was opened.

3. In many engineering systems with nonlinearity, high-frequency excitations are sources of the appearance of multifrequency-resonant regimes with high-frequency modes as well as low-frequency modes. It is visible from many experimental research results and also theoretical results (see Refs. [81–87]).

In the monographs written by Nayfeh [81–87], a coherent and unified treatment of analytical, computational, and experimental methods and concepts of modal nonlinear interactions is presented. This monograph is an obvious extension of Nayfeh's and Balachandran's well-known monograph titled by *Applied Nonlinear Dynamics* (1995). These methods are used to explore and unfold in a unified manner the fascinating complexities in nonlinear dynamical systems. Through the mechanisms discussed in this monograph, energy from high-frequency sources can be transferred to the low-frequency modes of supporting structures and foundations, and the result can be harmful large-amplitude oscillations that decrease their fatigue lives.

The interaction between amplitudes and phases of the different modes in the nonlinear systems with many degrees of the freedom as well as in the deformable body infinite numbers frequency vibrations with free and forced regimes is observed theoretically by averaging asymptotic methods Krilov–Bogoliyubov–Mitropolyskiy (1955, 1964, 1968, 1976 and 2003). This knowledge has great practical importance.

Application of the Krilov–Bogolyubov–Mitropolskiy asymptotic method as well as energy approach given in monographs by Mitopolskiy (see Refs. [73–80]) for study of the elastic bodies nonlinear oscillations and energy analysis of the elastic bodies oscillatory motions give new results listed in the previous part.

In the conclusion of this part, we can summarize the following: Oscillatory processes in dynamical systems depend on systems character; in such systems, energy is also transformed from one form to another and has different flows inside a dynamical system; transformation of kinetic energy into potential energy and vice versa occurs in conservative systems, but when linear systems are in question, the energy carried by a considered harmonic (mode) of adequate frequency remains constant during a dynamical process, as does the total systems mechanical energy; there is no mutual influence between harmonics, and the system may be presented by partial oscillators, the number of which is equal to the number of oscillations freedom degrees, or to the number of free vibrations own circular frequencies; during that the total mechanical energy of a single partial oscillator remains constant and the transformation of kinetic energy into potential occurs; in sash linear system, transfer energy between modes does not occurs (see Reference by Rašković (1965)).

When nonlinear conservative systems are in question, such conclusion as for linear systems would be incorrect. The theoretical and experimental studies reveal that the interactions between widely separated nonlinear modes result in various bifurcations, the coexistence of multiple attractors, and chaotic attractors. The theoretical results show also that damping may be destabilizing. The different types of nonlinear phenomena in single degree of freedom nonlinear system dynamics are investigated between other researchers.

4. An experimental and theoretical study of the response of a flexible cantilever beam to an external harmonic excitation near the beam's third natural frequency is presented and in addition. Malatkar and Nayfeh (2003) noted that the energy transfer between the third and first modes is very much dependent upon the closeness of the modulation (or Hopf bifurcation) frequency to the first-mode natural frequency. In earlier studies by Nayfeh and coworkers [81–87], the modulation frequency was close to the first-mode natural frequency, and therefore large first-mode swaying was observed. Nayfeh developed a reduced-order analytical model by discretizing the integral partial-differential equation of motion.

Identifying, evaluating, and controlling dynamical integrity measures in nonlinear mechanical oscillators are topics for researchers, presented in the Ref. [92]. Also some references by Hedrih [37] contain the energy transfer between coupled oscillators and a conclusion that energy transfer can be a measure of the dynamical integrity of hybrid systems as well as subsystems. Energy transfer in the complex system is subject of research published papers [66] and also [8, 13, 25, 32, 58].

In the paper by Lenci, S. and Rega, G., (2005) dimension reduction of homoclinic orbits of buckled beams via the nonlinear normal modes technique is presented. The problem of detecting the homoclinic orbits of an initially straight buckled beam is addressed. Two families of boundary conditions are identified and investigated in detail. A hierarchy of reduced order, single degree of freedom, models is determined. In the series of the papers [70, 71], the problem of detecting the homoclinic orbits applied to the different engineering system dynamics is investigated and obtained original research results. In the Refs. [68, 69], resonant nonlinear normal modes in the cases of two-to-one, three-to-one, and one-to-one internal resonances in undamped unforced one-dimensional systems with arbitrary linear, quadratic, and cubic nonlinearities are investigated for a class of shallow symmetric structural systems. Nonlinear orthogonality of the modes and activation of the associated interactions are clearly dual problems.

5. In the Refs. [51, 52], the expressions for the kinetic and potential energy as well as energy interaction between chains in the double DNA chain helix are obtained and analyzed for a linearized model. Corresponding expressions of the kinetic and potential energies of these uncoupled main chains are also defined for the eigen main chains of the double DNA chain helix. By obtained expressions, we concluded that there is no energy interaction between eigen main chains of the double DNA chain helix system. Time expressions of the main coordinates of the two eigen main chains are expressed by time, and eigen circular frequencies are obtained. Also, generalized coordinates of the double DNA chain helix are expressed by time correspond to the sets of the eigen circular frequencies. These data contribute to better understanding of biomechanical events of DNA transcription that occur parallel with biochemical processes. Considered as a linear mechanical system, DNA molecule as a double chain helix has its eigen circular frequencies and that is its characteristic. Mathematically, it is possible to decouple it into two chains with their set with corresponding eigen circular frequencies which are different. This may correspond to different chemical structure (the order of base pairs) of the complementary chains of DNA. We are free to propose that every specific set of base pair order has its eigen circular frequencies and its corresponding oscillatory energy, and it changes when DNA chains are coupled in the system of double chain helix. Oscillations of base pairs and corresponding oscillatory energy for specific set of base pairs may contribute to conformational chances of DNA double helix and its unzipping and folding.

1.3 Energy Exchange in Spring Pendulum System

For introducing to the problem of the energy transfer or transient in the hybrid nonlinear systems, it is useful to take, for simple analysis, into consideration the change energy between parts of the energy carrying on the generalized coordinates ϕ and ρ in the very known system, known under name *spring pendulum system*, with two degree of freedom. For the analysis of the energy in the spring pendulum, we can write the kinetic and potential energies in the forms:

$$E_k = \frac{1}{2} m \left[\dot{\rho}^2 + (\rho + \ell)^2 \dot{\phi}^2 \right]$$

and

$$E_p = \frac{1}{2}c\rho^2 + mg(\rho+\ell)(1-\cos\phi) \tag{1}$$

where: m is mass of the pendulum, ℓ length of pendulum string-neglected mass spring in the static equilibrium state of the pendulum, and c spring axial rigidity and ϕ and ρ are respectfully, angle and extension part of length of the string-spring of the pendulum with comparison of the sprig length in static equilibrium state of the pendulum, taken as the generalized coordinates of the system. For the linearized case for kinetic energy, after neglecting small member – part of kinetic energy on the generalized coordinate ϕ – we can taking into account the following expression:

* Expression $E_{k2} = \frac{1}{2}m(\rho+\ell)^2\dot{\phi}^2$ changes into approximation

$$E_{k2} \approx \frac{1}{2}m\left(\ell\dot{\phi}\right)^2. \tag{2}$$

Only for small oscillations – perturbations from equilibrium position – it is possible to use approximation of the expression for kinetic and potential energy in the form:

$$E_k \approx \frac{1}{2}m\left[\dot{\rho}^2 + \left(\ell\dot{\phi}\right)^2\right] \text{ and } E_p \approx \frac{1}{2}c\rho^2 + \frac{1}{2}mg\ell\phi^2 \tag{3}$$

For that linearized case, the generalized coordinates are normal coordinates of the small oscillations of the spring pendulum around equilibrium position $\rho = 0, \phi = 0$, and coordinates are decoupled. In this linearized case of the spring pendulum model, the energy carried on the these normal coordinates are uncoupled and transfer or transient of the total energy don't appeared between proper parts of the separate normal coordinate and on the separate processes defined by normal coordinates are conservative systems each with one degree of the freedom. In this case, in each of the coordinate, there are conversion of the energies from kinetic to potential, but the sum of the both of one normal coordinates is constant.

$$E_{k\rho} \approx \frac{1}{2}m\dot{\rho}^2 \text{ and } E_{p\rho} \approx \frac{1}{2}c\rho^2 \tag{4}$$

$$E_{k\phi} \approx \frac{1}{2}m\left(\ell\dot{\phi}\right)^2 \text{ and } E_{p\phi} \approx \frac{1}{2}mg\ell\phi^2 \tag{5}$$

This is visible from system of the differential equations in the linearized form:

$$\begin{aligned} &\ddot{\rho} + \omega_2^2\rho = 0 \text{ where } \omega_2^2 = \frac{c}{m} \\ &\ddot{\phi} + \omega_1^2\phi = 0 \text{ where } \omega_1^2 = \frac{g}{\ell}. \end{aligned} \tag{6}$$

but for the nonlinear case the interaction between coordinates is present and then energy transient appears.

$$\begin{aligned} E_k &= \frac{1}{2}m\left[\dot{\rho}^2 + \ell^2\dot{\phi}^2 + \rho^2\dot{\phi}^2 + 2\rho\ell\dot{\phi}^2\right] \qquad \text{and} \\ E_p &= \frac{1}{2}c\rho^2 + mg\ell(1-\cos\phi) + mg\rho(1-\cos\phi) \end{aligned} \tag{7}$$

We can separate the following parts:

1. Kinetic and potential energies carrying on the coordinate ρ are:

$$E_{k\rho} = \frac{1}{2}m\dot{\rho}^2 \text{ and } E_{p\rho} = \frac{1}{2}c\rho^2 + mg\rho \tag{8}$$

   By analyzing these previous expressions, we can see that with these expressions for decoupled oscillator with coordinate ρ, we have pure linear oscillator or harmonic oscillator with coordinate ρ and frequency $\omega_2^2 = \frac{c}{m}$, and separated process is isochronous.
2. Kinetic and potential energies carrying on the coordinate ϕ are

$$E_{k\phi} = \frac{1}{2}m\ell^2\dot{\phi}^2 \text{ and } E_{p\phi} = mg\ell(1-\cos\phi) \tag{9}$$

   By analyzing these previous expressions, we can see that with these expression for decoupled oscillator with coordinate ϕ, we have pure nonlinear oscillator with coordinate ϕ, and separated process is no isochronous. For a linearyzed case, this oscillator has eigen frequency $\omega_1^2 = \frac{g}{\ell}$.
3. Then, formally, we can conclude that in the spring pendulum, we have coupled two oscillators, one pure linear with one degree of freedom, and second nonlinear, also with one degree of freedom. In the hybrid system, these oscillators are coupled and mechanical energy of the coupling contain two parts: one kinetic energy and second potential energy. Then, in the coupling, hybrid connections with static and dynamic kinetic properties are introduced.

Kinetic and potential energies of the coordinate ϕ and ρ interaction in the nonlinear hybrid model are:

$$E_{k(\phi,\rho)} = \frac{1}{2}m[\rho+2\ell]\rho\dot{\phi}^2 \text{ and } E_{p(\phi,\rho)} = -mg\rho\cos\phi \tag{10}$$

For a nonlinear case, ordinary differential equations are in the following form:

$$\ddot{\rho} + \omega_2^2\rho = -g(1-\cos\phi) \tag{11}$$

$$\ddot{\phi} + \omega_1^2\phi = \omega_1^2(\phi - \sin\phi) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}(\rho+\ell) - \frac{1}{\ell^2}\rho(\rho+2\ell)\ddot{\phi} \tag{12}$$

or in nonlinear approximation forms for small oscillations around zero coordinates $\rho = 0, \phi = 0$ or around stable equilibrium position of the spring pendulum are

$$\ddot{\rho} + \omega_2^2\rho \approx -g\left(\frac{\phi^2}{2} - \frac{\phi^4}{24} + \frac{\phi^6}{6!} - \frac{\phi^8}{8!} + \ldots..\right) \tag{13}$$

$$\ddot{\phi} + \omega_1^2\phi \approx -\omega_1^2\left(\frac{\phi^3}{3} - \frac{\phi^5}{5!} + \frac{\phi^7}{7!} - \ldots.\right) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}(\rho+\ell) - \frac{1}{\ell^2}\rho(\rho+2\ell)\ddot{\phi} \tag{14}$$

If we introduce phase coordinate, then we can write:

$$
\begin{aligned}
&v = \dot{\rho} \\
&\dot{v} = -\omega_2^2 \rho - g\left(1 - \cos\phi\right) \\
&u = \dot{\phi} \\
&\dot{u} = -\omega_1^2 \phi + \omega_1^2 \left(\phi - \sin\phi\right) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}\left(\rho + \ell\right) - \frac{1}{\ell^2}\rho\left(\rho + 2\ell\right)\ddot{\phi}
\end{aligned}
\tag{15}
$$

or in the approximation

$$
\begin{aligned}
&v = \dot{\rho} \\
&\dot{v} \approx -\omega_2^2 \rho - g\left(\frac{\phi^2}{2} - \frac{\phi^4}{24} + \frac{\phi^6}{6!} - \frac{\phi^8}{8!} + \ldots..\right) \\
&u = \dot{\phi} \\
&\dot{u} \approx -\omega_1^2 \phi - \omega_1^2 \left(\frac{\phi^3}{3} - \frac{\phi^5}{5!} + \frac{\phi^7}{7!} - \ldots.\right) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}\left(\rho + \ell\right) - \frac{1}{\ell^2}\rho\left(\rho + 2\ell\right)\dot{u}
\end{aligned}
\tag{16}
$$

From system equations (11)–(12), as well from their approximations (13)–(14), we can see that their right-hand parts are nonlinear and are functions of generalized coordinates, as well as of the generalized coordinates first and second derivatives. Also we can see that generalized coordinates ϕ and ρ around their zero values, when $\rho = 0, \phi = 0$ at the stable equilibrium position of the spring pendulum, and that also they are main coordinates of the linearized model. It is of reason that the asymptotic averaged method is applicable for obtaining first asymptotic approximation of the particular solutions, and it is possible to use for energy analysis of the transfer energy between energies carried by generalized coordinates ϕ and ρ in this nonlinear system with two degree of freedom, but formally, we can take into account that we have two oscillators, one nonlinear and one linear each with one degree of freedom as two subsystems coupled in the hybrid system with two degree of freedom, by hybrid connection realized by statically and dynamical connections. This interconnection have two parts of energy interaction between subsystems expressed by kinetic and potential energies in the forms expressed by (10).

Taking into consideration some conclusion from considered system of the spring pendulum, we can conclude also that it is important to consider more simple case of the coupling between linear and nonlinear systems with one degree of freedom with different types of the coupling realized by simple static or dynamic elements, for to investigate hybrid phenomena in the coupled subsystems.

2 Energy Analysis and Free Vibration Nonlinear System

When nonlinear conservative systems are in question, such conclusion as for linear systems that no interaction between submotion components would be incorrect. The theoretical and experimental studies reveal that the interactions between widely separated modes result in various bifurcations, the coexistence of multiple attractors, and chaotic attractors.

Kinetic energy and potential energy in first asymptotic approximation for nonlinear conservative system nonlinear modes using normal coordinates of unperturbed corresponding linear system are (see Ref. [20]):

$$\mathbf{E}_k=\sum_{s=1}^{s=n}\mathbf{E}_{k\mathbf{s}}=\sum_{s=1}^{s=n}\left(\dot{\xi}_s^2\right)+g\left(\xi_1,\xi_2,\ldots,\xi_s,\xi_r,..,\xi_{n-1},\xi_n,\dot{\xi}_1,\dot{\xi}_2,\ldots,\dot{\xi}_s,\dot{\xi}_r,..,\dot{\xi}_{n-1},\dot{\xi}_n\right)$$

$$\mathbf{E}_p=\sum_{s=1}^{s=n}\left(\omega_s^2\xi_s^2\right)+f\left(\xi_1,\xi_2,\ldots,\xi_s,\xi_r,..,\xi_{n-1},\xi_n\right) \tag{17}$$

where

$$\xi_s=a_s\cos\left(\theta_s+\psi_s\right)\ \mathrm{s}=1,2,\ldots..\mathrm{n} \tag{18}$$

are first asymptotic approximations of normal coordinates, and a_s are amplitudes, and $\theta_s+\psi_s$ are phases as the functions of time and which are calculate from differential equations first approximations (see Ref. [78]).

2.1 Nonlinear Oscillator

Kinetic and potential energies and Rayleigh dissipative function of nonlinear oscillator with one degree of freedom and generalized coordinate x_1 are:

$$E_{k(1)}=\frac{1}{2}m_1\dot{x}_1^2,$$

$$E_{p(1)}=\frac{1}{2}c_1x_1^2\pm\frac{1}{4}\tilde{c}_1x_1^4$$

$$\Phi_{(1)}=\frac{1}{2}b_1\dot{x}_1^2 \tag{19}$$

where m_1 is masses, c_1 is the spring rigidity coefficient of the linear elasticity low, and $\tilde{c}_1$ the spring rigidity coefficient of the nonlinear elasticity low, upper sign (+) for hard and lower sign (−) for soft nonlinearity, b_1 coefficient of the system linear dumping force. For this nonlinear oscillator, it is right, $\frac{d}{dt}\left(E_{k(1)}+E_{p(1)}\right)=-2\Phi_{(1)}$, and for the case of the free vibrations.

For this case, differential equation is in the following form:

$$\ddot{x}_1+2\delta_1\dot{x}_2+\omega_1^2x_1=\mp\tilde{\omega}_{N1}^2x_1^3 \tag{20}$$

upper sign $(-)$ for hard and lower sign$(+)$ for soft nonlinearity.
where

$$\omega_1^2 = \frac{c_1}{m_1}, 2\delta_1 = \frac{b_1}{m_1}, \tilde{\omega}_{N1}^2 = \frac{\tilde{c}_1}{m_1}. \tag{21}$$

and characteristic equation of the basic liner equation, correspond to previous (20), have the following characteristic numbers: $\lambda_{1,2} = -\delta_1 \mp \mathbf{i}\sqrt{\omega_1^{\mathbf{2}} - \delta_1^2} = -\delta_1 \mp \mathbf{i}p_1$ for the small damping coefficient $\delta_1 < \omega_1$, and solution for free vibrations is in the form: $x_1(t) = R_{01}e^{-\delta_1 t}\cos(p_1 t + \alpha_{01})$. To obtain approximation by using averaged method, we propose solution in the following form:

$$x_1(t) = R_1(t)\, e^{-\delta_1 t}\cos\Phi_1(t) \tag{22}$$

where $R_1(t)$ and $\Phi_1(t)$ are unknown functions. Also, we can write: $\Phi_1(t) = p_1 t + \phi_1$. After averaging with respect to the full phase $\Phi_1(t)$, we obtain the following system of the averaged first-order differential equations:

$$\begin{aligned} \dot{R}_1(t) &= 0 \\ \dot{\phi}_1(t) &= \pm\frac{3}{8p_1}\tilde{\omega}_{N1}^2 R_1^2(t)\, e^{-2\delta_1 t} \end{aligned} \tag{23}$$

upper sign $(+)$ for hard and lower sign $(-)$ for soft nonlinearity.
After integration, we obtain for amplitude and phase the following first approximation:

$$\begin{aligned} R_1(t) &= R_{01} = const \\ \phi_1(t) &= \mp\frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}, \quad for \;\; \delta \neq 0 \end{aligned} \tag{24}$$

upper sign $(+)$ for hard and lower sign $(-)$ for soft nonlinearity, and for full phase $\Phi_1(t)$:

$$\Phi_1(t) = p_1 t \mp \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}, \quad for \;\; \delta \neq 0 \tag{25}$$

$(-)$ for strong and $(+)$ for soft nonlinearity, and solution in the first averaged approximation form is:

$$\begin{aligned} x_1(t) &= R_{01}e^{-\delta_1 t}\cos\Phi_1(t) \\ x_1(t) &= R_{01}e^{-\delta_1 t}\cos\left(p_1 t \mp \frac{3}{16\delta_1 p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t} + \alpha_{01}\right), \quad for \;\; \delta \neq 0 \end{aligned} \tag{26}$$

upper sign $(-)$ for hard and lower sign $(+)$ for soft nonlinearity, we can see that amplitude of the solution in the first averaged approximation form is in the form $R_{01}e^{-\delta_1 t}$ and that phase $\Phi_1(t)$ is also function of the time, and also frequency $\tilde{p}_1(t) = p_1 \mp \frac{3}{8p_1}\tilde{\omega}_{N1}^2 R_{01}^2 e^{-2\delta_1 t}$, for $\delta_1 \neq 0$, upper sign $(-)$ for hard and lower sign $(+)$ for soft nonlinearity., is changeable with time in the first asymptotic approximation obtained by averaged method.

By using previous obtained first asymptotic averaged approximation of the solution, we obtain Lyapunov exponent in the form:

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{2t} \ln\left[x_1^2(t) + \frac{1}{\omega_1^2}\dot{x}_1^2(t)\right] = -\delta_1 < 0 \tag{27}$$

or in the form

$$\tilde{\lambda}_1 = \lim_{t\to\infty} \frac{1}{2t} \ln\left[x_1^2(t) + \frac{\omega_{N1}^2}{\omega_1^2}x_1^4(t) + \frac{1}{\omega_1^2}\dot{x}_1^2(t)\right]$$

$$\tilde{\lambda}_1 = \lim_{t\to\infty} \frac{1}{2t} \ln\left[\frac{E_{sist}}{2m_1\omega_1^2}\right] = -\delta_1 < 0 \tag{28}$$

In our research, we can investigate system with small nonlinearity and small vibrations around periodic vibrations.

2.2 Linear Oscillator

Kinetic and potential energies and Rayleigh dissipative function (see Ref. by Hedrih (Stevanović) (2002) [31]) of linear oscillator with one degree of freedom and generalized coordinate x_2 are:

$$E_{k(2)} = \frac{1}{2}m_2\dot{x}_2^2,\ E_{p(2)} = \frac{1}{2}c_2x_2^2 \text{ and } \Phi_{(2)} = \frac{1}{2}b_2\dot{x}_2^2 \tag{29}$$

where m_2 is mass, c_2 is the spring rigidity coefficient of the linear elasticity low, b_2 coefficient of the system linear dumping force.

For this system, it is possible to show that: $\frac{d}{dt}\left(E_{k(2)} + E_{p(2)}\right) = -2\Phi_{(2)}$. For this case, the differential equation is in the following form: $\ddot{x}_2 + 2\delta_2\dot{x}_2 + \omega_2^2x_2 = 0$, where $\omega_2^2 = \frac{c_2}{m_2}$, $2\delta_2 = \frac{b_2}{m_2}$, and with characteristic numbers: $\lambda_{1,2} = -\delta_2 \mp \mathbf{i}\sqrt{\omega_2^2 - \delta_2^2}$ for the small damping coefficient $\delta_2 < \omega_2$. Solution for free vibrations is:

$$x_2(t) = R_0e^{-\delta_2t}\cos(p_2t + \alpha_2). \tag{30}$$

2.3 Hybrid Systems with Staitc Constraints

Kinetic and potential energies and Rayleigh dissipative function (see Ref. by Hedrih (Stevanović) (2002) [31]) of the hybrid system, containing two subsystems – one linear oscillator and one nonlinear oscillator, with two degree of freedom expressed by generalized coordinates x_1 and x_2 (see Fig. 1a*) are:

$$E_k = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \tag{31}$$

$$E_p = \frac{1}{2}c_1x_1^2 \pm \frac{1}{4}\tilde{c}_1x_1^4 + \frac{1}{2}c(x_1 - x_2)^2 + \frac{1}{2}c_2x_2^2 \tag{32}$$

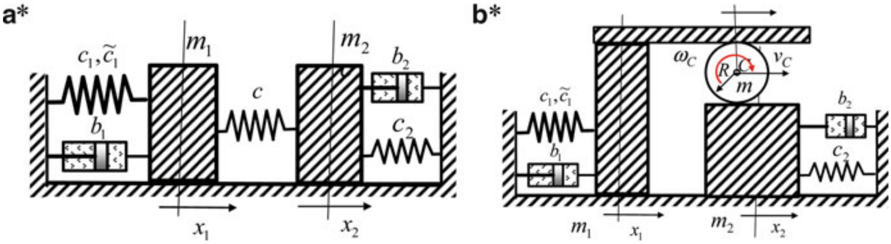

Fig. 1 Two hybrid systems containing coupled subsystems by $(\mathbf{a}^*)$ static constraint, coupled by linear spring with rigidity c and $(\mathbf{b}^*)$ dynamical constraint, coupled by rolling element of the mass m – dynamic coupling: one nonlinear (*left*) and second linear (*right*)

$$\Phi = \frac{1}{2}b_1\dot{x}_1^2 + \frac{1}{2}b_2\dot{x}_2^2 \tag{33}$$

where m_1 and m_2 are masses, c_1, c, and c_2 are the spring rigidity coefficients of the linear elasticity law, and $\tilde{c}_1$ the spring rigidity coefficient of the spring nonlinear elasticity law, where in (32) upper sign $(+)$ for hard and lower sign $(-)$ for soft nonlinearity. b_1 and b_2 coefficient of the system linear dumping forces. For this system, it is possible to show that: $\frac{d}{dt}(E_k + E_p) = -2\Phi$.

Energy interaction in this hybrid system, containing two coupled subsystems by statical constraint is potential energy of the spring for coupling nonlinear and linear subsystem and is expressed in the form:

$$E_{p(1,2)} = \frac{1}{2}c(x_2 - x_1)^2 \tag{34}$$

Coupled system of differential equations of the hybrid system containing two subsystems, one nonlinear and one linear, are in the forms:

$$\begin{aligned} &\ddot{x}_1 + 2\delta_1\dot{x}_2 + \left(\omega_1^2 + a_1^2\right)x_1 - a_1^2 x_2 = \mp\tilde{\omega}_{N1}^2 x_1^3 \\ &\ddot{x}_2 + 2\delta_2\dot{x}_2 + \left(\omega_2^2 + a_2^2\right)x_2 - a_2^2 x_1 = 0 \end{aligned} \tag{35}$$

where are: upper sign $(-)$ for hard and lower sign $(+)$ for soft nonlinearity; and…

$$\omega_i^2 = \frac{c_i}{m_i}, 2\delta_i = \frac{b_i}{m_i}, a_i^2 = \frac{c}{m_i}, \tilde{\omega}_{N1}^2 = \frac{\tilde{c}_1}{m_1}, i = 1,2. \tag{36}$$

Taking into account that consideration of the homogeneous system does not lose generality of the phenomena, next our considerations are applied to this homogeneous hybrid system.

For the basic linear equations of the coupled system of the differential equations of the hybrid system containing two subsystems, one linearized and one linear, are in the form:

$$\begin{aligned} &\ddot{x}_1 + 2\delta_1\dot{x}_1 + \left(\omega_1^2 + a_1^2\right)x_1 - a_1^2 x_2 = 0 \\ &\ddot{x}_2 + 2\delta_2\dot{x}_2 + \left(\omega_2^2 + a_2^2\right)x_2 - a_2^2 x_1 = 0 \end{aligned} \tag{37}$$

and for case that linearized and linear systems are equal ($\omega_1^2 = \omega_2^2$ and $\delta_1 = \delta_2$ and $a_1^2 = a_2^2$, we can define characteristic equation with roots – characteristic numbers:

$\lambda_{1,2} = -\delta \mp \mathbf{i} p_1$ and $\lambda_{3,4} = -\delta_1 \mp \mathbf{i} \tilde{p}_1$ for the small damping coefficient $\delta_1 < \omega_1$, it is possible to write:

$$\lambda_{1,2} = -\delta_1 \mp \mathbf{i}\sqrt{\omega_1^2 - \delta_1^2} = -\delta_1 \mp \mathbf{i} \tilde{p}_1 \quad \lambda_{3,4} = -\delta_1 \mp \mathbf{i}\sqrt{\omega_1^2 + 2a_1^2 - \delta_1^2} = -\delta_1 \mp \mathbf{i} \tilde{p}_1$$

where:

$p_1 = \sqrt{\omega_1^2 - \delta_1^2}$ for the small damping coefficient $\delta_1 < \omega_1$.

$\tilde{p}_2 = \tilde{p}_1 = \sqrt{\omega_1^2 + 2a_1^2 - \delta_1^2}$ for the small damping coefficient $\delta_1 < \omega_1$.

Corresponding solution of the linear-coupled subsystem into system, we can write in the following two-frequency form:

$$\begin{aligned} x_1(t) &= e^{-\delta t}\left[R_{01}\cos(p_1 t + \alpha_{01}) + R_{02}\cos(\tilde{p}_2 t + \alpha_{02})\right] \\ x_2(t) &= e^{-\delta t}\left[R_{01}\cos(p_1 t + \alpha_{01}) - R_{02}\cos(\tilde{p}_2 t + \alpha_{02})\right] \end{aligned} \tag{38}$$

where amplitudes R_{0i} and phases α_{0i} are constants depending of initial conditions.

By using averaged method, the first approximation of the solution of the hybrid system, containing coupled nonlinear and linear system, we propose in the forms:

$$\begin{aligned} x_1(t) &= e^{-\delta t}\left[R_1(t)\cos\Phi_1(t) + R_{21}(t)\cos\Phi_2(t)\right] \\ x_2(t) &= e^{-\delta t}\left[R_1(t)\cos\Phi_1(t) - R_{21}(t)\cos\Phi_2(t)\right] \end{aligned} \tag{39}$$

where amplitudes $R_i(t)$ and phases $\Phi_i(t),\ i = 1,2$ are unknown functions. Also, we can write: $\Phi_i(t) = p_i t + \phi_i$. Then after application averaging method and averaging obtained ordinary differential equations with respect to the full phase $\Phi_i(t)$, we obtain the following system of the first asymptotic approximation of the system differential equations for amplitudes $R_i(t)$ and phases $\Phi_i(t)$:

$$\begin{aligned} &\dot{R}_1(t) = 0 \\ &\dot{\phi}_1(t) = \pm\frac{3}{16p_1}\tilde{\omega}_{N1}^2\left[R_1^2(t) + 2R_2^2(t)\right]e^{-2\delta_1 t} \\ &\dot{R}_2(t) = 0 \\ &\dot{\phi}_2(t) = \pm\frac{3}{16\tilde{p}_2}\tilde{\omega}_{N1}^2\left[R_2^2(t) + 2R_1^2(t)\right]e^{-2\delta_1 t} \end{aligned} \tag{40}$$

where upper sign $(+)$ for hard and lower sign $(-)$ for soft nonlinearity.

After integration of the previous system of ordinary differential equations (40) in first asymptotic approximation in the case that damping is different them zero, $\delta_1 \neq 0$ we obtain the following expressions for two amplitudes $R_i(t)$ and two corresponding phases $\Phi_i(t)$, in first asymptotic approximation:

$$\begin{aligned}
&R_1(t) = R_{01} = const \\
&\phi_1(t) = \mp \frac{3}{32\delta p_1}\tilde{\omega}_{N1}^2\left[R_{01}^2 + 2R_{02}^2\right]e^{-2\delta_1 t} + \alpha_{01}, \quad \text{for } \delta_1 \neq 0 \\
&R_2(t) = R_{02} = const \\
&\phi_1(t) = \mp \frac{3}{32\delta \tilde{p}_2}\tilde{\omega}_{N1}^2\left[2R_{01}^2 + R_{02}^2\right]e^{-2\delta_1 t} + \alpha_{02}, \quad \text{for } \delta_1 \neq 0
\end{aligned} \tag{41}$$

where upper sign ($-$) for hard and lower sign ($+$) for soft nonlinearity. The first asymptotic approximation of the solutions in two frequency regime in averaged form of the hybrid system dynamics is in the following form are:

$$\begin{aligned}
x_1(t) = {} & e^{-\delta t}R_{01}\cos\left(p_1 t \mp \frac{3}{32\delta p_1}\tilde{\omega}_{N1}^2\left[R_{01}^2 + 2R_{02}^2\right]e^{-2\delta t} + \alpha_{01}\right) + \\
& + e^{-\delta t}R_{02}\cos\left(\tilde{p}_2 t \mp \frac{3}{32\delta \tilde{p}_2}\tilde{\omega}_{N1}^2\left[2R_{01}^2 + R_{02}^2\right]e^{-2\delta t} + \alpha_{02}\right)
\end{aligned}$$

for $\delta_1 \neq 0$

$$\begin{aligned}
x_2(t) = {} & e^{-\delta t}R_{01}\cos\left(p_1 t \mp \frac{3}{32\delta p_1}\tilde{\omega}_{N1}^2\left[R_{01}^2 + 2R_{02}^2\right]e^{-2\delta t} + \alpha_{01}\right) - \\
& - e^{-\delta t}R_{02}\cos\left(\tilde{p}_2 t \mp \frac{3}{32\delta \tilde{p}_2}\tilde{\omega}_{N1}^2\left[2R_{01}^2 + R_{02}^2\right]e^{-2\delta t} + \alpha_{02}\right)
\end{aligned}$$

for $\delta_1 \neq 0$ (42)

where upper sign ($-$) for hard and lower sign ($+$) for soft nonlinearity. We can see that amplitudes of the solution in the first approximation are in the form $R_{0i}e^{-\delta t}$ and that phases are also functions of the time, and also frequencies

$$p_1(t) = p_1 \mp \frac{3}{16\tilde{p}_1}\tilde{\omega}_{N1}^2\left[R_{01}^2 + 2R_{02}^2\right]e^{-2\delta t} \quad \text{and} \quad \tilde{p}_2(t) = \tilde{p}_2 \mp \frac{3}{16\tilde{p}_2}\tilde{\omega}_{N1}^2\left[2R_{01}^2 + R_{02}^2\right]e^{-2\delta t} \tag{43}$$

where upper sign ($-$) for hard and lower sign ($+$) for soft nonlinearity. Are changeable with time in the first approximation obtained by asymptotic averaged method.

By using previous first asymptotic approximation of the solution in the two frequency regime, we can obtain Lyapunov exponents in the forms:

$$\begin{aligned}
\lambda_1 &= \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_1^2(t) + \frac{1}{\omega_1^2}\dot{x}_1^2(t)\right] = -\delta < 0 \\
\lambda_2 &= \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_2^2(t) + \frac{1}{\omega_2^2}\dot{x}_2^2(t)\right] = -\delta < 0
\end{aligned} \tag{44}$$

Also, taking into account that system is nonlinear, we can obtain Lyapunov exponents in the following forms:

$$\tilde{\lambda}_1 = \lim_{t\to\infty} \frac{1}{2t} \ln \left[x_1^2(t) + \frac{\tilde{\omega}_{N1}^2}{\omega_1^2} x_1^4(t) + \frac{1}{\omega_1^2} \dot{x}_1^2(t) \right]$$

$$\tilde{\lambda}_1 = \lim_{t\to\infty} \frac{1}{2t} \ln \left[\frac{E_{subsist(1)}}{2m_1\omega_1^2} \right] = -\delta < 0 \tag{45}$$

For the nonhomogeneous case, we can define characteristic equation, with four roots: $\lambda_{1,2} = -\hat{\delta}_1 \mp \mathbf{i}\hat{p}_1$ and $\lambda_{3,4} = -\hat{\delta}_2 \mp \mathbf{i}\hat{p}_2$, and solution of the linear-coupled system, we can write in the following form:

$$x_1(t) = K_{21}^{(1)} e^{-\hat{\delta}_1 t} R_{01} \cos(\hat{p}_1 t + \alpha_{01}) + K_{21}^{(2)} e^{-\hat{\delta}_2 t} R_{02} \cos(\hat{p}_2 t + \alpha_{02})$$

$$x_2(t) = K_{22}^{(1)} e^{-\hat{\delta}_1 t} R_{01} \cos(\hat{p}_1 t + \alpha_{01}) + K_{22}^{(2)} e^{-\hat{\delta}_2 t} R_{02} \cos(\hat{p}_2 t + \alpha_{02}) \tag{46}$$

where $K_{2i}^{(s)}$ are cofactors of the system, and amplitudes and phases, R_{0i} and α_{0i}, are constants.

By using asymptotic averaged method, a first asymptotic approximation of the solution of the hybrid system, containing coupled nonlinear and linear system as subsystems, we propose solutions in the following forms:

$$x_1(t) = K_{21}^{(1)} e^{-\hat{\delta}_1 t} R_1(t) \cos\Phi_1(t) + K_{21}^{(2)} e^{-\hat{\delta}_2 t} R_{02} \cos\Phi_2(t)$$

$$x_2(t) = K_{22}^{(1)} e^{-\hat{\delta}_1 t} R_{01} \cos\Phi_1(t) + K_{22}^{(2)} e^{-\hat{\delta}_2 t} R_{02} \cos\Phi_2(t) \tag{47}$$

where $R_i(t)$ and $\Phi_i(t)$ are unknown functions. Also we can write: $\Phi_i(t) = \hat{p}_i t + \phi_i$. And all next is similar as in previous considered part.

2.4 Hybrid Systems with Dynamic Constraints

In Fig. 1b*, we can see a hybrid system containing two subsystems, one linear and one nonlinear coupled by dynamical constraint. Dynamical constraint consists of the one disk with mass m and mass inertia axial moment $\mathbf{J_C}$ with possibility of rolling between two masses m_1 and m_2 of the subsystems. In our research, we can investigate small nonlinearity in the subsystem, and also in the hybrid system and also small nonlinear vibrations around periodic regimes.

Kinetic energy of the coupling nonlinear and linear subsystems is in the following form:

$$E_{k(1,2)} = \frac{1}{2}\left(\hat{a}_{11}\dot{x}_1^2 + \hat{a}_{22}\dot{x}_2^2 + 2\dot{x}_1\dot{x}_2\hat{a}_{12}\right) \tag{48}$$

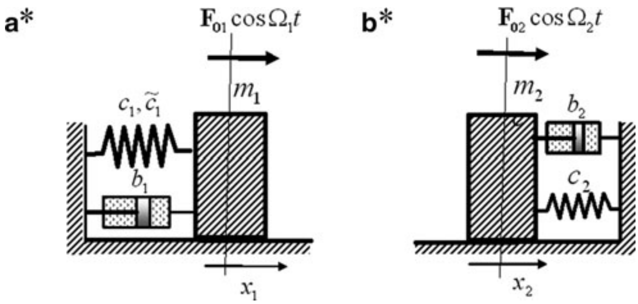

Fig. 2 Uncoupled subsystems: one nonlinear ($\mathbf{a}^*$) and second linear ($\mathbf{b}^*$)

where

$$\hat{a}_{11} = \frac{m}{4} + \frac{\mathbf{J_C}}{4R^2}, \hat{a}_{22} = \frac{m}{4} + \frac{\mathbf{J_C}}{4R^2}, \hat{a}_{12} = \frac{m}{4} - \frac{\mathbf{J_C}}{4R^2}. \tag{49}$$

Then we have a hybrid system with coupled dynamic, but also linear, constraint between two subsystems as a resultant dynamic of two subsystem dynamics in mutual interactions.

Kinetic and potential energies and Rayleigh energy dissipation function of the hybrid system, containing two subsystems – one linear oscillator and one nonlinear oscillator, with two degree of freedom expressed by generalized coordinates x_1 and x_2 (see Fig. 2a*) are:

$$E_k = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}\left[m\left(\frac{\dot{x}_1+\dot{x}_2}{2}\right)^2 + \mathbf{J_C}\left(\frac{\dot{x}_2-\dot{x}_1}{2R}\right)^2\right] \tag{50}$$

$$E_p = \frac{1}{2}c_1x_1^2 \pm \frac{1}{4}\tilde{c}_1x_1^4 + \frac{1}{2}c_2x_2^2$$

$$\Phi = \frac{1}{2}b_1\dot{x}_1^2 + \frac{1}{2}b_2\dot{x}_2^2 \tag{51}$$

where upper sign $(+)$ for hard and lower sign $(-)$ for soft nonlinearity. Also, where m_1 and m_2 are masses, c_1, c, and c_2 are the spring rigidity coefficients of the linear elasticity low, and $\tilde{c}_1$ the spring rigidity coefficient of the nonlinear elasticity low, b_1 and b_2 coefficient of the system linear dumping forces. For this system, it is possible to show that is: $\frac{d}{dt}(E_k+E_p) = -2\Phi$.

Energy interaction in this system is by kinetic energy of the rolling element for coupling nonlinear and linear subsystem and is expressed in the form:

$$E_k = \frac{1}{2}\left(\tilde{a}_{11}\dot{x}_1^2 + \tilde{a}_{22}\dot{x}_2^2 + 2\tilde{a}_{12}\dot{x}_1\dot{x}_2\right) \tag{52}$$

where

$$\tilde{a}_{11} = m_1 + \frac{m}{4} + \frac{\mathbf{J_C}}{4R^2} = a_{11} + \hat{a}_{11}, \tilde{a}_{22} = m_2 + \frac{m}{4} + \frac{\mathbf{J_C}}{4R^2} = a_{22} + \hat{a}_{22},$$

$$\tilde{a}_{12} = \frac{m}{4} - \frac{\mathbf{J_C}}{4R^2} = \hat{a}_{12} \tag{53}$$

Coefficient $\tilde{a}_{12} = \frac{m}{4} - \frac{\mathbf{J}_C}{4R^2}$ is coefficient of the subsystems coupling, and the constraint is dynamical. Then, this coefficient is coefficient of inertia. When this coefficient is equal to zero, then the system coordinate x_1 and x_2 are decoupled and there are not energy of the coupling, but there are energy of the influence of the dynamic constraint by additional members.

Kinetic energy of the first subsystem as a one part of the hybrid system is: $E_k = \frac{1}{2}\tilde{a}_{11}\dot{x}_1^2$. Kinetic energy of the second subsystem as a one part of the hybrid system is: $E_k = \frac{1}{2}\tilde{a}_{22}\dot{x}_2^2$. Kinetic energy of the coupling of the subsystems as a two parts of the hybrid system is: $E_k = \tilde{a}_{12}\dot{x}_1\dot{x}_2$.

Additional part of the kinetic energy of the first subsystem – reduction of the dynamic constraint to the first subsystem $E_{k(1)d} = \frac{1}{2}\hat{a}_{11}\dot{x}_1^2$.

Additional part of the kinetic energy of the second subsystem – reduction of the dynamic constraint to the second subsystem $E_{k(2)d} = \frac{1}{2}\hat{a}_{22}\dot{x}_2^2$. When the coefficient of subsystems coupling equals zero, $\tilde{a}_{12} = \left[\frac{m}{4} - \frac{\mathbf{J}_C}{4R^2}\right] = 0$, then subsystems do not have kinetic energy interaction, but have additional part of kinetic energy of the first subsystem – reduction of the dynamic constraint to the first subsystem and additional part of the kinetic energy of the second subsystem – reduction of the dynamic constraint to the second subsystem.

System of differential equations is based on the kinetic and potential energy and Rayleigh energy dissipation function and is obtained in the following form:

$$\ddot{x}_1 + \kappa_1\ddot{x}_2 + \tilde{\omega}_1^2 x_1 + 2\tilde{\delta}_1\dot{x}_1 = \mp\tilde{\tilde{\omega}}_{N1}^2 x_1^3$$
$$\ddot{x}_2 + \kappa_2\ddot{x}_1 + \tilde{\omega}_2^2 x_2 + 2\tilde{\delta}_2\dot{x}_2 = 0 \tag{54}$$

where upper sign $(-)$ for hard and lower sign $(+)$ for soft nonlinearity and following notations: $\kappa_1 = \frac{\tilde{a}_{12}}{\tilde{a}_{11}}, \kappa_2 = \frac{\tilde{a}_{12}}{\tilde{a}_{22}}, \tilde{\omega}_1^2 = \frac{c_1}{\tilde{a}_{11}}, \tilde{\omega}_2^2 = \frac{c_1}{\tilde{a}_{22}}, \tilde{\tilde{\omega}}_{N1}^2 = \frac{\tilde{c}_1}{\tilde{a}_{11}} = \tilde{\omega}_{N1}^2\frac{m_1}{\tilde{a}_{11}}, 2\tilde{\delta}_i = \frac{b_i}{\tilde{a}_{ii}}$, $i = 1,2$ are introduced.

For the basic linear equations of the linear dynamically coupled system of the differential equations of the hybrid system containing two subsystems, one linearized and one linear are in the form

$$\ddot{x}_1 + \kappa_1\ddot{x}_2 + \tilde{\omega}_1^2 x_1 + 2\tilde{\delta}_1\dot{x}_1 = 0$$
$$\ddot{x}_2 + \kappa_2\ddot{x}_1 + \tilde{\omega}_2^2 x_2 + 2\tilde{\delta}_2\dot{x}_2 = 0 \tag{55}$$

We can compose corresponding characteristic equation with four roots: $\lambda_{1,2} = -\hat{\delta}_1 \mp \mathbf{i}\hat{p}_1$ and $\lambda_{3,4} = -\hat{\delta}_2 \mp \mathbf{i}\hat{p}_2$. It is not difficult to obtain eigen amplitude numbers and solutions of the basic linear-coupled system, we can write in the following form:

$$x_1(t) = K_{21}^{(1)} e^{-\hat{\delta}_1 t} R_{01}\cos(\hat{p}_1 t + \alpha_{01}) + K_{21}^{(2)} e^{-\hat{\delta}_2 t} R_{02}\cos(\hat{p}_2 t + \alpha_{02})$$
$$x_2(t) = K_{22}^{(1)} e^{-\hat{\delta}_1 t} R_{01}\cos(\hat{p}_1 t + \alpha_{01}) + K_{22}^{(2)} e^{-\hat{\delta}_2 t} R_{02}\cos(\hat{p}_2 t + \alpha_{02}) \tag{56}$$

where $K_{2i}^{(s)}$ are cofactors of the system, and amplitudes R_{0i} and phases α_{0i}, are constants, depending of initial conditions.

By using asymptotic averaged method, a first asymptotic approximation of the solution of the hybrid system dynamics, containing dynamical coupled nonlinear and linear system, we propose solutions in the following forms:

$$x_1(t) = K_{21}^{(1)} e^{-\hat{\delta}_1 t} R_1(t) \cos \Phi_1(t) + K_{21}^{(2)} e^{-\hat{\delta}_2 t} R_2(t) \cos \Phi_2(t)$$
$$x_2(t) = K_{22}^{(1)} e^{-\hat{\delta}_1 t} R_1(t) \cos \Phi_1(t) + K_{22}^{(2)} e^{-\hat{\delta}_2 t} R_2(t) \cos \Phi_2(t) \tag{57}$$

where amplitudes $R_s(t)$ and phases $\Phi_s(t)$ are unknown functions. Also, we can write: $\Phi_i(t) = \hat{p}_i t + \phi_i$. After applying asymptotic averaging with respect to the full phases $\Phi_s(t)$, we obtain the system of the first asymptotic averaged approximation of the differential equations for amplitudes $R_i(t)$ and phases $\Phi_i(t)$. After integrating the system of averaged differential equations, we obtain first approximation of the amplitudes $R_i(t)$ and phases $\Phi_i(t)$ of the solution in the following form:

$$R_1(t) = R_{01} = const$$
$$\phi_1(t) = \mp \frac{3}{16 p_1 \left[K_{21}^{(1)} K_{22}^{(2)} - K_{22}^{(1)} K_{21}^{(2)} \right]} \tilde{\omega}_{N1}^2 \times \left\{ \frac{e^{-2\hat{\delta}_1 t}}{2\hat{\delta}_1} \left(K_{21}^{(1)} \right)^3 [R_{01}]^2 + \frac{e^{-2\hat{\delta}_2 t}}{\hat{\delta}_2} K_{21}^{(1)} \left[K_{21}^{(2)} \right]^2 [R_{02}]^2 \right\} + \alpha_{01}$$

for $\delta_1 \neq 0$

$$R_2(t) = R_{02} = const$$
$$\phi_2(t) = -\frac{3}{16 \hat{p}_2 \left[K_{21}^{(2)} K_{22}^{(1)} - K_{22}^{(2)} K_{21}^{(1)} \right]} \tilde{\omega}_{N1}^2 \times \left\{ \frac{e^{-2\hat{\delta}_1 t}}{\hat{\delta}_1} \left(K_{21}^{(1)} \right)^3 [R_{01}]^2 + \frac{e^{-2\hat{\delta}_2 t}}{2\hat{\delta}_2} K_{21}^{(1)} \left[K_{21}^{(2)} \right]^2 [R_{02}]^2 \right\} + \alpha_{02}$$

for $\delta_1 \neq 0$ (58)

where upper sign $(-)$ for hard and lower sign $(+)$ for soft nonlinearity. Solution in the first averaged asymptotic approximation is not difficult to compose by use expression (57) and (58).

By using previous first asymptotic approximation of the solution in the two frequency regime, we can obtain Lyapunov exponents in the forms:

$$\lambda_1 = \lim_{t \to \infty} \tfrac{1}{2t} \ln \left[x_1^2(t) + \tfrac{1}{\tilde{\omega}_1^2} \dot{x}_1^2(t) \right] = -\hat{\delta}_1 < 0$$
$$\lambda_2 = \lim_{t \to \infty} \tfrac{1}{2t} \ln \left[x_2^2(t) + \tfrac{1}{\tilde{\omega}_2^2} \dot{x}_2^2(t) \right] = -\hat{\delta}_2 < 0 \tag{59}$$

Also, taking into account that system is nonlinear, we can introduce first Lyapunov exponent in the forms:

$$\tilde{\lambda}_1 = \lim_{t\to\infty} \frac{1}{2t} \ln \left[x_1^2(t) + \frac{\tilde{\omega}_{N1}^2}{\tilde{\omega}_1^2} x_1^4(t) + \frac{1}{\tilde{\omega}_1^2} \dot{x}_1^2(t) \right]$$

$$\tilde{\lambda}_1 = \lim_{t\to\infty} \frac{1}{2t} \ln \left[\frac{E_{subsist(1)}}{2m_1 \tilde{\omega}_1^2} \right] = -\hat{\delta}_1 < 0 \tag{60}$$

3 Energy Analysis of Forced Nonlinear Systems

3.1 A Spring Pendulum

For introducing to the problem of the energy transfer or transient in the hybrid nonlinear system forced dynamics, it is useful to take, for simple analysis, into consideration the change energy between parts of the energy carrying on the generalized coordinates ϕ and ρ in the *spring pendulum system* with two degree of freedom excited by external excitations. For the analysis of the energy in the spring pendulum in the forced regime excited by external one frequency excitation – generalized forces $M_\phi(t) = M_0 \cos(\Omega_\phi t + \vartheta_\phi)$ and $F_\rho(t) = F_0 \cos(\Omega_\rho t + \vartheta_\rho)$ – we can write the kinetic and potential energies in the forms (1). By taking into account all comments and asymptotic approximation as in the introductory part of this paper, as well as corresponding expressions (2)–(5), system of the differential equations of the linearized system is in the following form:

$$\ddot{\rho} + \omega_2^2 \rho = h_{0\rho} \cos\left(\Omega_\rho t + \vartheta_\rho\right) \tag{61}$$

where

$$\omega_2^2 = \frac{c}{m},\ h_{0\rho} = \frac{F_0}{m}$$

$$\ddot{\phi} + \omega_1^2 \phi = h_{0\phi} \cos\left(\Omega_\phi t + \vartheta_\phi\right) \tag{62}$$

where $\omega_1^2 = \frac{g}{\ell}$, $h_{0\phi} = \frac{M_0}{m\ell^2}$.

Solutions of the linearized equations (61) an (62) are:

$$\rho(t) = R_2 \cos(\omega_2 t + \alpha_{02}) + \frac{h_{0\rho}}{\omega_2^2 - \Omega_\rho^2} \cos\left(\Omega_\rho t + \vartheta_\rho\right) \tag{63}$$

$$\phi(t) = R_1 \cos(\omega_1 t + \alpha_{01}) + \frac{h_{0\phi}}{\omega_1^2 - \Omega_\phi^2} \cos\left(\Omega_\phi t + \vartheta_\phi\right) \tag{64}$$

For that linearized case, both chosen coordinates are main coordinates of the linearized model, and from solutions (63)–(64), we can see that free and also, forced vibrations are uncoupled, and not interaction between free, and also forced modes of the vibrations. Then, we have two uncoupled oscillators with different eigen circular frequencies $\omega_1^2 = \frac{g}{\ell}$ and $\omega_2^2 = \frac{c}{m}$ and different forced external excitation frequencies Ω_ϕ and Ω_ρ and with possibilities of appearance two main uncoupled resonant regimes, when $\Omega^2_{\phi,resonant} = \omega_1^2 = \frac{g}{\ell}$ and $\Omega^2_{\rho,resonant} = \omega_2^2 = \frac{c}{m}$.

In this case, for linearized models and in the resonant cases, expressions for solutions are in the following forms:

$$\begin{aligned}\rho(t)|_{\Omega_{\rho,resonant}} &= \omega_2 = \rho_0 \cos\omega_2 t + \frac{\dot{\rho}_0}{\omega_2}\sin\omega_2 t + \\ &+ \frac{h_{0\rho}}{2\omega_2}\left[\omega_2 t \sin\left(\omega_2 t + \vartheta_\rho\right) - \sin\omega_2 t \sin\vartheta_\rho\right]\end{aligned} \tag{65}$$

$$\begin{aligned}\phi(t)|_{\Omega_{\phi,resonant}} &= \omega_1 = \phi_0 \cos\omega_1 t + \frac{\dot{\phi}_0}{\omega_1}\sin\omega_1 t + \\ &+ \frac{h_{0\phi}}{2\omega_1}\left[\omega_1 t \sin\left(\omega_1 t + \vartheta_\phi\right) - \sin\omega_1 t \sin\vartheta_\phi\right]\end{aligned} \tag{66}$$

But, for the nonlinear case the interaction between coordinates is present and then energy transient appears.

Expressions for kinetic and potential energies are in the same forms as presented and analyzed in first part for free vibrations and named by (6)–(10). Then, the expressions for coordinates are different and must be taken in the forms (65)–(66).

By analyzing corresponding expressions, we can see that with these expressions for decoupled oscillator with coordinate ρ, we have pure linear oscillator or harmonic oscillator with coordinate ρ and frequency $\omega_2^2 = \frac{c}{m}$, and separated process is isochronous. By analyzing these corresponding expressions, we can see that with these expressions for decoupled oscillators with coordinate ϕ, we have pure nonlinear oscillator with coordinate ϕ, and separated process is no isochronous. For a linearyzed case, this oscillator has eigen frequency $\omega_1^2 = \frac{g}{\ell}$.

For forced nonlinear case, differential equations of the system nonlinear oscillations are in the following form:

$$\ddot{\rho} + \omega_2^2\rho = -g(1-\cos\phi) + h_{0\rho}\cos\left(\Omega_\rho t + \vartheta_\rho\right) \tag{67}$$

$$\begin{aligned}\ddot{\phi} + \omega_1^2\phi &= \omega_1^2(\phi - \sin\phi) - \frac{2}{\ell^2}\dot{\rho}\dot{\phi}(\rho+\ell) - \\ &- \frac{1}{\ell^2}\rho(\rho+2\ell)\ddot{\phi} + h_{0\phi}\cos\left(\Omega_\phi t + \vartheta_\phi\right)\end{aligned} \tag{68}$$

or in nonlinear approximation forms for small oscillations around zero coordinates $\rho = 0, \phi = 0$ or of the around stable equilibrium position of the spring pendulum

$$\ddot{\rho}+\omega_2^2\rho \approx -g\left(\frac{\phi^2}{2}-\frac{\phi^4}{24}+\frac{\phi^6}{6!}-\frac{\phi^8}{8!}+\ldots..\right)+ \\ +h_{0\rho}\cos\left(\Omega_\rho t+\vartheta_\rho\right) \tag{69}$$

$$\ddot{\phi}+\omega_1^2\phi \approx -\omega_1^2\left(\frac{\phi^3}{3}-\frac{\phi^5}{5!}+\frac{\phi^7}{7!}-\ldots.\right)- \\ -\frac{2}{\ell^2}\dot{\rho}\dot{\phi}\left(\rho+\ell\right)-\frac{1}{\ell^2}\rho\left(\rho+2\ell\right)\ddot{\phi}+h_{0\phi}\cos\left(\Omega_\phi t+\vartheta_\phi\right) \tag{70}$$

If we introduce phase coordinate, then we can write:

$$\begin{aligned}
v &= \dot{\rho} \\
\dot{v} &= -\omega_2^2\rho-g\left(1-\cos\phi\right)+h_{0\rho}\cos\left(\Omega_\rho t+\vartheta_\rho\right) \\
u &= \dot{\phi} \\
\dot{u} &= -\omega_1^2\phi+\omega_1^2\left(\phi-\sin\phi\right)-\frac{2}{\ell^2}\dot{\rho}\dot{\phi}\left(\rho+\ell\right)- \\
&\quad -\frac{1}{\ell^2}\rho\left(\rho+2\ell\right)\dot{u}+h_{0\phi}\cos\left(\Omega_\phi t+\vartheta_\phi\right)
\end{aligned} \tag{71}$$

or in the approximation

$$\begin{aligned}
v &= \dot{\rho} \\
\dot{v} &\approx -\omega_2^2\rho-g\left(\frac{\phi^2}{2}-\frac{\phi^4}{24}+\frac{\phi^6}{6!}-\frac{\phi^8}{8!}+\ldots..\right)+h_{0\rho}\cos\left(\Omega_\rho t+\vartheta_\rho\right) \\
u &= \dot{\phi} \\
\dot{u} &\approx -\omega_1^2\phi-\omega_1^2\left(\frac{\phi^3}{3}-\frac{\phi^5}{5!}+\frac{\phi^7}{7!}-\ldots.\right)- \\
&\quad -\frac{2}{\ell^2}\dot{\rho}\dot{\phi}\left(\rho+\ell\right)-\frac{1}{\ell^2}\rho\left(\rho+2\ell\right)\dot{u}+h_{0\phi}\cos\left(\Omega_\phi t+\vartheta_\phi\right)
\end{aligned} \tag{72}$$

From system of the differential equations (67)–(68), as well as from their approximations (68)–(70), we can see that their right-hand parts are nonlinear and are functions of generalized coordinates, as well as of the generalized coordinates first and second derivatives with respect to time and function of time. Also, we can see that generalized coordinates ϕ and ρ around their zero values, when $\rho=0,\ \phi=0$ at the stable equilibrium position of the spring pendulum are also main coordinates of the linearized model. It is reason that the asymptotic averaged method is applicable for obtaining first asymptotic approximation of the solutions.

Then it is possible that first asymptotic approximations of the solutions of the system of nonlinear differential equations (67)–(68) take into account in the following asymptotic approximations for the small spring pendulum forced elongations in the form:

$$\begin{aligned}\rho &= a_\rho\left(t\right)\cos\left(\omega_1 t+\varphi_\rho\left(t\right)\right)\\ \phi &= a_\phi\left(t\right)\cos\left(\omega_2 t+\varphi_\phi\left(t\right)\right)\end{aligned} \tag{73}$$

where amplitudes $a_\rho(t)$ and $a_\varphi(t)$ and phases $\varphi_\rho(t)$ and $\varphi_\varphi(t)$ are defined by system of first order nonlinear differential equations in first asymptotic approximation in the following form:

$$\begin{aligned}
\dot{a}_\rho\left(t\right) &= \frac{h_{0\rho}}{\left(\omega_2+\Omega_\rho\right)}\sin\left(\varphi_\rho\left(t\right)-\vartheta_\rho\right)\\
\dot{\varphi}_\rho\left(t\right) &= \omega_2-\Omega_\rho-\frac{h_{0\rho}}{a_\rho\left(t\right)\left(\omega_2+\Omega_\rho\right)}\cos\left(\varphi_\rho\left(t\right)-\vartheta_\rho\right)\\
\dot{a}_\phi\left(t\right) &\approx -\frac{h_{0\phi}}{2\left(\omega_1+\Omega_\phi\right)}\sin\left(\varphi_\phi\left(t\right)-\vartheta_\phi\right)+\frac{h_{0\phi}}{3\left(\omega_1+\Omega_\phi\right)}\frac{a_\rho^2\left(t\right)}{\ell^2}\sin\left(\varphi_\phi\left(t\right)-\vartheta_\phi\right)\\
\dot{\varphi}_\phi\left(t\right) &\approx \omega_1-\Omega_\phi+\frac{\omega_1}{12}\left[1-\frac{a_\rho^2\left(t\right)}{2\ell^2}\right]-\frac{h_{0\phi}}{2a_\phi\left(t\right)\left(\omega_1+\Omega_\phi\right)}\cos\left(\varphi_\phi\left(t\right)-\vartheta_\phi\right)+\\
&\quad +\frac{h_{0\phi}}{3a_\phi\left(t\right)\left(\omega_1+\Omega_\phi\right)}\frac{a_\rho^2\left(t\right)}{\ell^2}\cos\left(\varphi_\phi\left(t\right)-\vartheta_\phi\right)
\end{aligned} \tag{74}$$

where $\Omega_\phi \approx \omega_1$ and $\Omega_\rho \approx \omega_2$ are external excitation frequencies in the resonant rages corresponding eigen frequencies of corresponding linearized system. Previous system of four nonlinear and first-order differential equation in the first asymptotic approximation are obtained by asymptotic Krilov–Bogoliyubov–Mitropolyskiy method and for small amplitudes of external excitations and in the resonant rages of the both frequencies.

Taking into consideration some conclusion from considered system of the spring pendulum, we can conclude, also, that it is important to consider more simple case of the coupling between linear and nonlinear systems with one degree of freedom with different types of the coupling realized by simple static or dynamic elements, for to investigate hybrid phenomena in the nonlinear system forced dynamics.

Also, it is possible to use for energy analysis of the transfer energy between energies carried by generalized coordinates ϕ and ρ in this nonlinear system forced dynamics with two degree of freedom, but formally, we can take into account that, we have two oscillators, one nonlinear and one linear each with one degree of freedom as two subsystems coupled in the hybrid system with two degree of freedom, by hybrid connection realized by statical and dynamical connections. This interconnection have two parts of energy interaction between subsystems expressed by kinetic and potential energy in the form (10).

Taking into consideration some conclusion for considered system of the spring pendulum forced oscillations, we can conclude also that it is important to consider more simple case of the coupling between linear and nonlinear systems with one degree of freedom with different types of the coupling realized by simple static or dynamic elements, for to investigate hybrid phenomena in the system forced dynamics.

3.2 A Nonlinear Oscillator

For to obtain asymptotic approximation of the nonlinear differential equation (73) by using asymptotic methods Krilov–Bogoliyubov–Mitropolyskiy, we propose solution in the first approximation in the following form:

$$x_1(t) = R_1(t)\, e^{-\delta_1 t} \cos \Phi_1(t) \tag{75}$$

where amplitude $R_1(t)$ and phase $\Phi_1(t)$ are unknown functions and defined by system of the first-order differential equations in the following form:

$$\dot{R}_1(t) = -\frac{h_{01} e^{\delta_1 t}}{p_1 + \Omega_1(\tau)} \sin \phi_1(t) \tag{76}$$

$$\dot{\phi}_1(t) = p_1 - \Omega_1 \pm \frac{3\tilde{\omega}_{N1}^2}{8p_1} R_1^2(t) e^{-2\delta_1 t} + \frac{h_{01} e^{\delta_1 t}}{R_1(t)\left[p_1 + \Omega_1(\tau)\right]} \sin \phi_1(t) \tag{77}$$

where upper sign $(+)$ for hard and lower sign $(-)$ for soft nonlinearity. Also, where: $\Phi_1(t) = p_1 t + \phi_1$, and for the case that frequency of external excitation is in the frequency interval of resonant range of the eigen frequency of the corresponding linearyzed system, $\Omega_1 \approx p_1$.

By using previous first asymptotic approximation of the solution (74)-(76)-(77) in the single frequency regime, we can obtain Lyapunov exponent in the form:

$$\lambda_1 = \lim_{t \to \infty} \frac{1}{2t} \ln \left[x_1^2(t) + \frac{1}{\omega_1^2} \dot{x}_1^2(t) \right] = -\delta_1 < 0 \tag{78}$$

In our research, we can investigate small nonlinearity and small vibrations around periodic vibrations in regimes of stationary resonant ranges, and far of resonant frequency range.

For forced vibrations and for work of the external excitation force and damping force, we can write that:

$$\mathbf{A}_{T_P(1)}^{F_w} = \int_0^{T_P} F_{W(1)}\left(\dot{x}_{p(1)}\right) dx_{p(1)}(t) = -\int_0^{T_P} b_1 \left(\dot{x}_{P1}\right)^2 dt \tag{79}$$

$$\mathbf{A}_{T_P(1)}^{F(t)} = \int_0^{T_P} F_{(1)}(t) dx_p(t) = -\int_0^{T_P} F_{(1)}(t) \dot{x}_{P1} dt \tag{80}$$

In linear systems $\mathbf{A}_{T_P(1)}^{F_w}$ and $\mathbf{A}_{T_P(1)}^{F(t)}$, these works for one period of the external excitation are equal, and in result of the appearance of the pure periodic forced vibrations with frequency of the external one frequency excitation. But in the nonlinear system when external excitation frequency is outside of the resonant frequency range intervals, and in the system appear pure periodic forced vibrations with external excitation frequency, then we can conclude that these works are equal. But, in nonlinear systems, it is evident that under the influence of the pure one-frequency external excitation, it is a possible appearance of different forced vibration regimes, as double periodic as well as chaotic like and stochastic like regimes, and this need to find relations between these works, of the external excitations and damping force. Also, it needs to investigate energy used to chaotic like and stochastic like forced regime appearance.

3.3 A Linear Oscillator

Expressions of kinetic and potential energies, and Raleigh energy dissipation function of linear oscillator, see Fig. 2b*, with one degree of freedom and generalized coordinate x_2 are same as expression (29). For this case, ordinary differential equation is in the following form: $\ddot{x}_2 + 2\delta_2\dot{x}_2 + \omega_2^2 x_2 = h_{02}\cos(\Omega_2 t + \vartheta_{02})$, where $\omega_2^2 = \frac{c_2}{m_2}$, $2\delta_2 = \frac{b_2}{m_2}$, $h_{02} = \frac{F_2}{m_2}$ and with characteristic eigen numbers: $\lambda_{1,2} = -\delta_2 \mp \mathbf{i}\sqrt{\omega_\mathbf{2}^\mathbf{2} - \delta_2^2}$ for the small damping coefficient $\delta_2 < \omega_2$.

In linear systems $\mathbf{A}_{T_P(1)}^{F_w}$ and $\mathbf{A}_{T_P(1)}^{F(t)}$, these work for one period of the one period of the external excitation are equal, and in result of the appearance of the pure periodic forced vibrations with frequency of the external one frequency vibrations:

$$\mathbf{A}_{T_P(2)}^{F_w} = \int_0^{T_P} F_{W(2)}\left(\dot{x}_{p(2)}\right) dx_{p(2)}(t) = -\int_0^{T_P} b_2\left(\dot{x}_{P2}\right)^2 dt \tag{81}$$

$$\mathbf{A}_{T_P(2)}^{F(t)} = \int_0^{T_P} F_{(2)}(t)dx_{p(2)}(t) = -\int_0^{T_P} F_2(t)\dot{x}_{P2}dt \tag{82}$$

3.4 Hybrid System with Static Constraints

Expressions for kinetic and potential energies, and Rayleigh function of energy dissipation of the hybrid system (see Fig. 3), containing two subsystems – one linear oscillator and one nonlinear oscillator, in results, with two degree of freedom are

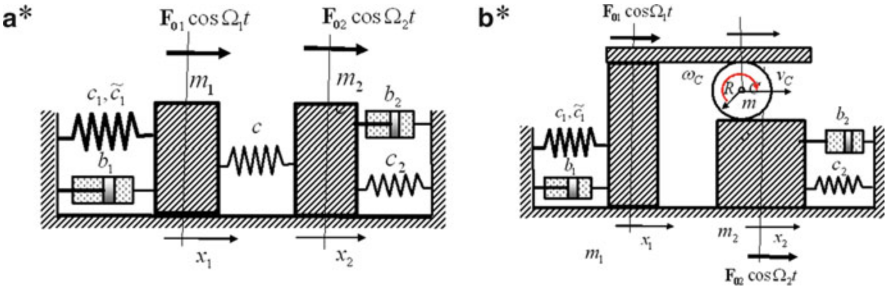

Fig. 3 Hybrid system containing coupled subsystems by ($\mathbf{a}^*$) static constraint, coupled by linear spring rigidity c and ($\mathbf{b}^*$) dynamical constraint, coupled by rolling element of the mass m – dynamic coupling: one nonlinear (*left*) and second linear (*right*) excited by external excitations

expressed by generalized coordinates x_1 and x_2 and in the forms (31), (32) and (33). For this system, it is possible to show that: $\frac{d}{dt}(E_k + E_p) = -2\Phi$ is for free vibrations and $\frac{d}{dt}(E_{k(1)} + E_{p(1)}) = -2\Phi_{(1)} + (\overrightarrow{F}_1, \overrightarrow{v}_1) + (\overrightarrow{F}_2, \overrightarrow{v}_2)$ is for forced vibrations.

Energy interaction in this hybrid system, containing two coupled subsystems by statically coupling (spring) element, is potential energy of the spring for coupling nonlinear and linear subsystems, and is expressed in the form (34).

System of the coupled ordinary differential equations of the hybrid system dynamics, containing two subsystems, one nonlinear and one linear are in the forms:

$$\ddot{x}_1 + 2\delta_1\dot{x}_2 + \left(\omega_1^2 + a_1^2\right)x_1 - a_1^2 x_2 = \mp\tilde{\omega}_{N1}^2 x_1^3 + h_{01}\cos\left(\Omega_1 t + \vartheta_{01}\right) \tag{83}$$

$$\ddot{x}_2 + 2\delta_2\dot{x}_2 + \left(\omega_2^2 + a_2^2\right)x_2 - a_2^2 x_1 = h_{02}\cos\left(\Omega_2 t + \vartheta_{02}\right) \tag{84}$$

where upper sign $(-)$ for hard and lower sign $(+)$ for soft nonlinearity, and are:

$$\omega_i^2 = \frac{c_i}{m_i},\, 2\delta_i = \frac{b_i}{m_i},\, a_i^2 = \frac{c}{m_i},\, \tilde{\omega}_{N1}^2 = \frac{\tilde{c}_1}{m_1},\, h_{0i} = \frac{F_{0i}}{m_i}\; i = 1,2. \tag{85}$$

Taking into account that consideration of the homogeneous system does not loose generality of the phenomena, our next consideration is to use this homogeneous hybrid system as a basic system to the nonhomogeneous with small nonlinearity members.

By use the corresponding basic linear differential equations of the coupled system of the differential equations of the hybrid system containing two subsystems, one linearized and one linear, we can obtain a characteristic equation with four characteristic numbers, same as in part 2.3. By using asymptotic method

Krilov–Bogoliyubov–Mitropolyskiy, we propose solution in the first approximation in the following form:

$$x_1(t) = e^{-\delta t}\left[R_1(t)\cos\Phi_1(t) + R_2(t)\cos\Phi_2(t)\right]$$
$$x_2(t) = e^{-\delta t}\left[R_1(t)\cos\Phi_1(t) - R_2(t)\cos\Phi_2(t)\right] \tag{86}$$

where amplitudes and phases, $R_i(t)$ and $\Phi_i(t), i = 1,2$ are unknown time functions and defined by system of the first order differential equations in the following form:

$$\begin{aligned}
\dot{R}_1(t) &= -\frac{h_{01}}{(p_1 + \Omega_1(\tau))}e^{\delta_1 t}\sin\phi_1(t)\\
\dot{\phi}_1 &= p_1 - \Omega_1 \pm \frac{1}{16p_1}\tilde{\omega}_{N1}^2 e^{\delta_1 t}\left[(R_1(t))^2 e^{-3\delta_1 t} + 3(R_2(t))^2 e^{-(2\delta_2+\delta_2)t}\right] -\\
&\quad - \frac{h_{01}}{(p_1 + \Omega_1(\tau))_1 R_1(t)}e^{\delta_1 t}\cos\phi_1(t)\\
\dot{R}_2(t) &= -\frac{h_{02}}{(p_2 + \Omega_2(\tau))}e^{\delta_2 t}\sin\phi_2(t)\\
\dot{\phi}_2 &= p_2 - \Omega_2 \pm \frac{\tilde{\omega}_{N1}^2}{16p_2}e^{\delta_2 t}\left[3(R_1(t))^2 e^{-(2\delta_1+\delta_2)t} + (R_2(t))^2 e^{-3\delta_2 t}\right] +\\
&\quad + \frac{h_{02}}{(p_2 + \Omega_2(\tau))R_2(t)}e^{\delta_2 t}\cos\phi_2(t)
\end{aligned} \tag{87}$$

where: upper sign $(+)$ for hard and lower sign $(-)$ for soft nonlinearity, and $\Phi_i(t) = p_i t + \vartheta_{0i} + \phi_i$, $\Omega_i \approx \hat{p}_i$, $i = 1,2$.

By using previous first asymptotic approximation of the two amplitudes and two phases of first asymptotic approximation of solution, we can obtain Lyapunov exponents in the forms:

$$\lambda_1 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_1^2(t) + \frac{1}{\omega_1^2}\dot{x}_1^2(t)\right] = -\delta_1 < 0$$
$$\lambda_2 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_2^2(t) + \frac{1}{\omega_2^2}\dot{x}_2^2(t)\right] = -\delta_2 < 0 \tag{88}$$

Also, taking into account that system is nonlinear:

$$\tilde{\lambda}_1 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[x_1^2(t) + \frac{\tilde{\omega}_{N1}^2}{\omega_1^2}x_1^4(t) + \frac{1}{\omega_1^2}\dot{x}_1^2(t)\right] = -\delta_1 < 0$$
$$\tilde{\lambda}_1 = \lim_{t\to\infty}\frac{1}{2t}\ln\left[\frac{E_{subsist(1)}}{2m_1\omega_1^2}\right] = -\delta_1 < 0 \tag{89}$$

3.5 Hybrid System with Dynamical Coupling (Rolling) Element

In Fig. 3b*, a hybrid system excited by two frequency external excitation, and containing two coupled oscillators, one linear and one nonlinear coupled by dynamically coupling in the form of rolling element is shown.

Expressions for kinetic and potential energies, and Rayleigh function of energy dissipation of the hybrid system (see Fig. 3b*), containing two subsystems – one linear oscillator and one nonlinear oscillator, in results, with two degree of freedom are expressed by generalized coordinates x_1 and x_2 and with dynamic coupling by rolling element, are same as expressions (48) (49), (51), (51), (52), and (53). This system is excited by external two frequency excitation $F_i(t) = F_{0i}\cos(\Omega_i t + \vartheta_i), i = 1,2$ applied to the subsystems into hybrid system.

For this system, it is possible to show that is: $\frac{d}{dt}(E_k + E_p) = -2\Phi$ for free vibrations, and for forced vibrations that is:

$$\frac{d}{dt}\left(E_{k(1)} + E_{p(1)}\right) = -2\Phi_{(1)} + \left(\vec{F}_1, \vec{v}_1\right) + \left(\vec{F}_2, \vec{v}_2\right). \tag{90}$$

The kinetic energy of the dynamical, coupling, by rolling element is expressed by (48)–(49). Coefficient $\tilde{a}_{12} = \frac{m}{4} - \frac{\mathbf{J_C}}{4R^2}$ is coefficient of the subsystems dynamical coupling, and this coefficient is coefficient of mass inertia with source in coupling rolling element. When this coefficient is equal to zero, then the system coordinate x_1 and x_2 are decoupled and there are not energy of the coupling, but there are energy of the influence of the dynamic coupling by mass of the additional member in the system. Then subsystems haven't kinetic energy interaction, but hybrid system have additional part of the kinetic energy of the first subsystem – reduction of the dynamic coupling element to the first subsystem and additional part of the kinetic energy of the first subsystem – reduction of the dynamic coupling element to the second subsystem.

Kinetic energy of the first subsystem as a one part of the hybrid system is: $E_k = \frac{1}{2}\tilde{a}_{11}\dot{x}_1^2$, kinetic energy of the second subsystem as a one part of the hybrid system kinetic energy is: $E_k = \frac{1}{2}\tilde{a}_{22}\dot{x}_2^2$. Kinetic energy of the coupling of the subsystems as a two parts of the hybrid system is: $E_k = \tilde{a}_{12}\dot{x}_1\dot{x}_2$. Additional part of the kinetic energy of the first subsystem – reduction of the dynamic coupling element to the first subsystem is $E_{k(1)d} = \frac{1}{2}\hat{a}_{11}\dot{x}_1^2$ and additional part of the kinetic energy of the second subsystem – reduction of the dynamic coupling element to the second subsystem is $E_{k(2)d} = \frac{1}{2}\hat{a}_{22}\dot{x}_2^2$.

After introducing the following notations:

$$\kappa_1 = \frac{\tilde{a}_{12}}{\tilde{a}_{11}},\ \kappa_2 = \frac{\tilde{a}_{12}}{\tilde{a}_{22}},\ \tilde{\omega}_1^2 = \frac{c_1}{\tilde{a}_{11}},\ \tilde{\omega}_2^2 = \frac{c_1}{\tilde{a}_{22}},$$

$$\tilde{\tilde{\omega}}_{N1}^2 = \frac{\tilde{c}_1}{\tilde{a}_{11}} = \tilde{\omega}_{N1}^2\frac{m_1}{\tilde{a}_{11}},\ 2\tilde{\delta}_i = \frac{b_i}{\tilde{a}_{ii}},\ h_{0i} = \frac{F_{0i}}{\tilde{a}_{ii}}\ i = 1,2 \tag{91}$$

the system of ordinary differential equations of the forced dynamics of the hybrid system obtain the following form:

$$
\begin{aligned}
&\ddot{x}_1+\kappa_1\ddot{x}_2+\tilde{\omega}_1^2 x_1+2\tilde{\delta}_1\dot{x}_1=\mp\tilde{\tilde{\omega}}_{N1}^2 x_1^3+h_{01}\cos\left(\Omega_1 t+\vartheta_{01}\right)\\
&\ddot{x}_2+\kappa_2\ddot{x}_1+\tilde{\omega}_2^2 x_2+2\tilde{\delta}_2\dot{x}_2=h_{02}\cos\left(\Omega_2 t+\vartheta_{02}\right)
\end{aligned}
\tag{92}
$$

where upper sign $(-)$ for hard and lower sign $(+)$ for soft nonlinearity. For the basic linear equations of the dynamically coupled system of the differential equations (92) of the hybrid system containing two subsystems, one linear and one linearized, have characteristic equation with four roots: $\lambda_{1,2}=-\hat{\delta}_1\mp\mathbf{i}\hat{p}_1$ and $\lambda_{3,4}=-\hat{\delta}_2\mp\mathbf{i}\hat{p}_2$ with discussion of their values. By using asymptotic method Krilov–Bogoliyubov–Mitropolyskiy, we propose solution in the first approximation in the form (52), where amplitudes $R_i(t)$ and phase $\Phi_i(t)$, $i=1,2$ are unknown functions and determined by system of the first-order differential equations in the following first asymptotic approximation form:

$$
\begin{aligned}
\dot{R}_1(t)&=e^{-\delta_1 t}\frac{h_{01}\left(K_{21}^{(2)}+\kappa_2 K_{22}^{(2)}\right)}{\tilde{\Delta}_{12}\left[\hat{p}_1+\Omega_1(\tau)\right]}\sin\phi_1(t)\\
\dot{\phi}_1(t)&=\hat{p}_1-\Omega_1(\tau)+\\
&\quad\mp\frac{3\tilde{\tilde{\omega}}_{N1}^2\left(K_{21}^{(2)}+\kappa_2 K_{22}^{(2)}\right)}{16\tilde{\Delta}_{12}\hat{p}_1}\left\{e^{-2\hat{\delta}_1 t}\left(K_{21}^{(1)}\right)^3\left[R_1(t)\right]^2\right.\\
&\qquad\qquad\left.+2e^{-2\hat{\delta}_2 t}K_{21}^{(1)}\left[K_{21}^{(2)}\right]^2\left[R_2(t)\right]^2\right\}-\\
&\quad-e^{-\delta_1 t}\frac{h_{01}\left(K_{21}^{(2)}+\kappa_2 K_{22}^{(2)}\right)}{\tilde{\Delta}_{12}\left[\hat{p}_1+\Omega_1(\tau)\right]R_1(t)}\sin\phi_1(t)\\
\dot{R}_2(t)&=-e^{\delta_2 t}\frac{h_{02}\left(K_{22}^{(1)}+\kappa_1 K_{21}^{(1)}\right)}{\tilde{\Delta}_{21}\left[\hat{p}_2+\Omega_2(\tau)\right]}\sin\phi_2(t)\\
\dot{\phi}_2(t)&=\hat{p}_2-\Omega_2(\tau)+\\
&\quad\mp\frac{3\tilde{\tilde{\omega}}_{N1}^2\left(K_{22}^{(1)}+\kappa_2 K_{21}^{(1)}\right)}{16\tilde{\Delta}_{21}\hat{p}_2}\left\{2e^{-2\hat{\delta}_1 t}\left(K_{21}^{(1)}\right)^2 K_{21}^{(2)}\left[R_1(t)\right]^2\right.\\
&\qquad\qquad\left.+e^{-2\hat{\delta}_2 t}\left[K_{21}^{(2)}\right]^3\left[R_2(t)\right]^2\right\}+\\
&\quad+e^{\delta_2 t}\frac{h_{02}\left(K_{22}^{(1)}+\kappa_1 K_{21}^{(1)}\right)}{\tilde{\Delta}_{21}\left[\hat{p}_2+\Omega_2(\tau)\right]R_2(t)}\cos\phi_2(t)
\end{aligned}
\tag{93}
$$

where: upper sign (+) for hard and lower sign (−) for soft nonlinearity, and also, $\Phi_i(t) = \hat{p}_i t + \phi_i,\ \Omega_i \approx \hat{p}_i,\ i = 1,2,\ \tau = \varepsilon t$ slow-changing time and determinants:

$$\begin{aligned}
&\tilde{\Delta}_{12} = \left[\left(K_{21}^{(1)} + \kappa_1 K_{22}^{(1)}\right)\left(K_{21}^{(2)} + \kappa_2 K_{22}^{(2)}\right) - \left(K_{22}^{(1)} + \kappa_2 K_{21}^{(1)}\right)\left(K_{21}^{(2)} + \kappa_1 K_{22}^{(2)}\right)\right] \\
&\tilde{\Delta}_{12} = \neq 0 \\
&\tilde{\Delta}_{21} = \left[\left(K_{21}^{(2)} + \kappa_1 K_{22}^{(2)}\right)\left(K_{21}^{(1)} + \kappa_2 K_{22}^{(1)}\right) - \left(K_{22}^{(2)} + \kappa_2 K_{21}^{(2)}\right)\left(K_{22}^{(1)} + \kappa_1 K_{21}^{(1)}\right)\right] \\
&\tilde{\Delta}_{21} \neq 0
\end{aligned} \tag{94}$$

By using previous system of first asymptotic approximation of amplitudes and phases of solution first asymptotic approximation, we can obtain Lyapunov exponents in the forms:

$$\begin{aligned}
\lambda_1 &= \lim_{t\to\infty} \frac{1}{2t} \ln\left[x_1^2(t) + \frac{1}{\tilde{\omega}_1^2}\dot{x}_1^2(t)\right] = -\hat{\delta}_1 < 0 \\
\lambda_2 &= \lim_{t\to\infty} \frac{1}{2t} \ln\left[x_2^2(t) + \frac{1}{\tilde{\omega}_2^2}\dot{x}_2^2(t)\right] = -\hat{\delta}_2 < 0
\end{aligned} \tag{95}$$

Also, taking into account that system is nonlinear, we can introduce first Lyapunov exponents in the forms:

$$\begin{aligned}
\tilde{\lambda}_1 &= \lim_{t\to\infty} \frac{1}{2t} \ln\left[x_1^2(t) + \frac{\tilde{\omega}_{N1}^2}{\tilde{\omega}_1^2}x_1^4(t) + \frac{1}{\tilde{\omega}_1^2}\dot{x}_1^2(t)\right] = -\hat{\delta}_1 < 0 \\
\tilde{\lambda}_1 &= \lim_{t\to\infty} \frac{1}{2t} \ln\left[\frac{E_{subsist(1)}}{2m_1\tilde{\omega}_1^2}\right] = -\hat{\delta}_1 < 0
\end{aligned} \tag{96}$$

4 Concluding Remarks on Energy Analysis

4.1 *Energy Analysis of Nonlinear System*

Numerical analysis of the series of the amplitude-frequency $[a_2(t) = e^{-\hat{\delta}_2 t} R_2(t),$ $\Omega_1(\tau) = \nu_1(\tau)]$ and $[a_2(t) = e^{-\hat{\delta}_2 t} R_2(t),\ \Omega_2(\tau) = \nu_2(\tau)]$ and phase-frequency curves $[\varphi_1(t) = \phi_1(t),\ \Omega_1(\tau) = \nu_1(\tau)]$ and $[\varphi_2(t) = \phi_2(t), \Omega_2(\tau) = \nu_2(\tau)]$ for stationary and nonstationary resonant regimes: ($\mathbf{a}^*$) amplitude-frequency curves for linear one-frequency stationary and nonstationary regime process for different velocities of forced excitation frequency change passing through the resonant range; ($\mathbf{b}^*$) amplitude-frequency curves for nonlinear-like one-frequency stationary and nonstationary regime process – oscillatory process for different velocities of forced excitation frequency change passing through the resonant range in both directions – increasing and decreasing frequency; (**c***) amplitude-frequency and (**d***) phase-frequency curves of a stationary resonant state of like two-frequency nonlinear oscillations of a nonlinear system with two degree of freedom obtained by integration of the system differential equations for the amplitudes and phases in the first asymptotic approximation show that interactions between modes appears. Numerical analysis illustrates the characteristic phenomena of a like two-frequency regime of coupled resonant states under stationary conditions for the system with small nonlinearity. We can notice the appearance of the singularity trigger with stable knots and homoclinic unstable saddle-type points along amplitude-frequency and phase-frequency characteristics for the resonant frequency interval of the resonant frequency interactions. The appearance of multiple resonant jumps, typical for multifrequency-coupled resonant states, is also noticeable. The appearance of homoclinic points of unstable saddle type points to the appearance of stochastic-like and chaotic-like processes in subsystems of hybrid system have source in coupled resonant states multifrequency oscillation regimes. This requires further study separate for each particular case.

Under the conditions of nonlinear system multifrequency forced oscillations, and by using the asymptotic method of Krilov–Bogolyubov–Mitropolyskiy, the appearance of "*own circular frequencies stroll*" may be noticed. Resonant frequency ranges dependent on the character of nonlinearity, and on the initial conditions and momentary adequate nonlinear harmonics and amplitudes and phases are formed in that way. That is the reason why, besides the notion "*resonant state*", we introduce notions "*passage through the coupled resonant state*s" and "*coupled resonant states*". With the appearance of the own circular frequency stroll, a mutual influence shown either in adequate harmonics amplitudes, frequencies, and phases increase or decrease appears.

As the energy (kinetic, potential, and the energy of dissipation caused by dissipative forces, as well as the energy of forced multifrequency forces work) "*carried*" by a nonlinear harmonics of a corresponding oscillations "*stroll*" frequency depends both on the amplitudes square and on the square of its time derivatives, or frequency, the harmonic amplitudes, phases, or frequencies change during the oscillatory process and regime itself as well as the interaction between them causes the energy change. *The appearance of energy transfer from one harmonic onto other or others of higher or lower frequencies can also be noticed here*.

On amplitude-frequency and phase-frequency multifrequency forced oscillations diagrams, we can notice the appearance of *one or more resonant jumps* which point to the appearance of *a resonant energy jumps*, both kinetic and potential, carried by a nonlinear harmonic. We can see that while one harmonic jumps to higher amplitudes, the other one to lower ones. The energy jumps indicated on energy-frequency graph of one nonlinear mode is similar as corresponding jumps in amplitude frequency graphs as well as in phase-frequency graphs.

On certain harmonics frequencies, a *resonant jump of energy carried by the observed nonlinear harmonics*, onto a lower or a higher value, happens. At the same time, similar resonant jumps of energy in the opposite or the same direction happen on other harmonics.

If the stationary multifrequency forced oscillations amplitudes and phases, an appearance of amplitude *trigger* and of *coupled amplitudes triggers* of coupled stationary singularities in an amplitudes combination stable–unstable–stable. That way, we have *a trigger and/or a coupled triggers of harmonic-energies* in a corresponding set of fixed frequencies from the harmonics' coupled resonant frequencies range.

The analysis of total mechanical energy of a nonlinear system is also significant, as well as the analysis of the kinetic and potential energy of each nonlinear harmonics, in the singular and bifurcation states, and especially of those corresponding two unstable and hyperbolically, as well as homoclinical orbits. The question about the analysis of energy and its transfer between harmonics under conditions under which *chaotic-like* and *stochastic-like* vibrations appear in *deterministic nonlinear dynamical system* remains open.

In nonlinear systems, we can observe the idea of equivalent systems exchange by the use of elementary linear simple oscillators which would be uncoupled and would make an equivalent replacement for a linearized system. And after that, we may, using *the asymptotic methods of nonlinear mechanics*, for instance, the method of Krilov–Bogolyubov–Mitropolyski, compose a system of necessary approximation of the first-order ordinary differential equations for nonlinear oscillations harmonics amplitudes and phases that are close to an unperturbed oscillations. From such a system of adequate approximation differential equations for amplitudes and phases that are *mutually coupled by non-linear members*, we may, using either quantitative or qualitative analysis, derive certain conclusions about the flows and *the transfer of energy* by following the phase and harmonics trajectories through the phase space of dynamical systems state.

A generalization of an analytical analysis of the transfer energy between linear and nonlinear oscillators for free vibrations with different type of coupling as a couple between two subsystems each of them with one degree of freedom is also important, but it is new task.

5 Energy Analysis of Hybrid Complex Structures

5.1 A Double Plate System

5.1.1 Partial Differential Equations

By using the model of double plate system with viscoelastic layer (similar as in Refs. Hedrih (2005, 2006)), we can consider the energy transfer between plates. For that reason, we use corresponding derived partial differential equations and

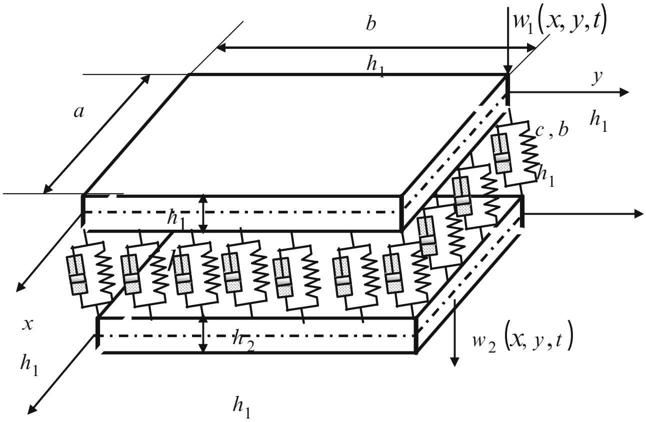

Fig. 4 Double plate system with viscoelastic layer: structure and noted corresponding kinetic parameters and coordinate systems

corresponding analytical results and expressions for solutions of the transversal displacements of the both plates vibrations. This double plate system is presented in Fig. 4.

The governing system of the coupled partial nonlinear differential equations for free double plates oscillations is in the following form (see Ref. [12]):

$$\frac{\partial^2 w_1(x,y,t)}{\partial t^2} + c^4_{(1)}\Delta\Delta w_1(x,y,t) - 2\delta_{(1)}\left[\frac{\partial w_2(x,y,t)}{\partial t} - \frac{\partial w_1(x,y,t)}{\partial t}\right] - \\ -a^2_{(1)}\left[w_2(x,y,t) - w_1(x,y,t)\right] = \\ = \pm\varepsilon\beta_{(1)}\left[w_2(x,y,t) - w_1(x,y,t)\right]^3 + \tilde{q}_{(1)}(x,y,t)$$

$$\frac{\partial^2 w_2(x,y,t)}{\partial t^2} + c^4_{(2)}\Delta\Delta w_2(x,y,t) + 2\delta_{(2)}\left[\frac{\partial w_2(x,y,t)}{\partial t} - \frac{\partial w_1(x,y,t)}{\partial t}\right] + \\ +a^2_{(2)}\left[w_2(x,y,t) - w_1(x,y,t)\right] = \\ = \pm\varepsilon\beta_{(2}\left[w_2(x,y,t) - w_1(x,y,t)\right]^3 + \tilde{q}_{(2)}(x,y,t) \quad (97)$$

where in first partial differential equation in the system (1) upper sign (+) for hard and lower sign (−) for soft nonlinearity, and in the second partial differential equation in the system (1) upper sign (−) for hard and lower sign (+) for soft nonlinearity. Also in the previous system of partial differential equations: $w_i(x,y,t),\ i = 1,2$ are plate small transverse deflections (with means, as has been discussed in books [88] and small compared to the plates thickness, $h_i,\ i = 1,2,$) and that plates vibrations occur only in the orthogonal direction with respect to the parallel middle surfaces of the plates passing through their parallel contours with same boundary plates conditions; $a^2_{(i)} = \frac{c}{\rho_i h_i}$, $2\delta_{(i)} = \frac{b}{\rho_i h_i}$ $i = 1,2$ and $c^4_{(i)} = \frac{\mathsf{D}_i}{\rho_i h_i}$, $i = 1,2$ with $\mathsf{D}_i = \frac{\mathbf{E}_i h^3}{12(1-\mu^2)}$, $i = 1,2$ corresponding bending cylindrical rigidities of the plates, and $\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$ is differential operator; E_i modulus of elasticity, μ_i Poisson's ratio and G_i shear modulus, ρ_i plate mass distribution. The plates are interconnected by a viscoelastic layer with constant surface stiffness c and with constant surface damping force coefficient b distributed along all plates' surfaces.

For the solutions of the governing system of the corresponding coupled partial differential equations (97) for forced double plate system oscillations, we take into account the eigen amplitude functions $\mathbf{W}_{(i)nm}(x,y),\ i=1,2,\ n,m=1,2,3,4,\ldots\infty$ and the time expansion with the coefficients in the form of the unknown time functions $T(t),\ i=1,2,\ n,m=1,2,3,4,\ldots\infty$ describing their time evolution:

$$w_i(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{(i)nm}(x,y)\, T_{(i)nm}(t),\ i=1,2 \tag{98}$$

where the eigen amplitude functions $\mathbf{W}_{(i)nm}(x,y),\ i=1,2,\ n,m=1,2,3,4,\ldots\infty$ are the same, for both plates in the system, as in the case with decoupled plates problem (see Ref. [12]). Then after introducing expression (98) into governing system of the coupled partial differential equations for forced double plates oscillations in the form (97): and after multiplying first and second equation with $W_{(i)sr}(x,y)dxdy$ and after integrating along the middle plate surface and taking into account orthogonality conditions and corresponding equal boundary conditions of the plates, we obtain the *mn*-family of the systems containing coupled two ordinary differential equations for determination series of the unknown time functions $T_{(i)nm}(t),\ i=1,2,\ n,m=1,2,3,4,\ldots\infty$.

We take into consideration, the case, when external distributed two-frequencies force is applied and distributed along upper surfaces of upper plate with both frequencies near eigen circular frequencies of coupled plate system presented by linearized model, $\Omega_{inm} \approx \hat{p}_{inm}, i=1,2$. In this case, the lower plate is free of load. We can conclude that external excitation frequencies are in the resonant frequency interval close to the resonant frequency of corresponding linear double plate system. We suppose that the functions of external excitation at *nm*-mode of oscillations are the two-frequency process in the form:

$$\tilde{q}_{(i)nm}(t) = h_{01nm}\cos[\Omega_{1nm}t + \phi_{1nm}] + h_{02nm}\cos[\Omega_{2nm}t + \phi_{2nm}] \tag{99}$$

For this case of the external two-frequency excitation time functions $T_{(i)nm}(t)$, describing their time evolution of the transversal displacements of the plate middle surface points are in the following forms (see Refs. by and [12, 54, 57, 58]):

$$T_{(i)nm}(t) = K_{inm}^{(1)} e^{-\hat{\delta}_{1nm}t} R_{1nm}(t)\cos\Phi_{1nm}(t) + K_{inm}^{(2)} e^{-\hat{\delta}_{2nm}t} R_{2nm}(t)\cos\Phi_{2nm}(t) \tag{100}$$

where K_{ijnm}^{s} cofactors of determinant corresponding to basic linear homegenous coupled system, $\hat{\delta}_{inm}$ real parts of the corresponding pair of the roots of the characteristic equation and amplitudes $R_{inm}(t)$ and full phases $\Phi_{inm}(t) = \Omega_{inm}t + \phi_{inm}(t)$ unknown time functions which, we are going to obtain using the Krilov–Bogolyubov–Mitropolyskiy asymptotic method (see Refs. [73–80]). It is taken into account that defined task satisfy all necessary conditions for applying asymptotic method Krilov–Bogolyubov–Mitropolskiy concerning small parameter and that external excitation frequencies $\Omega_{1nm} \approx \hat{p}_{1nm}$ and $\Omega_{2nm} \approx \hat{p}_{2nm}$ are in the resonant frequency intervals of the corresponding eigen frequencies of unperturbed linear system solution.

By applying the asymptotic method Krilov–Bogolyubov–Mitropolskiy (1965), we obtain the system of the first-order four ordinary differential equations according unknown amplitudes $R_{inm}(t) = e^{\hat{\delta}_{inm}t} a_{inm}(t)$ and full phases $\Phi_{inm}(t) = \Omega_{inm}t + \phi_{inm}(t)$ in the first asymptotic approximation in the following forms (see Refs. [6, 53, 67, 95]):

$$\dot{a}_{1nm}(t) = -\frac{\left(\delta_{(1)}K^{(2)}_{22nm} + \delta_{(2)}K^{(2)}_{21nm}\right)\left[K^{(1)}_{22nm} - K^{(1)}_{21nm}\right]}{\left(K^{(2)}_{22nm}K^{(1)}_{21nm} - K^{(2)}_{21nm}K^{(1)}_{22nm}\right)} a_{1nm}(t) +$$

$$+\frac{K^{(2)}_{22nm}h_{01nm}}{(\Omega_{1nm} + \hat{p}_{1nm})\left(K^{(2)}_{22nm}K^{(1)}_{21nm} - K^{(2)}_{21nm}K^{(1)}_{22nm}\right)} \cos\phi_{1nm}$$

$$\dot{\phi}_{1nm}(t) = \left(\hat{p}^{-}_{nm1}\Omega_{1nm}\right) -$$

$$\pm\frac{\varepsilon\Psi(W_{nm})\left(\beta_{(1)}K^{(2)}_{22} + \beta_{(2)}K^{(2)}_{21}\right)}{\left(K^{(2)}_{22}K^{(1)}_{21} - K^{(2)}_{21}K^{(1)}_{22}\right)\left(\Omega^{+}_{1nm}\hat{p}_{nm1}\right)}$$

$$\left[\left(K^{(1)}_{22nm} - K^{(1)}_{21nm}\right)^3 \frac{3}{8} a^2_{1nm}(t) + \frac{1}{2}\left(K^{(1)}_{22nm} - K^{(1)}_{21nm}\right)\left(K^{(2)}_{22nm} - K^{(2)}_{21nm}\right)^2 a^2_{2nm}(t)\right] -$$

$$-\frac{K^{(2)}_{22}h_{01nm}}{(\Omega_{1nm} + \hat{p}_{nm1})\left(K^{(2)}_{22nm}K^{(1)}_{21nm} - K^{(2)}_{21nm}K^{(1)}_{22nm}\right) a_{1nm}(t)} \sin\phi_{1nm}$$

$$\dot{a}_{2nm}(t) = -\frac{\left(\delta_{(1)}K^{(1)}_{22nm} + \delta_{(2)}K^{(1)}_{21nm}\right)\left[K^{(2)}_{22nm} - K^{(2)}_{21nm}\right]}{\left(K^{(1)}_{22nm}K^{(2)}_{21nm} - K^{(2)}_{22nm}K^{(1)}_{21nm}\right)} a_{2nm}(t) +$$

$$+\frac{K^{(1)}_{22nm}h_{02nm}}{(\Omega_{2nm} + \hat{p}_{2nm})\left(K^{(1)}_{22nm}K^{(2)}_{21nm} - K^{(2)}_{22nm}K^{(1)}_{21nm}\right)} \cos\phi_{2nm} \tag{101}$$

$$\dot{\phi}_{2nm}(t) = (\hat{p}_{2nm} - \Omega_{2nm}) -$$

$$\pm\frac{\varepsilon\Psi(W_{nm})\left(\beta_{(1)}K^{(1)}_{22nm} + \beta_{(2)}K^{(1)}_{21nm}\right)}{(\Omega_{2nm} + \hat{p}_{2nm})\left(K^{(1)}_{22nm}K^{(2)}_{21nm} - K^{(2)}_{22nm}K^{(1)}_{21nm}\right)}.$$

$$\left[\frac{1}{2}\left(K^{(1)}_{22nm}-K^{(1)}_{21nm}\right)^2\left(K^{(2)}_{22nm}-K^{(2)}_{21nm}\right)a^2_{1nm}(t)+\left(K^{(2)}_{22nm}-K^{(2)}_{21nm}\right)^3\frac{3}{8}a^2_{2nm}(t)\right]-$$

$$-\frac{K^{(1)}_{22nm}h_{02nm}}{\left(\Omega_{2nm}+\hat{p}_{2nm}\right)\left(K^{(1)}_{22nm}K^{(2)}_{21nm}-K^{(2)}_{22nm}K^{(1)}_{21nm}\right)a_{2nm}(t)}\sin\phi_{2nm}$$

where upper sign $(+)$ for hard and lower sign $(-)$ for soft nonlinearity, and also, $\Psi(W_{nm})=\frac{\int_0^a\int_0^b W^4_{(1)nm}(x,y)dxdy}{\int_0^a\int_0^b W^2_{(1)nm}(x,y)dxdy}$ is coefficient of influence of ideal elastic layer nonlinearity.

5.1.2 Kinetic Energy of Plates

The expressions of kinetic energies of the plates are in the following forms:

$$\mathbf{E}^{(i)}_k=\frac{1}{2}\iiint_V\rho_i\left(\frac{\partial w_i(x,y,t)}{\partial t}\right)^2 dzdA$$

$$\mathbf{E}^{(i)}_k=\frac{1}{2}\rho_i h_i\iint_A\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{(i)nm}(x,y)\dot{\mathrm{T}}_{(i)nm}(t)\right]^2 dA,\ i=1,2 \tag{102a}$$

or in the form:

$$\mathbf{E}^{(i)}_k=\frac{1}{2}\rho_i h_i\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}M_{(1)nm}(x,y)\left[\dot{\mathrm{T}}_{(i)nm}(t)\right]^2,\ i=1,2 \tag{102b}$$

where

$$M_{(i)nmsr}=\iint_A W_{(i)nm}(x,y)W_{(i)sr}(x,y)dA=\begin{cases}0 & sr\neq nm\\ M_{(1)nm} & sr=nm\end{cases} \tag{103}$$

The kinetic energy of the one plate we can express in the form of the sum by components $\mathbf{E}^{(i)}_{k,nm}$ belong to corresponding mn-family mode $n,m=1,2,3,4,\ldots.\infty$ in the following form:

$$\mathbf{E}^{(i)}_k=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbf{E}^{(i)}_{k,nm},\ i=1,2 \tag{104}$$

where the kinetic energy components $\mathbf{E}^{(i)}_{k,nm}$, $i=1,2$ belong to corresponding mn-family mode $n,m=1,2,3,4,\ldots.\infty$ was expressed by derivatives of the eigen component time functions belong to same corresponding mn-family mode

$$\mathbf{E}^{(i)}_{k,nm}=\frac{1}{2}\rho_i h_i M_{(1)nm}(x,y)\left[\dot{\mathrm{T}}_{(i)nm}(t)\right]^2=M_{(i)nmsr}\tilde{E}^{(i)}_{k,nm},\ i=1,2 \tag{105}$$

Also, we can introduce reduced component of the kinetic energy $\tilde{E}_{k,nm}^{(i)}.i = 1,2$ belong to corresponding mn-family mode $n,m = 1,2,3,4,\ldots.\infty$ in the following form:

$$\tilde{E}_{k,nm}^{(i)} = \frac{\mathbf{E}_{k,nm}^{(i)}}{M_{(1)nm}(x,y)_i} = \frac{\rho_i h_i}{2}\left[\dot{\mathrm{T}}_{(i)nm}(t)\right]^2, i = 1,2 \tag{106}$$

5.1.3 Potential Energy of Plates

The potential energy of the plate is equal to energy of the deformation of elastic plate in the vibration state and expression, we can write in the following form:

$$E_p = \mathbf{A_d} = \frac{1}{2}\iiint_V \left[\varepsilon_x\sigma_x + \varepsilon_y\sigma_y + \varepsilon_z\sigma_z + \gamma_{xy}\tau_{xy} + \gamma_{xz}\tau_{xz}|\gamma_{yz}\tau_{yz}\right] dV \tag{107}$$

where $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ are tensor strain components, $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$ are tensor stress components of the plate strain and stress vibration state. Tensor stress components τ_{zx} and τ_{zy} are small, as tensor strain components γ_{zx} and γ_{zy} are also small and can be neglected in the comparison with other members in expression for work of elastic deformation of the thin plate. Then, we can express the work of elastic plate deformation on the simpler form. Also, we take into account that plates are thin and that stress state is planar and that we can calculate with middle plate surface, and make averaging with respect to the middle plate surface (see Ref. [88]) and for the work of elastic plate deformation, we can write the following approximate expression:

$$\begin{aligned} E_p \approx \frac{D}{2}\iint_A \Bigg\{ &\left[\frac{\partial^2 w(x,y,t)}{\partial x^2}\right]^2 + \left[\frac{\partial^2 w(x,y,t)}{\partial y^2}\right]^2 \\ &+2\mu\left[\frac{\partial^2 w(x,y,t)}{\partial x^2}\right]\left[\frac{\partial^2 w(x,y,t)}{\partial y^2}\right] + 2(1-\mu)\left[\frac{\partial^2 w(x,y,t)}{\partial x\partial y}\right]^2\Bigg\} dA \end{aligned} \tag{108}$$

After introducing solutions (98) in previous expression (108) for expression of the potential energy of the plates, we obtain the following:

$$\mathbf{E}_p^{(i)} \approx \frac{\mathrm{D_i}}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\sum_{r=1}^{\infty} C_{nm,sr(i)}\mathrm{T}_{(i)nm}(t)\,\mathrm{T}_{(i)sr}(t), i = 1,2 \tag{109}$$

where,

$$C_{nm,sr(i)} = k_{nm}^4\begin{cases} 0 & sr \neq nm \\ M_{(i)nm} = \iint_A \left[W_{(i)sr}(x,y)\right]^2 dA & sr = nm \end{cases} \tag{110}$$

The potential energies of the separate plates are in the following forms:

$$\mathbf{E}_p^{(i)} \approx \frac{\mathrm{D}_i}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k_{nm}^4 M_{(i)mm} \left[\mathrm{T}_{(i)nm}(t)\right]^2 \tag{111}$$

or in the forms:

$$\mathbf{E}_p^{(i)} \approx \frac{\rho_i h_i}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{(i)nm}^2 M_{(i)nm} \left[\mathrm{T}_{(i)nm}(t)\right]^2 \tag{112}$$

where $\omega_{(i)nm}^2 = \frac{\mathrm{D}_i}{\rho_i h_i} k_{nm}^4$. The potential energy of the one plate, we can express in the form of the sum by components $\mathbf{E}_{p,nm}^{(i)}$ belong to corresponding mn-family mode $n, m = 1, 2, 3, 4, \ldots .\infty$ in the following form:

$$\mathbf{E}_p^{(i)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mathbf{E}_{p,nm}^{(i)},\ i = 1, 2 \tag{113}$$

where the energy components $\mathbf{E}_{k,nm}^{(i)}$, $i = 1, 2$ belong to corresponding mn-family mode $n, m = 1, 2, 3, 4, \ldots .\infty$ was expressed by derivatives of the component time functions belong to same corresponding mn-family mode

$$\mathbf{E}_{p,nm}^{(i)} \approx \frac{\rho_i h_i}{2} \omega_{(i)nm}^2 M_{nm} \left[\mathrm{T}_{(i)nm}(t)\right]^2 = M_{(i)nmsr} \tilde{\mathbf{E}}_{p,nm}^{(i)} \tag{114}$$

Also, we can introduce reduced component potential energy $\tilde{\mathbf{E}}_{p,nm}^{(i)} . i = 1, 2$ belong to corresponding mn-family mode $n, m = 1, 2, 3, 4, \ldots .\infty$

$$\tilde{\mathbf{E}}_{p,nm}^{(i)} = \frac{1}{2} \rho_i h_i \omega_{(i)nm}^2 M_{(1)nm} \left[\mathrm{T}_{(i)nm}(t)\right]^2 = \frac{\mathbf{E}_{p,nm}^{(i)}}{M_{(i)nmsr}}. \tag{115}$$

5.1.4 Potential Energy of Visco-Nonlinear Elastic Layer

For analysis of the double plate system with visco-nonlinear elastic layer, we can write expression for the potential energy of the constraints between coupled plates in the form of the energy of deformation of the distributed elastic layer neglected mass and properties of inertia and neglected kinetic energy. Then expression for the potential energy of the coupling of the plates is in the form:

$$\mathbf{E}_{p(a,b)\ell eyer} = \iint_A \left[\frac{c}{2}(w_2 - w_1)^2 \pm \frac{\tilde{c}}{4}(w_2 - w_1)^4\right] dA \tag{116}$$

After introducing solutions (98) in previous expression (116), and taking into account ortogonality conditions, for expression of the potential energy of the plate coupling, we obtain the following two next parts of expression for two pointed parts of the potential energy of the nonlinear coupling of the plates in the forms:

$$\mathbf{E}_{p(1,2)\ell ayer,linear} = \frac{1}{2}c\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} M_{(1)nm}\left[\mathrm{T}_{(2)nm}(t) - \mathrm{T}_{(1)nm}(t)\right]^2 \tag{117}$$

$$\mathbf{E}_{p,nm(1,2)\ell ayer,non-linear} = \pm\frac{1}{4}\tilde{c}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \tilde{M}_{(i)nmnm}\left[\mathrm{T}_{(2)nm}(t) - \mathrm{T}_{(1)nm}(t)\right]^4 \tag{118}$$

where upper sign (+) for hard and lower sign (−) for soft nonlinearity of the coupling layer. The potential energy of the nonlinear elastic properties of the layer between plates, we can express in the form of the sum by components $\mathbf{E}_{p,nm(1,2)layer,linear}$ and $\mathbf{E}_{p,nm(1,2)layer,non-linear}$ belong to corresponding *mn*-family mode $n,m = 1,2,3,4,\ldots.\infty$ in the form:

$$\mathbf{E}_{p(1,2)\ell ayer} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbf{E}_{p,nm(1,2)\ell ayer,linear} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\mathbf{E}_{p,nm(1,2)\ell ayer,non-linear} \tag{119}$$

where the energy components $\mathbf{E}_{p,nm(1,2)\ell ayer,linear}$ and $\mathbf{E}_{p,nm(1,2)\ell ayer,non-linear}$ belong to corresponding *mn*-family mode was expressed by the eigen component time functions belong to same corresponding *mn*-family mode. Also, we can introduce reduced components of the potential energy of the light distributed elastic layer $\tilde{\mathbf{E}}_{p,nm(1,2)\ell ayer,linear}$ and $\tilde{\mathbf{E}}_{p,nm(1,2)\ell ayer,non-linear}$, belong to corresponding *mn*-family mode like as:

$$\tilde{\mathbf{E}}_{p,nm(1,2)\ell ayer,linear} = \frac{1}{2}c\left[\mathrm{T}_{(2)nm}(t) - \mathrm{T}_{(1)nm}(t)\right]^2 = \frac{\mathbf{E}_{p,nm(1,2)\ell ayer,linear}}{M_{(i)nm,nm}} \tag{120}$$

$$\tilde{\mathbf{E}}_{p,nm(1,2)\ell ayer,non-linear} = \pm\frac{1}{4}\tilde{c}\left[\mathrm{T}_{(2)nm}(t) - \mathrm{T}_{(1)nm}(t)\right]^4 = \frac{\mathbf{E}_{p,nm(1,2)\ell ayer,non-linear}}{\tilde{M}_{(i)nmnm}} \tag{121}$$

where upper sign (+) for hard and lower sign (-) for soft nonlinearity of the coupling layer.

5.1.5 Rayleigh Energy Dissipation

For analysis of the double plate system with visco-nonlinear elastic layer, we can write expression for the Rayleigh function of the energy dissipation of the constraint between coupled plates in the form of the power of the damping force depending of velocity of the deformation of the distributed visco-nonlinear elastic layer neglected mass and properties of inertia and neglected kinetic energy. Then expression for the Rayleigh function of the energy dissipation in the viscoelastic layer of the plate system is in the form:

$$\Phi_{(1,2)\ell eyer} = \iint_A \frac{1}{2} b \left(\dot{w}_2 - \dot{w}_1\right)^2 dA \tag{122}$$

$$\Phi_{(1,2)\ell eyer} = \frac{1}{2} b \iint_A \left(\frac{\partial w_2 (x,y,t)}{\partial t} - \frac{\partial w_2 (x,y,t)}{\partial t} \right)^2 dA, i = 1,2$$

$$\Phi_{(1,2)\ell eyer} = \frac{1}{2} b \iint_A \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{(i)nm} (x,y) \left(\dot{\mathrm{T}}_{(1)nm} (t) - \dot{\mathrm{T}}_{(2)nm} (t) \right) \right]^2 dA$$

or in the form:

$$\Phi_{(1,2)\ell eyer} = \frac{1}{2} b \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} M_{(1)nm} (x,y) \left[\dot{\mathrm{T}}_{(2)nm} (t) - \dot{\mathrm{T}}_{(1)nm} (t) \right]^2,$$

$$i = 1,2 \tag{123}$$

$$\Phi_{(1,2)\ell ayer} = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} M_{(1)nm} \tilde{\Phi}_{nm(1,2)\ell ayer}$$

where

$$b_{(i)nmsr} = b \iint_A \mathrm{W}_{(i)nm} (x,y) \mathrm{W}_{(i)sr} (x,y) dA = \begin{cases} 0 & sr \neq nm \\ bM_{(1)nm} & sr = nm \end{cases}$$

$$\tilde{\Phi}_{nm(1,2)\ell ayer} = \frac{1}{2} b \left[\dot{\mathrm{T}}_{(2)nm} (t) - \dot{\mathrm{T}}_{(1)nm} (t) \right]^2 = \frac{\Phi_{nm(1,2)\ell ayer}}{M_{(1)nm}} \tag{124}$$

5.1.6 Lyapunov Exponents and Concluding Remarks

For each of the eigen plate time functions $T_{(1)nm}(t)$ and $T_{(2)nm}(t)$ and time processes in nm-mode, we can define Lyapunov exponents in the form:

$$\lambda_{nm(i)} = \lim_{t \to \infty} \frac{1}{2t} \ln \left\{ \left[T_{(i)nm} (t) \right]^2 + \frac{1}{\tilde{\omega}^2_{(i)nm}} \left[\dot{T}_{(i)nm} (t) \right]^2 \right\}$$

$$\lambda_{nm(i)} = \lim_{t \to \infty} \frac{1}{2t} \ln \frac{2\tilde{\tilde{\mathbf{E}}}_{nm(i)}}{\tilde{\omega}^2_{(i)nm}} = -\left(\tilde{\delta}_{nm(1)} + \tilde{\delta}_{nm(2)} \right), \tag{125}$$

$$i = 1,2, n,m = 1,2,3,4,\ldots.\infty$$

Also, by using analogy, we can define Lyapunov exponents of the plate energy interaction in the following form:

$$\lambda_{nm(1,2)} = \lim_{t\to\infty} \frac{1}{2t} \ln \frac{2\tilde{\tilde{\mathbf{E}}}_{nm(1,2)}}{\omega_{(1)nm}\omega_{(2)nm}} = -\left(\tilde{\delta}_{nm(1)} + \tilde{\delta}_{nm(2)}\right) < 0, \qquad (126)$$

$$n, m = 1, 2, 3, 4, \ldots \infty$$

5.1.7 Concluding Remarks

For the case of the free vibrations of conservative system, these Lyapunov exponents are equal to zero. But by using this energy approach, we can introduce Lyapunov exponents of this type and way for coupled hybrid systems with different type of the material properties, as it is visco-nonlinear elastic or creep, and to use for investigation of the stability process, or deformable forms of the deformable body motion in the hybrid systems. Then, we can see that these Lyapunov exponents are measures of the processes integrity or system motion integrity.

For the second case of a model of the double plate system with discontinuity in elastic layer considered as a model of the interface crack between two plates connected by thin elastic layer of the Winkler type and by using obtained results presented in Ref. by Hedrih (2006b), it is easy to conduct energy analysis of the transfer energy also using consideration from this paper and corresponding solutions from cited paper. For that case, it is necessary to take into account that all *nm*-families of the modes are in mutual interaction, because discontinuity in the elastic layer is special type of the strong nonlinearity. In that case, defined Lyapunov exponents obtain important role in analysis of the transfer energy between plates, including interaction between different nonlinear modes.

5.2 Energy Exchange in an Axially Moving Double-Belt System

In this part, as an author's new research result, an analytical study of the energy transfer between two coupled like-string belts interconnected by light pure elastic layer in the axially moving sandwich double belt system, in the free vibrations is presented (see Fig. 5, and Refs. [9, 40]).

On the basis of the obtained analytical expressions for the kinetic and potential energy of the belts and potential energy of the light pure elastic distributed layer, numerous conclusions are derived. For the pure linear elastic double belt system, no transfer energy between different eigen modes of transversal vibrations of the axially moving double belt system, but in each from the set of the infinite numbers eigen modes, there are transfer energy between belts, and free transversal vibrations are like two-frequency, when change of the potential energy of the booth belts are four frequency, and potential energy interaction is one frequency in the each eigen mode. Changes of the kinetic energy of the both belts of the sandwich double

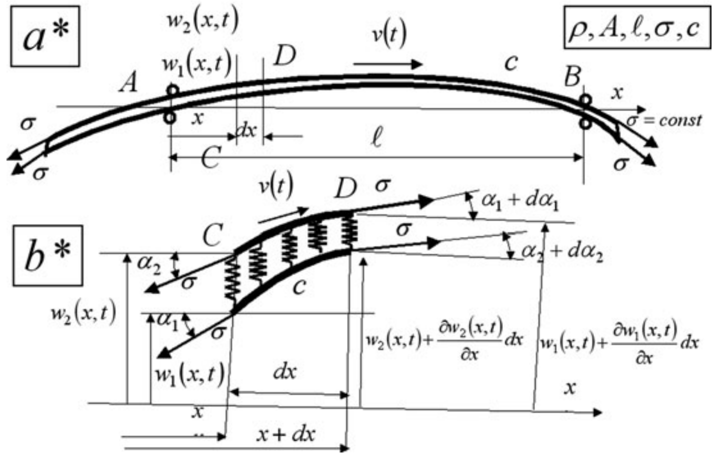

Fig. 5 Transversal vibrations of the axially moving sandwich belt system. ($\mathbf{a}^*$) Kinetic parameters of the transversal vibrations of the axially moving sandwich belt system. ($\mathbf{b}^*$) Elementary segment of the axially moving sandwich belt system with length *dx* and notations of the kinetics parameters

axially moving belt system is two frequency like oscillatory regimes with two-time multiplicities of the eigen frequencies of the corresponding eigen amplitude mode.

For concluding remarks, we can summarize research results obtained by energy analysis of the axially moving double belt system directing attentions of the reader to the author References (for detail see Refs. [9, 40]). Analogy values of the kinetic energy $\tilde{E}_{k(i)}(\eta)$ and potential energy $\tilde{E}_{p(i)}(\eta)$ as well as Raleigh function $\tilde{\Phi}_{(i)}(\eta)$ of the energy dissipation of the one belt of the considered axially moving sandwich double belt system are *four frequency function with respect to* η *- coordinate in each of s-eigen modes o*f the double belt system transversal vibrations with infinite number of possible modes. These frequencies are double values of the both eigen frequencies of the corresponding s-mode $2\tilde{p}_s$ and $2\tilde{\tilde{p}}_s,\ s = 1,2,3,4,\ldots\ldots.$, and values of *the sum and of the difference of the two corresponding eigen s-mode* of the double belt system transversal vibrations $\tilde{p}_s + \tilde{\tilde{p}}_s$ and $\tilde{\tilde{p}}_s - \tilde{p}_s,\ s = 1,2,3,4,\ldots\ldots..$

Analogy values of the system total kinetic energy $\tilde{E}_k(\eta)$ and system total potential energy $\tilde{E}_p(\eta)$ as well as Raleigh function $\tilde{\Phi}(\eta)$ of the energy dissipation of the considered axially moving sandwich double belt system are *two frequency functions with respect to* η*-coordinate in each s -eigen mode o*f the double belt system transversal vibrations infinite number of possible modes. These frequencies are double values of the both eigen frequencies of the corresponding s-mode $2\tilde{p}_s$ and $2\tilde{\tilde{p}}_s,\ s = 1,2,3,4,\ldots\ldots..$of the basic belts dynamics.

Analogy value of the potential energy $\tilde{E}_{p(1,2)}(\eta)$ of the elastic layer between belts of the axially moving double belt system – reduced analogue value of the potential energy of the *interaction between belts (two subsystems coupled by elastic layer) in the hybrid system* i*s one frequency function o*f η *- coordinate in each eigen s -mode o*f the axially moving double belt system. These frequencies are double values of the higher of two s-eigen frequencies of the corresponding s-mode and expressed by $2\tilde{\tilde{p}}_s,\ s = 1,2,3,4,\ldots\ldots..$

Research results presented in this paper are advances to the previous published results in the author-cited papers containing analytical and numerical research results concerning free and forced transversal vibrations of the axially moving sandwich double belt system. To the present question concerning the main aim of this research and about the usefulness of the obtained results can be answered that the primary and main aim of this research is in theoretical and methodological usefulness for university teaching process in the subject of Elastodynamics, as analytical results for introducing students with mechanisms of the transfer energy in the hybrid systems between subsystems, as well as about energy transformation inside of the sets of eigen modes.

Considered axially moving, sandwich double belt system is a hybrid simple pure elastic and pure rheolinear systems with elegant possibilities to make an analysis of the analogy in the plane $\xi = x, \eta = \frac{v_0}{c_0^2 - v_0^2}x + t$.

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