Chapter 2
Kinematics

2.1 Coordinate Frames

The first step in the study of the kinematics and dynamics of a rigid body is to have a clear description of its position and orientation and their changes with time. This is achieved by using coordinate frames, vectors, matrices, and other mathematical tools.

Several standard coordinate frames exist including

- *Cartesian* coordinate frames
- *Spherical* coordinate frames and
- *Cylindrical* coordinate frames.

A Cartesian or rectangular coordinate frame consists of an origin and three mutually perpendicular axes arranged according to the right-hand rule. In this book, without an explicit explanation, a coordinate frame or simply a frame means a Cartesian coordinate frame.

As depicted in Fig. 2.1, coordinate frame $\{A\} : O_A\hat{x}_A\hat{y}_A\hat{z}_A$ is attached to a rigid body (A). It has an origin O_A and three axes along unit vectors $\hat{x}_A$, $\hat{y}_A$, and $\hat{z}_A$. Frame $\{B\} : O_B\hat{x}_B\hat{y}_B\hat{z}_B$ is attached to another body (B). It has an origin O_B and three axes along unit vectors $\hat{x}_B$, $\hat{y}_B$, and $\hat{z}_B$. Frames like $\{A\}$ and $\{B\}$ that are attached to rigid bodies are usually called *body frames* or simply *frames*. A special coordinate frame is the so-called *universe frame* or *inertial frame of reference*, $\{U\} : OXYZ$, in which Newton's law of motion is valid. In the context of engineering applications, the universe frame is always assumed to be fixed on the Earth and its axes are denoted by the special base vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$.

Though the Cartesian coordinate frame is the most popular one for studying rigid-body motions, cylindrical and spherical coordinate frames are more convenient for describing circular motions.

Figure 2.2 shows a cylindrical coordinate frame $\{A\}$ defined in universe frame $\{U\}$. It consists of an origin O_A and three mutually perpendicular axes (called

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Fig. 2.1 Coordinate frames

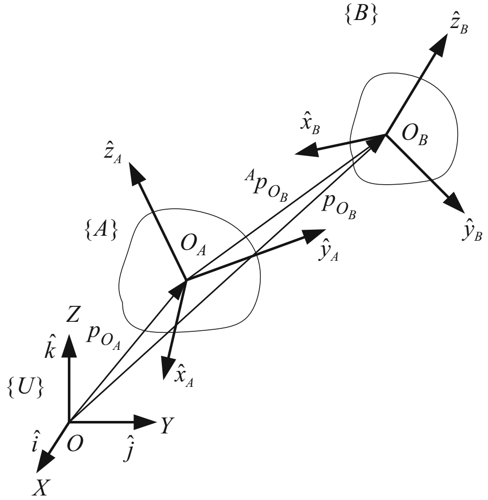
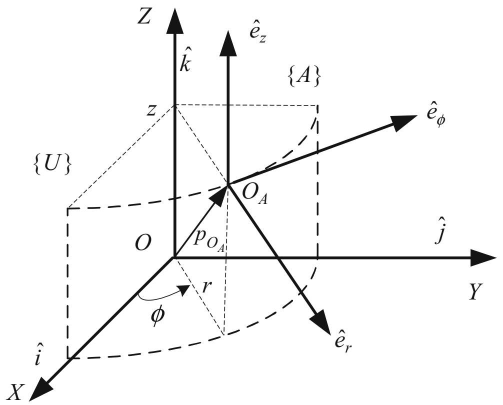


Fig. 2.2 Cylindrical coordinate frame



cylindrical axes), $\hat{e}_z$, $\hat{e}_r$, and $\hat{e}_\phi$:

$$
\begin{aligned}
\hat{e}_r &= \cos\phi\ \hat{\imath} + \sin\phi\ \hat{\jmath},\\
\hat{e}_z &= \hat{k},\\
\hat{e}_\phi &= \hat{e}_z \times \hat{e}_r = -\sin\phi\ \hat{\imath} + \cos\phi\ \hat{\jmath},\ \text{right-hand rule},\\
p_{O_A} &= r\hat{e}_r + z\hat{e}_z,
\end{aligned}
\tag{2.1}
$$

where ϕ is the angle between the plane formed by axis Z and p_{O_A} and the plane XZ, r is the length of the projection of p_{O_A} on the XY plane or the radius of the cylinder, and z is the z coordinate of point O_A in $\{U\}$. r, ϕ, and z are called the cylindrical coordinates of point O_A.

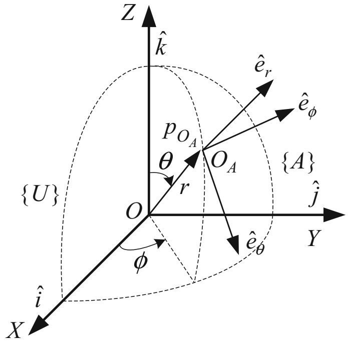


Fig. 2.3 Spherical coordinate frame

A unique feature of a cylindrical coordinate frame is that its axes change with the position of its origin, whereas in a Cartesian coordinate frame, the origin has no effect on the directions of the axes of the frame.

There is a one-to-one mapping between the Cartesian coordinates and the cylindrical coordinates of a point. For point O_A, the mapping from its cylindrical coordinates to its Cartesian coordinates can be derived from expanding (2.1):

$$
\begin{aligned}
p_{O_A} &= r\cos\phi\ \hat{i} + r\sin\phi\ \hat{j} + z\hat{k},\\
x_{O_A} &= r\cos\phi,\\
y_{O_A} &= r\sin\phi.\\
z_{O_A} &= z
\end{aligned}
$$

Given the Cartesian coordinates $(x_{O_A},\ y_{O_A},\ z_{O_A})$ of point O_A, the corresponding cylindrical coordinates are

$$
\begin{aligned}
r &= \sqrt{x_{O_A}^2 + y_{O_A}^2},\\
\phi &= \tan^{-1}\frac{y_{O_A}}{x_{O_A}}, \quad x_{O_A} \neq 0,\\
z &= z_{O_A}.
\end{aligned}
$$

If $x_{O_A} = 0$ and $y_{O_A} \neq 0$, then $\phi = \pm\frac{\pi}{2}$. If $x_{O_A} = y_{O_A} = 0$, then O_A is located on the Z axis and ϕ is not defined.

Figure 2.3 shows a spherical frame $\{A\} : O_A\hat{e}_\theta\hat{e}_\phi\hat{e}r$ in the universe coordinate frame $\{U\}$. The frame is defined by

$$
\begin{aligned}
p_{O_A} &= r\hat{e}_r,\\
\hat{e}_r &= \sin\theta\cos\phi\ \hat{i} + \sin\theta\sin\phi\ \hat{j} + \cos\theta\ \hat{k},\\
\hat{e}_\theta &= \cos\theta\cos\phi\ \hat{i} + \cos\theta\sin\phi\ \hat{j} - \sin\theta\ \hat{k},\\
\hat{e}_\phi &= \hat{e}_r \times \hat{e}_\theta = -\sin\phi\ \hat{i} + \cos\phi\ \hat{j},\ \text{right-hand rule.}
\end{aligned}
$$

The spherical coordinates include r, θ, and ϕ, where r is the radius of the sphere, θ is the colatitude, and ϕ is the azimuth angle. Like a cylindrical coordinate frame, the axes of a spherical coordinate frame depend on its origin. The spherical coordinates and the Cartesian coordinates of origin O_A are related through

$$
\begin{aligned}
r &= \sqrt{x_{O_A}^2 + y_{O_A}^2 + z_{O_A}^2},\\
\theta &= \tan^{-1} \frac{z_{O_A}}{\sqrt{x_{O_A}^2 + y_{O_A}^2}}, \quad x_{O_A} \neq 0, \quad y_{O_A} \neq 0,\\
\phi &= \tan^{-1} \frac{y_{O_A}}{x_{O_A}}, \quad x_{O_A} \neq 0.
\end{aligned}
$$

2.2 Position and Orientation

The positions and orientations of rigid bodies A and B are fully described by$\{B\}$ and $\{A\}$, the frames attached to them. When O_B, $\hat{x}_B$, $\hat{y}_B$, and $\hat{z}_B$ are *observed* and *described* in $\{A\}$, they are denoted as vectors ${}^A p_{O_B}$, ${}^A\hat{x}_B$, ${}^A\hat{y}_B$, and ${}^A\hat{z}_B$.

Many key quantities describing the motion of a rigid body like position, velocity, and acceleration are presented in the form of a vector. The vector should contain the following information:

- The coordinate frame, called the *observation frame*, in which the vector is observed. In this frame, the geometrical entity of the vector, a directional line, is formulated and observed by one who keeps a *fixed* position and orientation with respect to the frame.
- The coordinate frame, called the *description frame*, in which the vector is described. In this frame, the directional line observed in the observation frame is presented by its projections (x, y, and z components) along the axes of the frame. In other words, a numerical or algebraic expression of the vector, a row or a column of numbers or variables, is formulated.

Take the position vector of the origin of $\{B\}$, O_B in Fig. 2.1, as an example. If the observation frame is $\{A\}$, then the vector is observed as the directional line $\overline{O_A O_B}$. If the observation frame changes to $\{U\}$, then the vector is observed as the directional line $\overline{O O_B}$. What happens if the observation frame is $\{B\}$? The directional line is now degenerated to a point, and a zero (null) vector is formed. For each directional line observed, its x, y, and z components along the axes of description frame $\{A\}$ are different from those along the axes of another description frame $\{U\}$, generating different vector expressions.

It is clumsy to explain in words the observation frame and the description frame for the definition of a vector. We must find a way to embed them in a neat and compact notation for the vector.

2.2.1 Vector Notations

Let p_B be the position vector of point B. Define

$$^{C}p_{B/A}$$

as the vector p_B observed in frame $\{A\}$ and described in frame $\{C\}$. This notation consists of three parts:

- p_B: the name of the vector that is at the center of the notation. It refers to the (kinematic or dynamic) *property* that the vector stands for and the *physical entity* with which that property is associated. Here, "p" stands for *position*, and "B" refers to point B. Depending on the way the vector is defined, the number of symbols in the name varies.
- A: the name of the observation frame that forms the right subscript of the vector name. It is separated from the vector name with a slash.
- C: the name of the observation frame that is the left superscript of the vector name.

The notation can be simplified in the following special cases:

- When a universe frame is involved, its name is not shown in the notation. For example:
  - $p_{B/A}$ means the position of B observed in frame $\{A\}$ but described in universe frame $\{U\}$.
  - p_B means the position of B observed and described in universe frame $\{U\}$.
  - $^{A}p_{B/}$ means the position of B observed in universe frame $\{U\}$ but described in frame $\{A\}$.
- When the observation frame and the description frame are the same, the right subscript (the name of the observation frame) is not shown. For example:
  - $^{A}p_B$ means the position of B observed and described in frame $\{A\}$.

Remark. The above notations are defined using position vector as an example, but they can be used to represent any other types of vectors. If p is replaced by v, ω, or a, it represents a linear velocity, an angular velocity, or a linear acceleration, respectively.

In some cases when no confusion would be caused without explicitly listing the observation frame and the description frame, it is acceptable to omit the names of the frames to simplify the notation.

The following example is presented to show the application of the notations defined above.

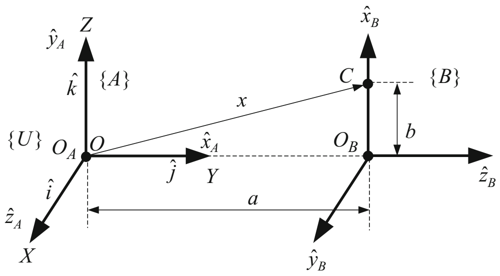


Fig. 2.4 Observation and description of a vector in different frames

Example 2.1. Universe frame $\{U\}$ and two body frames $\{A\}$ and $\{B\}$ are set up as shown in Fig. 2.4. $\{A\}$ and $\{U\}$ share the same origin, and $\hat{x}_A = \hat{j}$, $\hat{y}_A = \hat{k}$, and $\hat{z}_A = \hat{i}$. The origin of frame $\{B\}$ is on axis $\hat{j}$ with a distance a from O_A, and $\hat{x}_B = \hat{k}$, $\hat{y}_B = \hat{i}$, and $\hat{z}_B = \hat{j}$. Point C is on axis $\hat{z}_B$.

The expressions for the positions of O_B and C are determined by the choice of observation frame and description frame. For the position of point O_B:

$$\begin{aligned}
{}^A p_{O_B} &= [a\ 0\ 0]^{\mathrm{T}}, \text{ observed in } \{A\} \text{ and described in } \{A\};\\
p_{O_{B/A}} &= [0\ a\ 0]^{\mathrm{T}}, \text{ observed in } \{A\} \text{ and described in } \{U\};\\
p_{O_B} &= [0\ a\ 0]^{\mathrm{T}}, \text{ observed in } \{U\} \text{ and described in } \{U\};\\
{}^B p_{O_{B/}} &= [0\ 0\ a]^{\mathrm{T}}, \text{observed in } \{U\} \text{ and described in } \{B\};\\
{}^B p_{O_B} &= [0\ 0\ 0]^{\mathrm{T}}, \text{observed in } \{B\} \text{ and described in } \{B\}. \text{ This is the origin of } \{B\}!
\end{aligned}$$

For the position of point C:

$$\begin{aligned}
{}^B p_C &= [b\ 0\ 0]^{\mathrm{T}}, \text{ observed in } \{B\} \text{ and described in } \{B\};\\
p_{C/B} &= [0\ 0\ b]^{\mathrm{T}}, \text{ observed in } \{B\} \text{ and described in } \{U\};\\
p_C &= [0\ a\ b]^{\mathrm{T}}, \text{ observed in } \{U\} \text{ and described in } \{U\}.
\end{aligned}$$

Let x be the vector corresponding to the directional line $\overline{OC}$; then

$$\begin{aligned}
{}^A x &= [a\ b\ 0]^{\mathrm{T}}, \text{ observed and described in } \{A\};\\
x &= [0\ a\ b]^{\mathrm{T}}, \text{ observed and described in } \{U\};\\
x_{/A} &= x_{/O_A} = [0\ a\ b]^{\mathrm{T}}, \text{ observed in } \{A\} \text{ and described in } \{U\};\\
{}^B x_{/A} &= {}^B x_{/O_A} = [b\ 0\ a]^{\mathrm{T}}, \text{ observed in } \{A\} \text{ and described in } \{B\};\\
{}^B x_{/} &= [b\ 0\ a]^{\mathrm{T}}, \text{ observed in } \{U\} \text{ and described in } \{B\}.
\end{aligned}$$

2.2.2 Orientation

Referring to Fig. 2.1, the orientation of bodies A and B with respect to universe frame $\{U\}$ are governed by the directions of the unit vectors along their orthogonal axes. Using them as column vectors, *rotation matrices* are formed:

$$R_A = [\hat{x}_A \ \hat{y}_A \ \hat{z}_A] \in R^{3\times 3},$$
$$R_B = [\hat{x}_B \ \hat{y}_B \ \hat{z}_B] \in R^{3\times 3}.$$

As the column vectors are orthogonal to each other, R_A and R_B are orthogonal matrices, $R_A^{\mathrm{T}} R_A = I^3$, and $R_B^{\mathrm{T}} R_B = I^3$.

Similarly, the orientation of $\{B\}$ with respect to $\{A\}$ can be described compactly by putting base vectors ${}^A\hat{x}_B$, ${}^A\hat{y}_B$, and ${}^A\hat{z}_B$ in a rotation matrix:

$$ {}^A_BR = [{}^A\hat{x}_B \ {}^A\hat{x}_B \ {}^A\hat{z}_B] \in R^{3\times 3}. \tag{2.2}$$

Note ${}^A\hat{x}_B$, ${}^A\hat{y}_B$, and ${}^A\hat{z}_B$ are the base vectors of $\{B\}$ observed and described in $\{A\}$, and they are presented in the notations introduced in Sect. 2.2.1.

Treating ${}^A\hat{x}_B$ as a combination of the projections of $\hat{x}_B$ onto the three axes of frame $\{A\}$, we have

$$ {}^A\hat{x}_B = (\hat{x}_B^{\mathrm{T}}\hat{x}_A)\hat{x}_A + (\hat{x}_B^{\mathrm{T}}\hat{y}_A)\hat{y}_A + (\hat{x}_B^{T}\hat{z}_A)\hat{z}_A.$$

This also applies to ${}^A\hat{y}_B$ and ${}^A\hat{z}_B$ such that

$$ {}^A\hat{y}_B = (\hat{y}_B^{\mathrm{T}}\hat{x}_A)\hat{x}_A + (\hat{y}_B^{\mathrm{T}}\hat{y}_A)\hat{y}_A + (\hat{y}_B^{\mathrm{T}}\hat{z}_A)\hat{z}_A,$$
$$ {}^A\hat{z}_B = (\hat{z}_B^{\mathrm{T}}\hat{x}_A)\hat{x}_A + (\hat{z}_B^{\mathrm{T}}\hat{y}_A)\hat{y}_A + (\hat{z}_B^{\mathrm{T}}\hat{z}_A)\hat{z}_A.$$

Substituting them into (2.2), we have

$$ {}^A_BR = \begin{bmatrix} \hat{x}_B^{\mathrm{T}}\hat{x}_A & \hat{y}_B^{\mathrm{T}}\hat{x}_A & \hat{z}_B^{\mathrm{T}}\hat{x}_A \\ \hat{x}_B^{\mathrm{T}}\hat{y}_A & \hat{y}_B^{\mathrm{T}}\hat{y}_A & \hat{z}_B^{\mathrm{T}}\hat{y}_A \\ \hat{x}_B^{\mathrm{T}}\hat{z}_A & \hat{y}_B^{\mathrm{T}}\hat{z}_A & \hat{z}_B^{\mathrm{T}}\hat{z}_A \end{bmatrix}.$$

What follows is an important relation between R_A, R_B, and ${}^A_BR$:

$$R_B = R_A \, {}^A_BR. \tag{2.3}$$

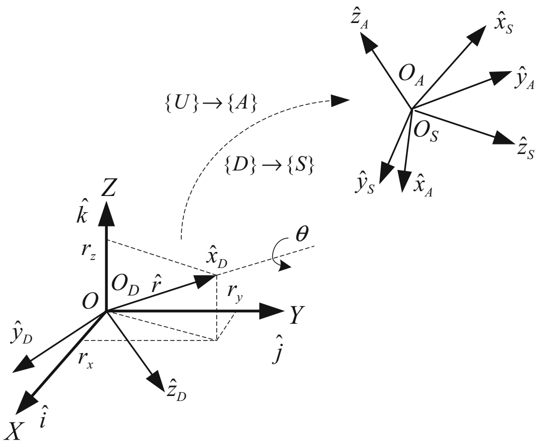


Fig. 2.5 Relation between a rotation matrix with corresponding rotation axis and angular displacement

This can be proved by expanding the right-hand side of (2.3):

$$
\begin{aligned}
R_A\,{}^A_BR &= [\hat{x}_A\ \hat{y}_A\ \hat{z}_A] \begin{bmatrix} \hat{x}_B^{\mathrm{T}}\hat{x}_A & \hat{y}_B^{\mathrm{T}}\hat{x}_A & \hat{z}_B^{\mathrm{T}}\hat{x}_A \\ \hat{x}_B^{\mathrm{T}}\hat{y}_A & \hat{y}_B^{\mathrm{T}}\hat{y}_A & \hat{z}_B^{\mathrm{T}}\hat{y}_A \\ \hat{x}_B^{\mathrm{T}}\hat{z}_A & \hat{y}_B^{\mathrm{T}}\hat{z}_A & \hat{z}_B^{\mathrm{T}}\hat{z}_A \end{bmatrix} \\
&= \left[\hat{x}_A\hat{x}_A^{\mathrm{T}} + \hat{y}_A\hat{y}_A^{\mathrm{T}} + \hat{z}_A\hat{z}_A^{\mathrm{T}}\right][\hat{x}_B\ \hat{y}_B\ \hat{z}_B] \\
&= [\hat{x}_A\ \hat{y}_A\ \hat{z}_A]\,[\hat{x}_A\ \hat{y}_A\ \hat{z}_A]^{\mathrm{T}}\,R_B \\
&= (R_A\ R_A^{\mathrm{T}})R_B = I^3 R_B = R_B.
\end{aligned}
$$

The rotation matrix provides a compact and systematic way to describe a body's rotation, but it contains nine parameters that are not independent of each other as they need to meet the requirement of an orthogonal matrix. In what follows, an alternative expression for the orientation of a body is presented by specifying the angular displacement and the axis of rotation.

As shown in Fig. 2.5, which is derived from Fig. 2.1, assume the orientation of $\{A\}$ is generated by $\{U\}$ rotating about axis $\hat{r} = [r_x\ r_y\ r_z]^{\mathrm{T}}$ at angle θ. Define a new frame $\{D\}$ such that its origin is at O (origin of $\{U\}$) and its three axes are

$$
\begin{aligned}
\hat{x}_D &= \hat{r}, \\
\hat{y}_D &= \frac{\hat{r} \times \hat{k}}{\|\hat{r} \times \hat{k}\|}, \\
\hat{z}_D &= \hat{x}_D \times \hat{y}_D = \frac{\hat{r} \times (\hat{r} \times \hat{k})}{\|\hat{r} \times \hat{k}\|}.
\end{aligned}
$$

Note that $\hat{r}$ cannot align with $\hat{k}$; otherwise $\hat{y}_D$ and $\hat{z}_D$ would not be defined.

The rotation matrix of frame $\{D\}$ with respect to $\{U\}$ is

$$ {}^{U}_{D}R = [\hat{x}_D \; \hat{y}_D \; \hat{z}_D] \,, \tag{2.4} $$

and its inverse is the rotation matrix describing the orientation of U with respect to $\{D\}$,

$$ {}^{D}_{U}R = {}^{U}_{D}R^{-1} = {}^{U}_{D}R^{\mathrm{T}}. $$

After $\{U\}$ rotates around vector $\hat{r}$ at an angle θ, $\{U\}$ changes to $\{A\}$, and $\{D\}$ to $\{S\}$. As $\{D\}$ is fixed with respect to $\{U\}$, the relative orientation between $\{A\}$ and $\{S\}$ is the same as that between $\{U\}$ and $\{D\}$:

$$ \begin{aligned} {}^{A}_{S}R &= {}^{U}_{D}R, \\ {}^{S}_{A}R &= {}^{D}_{U}R. \end{aligned} \tag{2.5} $$

But

$$ R_A = R_S \, {}^{S}_{A}R. $$

From (2.5),

$$ R_A = {}^{U}_{S}R {}^{D}_{U}R = ({}^{U}_{D}R \; {}^{D}_{S}R) \; {}^{D}_{U}R. \tag{2.6} $$

As $\{S\}$ is produced by rotating $\{D\}$ around its X axis by θ,

$$ {}^{D}_{S}R = R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}. \tag{2.7} $$

Substituting (2.4) and (2.7) into (2.6), we obtain

$$ \begin{aligned} R_A &= \begin{bmatrix} r_x & \frac{r_y}{\sqrt{1-r_z^2}} & \frac{r_x r_z}{\sqrt{1-r_z^2}} \\ r_y & -\frac{r_x}{\sqrt{1-r_z^2}} & \frac{r_y r_z}{\sqrt{1-r_z^2}} \\ r_z & 0 & -\sqrt{1-r_z^2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r_x & r_y & r_z \\ \frac{r_y}{\sqrt{1-r_z^2}} & -\frac{r_x}{\sqrt{1-r_z^2}} & 0 \\ \frac{r_x r_z}{\sqrt{1-r_z^2}} & \frac{r_y r_z}{\sqrt{1-r_z^2}} & -\sqrt{1-r_z^2} \end{bmatrix} \\ &= \begin{bmatrix} r_x^2(1-\cos\theta)+\cos\theta & r_x r_y(1-\cos\theta)-r_z\sin\theta & r_x r_z(1-\cos\theta)+r_y\sin\theta \\ r_x r_y(1-\cos\theta)+r_z\sin\theta & r_y^2(1-\cos\theta)+\cos\theta & r_y r_z(1-\cos\theta)-r_x\sin\theta \\ r_x r_z(1-\cos\theta)-r_y\sin\theta & r_y r_z(1-\cos\theta)+r_x\sin\theta & r_z^2(1-\cos\theta)+\cos\theta \end{bmatrix}. \end{aligned} $$

Note that $r_x^2 + r_y^2 + r_z^2 = 1$ is used to simplify the above expressions.

To show the axis and the angle associated with the rotation, the matrix is denoted as $R_r(\theta)$, which is in the same form as that for basic rotation matrices ($R_x(\theta)$, $R_y(\theta)$, and $R_z(\theta)$). To make the mathematical expressions more compact, the following abbreviations of some common trigonometric functions are used:

$$ \begin{aligned} c_\theta &\stackrel{\Delta}{=} \cos\theta \\ s_\theta &\stackrel{\Delta}{=} \sin\theta \end{aligned} $$

and then

$$R_A = R_r(\theta) \begin{bmatrix} r_x^2(1-c_\theta)+c_\theta & r_x r_y(1-c_\theta)-r_z s_\theta & r_x r_z(1-c_\theta)+r_y s_\theta \\ r_x r_y(1-c_\theta)+r_z s_\theta & r_y^2(1-c_\theta)+c_\theta & r_y r_z(1-c_\theta)-r_x s_\theta \\ r_x r_z(1-c_\theta)-r_y s_\theta & r_y r_z(1-c_\theta)+r_x s_\theta & r_z^2(1-c_\theta)+c_\theta \end{bmatrix}. \tag{2.8}$$

$R_r(\theta)$ meets all the requirements of a rotation matrix, and it becomes a basic rotation matrix if $\hat{r}$ is replaced by one of the principal axes ($\hat{i}$, $\hat{j}$, or $\hat{k}$) of $\{U\}$.

It can be shown that $\hat{r}$ is the eigenvector of $R_r(\theta)$, just as $\hat{x}$ is the eigenvector of $R_x(\theta)$. This can be verified from the fact that the rotation axis does not change during rotation.

The relation in (2.8) can also be used to find $\hat{r}$ and θ associated with the rotation matrix

$$R_A = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. \tag{2.9}$$

Equalizing (2.9) and (2.8) elementwise, we have

$$\theta = \cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right), \tag{2.10}$$

$$\hat{r} = \frac{[r_{32}-r_{23} \;\; r_{13}-r_{31} \;\; r_{21}-r_{12}]^{\mathrm{T}}}{2\sin\theta}, \quad \sin\theta \neq 0. \tag{2.11}$$

2.2.3 Position

Referring to Fig. 2.1, the position of body $\{A\}$ with respect to universe frame $\{U\}$ is given by vector p_{O_A}. Although any point in the body can be chosen as the origin (O_A) of the body frame, it will be made clear later that the choice of O_A affects the description of dynamics of the body, and a good candidate for O_A is the center of mass (COM).

The relation between the positions of two bodies are specified by that of the positions of the origins of their body frames. The position relation of frames $\{A\}$ and $\{B\}$ in Fig. 2.1 is given by

$$p_{O_B} = p_{O_A} + p_{O_{B/A}}, \tag{2.12}$$

which follows the rule of vector addition. Note all the vectors are described in universe frame $\{U\}$.

Express ${}^A p_{O_B}$ with its column vectors:

$${}^A p_{O_B} = [x_A \; y_A \; z_A]^{\mathrm{T}}$$

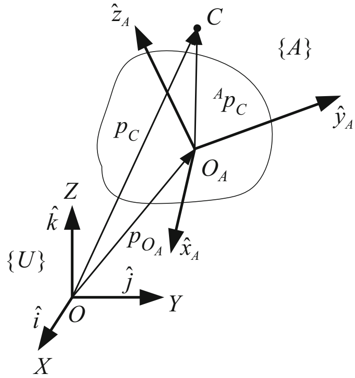


Fig. 2.6 Position of a point in body frame

then

$$\begin{aligned} p_{O_{B/A}} &= x_A \hat{x}_A + y_A \hat{y}_A + z_A \hat{z}_A \\ &= [\hat{x}_A \ \hat{y}_A \ \hat{z}_A][x_A \ y_A \ z_A]^{\mathrm{T}} \\ &= R_A \, {}^A p_{O_B}. \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.12), we have

$$p_{O_B} = p_{O_A} + R_A \, {}^A p_{O_B}. \tag{2.14}$$

The position of any point in the body frame can be obtained in the same way. Referring to Fig. 2.6, the position of point C is given by

$$p_C = p_{O_A} + p_{C/A},$$

where $p_{C/A} = R_A {}^A p_C$.

Remark. Equation (2.13) can be generalized to describe the relation between two vectors observed in the same frame (e.g., $\{A\}$), but described in different frames (e.g., $\{A\}$ and $\{U\}$). One vector is transformed into another by the rotation matrix (e.g., R_A) between the description frames. If the universe frame in (2.13) is replaced with another frame $\{D\}$, then

$${}^D p_{O_{B/A}} = {}^D_A R \, {}^A p_{O_B}. \tag{2.15}$$

In what follows, several examples are provided to show the applications of the concepts and methods presented above. They include the establishment of basic rotation matrices, the steps to set up coordinate frames and establish their relations in a mechanism consisting of multiple rigid bodies, and position and orientation analysis for rigid bodies.

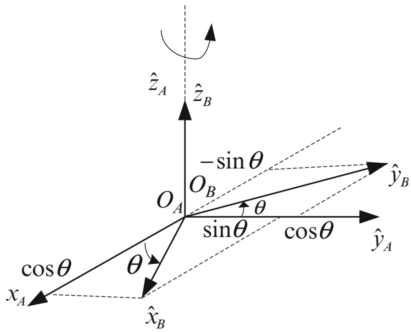


Fig. 2.7 Example 2.2

Example 2.2. In this example, the rotation matrix (basic rotation matrix) for a frame rotating about one of its principal axes is to be obtained. As shown in Fig. 2.7, frame $\{A\} : O_A\hat{x}_A\hat{y}_A\hat{z}_A$ rotates around axis $\hat{z}_A$ at angle θ. As a result, a new frame, $\{B\} : O_B\hat{x}_B\hat{y}_B\hat{z}_B$, is generated. Before proceeding further, note the following rules about the measurement of a rotation angle:

- The angle is positive for an anticlockwise rotation.
- The unit of the angle is radian.
- The range of the angles is normalized in the interval $[-\pi \;\; \pi]$.

It is clear that

$$\begin{aligned} O_B &= O_A, \\ \hat{x}_B &= \cos\theta\; \hat{x}_A + \sin\theta\, \hat{y}_A, \\ \hat{y}_B &= -\sin\theta\, \hat{x}_A + \cos\theta\, \hat{y}_A, \\ \hat{z}_B &= \hat{z}_A. \end{aligned}$$

As such

$$ {}^{A}_{B}R = R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.16}$$

Note $R_z(\theta)$ is used to denote any rotation matrix produced by rotating the frame around its z axis. Following the same procedure, the rotation matrices corresponding to rotations around the X and Y axes are derived:

$$ R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}. \tag{2.17}$$

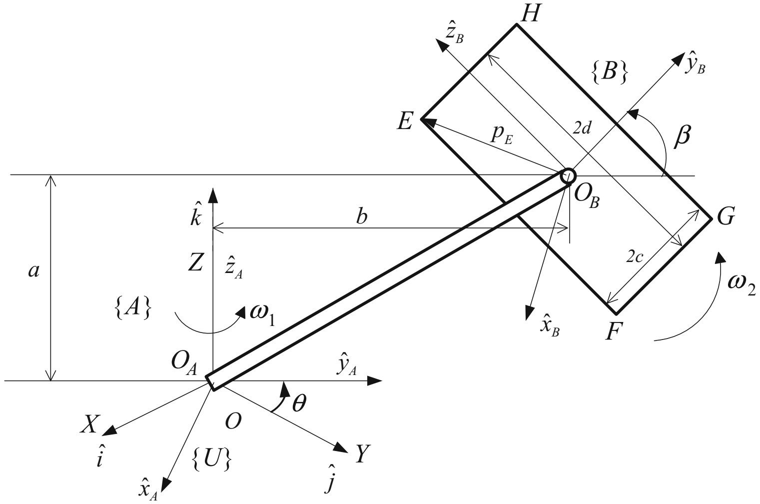


Fig. 2.8 Example 2.3

Example 2.3. As schematically shown in Fig. 2.8, a mechanism consists of a rod ($\overline{O_A O_B}$) and a rectangle box ($EFGH$). The rod rotates with speed ω_1 about the fixed axis Z perpendicular to the ground and the box rotates with speed ω_2 about the joint axis with the rod, x_B, which goes through the center of the box. The dimensions of the rod and the box are shown in the figure. Our task is to determine the positions and orientation of all the rigid bodies and their relations as the functions of the angular displacements of the rod and the box about their rotation axes, respectively. Note in this example and the rest of the book, ISO units are used for measurements without explicit explanation.

First, the following coordinate frames are set up:

- Universe coordinate frame $\{U\} : OXYZ$. Its origin O is at the bottom of the rod (O_A). Three fixed principal axes X, Y, and Z are along the base vectors $\hat{i}$, $\hat{j}$, and k.
- Body frame of the rod $\{A\} : O_A\hat{x}_A\hat{y}_A\hat{z}_A$. Its origin O_A coincides with O, and $\hat{z}_A$ coincides with the Z axis. The $\hat{y}_A$ axis is the projection of the rod on the XOY plane. Its angle θ with the Y axis describes the angular displacement of the rod around the Z axis. The $\hat{x}_A$ axis is then decided by the right-hand rule.
- Body frame of the box $\{B\} : O_B\hat{x}_B\hat{y}_B\hat{z}_B$. Its origin O_B is at the joint at the end of the rod, O_B. The $\hat{x}_B$ axis is along the rotation axis, which is parallel to $\hat{x}_A$. The $\hat{y}_B$ and $\hat{z}_B$ axes are perpendicular to the sides GH and EH of the rectangular box, respectively.

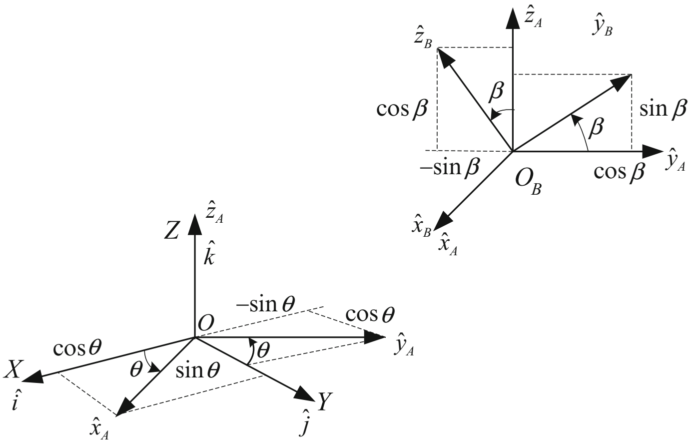


Fig. 2.9 Example: relations between the axes of the frames

The relations between the axes of $\{A\}$, $\{B\}$, and $\{U\}$ are sketched in Fig. 2.9.

Observing and describing $\{A\}$ in $\{U\}$, we have

$$
\begin{aligned}
p_{O_A} &= [0\ 0\ 0]^{\mathrm{T}}, \qquad (2.18)\\
\hat{x}_A &= \cos\theta\ \hat{i} + \sin\theta\ \hat{j} + 0\hat{k} = [\cos\theta\ \ \sin\theta\ \ 0]^{\mathrm{T}},\\
\hat{y}_A &= -\sin\theta\ \hat{i} + \cos\theta\ \hat{j} + 0\hat{k} = [-\sin\theta\ \ \cos\theta\ \ 0]^{\mathrm{T}},\\
\hat{z}_A &= \hat{k} = [0\ 0\ 1]^{\mathrm{T}}, \quad \text{and}
\end{aligned}
$$

$$
R_A = [\hat{x}_A\ \hat{y}_A\ \hat{z}_A] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (2.19)
$$

Observing and describing $\{B\}$ in $\{A\}$, we have

$$
\begin{aligned}
{}^{A}p_{O_B} &= [0\ b\ a]^{\mathrm{T}}, \qquad (2.20)\\
{}^{A}\hat{x}_B &= [1\ 0\ 0]^{\mathrm{T}},\ \hat{x}_B // \hat{x}_A,\\
{}^{A}\hat{y}_B &= [0\ \ \cos\beta\ \ \sin\beta]^{\mathrm{T}}\\
{}^{A}\hat{z}_B &= [0\ -\sin\beta\ \ \cos\beta]^{\mathrm{T}}, \quad \text{and}
\end{aligned}
$$

$$
{}^{A}_{B}R = [{}^{A}\hat{x}_B\ {}^{A}\hat{y}_B\ {}^{A}\hat{z}_B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}. \qquad (2.21)
$$

Note rotation matrices R_A and ${}^A_B R$ can also be obtained using basic rotation matrices defined by (2.16) and (2.17):

$$R_A = R_z(\theta),\ {}^A_B R = R_x(\beta)$$

Now examine the description of frame $\{B\}$ in universe frame $\{U\}$:

$$p_{O_B} = R_A\ {}^A p_{O_B} = [-b \sin\theta\ b\cos\theta\ a]^{\mathrm{T}}, \tag{2.22}$$

$$\hat{x}_B = \hat{x}_A = R_A\ {}^A\hat{x}_B,$$

$$\hat{y}_B = 0\ \hat{x}_A + \cos\beta\ \hat{y}_A + \sin\beta\ \hat{z}_A = R_A[0\ \cos\beta\ \ \sin\beta]^{\mathrm{T}} = R_A\ {}^A\hat{y}_B,$$

$$\hat{z}_B = 0\ \hat{x}_A - \sin\beta\ \hat{y}_A + \cos\beta\ \hat{z}_A = R_A[0\ -\sin\beta\ \cos\beta]^{\mathrm{T}} = R_A\ {}^A\hat{z}_B, \quad \text{and}$$

$$R_B = [\hat{x}_B\ \hat{y}_B\ \hat{z}_B] = R_A\ {}^A_B R = \begin{bmatrix} \cos\theta & -\sin\theta\cos\beta & \sin\theta\sin\beta \\ \sin\theta & \cos\theta\cos\beta & -\cos\theta\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}. \tag{2.23}$$

Thus (2.18), (2.19), (2.22), and (2.23) give the positions and orientations of the rod and the box as the functions of the rotation angles θ and β, respectively.

Example 2.4. This example is the continuation of Example 2.2.3. Our task is to (1) find the position of point E and an end of edge EH of the box and (2) derive the rotational axis of frame $\{B\}$ and the angular displacement of the frame around this axis with respect to the universe frame $\{U\} : OXYZ$ shown in Fig. 2.8.

From Fig. 2.8 the expressions for the position of E in frames $\{B\}$ and $\{U\}$ respectively are

$${}^B p_E = [0\ \ -c\ d]^{\mathrm{T}}, \tag{2.24}$$

$$p_E = p_{O_B} + R_B\ {}^B p_E. \tag{2.25}$$

Terms p_{O_B} and R_B are derived in (2.22) and (2.23), respectively, and ${}^B p_E$ is given in (2.24). Substituting them into (2.25), we have

$$p_E = \begin{bmatrix} -(b + c\cos\beta - d\ \sin\beta)\sin\theta \\ (b - c\cos\beta - d\ \sin\beta)\cos\theta \\ a - c\sin\beta + d\ \cos\beta \end{bmatrix}.$$

The equivalent rotational axis ($\hat{r}_B$) and the angular displacement (θ_B) for frame $\{B\}$ with respect to frame $\{U\}$ can be found from applying (2.10) and (2.11) on the rotation matrix R_B in (2.23):

$$\theta_B = \cos^{-1}\left(\frac{\cos\theta(1+\cos\beta) + \cos\beta - 1}{2}\right)$$

$$\hat{r} = \frac{[\sin\beta(1+\sin\theta)\ \ \sin\beta\sin\theta\sin\theta(1+\cos\beta)]^{\mathrm{T}}}{2\sin\theta_B},\ \sin\theta_B \neq 0.$$

2.3 Velocity

In the previous sections, the methods for determining the position and orientation of a rigid body were discussed. If position and orientation change with time, two types of velocities are defined to quantify their rate of change:

- *Linear velocity*: This is the velocity of a *fixed point* in a body. It is normally referred to as the velocity of the origin of the body frame.
- *Angular velocity*: This refers to the angular velocity of the body around an *instantaneous* rotation axis. It is related to the rate of change of the a frame's rotation matrix with respect to time.

The magnitude of the velocity is called the *speed*. For the linear velocity v and the angular velocity ω, the corresponding linear and angular speeds are $\|v\|$ and $\|\omega\|$, respectively.

2.3.1 Linear Velocity

Referring to Fig. 2.10, v_A is the linear velocity of a body represented by body frame $\{A\}$. Using the notations introduced in Sect. 2.2.1, it is observed and described in universe frame $\{U\}$.

By definition,

$$v_A = \dot{p}_{O_A} = \frac{\mathrm{d}p_{O_A}}{\mathrm{d}t}.$$

By default, velocity means *instantaneous* velocity, which is the velocity at the instant of concern. Sometimes we might only be concerned with the *average*

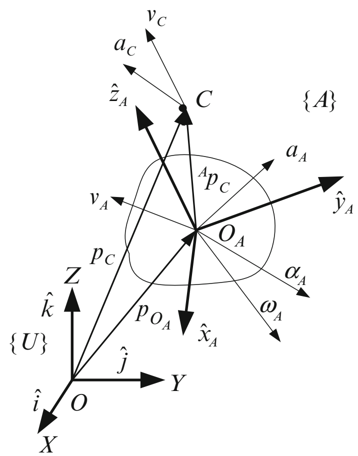


Fig. 2.10 Linear and angular velocity of a rigid body

velocity, the ratio between the displacement and the time duration. For example, the average velocity of frame $\{A\}$ from time instant t_i to t_f is defined as

$$\overline{v}_A = \frac{p_{O_A}(t_f) - p_{O_A}(t_i)}{t_f - t_i}.$$

The velocity of a point (C) in a body is given by

$$v_C = \dot{p}_C = \frac{\mathrm{d}p_C}{\mathrm{d}t}$$

As

$$\begin{aligned} p_C &= p_{O_A} + p_{C/A}, \\ p_{C/A} &= R_A\,{}^A p_C, \end{aligned}$$

then

$$\begin{aligned} v_C &= \dot{p}_{O_A} + R_A\,{}^A v_C + \dot{R}_A\,{}^A p_C \\ &= v_A + v_{C/A} + \dot{R}_A\,{}^A p_C. \end{aligned} \tag{2.26}$$

v_C consists of the following three parts:

- $v_A = \dot{p}_{O_A}$: due to the linear motion (translation) of frame $\{A\}$;
- $v_{C/A} = R_A\,{}^A v_C$: due to the motion of point C with respect to frame $\{A\}$;
- $\dot{R}_A\,{}^A p_C$: due to the rotation of frame $\{A\}$. In some books, this is called the *apparent velocity*. $\dot{R}_A$ is the rate of change with time of the orientation of frame $\{A\}$. It will be shown later that it is a skew-symmetric matrix from which the angular velocity of a body can be obtained.

There are two special cases regarding the velocity of point C.

- When $\dot{R}_A = 0$ and $v_A \neq 0$:

$$v_C = v_A + R_A\,{}^A v_C = v_A + v_{C/A}.$$

In this case, frame $\{A\}$ only translates with velocity v_A in $\{U\}$. The velocity of C is the sum of the linear velocity of frame $\{A\}$ and velocity of C with respect to $\{A\}$.

- When $\dot{R}_A \neq 0$ and $v_A = 0$:

$$v_C = R_A\,{}^A v_C + \dot{R}_A\,{}^A p_C = v_{C/A} + \dot{R}_A\,{}^A p_C.$$

In this case, frame $\{A\}$ only rotates in $\{U\}$. The velocity of C is the sum of the linear velocity caused by the rotation of frame $\{A\}$ and the velocity of C with respect to $\{A\}$.

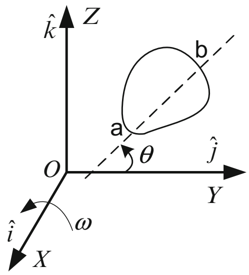


Fig. 2.11 A rigid body rotating around a fixed axis

2.3.2 *Angular Velocity*

Angular velocity is a concept unique to rigid bodies. It can be easily understood if the rigid body rotates about a *fixed* axis. As shown in Fig. 2.11, a rigid body rotates about the X axis of the universe frame. Its angular displacement (θ) is represented by the angle between any fixed line (e.g., ab) on the body projected on the plane perpendicular to the X axis (YZ plane) and a principal axis ($\hat{y}$) on the same plane. The angular velocity ω is then given by

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\imath} = \dot{\theta}\hat{\imath}.$$

However, it is not so straightforward to describe the angular velocity if the rotation axis also changes with time. As shown in (2.26), angular velocity is related to the rate of change of the rotation matrix with respect to time. This provides a mechanism to find it.

According to the first principle of differentiation, the derivative of R_A at time instant t is

$$\dot{R}_A(t) = \lim_{\Delta t \to 0} \frac{R_A(t + \Delta t) - R_A(t)}{\Delta t}. \tag{2.27}$$

Assuming that the instantaneous rotation axis is $\hat{r}$ and the angular displacement is $\Delta\theta$ during time duration Δt, then

$$R_A(t + \Delta t) = R_r(\Delta\theta) R_A(t), \tag{2.28}$$

where

$$R_r(\Delta\theta) = (1 - \cos\Delta\theta)\begin{bmatrix} r_x^2 & r_x r_y & r_x r_z \\ r_x r_y & r_y^2 & r_y r_z \\ r_x r_z & r_y r_z & r_z^2 \end{bmatrix} + \begin{bmatrix} \cos\Delta\theta & -r_z \sin\Delta\theta & r_y \sin\Delta\theta \\ r_z \sin\Delta\theta & \cos\Delta\theta & -r_x \sin\Delta\theta \\ -r_y \sin\Delta\theta & r_x \sin\Delta\theta & \cos\Delta\theta \end{bmatrix}$$

as defined in (2.8).

When $\varDelta t \to 0$, $\varDelta\theta \to 0$,

$$\cos\varDelta\theta \sim 1, \quad \sin\varDelta\theta \sim \varDelta\theta,$$

and $R_r(\varDelta\theta)$ is rewritten as

$$R_r(\varDelta\theta)= \begin{bmatrix} 1 & -r_z\varDelta\theta & r_y\varDelta\theta \\ r_z\varDelta\theta & 1 & -r_x\varDelta\theta \\ -r_y\varDelta\theta & r_x\varDelta\theta & 1 \end{bmatrix}. \tag{2.29}$$

Substituting (2.28) and (2.29) into (2.27), we have

$$\begin{aligned} \dot{R}_A(t) &= \lim_{\varDelta t\to 0} \begin{bmatrix} 0 & -r_z\frac{\varDelta\theta}{\varDelta t} & r_y\frac{\varDelta\theta}{\varDelta t} \\ r_z\frac{\varDelta\theta}{\varDelta t} & 0 & -r_x\frac{\varDelta\theta}{\varDelta t} \\ -r_y\frac{\varDelta\theta}{\varDelta t} & r_x\frac{\varDelta\theta}{\varDelta t} & 0 \end{bmatrix} R_A(t) \\ &= \begin{bmatrix} 0 & -r_z\dot{\theta} & r_y\dot{\theta} \\ r_z\dot{\theta} & 0 & -r_x\dot{\theta} \\ -r_y\dot{\theta} & r_x\dot{\theta} & 0 \end{bmatrix} R_A(t). \end{aligned} \tag{2.30}$$

Define

$$\omega_A = \dot{\theta}\hat{r} = [\omega_x\ \omega_y\ \omega_z]^{\mathrm{T}}, \tag{2.31}$$

where

$$\omega_x = r_x\dot{\theta}, \quad \omega_y = r_y\dot{\theta}, \quad \omega_z = r_z\dot{\theta}$$

and

$$\varOmega_A = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{2.32}$$

Equation (2.30) is rewritten as

$$\dot{R}_A(t) = \varOmega_A R_A(t) \tag{2.33}$$

and

$$\varOmega_A = \dot{R}_A(t) R_A^{-1}(t) = \dot{R}_A(t) R_A^{\mathrm{T}}(t). \tag{2.34}$$

Note that $R_A^{-1}(t) = R_A^{\mathrm{T}}(t)$ is used in the above derivation.

Also note that $\varOmega_A$ is a skew-symmetric matrix:

$$\varOmega_A^{\mathrm{T}} + \varOmega_A = 0$$

Matrix Ω_A is fully determined by the components of ω_A. Usually this relationship is denoted as

$$\Omega_A \triangleq [\omega_A \times] \quad \text{or} \quad \Omega_A \triangleq S(\omega_A).$$

The second notation stands for a skew-symmetric matrix determined by ω_A as defined in (2.31).

Consider a body rotating around the X axis as shown in Fig. 2.11. The rotation matrix is given by (2.17), which is reproduced below:

$$R_A = R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

Then

$$\Omega_A = \dot{R}_A R_A^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\dot{\theta}\sin\theta & -\dot{\theta}\cos\theta \\ 0 & \dot{\theta}\cos\theta & -\dot{\theta}\sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix}.$$

So

$$\omega_A = \dot{\theta}[1\ 0\ 0]^{\mathrm{T}} \quad \text{or} \quad \omega_A = \dot{\theta}\hat{i},$$

which is exactly the angular velocity around the X axis.

Equation (2.34) can also be derived from the fact that

$$R_A(t)R_A^{\mathrm{T}}(t) = I^3.$$

Differentiating it with time t, we have

$$\dot{R}_A(t)R_A^{\mathrm{T}}(t) + R_A(t)\dot{R}_A^{\mathrm{T}}(t) = 0$$

or

$$(\dot{R}_A(t)R_A^{\mathrm{T}}(t))^{\mathrm{T}} + \dot{R}_A(t)R_A^{\mathrm{T}}(t) = 0.$$

Define

$$\Omega_A \triangleq \dot{R}_A(t)R_A^{\mathrm{T}}(t);$$

then

$$\Omega_A^{\mathrm{T}} + \Omega_A = 0.$$

Thus Ω_A is a skew-symmetric matrix.

Next, we discuss the angular velocity of one frame relative to another frame. Taking frames $\{A\}$ and $\{B\}$ as an example, we have

$$
\begin{aligned}
{}^{A}\Omega_{B} &= [{}^{A}\omega_{B}\times] = {}^{A}_{B}\dot{R}\,{}^{A}_{B}R^{\mathrm{T}} \\
{}^{A}_{B}\dot{R} &= {}^{A}\Omega_{B}\,{}^{A}_{B}R,
\end{aligned} \tag{2.35}
$$

where ${}^{A}\omega_{B}$ is the angular velocity of $\{B\}$ with respect to $\{A\}$, and ${}^{A}\Omega_{B}$ is the corresponding skew-symmetric matrix.

There is an interesting relation among ω_B, ω_A, and ${}^{A}\omega_{B}$:

$$
\omega_B = \omega_A + \omega_{B/A} \tag{2.36}
$$

$$
= \omega_A + R_A\,{}^{A}\omega_{B}. \tag{2.37}
$$

This means that if the relative angular velocity of body B with respect to body A and the angular velocity of the latter have the same description frame ($\{U\}$), a simple addition of these velocities is the angular velocity of body B with respect to the description frame ($\{U\}$). The following is the proof of this relation.

Note

$$
R_B = R_A\,{}^{A}_{B}R.
$$

Differentiating the above and multiplying it by R_B^{T}, we have

$$
\begin{aligned}
\dot{R}_B R_B^{\mathrm{T}} = \Omega_B &= (\dot{R}_A\,{}^{A}_{B}R + R_A\,{}^{A}_{B}\dot{R})R_B^{\mathrm{T}} \\
&= (\Omega_A R_A\,{}^{A}_{B}R + R_A\,{}^{A}\Omega_{B}\,{}^{A}_{B}R)R_B^{\mathrm{T}} \\
&= (\Omega_A R_B + R_A\,{}^{A}\Omega_{B}\,{}^{A}_{B}R)R_B^{\mathrm{T}} \\
&= \Omega_A + R_A\,{}^{A}\Omega_{B}(R_B\,{}^{B}_{A}R)^{\mathrm{T}} \\
&= \Omega_A + R_A\,{}^{A}\Omega_{B}R_A^{\mathrm{T}}.
\end{aligned} \tag{2.38}
$$

From (1.16), for any vector $x \in R^3$,

$$
\begin{aligned}
R_A\,{}^{A}\Omega_{B}R_A^{\mathrm{T}}x &= R_A\,({}^{A}\omega_{B} \times (R_A^{\mathrm{T}}x)) \\
&= (R_A{}^{A}\omega_{B}) \times (R_A\,R_A^{\mathrm{T}}x) \\
&= (R_A{}^{A}\omega_{B})x \\
&= S(R_A\,{}^{A}\omega_{B})x,
\end{aligned}
$$

which means that $R_A\,{}^{A}\Omega_{B}R_A^{\mathrm{T}} = R_A{}^{A}\omega_{B}$. Substituting it into (2.38), we have

$$
[\omega_B\times] = [\omega_A\times] + [R_A\,{}^{A}\omega_{B}\times],
$$

and then (2.36) and (2.37) are derived.

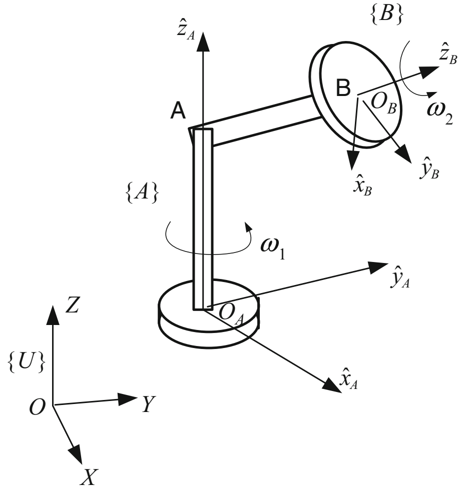


Fig. 2.12 Relation between angular velocities

This relation is also valid if the description frame is any frame other than the universe frame $\{U\}$. For example, if the description frame is $\{D\}$, then

$$^D\omega_B = {}^D\omega_A + {}^D\omega_{B/A}. \tag{2.39}$$

What follows is an example to show the application of this relation. Referring to Fig. 2.12, a plate is rotating with an angular speed of ω_2 about bar AB of the frame. At the mean time, the frame rotates with an angular speed of ω_1 about axis $\hat{z}_A$. The bar and the plate are represented by body frames $\{A\}$ and $\{B\}$, respectively. Assume that AB is parallel to $\hat{y}_A$; then

$$\omega_A = \omega_1 \hat{z}_A$$
$$\omega_{B/A} = \omega_2 \hat{x}_B = \omega_2 \hat{y}_A$$

and

$$\omega_B = \omega_A + \omega_{B/A} = \omega_1 \hat{z}_A + \omega_2 \hat{y}_A.$$

2.3.3 Relation Between Linear and Angular Velocities

Consider the velocity of point C in Fig. 2.10. From its expression given in (2.26) and the definition of angular velocity, we have

$$\begin{aligned} v_C &= v_A + R_A\,{}^A v_C + \Omega_A\,R_A\,{}^A p_C \\ &= v_A + v_{C/A} + \omega_A \times p_{C/A}. \end{aligned} \tag{2.40}$$

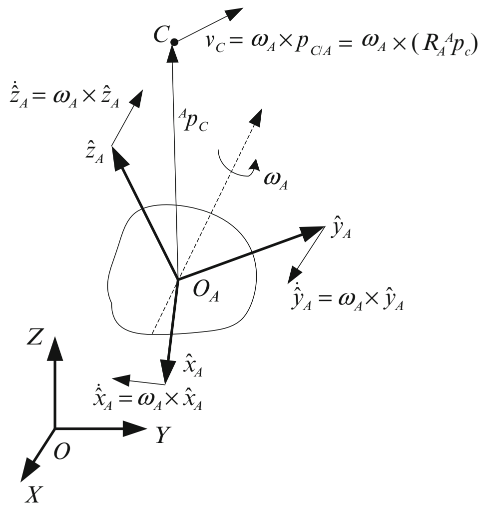


Fig. 2.13 Relation between linear and angular velocities in a rotating body

If C is fixed on a body that only rotates, ${}^A p_C$ is a constant, ${}^A v_C = 0$, $v_A = 0$, and

$$v_C = \dot{p}_{C/A} = \omega_A \times p_{C/A}. \tag{2.41}$$

This shows that the linear velocity of a point fixed on a rotating body is a cross product of the body's angular velocity and the position vector of that point.

This conclusion can also be found in (2.30) and (2.33), where

$$\dot{\hat{x}}_A = \Omega_A \hat{x}_A = \omega_A \times \hat{x}_A, \tag{2.42}$$

$$\dot{\hat{y}}_A = \Omega_A \hat{y}_A = \omega_A \times \hat{y}_A, \tag{2.43}$$

$$\dot{\hat{z}}_A = \Omega_A \hat{z}_A = \omega_A \times \hat{z}_A. \tag{2.44}$$

Each of them corresponds to the linear velocity of the tip of the corresponding principal axis rotating with angular velocity ω_A.

The above relations are also depicted schematically in Fig. 2.13. An interesting relation between a vector and its derivative can be found from (2.41)–(2.44). Vectors $p_{C/A}$, $\hat{x}_A$, $\hat{y}_A$, and $\hat{z}_A$ are all *fixed* with respect to frame $\{A\}$. They are all described in universe frame $\{U\}$. Using b to represent those vectors, their derivatives have the same expression:

$$\dot{b} = \Omega_A b = \omega_A \times b. \tag{2.45}$$

This relation is also schematically depicted in Fig. 2.14.

In a more general case when the motion of vector b with respect to frame $\{A\}$ is considered, the derivative of b corresponds to the last two terms of (2.40) if treating b as $p_{C/A}$:

$$\dot{b} = \dot{b}_{/A} + \omega_A \times b. \tag{2.46}$$

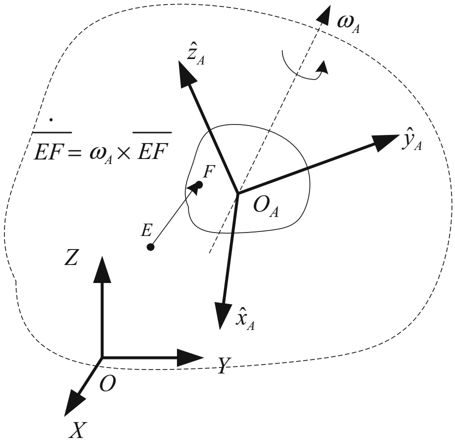


Fig. 2.14 Derivative of a vector fixed in a frame

According to the vector notations defined in Sect. 2.2.1, $\dot{b}_{/A}$ refers to the derivative of b observed in $\{A\}$ but described in $\{U\}$. In some books, it is denoted as $\frac{\delta b}{\delta t}$ to differentiate it from $\dot{b} = \frac{\mathrm{d}b}{\mathrm{d}t}$.

For $b = p_{C/A}$ in (2.40),

$$\dot{b}_{/A} = \dot{p}_{C/A} = v_{C/A} = R_A{}^A\dot{p}_C.$$

The relations presented in (2.45) and (2.46) play a very important role in studying the kinematics and dynamics of rigid bodies.

2.4 Acceleration

Velocity is the rate of change of position or orientation with time. Likewise, acceleration is the measurement of the rate of change of velocity with time. There are two types of acceleration, linear acceleration corresponding to linear velocity and angular acceleration corresponding to angular velocity.

Take frame $\{A\}$ in Fig. 2.10 as an example. Its linear and angular accelerations, represented by a_A and α_A, respectively, are the time derivatives of its linear and angular velocities, respectively:

$$a_A = \dot{v}_A,$$

$$\alpha_A = \dot{\omega}_A.$$

Consider

$$\omega_{B/A} = R_A{}^A\omega_B.$$

Differentiating it with respect to time t, we have

$$\begin{aligned}\dot{\omega}_{B/A} &= \dot{R}_A {}^A\omega_B + R_A {}^A\dot{\omega}_B \\ &= \omega_A \times (R_A {}^A\omega_B) + R_A {}^A\alpha_B \\ &= \omega_A \times \omega_{B/A} + \alpha_{B/A}, \end{aligned} \tag{2.47}$$

where

$$\begin{aligned} {}^A\alpha_B &\stackrel{\Delta}{=} {}^A\dot{\omega}_B, \\ \alpha_{B/A} &\stackrel{\Delta}{=} R_A {}^A\alpha_B. \end{aligned}$$

Differentiating (2.37) with respect to time t and considering (2.47), we have the angular acceleration of frame $\{B\}$:

$$\alpha_B = \dot{\omega}_B = \alpha_A + \omega_A \times \omega_{B/A} + \alpha_{B/A}.$$

Next let us examine the acceleration of point C with respect to $\{U\}$ as shown in Fig. 2.10. From the definition, its acceleration is obtained by differentiating v_C with respect to time t:

$$a_C = \dot{v}_C.$$

v_C is given in (2.40), which is reproduced below for completeness:

$$v_C = v_A + R_A {}^Av_C + \omega_A \times (R_A {}^Ap_C).$$

Taking the time derivative of v_C, we have

$$a_C = a_A + R_A {}^A\dot{v}_C + \dot{R}_A {}^Av_C + \dot{\omega}_A \times (R_A {}^Ap_C) + \omega_A \times (R_A {}^A\dot{p}_C + \dot{R}_A {}^Ap_C). \tag{2.48}$$

Defining

$${}^Aa_C \stackrel{\Delta}{=} {}^A\dot{v}_C$$

as the acceleration of point C relative to $\{A\}$, and noting that

$$\begin{aligned} \dot{R}_A {}^Av_C &= \Omega_A R_A {}^Av_C = \omega_A \times (R_A {}^Av_C), \\ \dot{R}_A {}^Ap_C &= \Omega_A R_A {}^Ap_C = \omega_A \times (R_A {}^Ap_C), \\ {}^A\dot{p}_C &= {}^Av_C, \end{aligned}$$

(2.48) can be rewritten as

$$\begin{aligned} a_C &= a_A + R_A {}^Aa_C + 2\omega_A \times (R_A {}^Av_C) + \alpha_A \times (R_A {}^Ap_C) + \omega_A \times (\omega_A \times (R_A {}^Ap_C)) \\ &= a_A + a_{C/A} + 2\omega_A \times v_{C/A} + \alpha_A \times p_{C/A} + \omega_A \times (\omega_A \times p_{C/A}). \end{aligned} \tag{2.49}$$

This is a comprehensive expression of the acceleration of a point that consists of the following parts:

- a_A: acceleration of frame $\{A\}$;
- $a_{C/A} = R_A\, {}^A a_C$: acceleration of C with respect to $\{A\}$ but described in universe frame $\{U\}$;
- $2\omega_A \times v_{C/A}$: *Coriolis acceleration* reflecting the effect of interaction between the linear motion of point C and the rotation of frame $\{A\}$;
- $\alpha_A \times p_{C/A}$: acceleration resulting from the angular acceleration of frame $\{A\}$; it corresponds to the *tangent acceleration* of a particle in circular motion;
- $\omega_A \times (\omega_A \times p_{C/A})$: acceleration related to the angular velocity of frame $\{A\}$; it corresponds to the *centripetal acceleration* of a particle in circular motion.

If C is the origin of a frame, then (2.49) can also be used to calculate the linear acceleration of the frame.

Up to now, we have covered all the essential concepts of position, velocity, and acceleration of a body frame and a point. As a summary, our steps taken for introducing these concepts are listed below.

Step 1: Set up coordinate frames for observing and describing the position and orientation of the body and the position of the point (Sects. 2.1 and 2.2.1).
Step 2: Obtain the position of a point, or position and orientation of a body, as functions of time and system parameters (Sect. 2.2).
Step 3: Determine the linear velocity from the differentiation of position with respect to time (Sect. 2.3).
Step 4: Determine the angular velocity from the product of the differentiation of the rotational matrix with respect to time and its transpose (Sect. 2.3).
Step 5: Determine the linear acceleration and angular acceleration differentiating the linear and angular velocities, respectively (Sect. 2.4).

The above steps are based on the first principle of differential relations among position, velocity, and acceleration and can be used to analyze the kinematics of rigid bodies. Alternatively, the formulas developed in the above sections can be readily used. They are grouped below for easy reference.

- For point C in frame $\{A\}$:

$$p_C = p_{O_A} + p_{C/A}, \tag{2.50}$$

$$v_C = v_A + v_{C/A} + \omega_A \times p_{C/A}, \tag{2.51}$$

$$a_C = a_A + a_{C/A} + 2\omega_A \times v_{C/A} + \alpha_A \times p_{C/A} + \omega_A \times (\omega_A \times p_{C/A}). \tag{2.52}$$

- For frame $\{B\}$ relative to frame $\{A\}$:
  - Angular motion:

$$\dot{R}_A = \Omega_A R_A, \tag{2.53}$$

$$\Omega_A = \dot{R}_A R_A^{\mathrm{T}}, \tag{2.54}$$

$$\Omega_A = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \tag{2.55}$$

$$\omega_A = [\omega_x \ \omega_y \ \omega_z]^{\mathrm{T}}, \tag{2.56}$$

$$R_B = R_A \, {}^A_B R, \tag{2.57}$$

$$\omega_{B/A} = R_A \, {}^A\omega_B, \tag{2.58}$$

$$\omega_B = \omega_A + \omega_{B/A}, \tag{2.59}$$

$${}^A\alpha_B = {}^A\dot{\omega}_B, \tag{2.60}$$

$$\alpha_{B/A} = R_A \, {}^A\alpha_B, \tag{2.61}$$

$$\dot{\omega}_{B/A} = \omega_A \times \omega_{B/A} + \alpha_{B/A}, \tag{2.62}$$

$$\alpha_B = \dot{\omega}_B = \alpha_A + \omega_A \times \omega_{B/A} + \alpha_{B/A}. \tag{2.63}$$

  – Linear motion: This is also governed by (2.50)–(2.52), replacing point C with O_B.

- For vector b in frame $\{A\}$:
  – When b is fixed with respect to $\{A\}$:

$$\dot{b} = \omega_A \times b. \tag{2.64}$$

  – When b changes with respect to $\{A\}$:

$$\dot{b} = \dot{b}_{/A} + \omega_A \times b, \tag{2.65}$$

$$\dot{b}_{/A} = R_A {}^A\dot{b}. \tag{2.66}$$

Remark.

- The vector notations introduced in Sect. 2.2.1 are used for kinematic analysis in this chapter. For example, $p_{C/A}$ stands for the position of C observed in $\{A\}$ but described in $\{U\}$. It is related to the position of C observed and described in $\{A\}$ through the following transformation:

$$p_{C/A} = R_A \, {}^A p_C,$$

  where R_A is the rotation matrix of $\{A\}$ with respect to $\{U\}$. This rule is also applicable for other vectors like $v_{C/A}$, $a_{C/A}$, $\omega_{B/A}$, and $\alpha_{B/A}$.
- The terms in these formulas are related to physical properties like rotational axes and basic kinematic variables (relative position, velocities, and accelerations, ${}^A p_C$, ${}^A v_C$, ${}^A a_C$, ${}^A\omega_B$, ${}^A\alpha_B$, etc.), which in many applications can be derived from analyzing simple linear or circular motions.

Special Cases

- *Circular motion*
  Referring to Fig. 2.2, a cylindrical coordinate frame is set up to describe circular motions. The position of O_A is

$$p_{O_A} = r\hat{e}_r + z\hat{e}_z,$$

where

$$\begin{aligned}
\hat{e}_r &= \cos\phi\ \hat{i} + \sin\phi\ \hat{j}, \\
\hat{e}_z &= \hat{k}, \\
\hat{e}_\phi &= \hat{e}_z \times \hat{e}_r = -\sin\phi\ \hat{i} + \cos\phi\ \hat{j}
\end{aligned} \tag{2.67}$$

are the axes of the cylindrical frame attached to point O_A.

The following relations between $\dot{\hat{e}}_r$ and $\dot{\hat{e}}_\phi$ can be derived directly from the above equations:

$$\dot{\hat{e}}_r = -\dot{\phi}\sin\phi\ \hat{i} + \dot{\phi}\cos\phi\ \hat{j} = \dot{\phi}\hat{e}_\phi, \tag{2.68}$$

$$\dot{\hat{e}}_\phi = -\dot{\phi}\cos\phi\ \hat{i} - \dot{\phi}\sin\phi\ \hat{j} = -\dot{\phi}\hat{e}_r. \tag{2.69}$$

So the derivative of $\hat{e}_r$ is a vector aligning with $\hat{e}_\phi$, and vice versa.

Differentiating p_{O_A} with respect to time t and considering relations in (2.68) and (2.69), we have the velocity and acceleration of O_A:

$$\begin{aligned}
v_A &= \dot{r}\hat{e}_r + r\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z, \\
a_A &= (\ddot{r} - r\dot{\phi}^2)\hat{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z.
\end{aligned} \tag{2.70}$$

In a circular motion, r is constant and $\dot{r} = 0$, and the above equation is simplified to

$$v_A = r\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z, \tag{2.71}$$

$$a_A = -r\dot{\phi}^2\hat{e}_r + r\ddot{\phi}\hat{e}_\phi + \ddot{z}\hat{e}_z. \tag{2.72}$$

Velocity v_A contains two components: *tangent velocity* ($r\dot{\phi}\hat{e}_\phi$) and velocity along $\hat{e}_z$ ($\dot{z}\hat{e}_z$).

Acceleration a_A contains three components: *tangent acceleration* ($r\ddot{\phi}\hat{e}_\phi$), *normal or centripetal acceleration* ($-r\dot{\phi}^2\hat{e}_r$), and acceleration along $\hat{e}_z$ ($\ddot{z}\hat{e}_z$). The unit tangent and normal vectors correspond to $\hat{e}_r$ and $\hat{e}_\phi$, respectively. Obviously (2.70) and (2.72) are similar to (2.52) in terms of format.

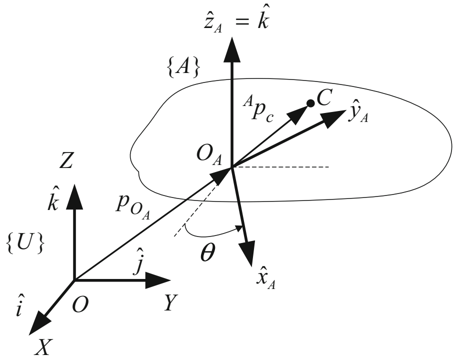


Fig. 2.15 A rigid body in planar motion

Since $\hat{e}_z$ is the fixed rotational axis of frame A,

$$\omega_A = \dot{\phi}\hat{e}_z,$$
$$\alpha_A = \ddot{\phi}\hat{e}_z.$$

- *Planar motion*
  In many engineering applications, a rigid body's motion is constrained in a two-dimensional plane. The body's rotation axis is fixed and is perpendicular to the plane. Its angular velocity and angular acceleration are along the rotation axis, and position, linear velocity, and linear acceleration lie on the same plane.
  Referring to Fig. 2.15, axis $\hat{z}_A$ of the body frame is chosen to be in the same direction of axis Z of the universe frame, and plane $\hat{x}_A\hat{y}_A$ is parallel to plane XY. The angle, θ, is the angular displacement of axis X (Y) around the $\hat{z}_A$ $(\hat{k})$ axis to make itself align with $\hat{x}_A$ $(\hat{y}_A)$. The rotation matrix of the body is

$$R_A = R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

  The angular velocity and the angular acceleration of the body are

$$\omega_A = [0\ 0\ \dot{\theta}]^{\mathrm{T}} = \dot{\theta}\hat{k},$$
$$\alpha_A = \dot{\omega}_A = [0\ 0\ \ddot{\theta}]^{\mathrm{T}} = \ddot{\theta}\hat{k}.$$

- *Motion around a fixed point*
  In this case, the origin of the body frame is chosen to be the fixed point, $v_A = 0$ and $a_A = 0$.

In what follows, several examples will be provided to apply the kinematic analysis methods described in this section.

Example 2.5. Referring to Fig. 2.8, the rod rotates about axis $\hat{z}_A$ (fixed) with angular speed ω_1, which increases at rate $\dot{\omega}_1$, and the box rotates around axis $\hat{x}_B$ at constant speed ω_2. Our task is to find

- The linear and angular velocities and linear and angular accelerations of the box and
- The velocity and acceleration of point E

with respect to universe frame $\{U\}$.

Here the rigid body of concern is $\{B\}$ and the point of concern is E. The first step is to derive the descriptions of the orientation of $\{B\}$ and the position of E with respect to frame $\{U\}$. This was completed in Examples 2.2.3 and 2.2.4. The relevant terms were derived there and are reproduced below for completeness:

- Rotation matrix of $\{B\}$ with respect to $\{U\}$

$$R_B = \begin{bmatrix} \cos\theta & -\sin\theta\cos\beta & \sin\theta\sin\beta \\ \sin\theta & \cos\theta\cos\beta & -\cos\theta\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}; \tag{2.73}$$

- Position of $\{B\}$ with respect to $\{U\}$:

$$p_{O_B} = [-b\sin\theta \; b\cos\theta \; a]^{\mathrm{T}}; \tag{2.74}$$

- Position of frame $\{B\}$ with respect to $\{A\}$:

$${}^{A}p_{O_B} = [0 \; b \; a]^{\mathrm{T}}; \tag{2.75}$$

- Position of $\{A\}$ with respect to $\{U\}$:

$$p_{O_A} = [0 \; 0 \; 0]^{\mathrm{T}}; \tag{2.76}$$

- Rotation matrix of frame $\{A\}$ with respect to $\{U\}$:

$$R_A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{2.77}$$

- Rotation matrix of frame $\{B\}$ with respect to $\{A\}$:

$${}^{A}_{B}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}; \tag{2.78}$$

- Position of E with respect to $\{B\}$:

$${}^{B}p_{E} = [0 \; -c \; d]^{\mathrm{T}}; \tag{2.79}$$

- Position of E with respect to $\{U\}$:

$$p_E = \begin{bmatrix} -(b + c\cos\beta - d\sin\beta)\sin\theta \\ (b - c\cos\beta - d\sin\beta)\cos\theta \\ a - c\sin\beta + d\cos\beta \end{bmatrix}. \tag{2.80}$$

From the given conditions,

$$\omega_1 = \dot{\theta}, \tag{2.81}$$

$$\omega_2 = \dot{\beta}. \tag{2.82}$$

The linear velocity of frame $\{B\}$ (box) is derived by differentiating p_{O_B} (2.74) with respect to time:

$$v_B = \dot{p}_{O_B} = -b\dot{\theta}[\cos\theta \ \ \sin\theta \ \ 0]^{\mathrm{T}} = -b\omega_1[\cos\theta \ \ \sin\theta \ \ 0]^{\mathrm{T}},$$

and the linear acceleration of the box is

$$\begin{aligned} a_B = \dot{v}_B &= -b\dot{\omega}_1[\cos\theta \ \ \sin\theta \ \ 0]^{\mathrm{T}} - b\omega_1[-\dot{\theta}\sin\theta \ \ \dot{\theta}\cos\theta \ \ 0]^{\mathrm{T}} \\ &= -b\dot{\omega}_1[\cos\theta \ \ \sin\theta \ \ 0]^{\mathrm{T}} - b\omega_1[-\omega_1\sin\theta \ \ \omega_1\cos\theta \ \ 0]^{\mathrm{T}} \\ &= b[-\dot{\omega}_1\cos\theta + \omega_1^2\sin\theta \ \ -\dot{\omega}_1\sin\theta - \omega_1^2\cos\theta \ \ 0]^{\mathrm{T}}. \end{aligned}$$

First examine the angular velocities and acceleration of $\{B\}$. From the definition of angular velocity,

$$\Omega_B = [\omega_B\times] = \dot{R}_B R_B^{\mathrm{T}}, \tag{2.83}$$

where R_B is given in (2.73). Differentiating R_B with respect to time and noting $\dot{\theta} = \omega_1$ and $\dot{\beta} = \omega_2$, we have

$$\dot{R}_B = \begin{bmatrix} -\omega_1\sin\theta & -\omega_1\cos\theta\cos\beta + \omega_2\sin\theta\sin\beta & \omega_1\cos\theta\sin\beta + \omega_2\sin\theta\cos\beta \\ \omega_1\cos\theta & -\omega_1\sin\theta\cos\beta - \omega_2\cos\theta\sin\beta & \omega_1\sin\theta\sin\beta - \omega_2\cos\theta\cos\beta \\ 0 & \omega_2\cos\beta & -\omega_2\sin\beta \end{bmatrix}.$$

Note (2.81) and (2.82) are used in the above derivation.

Substituting the above into (2.83), we have

$$\Omega_B = [\omega_B\times] = \begin{bmatrix} 0 & -\omega_1 & \omega_2\sin\theta \\ \omega_1 & 0 & -\omega_2\cos\theta \\ -\omega_2\sin\theta & \omega_2\cos\theta & 0 \end{bmatrix}.$$

From the definition (2.31) of angular velocity,

$$\omega_B = [\omega_2\cos\theta \ \ \omega_2\sin\theta \ \ \omega_1]^{\mathrm{T}}. \tag{2.84}$$

Differentiating ω_B and noting that ω_2 is a constant, the angular acceleration of $\{B\}$ is obtained:

$$\begin{aligned}\alpha_B = \dot{\omega}_B &= [-\omega_2\dot{\theta}\sin\theta \;\; \dot{\theta}\omega_2\cos\theta \;\; \dot{\omega}_1]^{\mathrm{T}} \\ &= [-\omega_1\omega_2\sin\theta \;\; \omega_1\omega_2\cos\theta \;\; \dot{\omega}_1]^{\mathrm{T}}.\end{aligned}$$

Next, examine the velocity and acceleration of point E. Its velocity is derived directly by differentiating its position, p_E (2.80), with respect to time:

$$v_E = \dot{p}_E = \begin{bmatrix} -(b + c\cos\beta - d\sin\beta)\cos\theta\omega_1 + (c\sin\beta + d\cos\beta)\sin\theta\omega_2 \\ -(b - c\cos\beta - d\sin\beta)\sin\theta\omega_1 + (c\sin\beta - d\cos\beta)\cos\theta\omega_2 \\ -(d\sin\beta + c\cos\beta)\omega_2 \end{bmatrix}.$$

The acceleration of point E is obtained by differentiating v_E with respect to time:

$$a_E = [a_{E_x} \; a_{E_y} \; a_{E_z}]^{\mathrm{T}},$$

where

$$\begin{aligned} a_{E_x} &= 2\omega_1\omega_2(d\cos\beta + c\sin\beta) + \omega_2^2\sin\theta(c\cos\beta - d\sin\beta) \\ &\quad +(\omega_1^2\sin\theta - \dot{\omega}_1\cos\theta)(b + c\cos\beta - d\sin\beta), \\ a_{E_y} &= 2\omega_1\omega_2\sin\theta(d\cos\beta - c\sin\beta) + \omega_2^2\cos\theta(c\cos\beta + d\sin\beta) \\ &\quad +(\omega_1\cos\theta + \dot{\omega}_1\sin\theta)(c\cos\beta - b + d\sin\beta), \\ a_{E_z} &= -\omega_2^2(d\cos\beta - c\sin\beta). \end{aligned}$$

Note that $\dot{\theta} = \omega_1$, $\dot{\beta} = \omega_2$, and $\dot{\omega}_2 = 0$ are used in the above derivation.

In the above, all the quantities in the task list are derived mainly based on the first principles of differentiation of the expressions for positions or rotation matrices. In what follows, the velocities and accelerations of frame $\{B\}$ will be obtained from studying its relations relative to other bodies in the mechanism, and the formula listed from (2.50) to (2.52) will be used for this purpose.

From the conditions given, the following terms are obtained:

- Angular velocity of the rod ($\{A\}$) with respect to $\{U\}$:

$$\omega_A = [0\;0\;\omega_1]^{\mathrm{T}}; \tag{2.85}$$

- Angular acceleration of the rod ($\{A\}$) with respect to $\{U\}$:

$$\alpha_A = [0\;0\;\dot{\omega}_1]^{\mathrm{T}}; \tag{2.86}$$

- linear velocity of the rod ($\{A\}$) with respect to $\{U\}$:

$$v_A = 0; \tag{2.87}$$

- linear acceleration of the rod ($\{A\}$) with respect to $\{U\}$:

$$a_A = 0; \tag{2.88}$$

- Angular velocity of the box ($\{B\}$) with respect to the rod ($\{A\}$):

$${}^A\omega_B = [\omega_2 \; 0 \; 0]^{\mathrm{T}}; \tag{2.89}$$

- Angular acceleration of the box ($\{B\}$) with respect to the rod ($\{A\}$):

$${}^A\alpha_B = 0. \tag{2.90}$$

As $v_A = 0$ and ${}^Av_B = 0$, then

$$v_B = \omega_A \times (R_A \, {}^Ap_{OB}). \tag{2.91}$$

Substituting (2.85), (2.77), and (2.75) into the above equations, and after some matrix and vector manipulations, we have

$$v_B = -b\omega_1[\cos\theta \;\; \sin\theta \; 0]^{\mathrm{T}}.$$

From (2.52), the linear acceleration of $\{B\}$ is given by

$$\begin{aligned} a_B &= a_A + R_A \, {}^Aa_B + 2\omega_A \times (R_A \, {}^Av_B) + \alpha_A \times (R_A \, {}^Ap_{O_B}) \\ &+ \omega_A \times (\omega_A \times (R_A \, {}^Ap_{O_B})). \end{aligned} \tag{2.92}$$

As $a_A = 0$, ${}^Av_B = 0$, and ${}^Aa_B = 0$,

$$a_B = b[-\dot{\omega}_1 \cos\theta + \omega_1^2 \sin\theta \;\; -\dot{\omega}_1 \sin\theta - \omega_1^2 \cos\theta \;\; 0]^{\mathrm{T}}.$$

Note that O_B is in a circular motion with $\hat{z}$ being the rotational axis and b the radius. v_B and a_B can also be solved using (2.71) and (2.72):

$$v_B = b\omega_1 \hat{e}_\theta, \tag{2.93}$$

$$a_B = b\dot{\omega}_1 \hat{e}_\theta - b\omega_1^2 \hat{e}_r. \tag{2.94}$$

Here $\hat{e}_\theta$ is the unit tangent vector corresponding to angular displacement θ. It is equivalent to $\hat{e}_\phi$ in (2.67):

$$\hat{e}_\theta = -\hat{x}_A = [\cos\theta \;\; \sin\theta \; 0]^{\mathrm{T}},$$

$$\hat{e}_r = \hat{y}_A = [-\sin\theta \;\; \cos\theta \; 0]^{\mathrm{T}}.$$

Substituting them into (2.93) and (2.94), v_B and a_B are derived.

Describe in the universe frame the angular velocities of frame $\{A\}$, and the relative angular velocity of $\{B\}$ with respect to $\{A\}$:

$$\omega_A = \omega_1 \hat{k},$$

$$\omega_{B/A} = R_A \, {}^A\omega_B = \omega_2 \hat{x}_B;$$

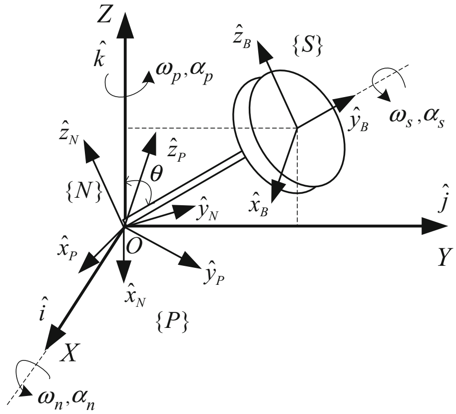


Fig. 2.16 Example 2.6

then ω_B can be obtained through application of (2.59):

$$\omega_B = \omega_A + \omega_{B/A} = \omega_1 \hat{k} + \omega_2 \hat{x}_B = [\omega_2 \cos\theta \ \omega_2 \sin\theta \ \omega_1]^{\mathrm{T}}.$$

The angular acceleration of $\{B\}$ is given by

$$\alpha_B = \alpha_A + \omega_A \times (R_A \, {}^A\omega_B) + R_A \, {}^A\alpha_B. \tag{2.95}$$

Noting that ${}^A\alpha_B = 0$, and substituting them together with other known terms into (2.95), we have

$$\begin{aligned} a_B &= b[-\dot{\omega}_1 \cos\theta + \omega_1^2 \sin\theta \ \ -\dot{\omega}_1 \sin\theta - \omega_1^2 \cos\theta \ 0]^{\mathrm{T}}, \\ \alpha_B &= [-\omega_1\omega_2 \sin\theta \ \omega_1\omega_2 \cos\theta \ \dot{\omega}_1]^{\mathrm{T}}. \end{aligned}$$

Example 2.6. This example is about the motion of a gyrotop (Fig. 2.16), which consists of three rotations called *spin*, *nutation*, and *precession*. At the instant of interest, the spin axis is on the YOZ plane and has an angle θ with respect to the Z axis of the universe frame $\{U\} : OXYZ$. The direction of precession and nutation are along the Z and X axes. The spin axis is chosen as a principal axis ($\hat{y}_B$) of the body frame of the gyrotop. The angular speeds for the three rotations are ω_s, ω_n, and ω_p, respectively, and the magnitudes of their angular accelerations are α_s, α_n, and ω_p, respectively. Our task is to determine the angular velocity and angular acceleration of the gyrotop with respect to the universe frame.

Assume that the rotation of the gyrotop proceeds in the order precession (P), nutation (N), and spin (S). At the start, the body frame of the gyrotop is the universe frame. The frame generated from each rotation can be described with respect to that generated from the previous rotation, and they are given as $R_P = \mathrm{Rot}_Z(\theta_P)$ (for frame $\{P\}$ generated from precession), ${}^P_N R = \mathrm{Rot}_X(\theta_N)$ (for frame $\{N\}$ generated from nutation), and ${}^N_S R = \mathrm{Rot}_Y(\theta_S)$ (for frame $\{S\}$ generated from spin), where

θ_P, θ_N, and θ_S are the angular displacements about the rotational axes $\hat{k}$, $\hat{i}$, and $\hat{y}_B$, respectively. Frame $\{S\}$ is actually the body frame of the gyrotop where the axis $\hat{y}_B$ aligns with the spin axis. The corresponding angular velocities described in the universe frames are denoted by ω_P, $R_P{}^P\omega_N$, and $R_N{}^N\omega_S$, respectively.
From the given conditions:

$$\hat{y}_B = [0 \;\; \sin\theta \;\; \cos\theta]; \tag{2.96}$$

$$\omega_P = [0\; 0\; \omega_p]^{\mathrm{T}}, \text{ angular velocity for precession;} \tag{2.97}$$

$$\dot{\omega}_P = [0\; 0\; \alpha_p]^{\mathrm{T}}, \text{ angular acceleration for precession;} \tag{2.98}$$

$$\omega_{N/P} = R_P\, {}^P\omega_N = [\omega_n \;\; 0 \; 0]^{\mathrm{T}}, \text{ angular velocity for nutation;} \tag{2.99}$$

$$\dot{\omega}_{N/P} = R_P\, {}^P\dot{\omega}_N = [\alpha_n \;\; 0 \; 0]^{\mathrm{T}}, \text{ angular acceleration for nutation;} \tag{2.100}$$

$$\omega_{S/N} = R_N{}^N\omega_S = \omega_s\,\hat{y}_B, \text{ angular velocity for spin;} \tag{2.101}$$

$$\dot{\omega}_{S/N} = R_N\, {}^N\dot{\omega}_S = \alpha_s\,\hat{y}_B, \text{ angular acceleration for spin.} \tag{2.102}$$

From (2.36) and (2.39) while considering (2.13) and (2.15):

$$\omega_S = \omega_P + \omega_{S/P} = \omega_P + R_P{}^P\omega_S, \tag{2.103}$$

$${}^P\omega_S = {}^P\omega_N + {}^P\omega_{S/N} = {}^P\omega_N + {}^P_N R\, {}^N\omega_S. \tag{2.104}$$

Substituting (2.104) into (2.103), we have

$$\omega_S = \omega_P + \omega_{N/P} + \omega_{S/N}. \tag{2.105}$$

Using the terms in (2.97), (2.99), and (2.101), we have

$$\omega_S = [\omega_n \;\; \omega_s \sin\theta \;\; \omega_p + \omega_s \cos\theta]^{\mathrm{T}},$$

which is the angular velocity of the gyrotop with respect to the universe frame.

Differentiating (2.103) with respect to time t gives the angular acceleration of the gyrotop:

$$\alpha_S = \dot{\omega}_P + \dot{\omega}_{S/P} + \omega_P \times \omega_{S/P}.$$

Substituting (2.104) into the above equation and noting that $\omega_{S/P} = R_P{}^P\omega_S$,

$$\alpha_S = \dot{\omega}_P + \dot{\omega}_{S/P} + \omega_P \times (\omega_{N/P} + \omega_{S/N}). \tag{2.106}$$

All the terms except for $\dot{\omega}_{S/P}$ on the right-hand side of the above equation are known. Differentiating (2.104) with respect to time t and premultiplying it by R_P, we have

$$\begin{aligned}\dot{\omega}_{S/P} &= R_P{}^P\dot{\omega}_S = R_P({}^P\dot{\omega}_N + {}^P_N R\, {}^N\dot{\omega}_S + {}^P\omega_N \times ({}^P_N R\, {}^N\omega_S)) \\ &= \dot{\omega}_{N/P} + \dot{\omega}_{S/N} + R_P({}^P\omega_N \times ({}^P_N R\, {}^N\omega_S)).\end{aligned} \tag{2.107}$$

According to (1.16),

$$R_P({}^P\omega_N \times ({}^P_N R\ {}^N\omega_S)) = (R_P\ {}^P\omega_N) \times (R_P\ {}^P_N R\ {}^N\omega_S) = (R_P\ {}^P\omega_N) \times (R_N\ {}^N\omega_S)$$
$$= \omega_{N/P} \times \omega_{S/N},$$

and (2.107) becomes

$$\dot{\omega}_{S/P} = \dot{\omega}_{N/P} + \dot{\omega}_{S/N} + \omega_{N/P} \times \omega_{S/N}.$$

Substituting it into (2.106), we have

$$\alpha_S = \dot{\omega}_P + \dot{\omega}_{N/P} + \dot{\omega}_{S/N} + \omega_{N/P} \times \omega_{S/N} + \omega_P \times (\omega_{N/P} + \omega_{S/N}). \tag{2.108}$$

Inserting the values listed from (2.97) to (2.102) into the above equation, we have the angular acceleration of the gyrotop with respect to the universe frame:

$$\alpha_S = [\alpha_{Sx}\ \alpha_{Sy}\ \alpha_{Sz}]^{\mathrm{T}},$$

where

$$\alpha_{Sx} = \alpha_n + \alpha_s - \omega_p \omega_s \sin\theta,$$
$$\alpha_{Sy} = \omega_p(\omega_n + \omega_s) + \alpha_s \sin\theta - \omega_n \omega_s \cos\theta,$$
$$\alpha_{Sz} = \alpha_p + \alpha_s \cos\theta + \omega_n \omega_s \sin\theta.$$

Remark.

- Our approach for the above velocity and acceleration analysis is based mainly on the manipulations of vectors and rotation matrices related to rigid-body motions. They are denoted with the notations defined in Sect. 2.2.1. The advantage of this method is that once the coordinate frames are set up, kinematic analysis can be done systematically either through the existing procedure and formulas or the first principle of relating the derivative of a rotation matrix to the angular velocity and acceleration of the body.
- Equations (2.105) and (2.108) in this example are the kinematic equations governing the motion of a generic gyrotop.

Example 2.7. A slider crank mechanism is schematically shown in Fig. 2.17. Given that the crank (Link 1) is at angle φ with angular speed ω, determine what its angular acceleration should be in order for the piston (slider) to accelerate to the left at α_x. The dimensions of the mechanism are shown in the figure.

Set up the universe coordinate frame $\{U\} : OXYZ$, where axes X and Y form the plane for the motion of the mechanism and axis Z is perpendicular to the plane according to the right-hand rule. The XY plane is on the paper, and thus axis Z (denoted as a black dot) points out of the paper. From the given conditions, the angular velocity of Link 1 is

$$\omega_1 = \omega\hat{k}. \tag{2.109}$$

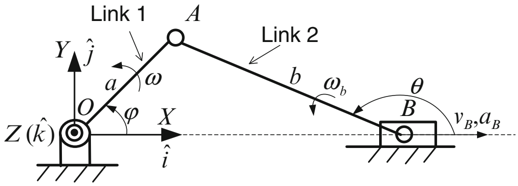


Fig. 2.17 Example 2.7

The position of joint A is

$$p_A = a\cos\varphi\hat{i} + a\sin\varphi\hat{j}. \tag{2.110}$$

Link 1 is in a circular motion, so the velocity of A is

$$v_A = \omega_1 \times p_A = \omega\hat{k} \times (a\cos\varphi\hat{i} + a\sin\varphi\hat{j}) = a\omega(\cos\varphi\hat{j} - \sin\varphi\hat{i}).$$

The velocity of joint B is

$$v_B = v_A + \omega_2 \times p_{B/A}, \tag{2.111}$$

where $\omega_2 = \omega_b\hat{k}$ is the angular velocity of Link 2 and $p_{B/A}$ is the position vector from A to B.

From the figure,

$$p_{B/A} = b\cos\theta\hat{i} + b\sin\theta\hat{j}, \tag{2.112}$$

$$p_B = p_A + p_{B/A} = (a\cos\varphi + b\cos\theta)\hat{i} + (a\sin\varphi + b\sin\theta)\hat{j}.$$

Substituting (2.112) into (2.111), we have

$$v_B = -(a\omega_1\sin\varphi + b\omega_b\sin\theta)\hat{i} + (a\omega\cos\varphi + b\omega_b\cos\varphi)\hat{j}.$$

Next we will solve ω_b and θ from the available conditions.

Considering that the slider can only travel along the X axis, the Y elements of both v_B and p_B are zero:

$$a\sin\varphi + b\sin\theta = 0,$$

$$a\omega\cos\varphi + b\omega_b\cos\varphi = 0.$$

Solving for θ and ω_b from the above equations, we have

$$\theta = \arcsin\left(-\frac{a\sin\varphi}{b}\right), \tag{2.113}$$

$$\omega_b = -\frac{a\cos\varphi}{b\cos\theta}\omega. \tag{2.114}$$

Now we are ready to perform acceleration analysis. Assume the acceleration of Link 1 is

$$\alpha_1 = \alpha\hat{k}. \tag{2.115}$$

As Link 1 is in a circular motion around axis Z, the acceleration of A is

$$\alpha_A = \alpha_1 \times p_A + \omega_1 \times (\omega_1 \times p_A).$$

Substituting expressions for p_A (2.110), ω_1 (2.109), and α_1 (2.115) into the above equation, we have

$$\alpha_A = -a(\alpha \sin\varphi + \omega^2 \cos\varphi)\hat{i} + a(\alpha \cos\varphi - \omega^2 \sin\varphi)\hat{j}. \tag{2.116}$$

The acceleration of the slider, a_B, is related to the acceleration of A (a_A) through

$$a_B = a_A + a_{B/A}, \tag{2.117}$$

where

$$a_{B/A} = \alpha_2 \times p_{B/A} + \omega_2 \times (\omega_2 \times p_{B/A})$$

is the acceleration of B with respect to the body frame at A. Considering that $\alpha_2 = \alpha_b \hat{k}$ is the angular acceleration of Link 2, and substituting expressions of ω_2 and $p_{B/A}$ into the above equation, we have

$$a_{B/A} = -b(\alpha_b \sin\theta + \omega_b^2 \cos\theta)\hat{i} + b(\alpha_b \cos\theta - \omega_b^2 \sin\theta)\hat{j}. \tag{2.118}$$

Then from (2.116) to (2.118) we have

$$\begin{aligned} a_B = &-(a\alpha \sin\varphi + a\omega^2 \cos\varphi + b\alpha_b \sin\theta + b\omega_b^2 \cos\theta)\hat{i} \\ &+ (a\alpha \cos\varphi - a\omega^2 \sin\varphi + b\alpha_b \cos\theta - b\omega_b^2 \sin\theta)\hat{j}, \end{aligned}$$

where, except for α and α_b, all the terms on the right-hand side of the equation are known.

From $a_B = -\alpha_x \hat{i}$, it follows that

$$\begin{aligned} a\alpha \sin\varphi + a\omega^2 \cos\varphi + b\alpha_b \sin\theta + b\omega_b^2 \cos\theta &= \alpha_x, \\ a\alpha \cos\varphi - a\omega^2 \sin\varphi + b\alpha_b \cos\theta - b\omega_b^2 \sin\theta &= 0, \end{aligned}$$

and

$$\begin{aligned} \alpha &= \frac{p_2 \sin\theta - p_1 \cos\theta}{a \sin(\theta + \varphi)}, \\ p_1 &= \alpha_x - a\omega^2 \cos\varphi - b\omega_b^2 \cos\theta, \\ p_2 &= a\omega^2 \sin\varphi + b\omega_b^2 \sin\theta, \end{aligned}$$

where θ and ω_b are given in (2.113) and (2.114), respectively. This is the angular acceleration needed for the crank (Link 1) to make the slider accelerate at a_x to the left.

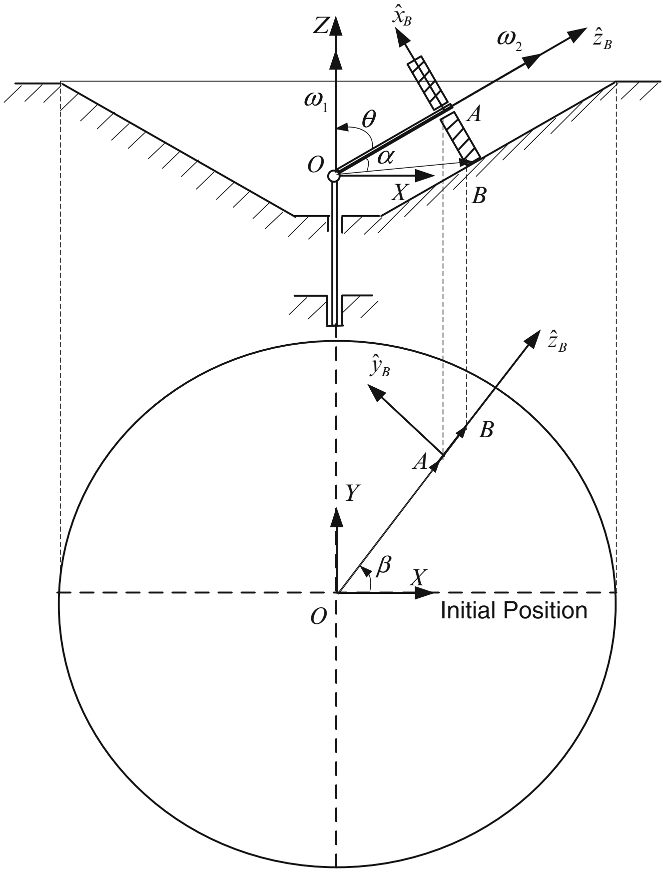


Fig. 2.18 Example 2.8

Example 2.8. As shown in Fig. 2.18, axis OA is driven by a motor (not shown) to rotate at a *constant* angular speed ω_1 about vertical axis Z. The wheel of radius r rotates around axis OA and travels along the cone surface parallel to axis OA. Point B is the contact point between the wheel and the surface. $\|\overline{OA}\| = l$, the angle between axes $\overline{OA}$ and Z, is θ, and $\overline{OB}$ formsan angle α with $\overline{OA}$. The plane formed by $\overline{OA}$ and $\overline{OB}$ has an angle displacement β from its initial position. Find the wheel's angular velocity and its angular speed about axis $\overline{OA}$.

To describe the motion of the system, set up universe frame $OXYZ$ at joint O and body frame $A\hat{x}_B\hat{y}_B\hat{z}_B$ at point A shown in the figure.

From the assumption, the angular velocity of $\overline{OA}$ around the Z axis is

$$\omega = \omega_1\hat{k}.$$

The velocity of point A is thus

$$v_A = \omega_1 \hat{k} \times \overline{OA}.$$

Assume the angular velocity of the wheel is ω_w; then the velocity of point A can also be expressed as

$$v_A = v_B + \omega_w \times \overline{BA},$$

where v_B is the velocity of contact point B between the wheel and the surface. Since normally there is no slippage between the wheel and the surface, $v_B = 0$, and thus

$$v_A = \omega_w \times \overline{BA}$$

and

$$\omega_1 \hat{k} \times \overline{OA} = \omega_w \times \overline{BA}. \tag{2.119}$$

Let the angular speed of the wheel about axis OA be ω_2. From (2.59) we have

$$\omega_w = \omega_1 \hat{k} + \omega_2 \hat{z}_B = \omega_1 \hat{k} + \omega_2 \frac{\overline{OA}}{l}. \tag{2.120}$$

Note that $\hat{z}_B = \frac{\overline{OA}}{\|\overline{OA}\|} = \frac{\overline{OA}}{l}$ is used in the above equation.

Substituting the above into (2.119), we have

$$\omega_1 \hat{k} \times (\overline{OA} - \overline{BA}) = \omega_1 \hat{k} \times \overline{OB} = \omega_2 \times \left(\frac{\overline{OA} \times \overline{BA}}{l} \right).$$

Taking the norm of each side of the above equation, and considering that $\overline{OA}$ is perpendicular to $\overline{BA}$, $\|\overline{OA}\| = l$, and $\|BA\| = r$, and the angle between axis Z and $\overline{OB}$ is $\alpha + \beta$, we have

$$\omega_1 \|\overline{OB}\| \sin(\alpha + \theta) = \omega_2 r.$$

But $\|\overline{OB}\| \sin \alpha = r$, so from the above equation we have

$$\omega_2 = \frac{\sin(\alpha + \theta)}{\sin \alpha} \omega_1, \quad \alpha \neq 0, \ \pi. \tag{2.121}$$

Note that α cannot be zero or π when the wheel has a nonzero radius.

Then from (2.120) to (2.121) we have the angular velocity of the wheel:

$$\omega_w = \omega_1 \left(\hat{k} + \frac{\sin(\alpha + \theta)}{\sin \alpha} \hat{z}_B \right).$$

Consider that

$$\hat{z}_B = \sin\theta\cos\beta\hat{i} + \sin\theta\sin\beta\,\hat{j} + \cos\theta\hat{k};$$

then

$$\omega_w = \omega_{w_x}\hat{i} + \omega_{w_y}\hat{j} + \omega_{w_z}\hat{k},$$

where

$$\begin{aligned}
\omega_{w_x} &= \frac{\sin(\alpha+\theta)\sin\theta\cos\beta}{\sin\alpha}\omega_1, \\
\omega_{w_y} &= \frac{\sin(\alpha+\theta)\sin\theta\sin\beta}{\sin\alpha}\omega_1, \\
\omega_{w_z} &= (1+\cos\theta)\omega_1.
\end{aligned}$$

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