Chapter 2
The Simplest Schemes of Generalized Solution of Linear Operator Equation

Let E, F be Banach spaces and $\mathscr{L}$ be a linear operator with an everywhere dense domain $D(\mathscr{L}) \subset E$, which acts from E into F. Let us consider an operator equation

$$\mathscr{L}u = f, \qquad u \in D(\mathscr{L}), f \in F \tag{2.1}$$

and an adjoint equation

$$\mathscr{L}^* \varphi = l, \qquad \varphi \in D(\mathscr{L}^*), l \in E^*, \tag{2.2}$$

where E^* and F^* are conjugate Banach spaces to E and F, respectively, $\mathscr{L}^*$ is an adjoint operator to $\mathscr{L}$. Suppose that the range $R(\mathscr{L}) \subset F$ of $\mathscr{L}$ is an everywhere dense set in F and (2.1) is uniquely solvable over $R(\mathscr{L})$, i.e. the null space $\mathrm{Ker}(\mathscr{L})$ of $\mathscr{L}$ consists only of the zero element θ : $\mathrm{Ker}(\mathscr{L}) = \theta$. Thus, $\mathscr{L}$ sets a one-to-one mapping between $D(\mathscr{L})$ and $R(\mathscr{L})$. Note that the continuity of $\mathscr{L}$ is not supposed.

The aim of this chapter is to give a "meaningful" definition of the solution of (2.1) when $f \notin R(\mathscr{L})$.

2.1 Strong Generalized Solution

Let us introduce one more norm on the linear set $D(\mathscr{L})$ in the space E. Since $\mathscr{L} : E \to F$ is a linear injective operator with a domain $D(\mathscr{L}) \subset E$, the function

$$D(\mathscr{L}) \ni u \to \|\mathscr{L}u\|_F \in \mathbb{R}$$

has all properties of a norm on $D(\mathscr{L})$. Hence, $D(\mathscr{L})$ with this norm turns into a normed space, which may be incomplete. Let $\bar{E}$ be a completion of this normed space.

The fact that $\|u\|_{\bar{E}} = \|\mathscr{L}u\|_F$ for all $u \in D(\mathscr{L})$ allows to extend $\mathscr{L}$ from $D(\mathscr{L})$ onto $\bar{E}$. Indeed, if u is an arbitrary element from $\bar{E}$, then the density of $D(\mathscr{L})$ in $\bar{E}$ implies that there is such a sequence $u_i \in D(\mathscr{L})$ that $u_i \to u$ in $\bar{E}$ as $i \to \infty$.

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Since u_i is a convergent sequence in $\bar{E}$ and hence it is a Cauchy sequence in $\bar{E}$, and $\|u_i - u_j\|_{\bar{E}} = \|\mathscr{L}u_i - \mathscr{L}u_j\|_F$ then $\mathscr{L}u_i$ is a Cauchy sequence in F. However, F is a complete normed space. Thus, there is such an element f in F that $\mathscr{L}u_i \to f$ in F as $i \to \infty$. Determine a value of the operator $\mathscr{L}$ on the element u in the following way: $\bar{\mathscr{L}}u = f$, where $\bar{\mathscr{L}} : \bar{E} \to F$ is an extended operator defined on the entire space $\bar{E}$. Note that the value $\bar{\mathscr{L}}u$ is defined correctly, i.e. the element $f = \bar{\mathscr{L}}u \in F$ does not depend on the selection of the sequence u_i. Thus, the operator $\bar{\mathscr{L}} : \bar{E} \to F$, $D(\bar{\mathscr{L}}) = \bar{E}$ is an extension of $\mathscr{L}$ on the whole space $\bar{E}$.

Definition 2.1. *A strong generalized solution* of (2.1) is such an element $u \in \bar{E}$ that equality (2.1) holds for the extended operator $\bar{\mathscr{L}}$.

Remark 2.1. If $E = F$ is a Hilbert space and $\mathscr{L}$ is a symmetric operator then the operator $\bar{\mathscr{L}}$ is called a self-adjoint extension of operator by Friedrichs.

As mentioned before, the concept of a strong generalized solution $\bar{u}$ arises when a right-hand side f of (2.1) does not belong to the range $R(\mathscr{L})$ of $\mathscr{L}$. In this case, the ordinary (classical) solution does not exist. The word “strong” means that the topology of the space $\bar{E}$ is normed.

Let us study some properties of $\bar{\mathscr{L}}$. It follows from the linearity of $\mathscr{L}$ that the operator $\bar{\mathscr{L}}$ is linear also. Let us prove that $\bar{\mathscr{L}}$ is an injective operator. Indeed, if $u \in \bar{E}$ is such an element that $\bar{\mathscr{L}}u = 0$, then selecting a sequence $u_i \in D(\mathscr{L})$ converging to u in $\bar{E}$ as $i \to \infty$ we have that $\mathscr{L}u_i \to \bar{\mathscr{L}}u = 0$ in F as $i \to \infty$. The last statement can be rewritten as $\|\mathscr{L}u_i\|_F \to 0$ or $\|u_i\|_{\bar{E}} \to 0$. Hence, $\|u\|_{\bar{E}} = 0$. Thus, the injectivity of the operator $\bar{\mathscr{L}}$ is proven. In addition, the equality $\|\mathscr{L}u\|_F = \|u\|_{\bar{E}}$, which holds for an arbitrary $u \in D(\mathscr{L})$, clearly holds for all $u \in \bar{E}$ (taking into account the replacement of $\mathscr{L}$ by $\bar{\mathscr{L}}$). From $\|\bar{\mathscr{L}}u\|_F = \|u\|_{\bar{E}}$, where $u \in \bar{E}$, it follows that the operator $\bar{\mathscr{L}}$ is continuous and coercive.

The properties of the operator $\bar{\mathscr{L}}$ can be proven in another way. Indeed, the operator $\mathscr{L}$ is a one-to-one map between $D(\mathscr{L})$ and $R(\mathscr{L})$. In addition, if $(D(\mathscr{L})$ is a normed space with the norm $\|u\|_{\bar{E}}$ and $R(\mathscr{L})$ is a normed space with the norm $\|f\|_F)$ then the completion of $D(\mathscr{L})$ coincides with $\bar{E}$ and the completion $R(\mathscr{L})$ coincides with F (remember that $R(\mathscr{L})$ is a dense subset of F). On the other hand, granting the equality $\|\mathscr{L}u\|_F = \|u\|_{\bar{E}}$, which holds for all $u \in D(\mathscr{L})$, we have that the operator $\mathscr{L}$ is an isometry between the normed spaces $D(\mathscr{L})$ and $R(\mathscr{L})$. Hence, their completions are isometrical. This isometry defines the completion $\bar{\mathscr{L}}$ of the operator $\mathscr{L}$. Thus, the operator $\bar{\mathscr{L}}$ sets an isometry between $\bar{E}$ and F. This implies the above-mentioned properties of $\bar{\mathscr{L}}$. The foregoing implies the following theorem.

Theorem 2.1. *For any $f \in F$ there exists a unique strong generalized solution of* (2.1) *in the sense of Definition 2.1.*

If $f \in R(\mathscr{L})$, then a strong generalized solution $\bar{u}$ turns into a classic solution. It is also clear that the classic solution is strong, and it is classic if $\bar{u} \in D(\mathscr{L})$.

Let us clarify the relations between the spaces E and $\bar{E}$. Since $D(\mathscr{L})$ is a dense linear subset of E (of course, in the sense of the norm of the space E), then the set E may be obtained by completing $D(\mathscr{L})$ with respect to the norm $\|u\|_E$. Thus, the spaces E and $\bar{E}$ may be considered as completions of the same linear set $D(\mathscr{L})$

with respect to the two different norms: $\|u\|_E$ and $\|u\|_{\bar{E}}$. Unfortunately, in general case, elements of the spaces E and $\bar{E}$ are incomparable. It is explained by the fact that, on one hand, the operator $\mathscr{L} : E \to F$ can be unbounded and, from the other hand, it can be non-coercive, even though it is a linear injective operator. This means that in general case the norms $\|u\|_E$ and $\|\mathscr{L}u\|_F = \|u\|_{\bar{E}}$ can induce incomparable topologies on $D(\mathscr{L})$.

When $\mathscr{L} : E \to F$ is a linear continuous operator the case is more simple. Then the topology induced on $D(\mathscr{L})$ by the norm $\|u\|_{\bar{E}}$ is weaker than the topology of the space E.[1]

Consider another possibility. Let an operator $\mathscr{L} : E \to F$ be coercive, i.e. there exists such a constant $c > 0$ that

$$\|u\|_E \le c\|\mathscr{L}u\|_F = c\|u\|_{\bar{E}} \tag{2.3}$$

for all $u \in D(\mathscr{L})$. In this case, the norms $\|u\|_E$ and $\|\mathscr{L}u\|_F = \|u\|_{\bar{E}}$ are comparable over $D(\mathscr{L})$ (the topology of the space $\bar{E}$ is stronger than the topology of the space E on $D(\mathscr{L})$) and there is a relation between elements of E and $\bar{E}$.

Theorem 2.2. *Let $\mathscr{L}$ be a closable coercive operator. Then there exists a dense continuous embedding $\bar{E} \subset E$.*

Proof. Since the spaces $\bar{E}$ and E are the completions of the linear set $D(\mathscr{L})$ with respect to two norms and (2.3) holds, then in order to prove the theorem, it is enough to check the condition:

(π) if $u_i \in D(\mathscr{L})$ and $u_i \to u$ in $\bar{E}$, $u_i \to 0$ in E, then $u = 0$.

However, this condition can be rewritten in the following way:

(π) if $u_i \in D(\mathscr{L})$ and $\mathscr{L}u_i \to f$ in F, $u_i \to 0$ in E, then $f = 0$.

The last condition is clear, since the operator $\mathscr{L}$ is closable. □

Thus, we ascertained that $\bar{E} \subset E$, i.e. an arbitrary strong generalized solution of (2.1) is an element of the space E.

2.2 Strong Near-Solution

Suppose that the right-hand side of (2.1), i.e. the element f, does not belong to the range $R(\mathscr{L})$ of an operator $\mathscr{L}$. Since $R(\mathscr{L})$ is everywhere dense in F and (2.1)

[1] Note that studying of a closable operator $\mathscr{L} : E \to F$ can be reduced (at least theoretically) to studying of a linear continuous operator $\mathscr{L}_1$ defined on the same set $D(\mathscr{L})$, but with respect to another norm. Indeed, introducing in $D(\mathscr{L})$ a graph norm

$$\|u\|_\Gamma = \|u\|_E + \|\mathscr{L}u\|_F,$$

with respect to which the linear set $D(\mathscr{L})$ is Banach, we have that the operator $\mathscr{L}_1 : D(\mathscr{L}) \to F$ is linear and continuous ($\mathscr{L}_1 u = \mathscr{L}u$, $u \in D(\mathscr{L}) = D(\mathscr{L}_1)$).

is uniquely solvable, then there exists a sequence $f_n \in R(\mathscr{L})$ such that $f_n \to f$ as $n \to \infty$, and a sequence $u_n = \mathscr{L}^{-1}(f_n)$ convergent to some element $\bar{u} \in \bar{E}$ in $\bar{E}$.

Definition 2.2. A sequence of elements $u_n \in D(\mathscr{L})$ is called a *strong near-solution* of the operator equation (2.1), if $f_n = \mathscr{L} u_n \to f$ as $n \to \infty$ in the metric of the space F. An element $\bar{u} \in \bar{E}$ is called the *strong limit element* of the near-solution.

The concept of a "near-solution" is justified by the following arguments. In many important practical cases, it is impossible or almost impossible to determine the right-hand side f of (2.1) absolutely exactly; therefore, we have to consider its ε-approximation, i.e. and element $f \in R(\mathscr{L})$ such that $\rho(f, f_\varepsilon) = ||f - f_\varepsilon|| < \varepsilon$. In this case, there exists an element $u_\varepsilon = \mathscr{L}^{-1}(f_\varepsilon)$ from the domain of the operator $\mathscr{L}$, which can be considered as "ε - approximation" of the solution of (2.1), i.e. the right-hand (2.1) is closely approximated by its image $\mathscr{L} u_\varepsilon = f_\varepsilon$. If the elements u_ε "become stabilize" as $\varepsilon \to 0$, i.e. if they converge in some topology (in $\bar{E}$) to the fixed element $\bar{u} \notin D(\mathscr{L})$, then it is naturally to consider the element u_ε as an "ε -solution" or "near-solution". Note that in many cases the "accuracy" of a solution u_ε is defined by the closeness of its image $\mathscr{L} u_\varepsilon = f_\varepsilon$ to the element f, i.e. by a norm of the space $\bar{E}$.

The definitions of a strong generalized solution and a near-solution of (2.1) imply that these concepts are equivalent, i.e. an element $u \in \bar{E}$ is a strong generalized solution of the operator equation (2.1) iff it is a strong limit element of a near-solution.

2.3 Weak Generalized Solution

Let us consider a definition of a generalized solution of an operator equation in a linear topological space with a topology which is not necessarily induced by a norm.

As before, suppose that $\mathscr{L} : E \to F$ is a linear injective operator, which acts between Banach spaces E, F with everywhere dense domain and range in E and F, respectively. In addition, suppose that $D(\mathscr{L}^*)$ is a total subset of F^* in a duality (F, F^*), and $R(\mathscr{L}^*)$ is a total subset of E^* in a duality (E, E^*). Note that the totality property of $R(\mathscr{L}^*)$ may be replaced by one of the following conditions:

(a) The space E is reflexive (if the space F is reflexive also, then the set $D(\mathscr{L}^*)$ is strongly dense in F^*).
(b) The operator $\mathscr{L}$ is continuous, i.e. $D(\mathscr{L}) = E$;

In condition (a), the totality of $R(\mathscr{L}^*)$ follows from [40], and in case (b) it follows from the formulae

$$R(\mathscr{L}^*)^\circ \cap D(\mathscr{L}) = \mathrm{Ker}(\mathscr{L}), \tag{2.4}$$

where $R(\mathscr{L}^*)^\circ \subset E$ is a polar of the set $R(\mathscr{L}^*) \subset E^*$ in a duality (E, E^*). Let us prove (2.4) for an arbitrary linear operator. Since $R(\mathscr{L}^*)$ is a linear set, then

$$\begin{aligned} R(\mathscr{L}^*)^\circ \cap D(\mathscr{L}) &= \{u \in E : u \in D(\mathscr{L}), l(u) = 0, \forall l \in R(\mathscr{L}^*)\} \\ &= \{u \in E : u \in D(\mathscr{L}), \varphi(\mathscr{L} u) = 0, \forall \varphi \in D(\mathscr{L}^*)\} \end{aligned}$$

Since $D(\mathscr{L}^*)$ is a total linear subspace, then

$$R(\mathscr{L}^*)^\circ \cap D(\mathscr{L}) = \{u \in E : u \in D(\mathscr{L}), \mathscr{L}u = 0\} = \mathrm{Ker}(\mathscr{L}).$$

Therefore, formulae (2.4) is proved. From (2.4) it is follows that

$$(R(\mathscr{L}^*)^\circ \cap D(\mathscr{L}))^\circ = (\mathrm{Ker}(\mathscr{L}))^\circ.$$

If $\mathscr{L}$ is a continuous injective operator, then $D(\mathscr{L}) = E$, $\mathrm{Ker}(\mathscr{L}) = \varnothing$. Therefore,

$$(R(\mathscr{L}^*))^{\circ\circ} = (\mathrm{Ker}(\mathscr{L}))^\circ = E^*.$$

So, a bipolar of the set $R(\mathscr{L}^*)$, i.e. a weak closure $R(\mathscr{L}^*)$ coincides with E^*; hence, $R(\mathscr{L}^*)$ is total in E^*.

Finally, we see that the set of functionals $R(\mathscr{L}^*) \subset E^*$ is a total linear manyfold with respect to the duality (E^*, E); the linear subspaces F and $D(\mathscr{L}^*)$ are in duality also.

Denote by $\widetilde{E}$ a completion of a space E with respect to a topology $\sigma(E, R(\mathscr{L}^*))$. Since the sets E and $R(\mathscr{L}^*)$ are in duality, then the space $\widetilde{E}$ is a Hausdorff locally convex topological vector space. Each of the functionals $l \in E^*$ which has the form $l = \mathscr{L}^*\varphi$, where $\varphi \in D(\mathscr{L}^*)$, allows a unique extension by continuity on the whole space $\widetilde{E}$, which we will denote as $\widetilde{l}$. A conjugate space to $\widetilde{E}$ is a space consisting of various functionals $\widetilde{l}$, where $l = \mathscr{L}^*\varphi$, $\varphi \in D(\mathscr{L}^*)$.

Let us consider an arbitrary continuous linear functional $\varphi \in D(\mathscr{L}^*)$. Then (2.1) implies that

$$\varphi(\mathscr{L}u) = \varphi(f), \quad l(u) = (\mathscr{L}^*\varphi)(u) = \varphi(f). \tag{2.5}$$

Definition 2.3. *A weak generalized solution* of the operator equation (2.1) is an element $u \in \widetilde{E}$, which satisfies the relation

$$\widetilde{l}(u) = \varphi(f) \qquad \text{for all } \varphi \in D(\mathscr{L}^*), \tag{2.6}$$

where $l = \mathscr{L}^*\varphi$.

A weak generalized solution $u \in \widetilde{E}$, as a strong generalized solution of (2.1) also arises when the right-hand side of (2.1), i.e. the element f, does not belong to the range $R(\mathscr{L})$ of the operator $\mathscr{L}$ and a classic solution does not exist.

Relations (2.5) imply that any classic solution is a weak solution also. On the other hand, if $f \in R(\mathscr{L})$, then a weak generalized solution u turns into a classic one. Indeed, let $f \in R(\mathscr{L})$ and $u \in \widetilde{E}$ be a weak generalized solution. Therefore, for all $\varphi \in D(\mathscr{L}^*)$ we have

$$\widetilde{l}(u) = \varphi(f), \quad l = \mathscr{L}^*\varphi.$$

Moreover, there exists such an element $u_1 \in D(\mathscr{L})$ that $\mathscr{L}u_1 = f$. This element u_1 is a weak generalized solution, i.e.

$$\widetilde{l}(u_1) = \varphi(f), \quad l = \mathscr{L}^*\varphi$$

for all $\varphi \in D(\mathscr{L}^*)$. Thus, for all $\varphi \in D(\mathscr{L}^*)$ the equality $\widetilde{l}(u_1) = \widetilde{l}(u)$ holds, where $l = \mathscr{L}^*\varphi$. Since the set of all functionals $\widetilde{l}$, where $l = \mathscr{L}^*\varphi$, $\varphi \in D(\mathscr{L}^*)$, coincides with the space $\widetilde{E}^*$, then $u = u_1$ in $\widetilde{E}$, and therefore in E.

Analogously, if a weak generalized solution $\widetilde{u}$ belongs to $D(\mathscr{L})$, then it is a classic solution.

2.4 Weak Near-Solution

Analogous to a strong near-solution let us introduce a weak near-solution.

Definition 2.4. A sequence $u_n \subset D(\mathscr{L})$ is called a *weak near-solution* of the operator equation (2.1) if $f_n = \mathscr{L}u_n \to f$ as $n \to \infty$ with respect to the metric of the space F and $u_n = \mathscr{L}^{-1}(f_n) \to \widetilde{u} \in \widetilde{E}$ as $n \to \infty$ with respect to the weak topology $\sigma(\widetilde{E}, R(\mathscr{L}^*))$; an element $\widetilde{u} \in \widetilde{E}$ is called a *Limit element!weak*.

As it will be proved below, the effect of stabilizing of a sequence of elements u_n in the space $\widetilde{E}$ is a corollary of the convergence of f_n to f; so, there exists an analogy between string and weak near-solutions.

Consider the relation between a weak generalized solution and a weak near-solution. Let us prove that u is a weak generalized solution of the operator equation (2.1) iff it is a weak limit element of a near solution. Indeed, let u be a limit element of a near-solution, then $u \in \widetilde{E}$ and there exists a sequence $\{f_n\} \subset R(\mathscr{L})$ such that f_n tends to f as $n \to \infty$ with respect to the metric of F, and $u_n = \mathscr{L}^{-1}(f_n) \in D(\mathscr{L})$ tends to u as $n \to \infty$ in the topology $\sigma(\widetilde{E}, R(\mathscr{L}^*))$. Therefore, for all $\varphi \in D(\mathscr{L}^*) \subset F^*$ we have that $\varphi(\mathscr{L}u_n) = \varphi(f_n)$, hence

$$l(u_n) = (\mathscr{L}^*\varphi)(u_n) = \varphi(f_n) \to \varphi(f)$$

as $n \to \infty$, where $l = \mathscr{L}^*\varphi \in R(\mathscr{L}^*)$.

In addition, since $l \in R(\mathscr{L}^*)$, then $l(u_n) = \widetilde{l}(u_n) \to \widetilde{l}(u)$ as $n \to \infty$. Thus, we have that

$$\widetilde{l}(u) = \varphi(f)$$

for all $\varphi \in D(\mathscr{L}^*)$ such that $l = \mathscr{L}^*\varphi$, i.e. u is a weak generalized solution of (2.1).

Conversely, let us suppose that u is a weak solution of (2.1), i.e.

$$\widetilde{l}(u) = \varphi(f) \qquad \text{for all } \varphi \in D(\mathscr{L}^*),$$

where $l = \mathscr{L}^*\varphi$.

Let $\{f_n\}$ be an arbitrary sequence from $R(\mathscr{L})$ convergent to f as $n \to \infty$ with respect to the norm of F. Denote $\mathscr{L}^{-1}(f_n)$ as u_n. Then for an arbitrary functional $\varphi \in D(\mathscr{L}^*)$ such that $l = \mathscr{L}^*\varphi$ we have

$$l(u_n) = (\mathscr{L}^*\varphi)(u_n) = \varphi(\mathscr{L}x_n) = \varphi(f_n) \to \varphi(f)$$

as $n \to \infty$.

Thus, for any functional $\in R(\mathscr{L}^*)$ we have that $l(u_n) \to \varphi(f) = \widetilde{l}(u)$ as $n \to \infty$. Therefore, the sequence u_n converges to u with respect to the topology $\sigma(E, R(\mathscr{L}^*))$, hence u is a limit element of a near-solution $\{u_n\}$.

2.5 Existence and Uniqueness of a Weak Generalized Solution of a Linear Operator Equation

In this section, we prove the theorem on existence and uniqueness of a weak generalized solution of the operator equation (2.1) on the assumptions stated above, i.e. if $\mathscr{L}$ is a linear operator with dense domain $D(\mathscr{L})$ and dense range $R(\mathscr{L})$, (2.1) is uniquely solvable, and the sets $D(\mathscr{L}^*)$ and $R(\mathscr{L}^*)$ are total in the spaces F^* and E^* with respect to the corresponding weak topologies.

Let us start with the relatively simple problem of uniqueness. Suppose that the operator equation (2.1) in addition to a weak generalized solution $u \in \widetilde{E}$ has another weak generalized solution $\widetilde{u} \in \widetilde{E}$ $(u \neq \widetilde{u})$, then

$$\widetilde{l}(u) = \varphi(f) = \widetilde{l}(\widetilde{u})$$

for all $\varphi \in D(\mathscr{L}^*)$, $l = \mathscr{L}^* \varphi$.

Since the set of the functionals $\widetilde{l}$ coincides with the conjugate space $\widetilde{E}^*$, then $u = \widetilde{u}$, and we have a contradiction. Thus, the operator equation (2.1) may not have more than one weak generalized solution.

Now, let us consider the problem of existence of a weak generalized solution. Suppose that the right-hand side of the operator equation (2.1), i.e. the element f does not belong to the range $R(\mathscr{L})$ of the operator $\mathscr{L}$. Since (2.1) is densely solvable, then there exists such a sequence of elements f_n from $R(\mathscr{L})$ that $f_n \to f$ as $n \to \infty$ with respect to the norm F. Let us prove that the sequence $u_n = \mathscr{L}^{-1}(f_n)$ is a weak near-solution, and its limit element u belongs to $\widetilde{E}$. For this purpose let us consider the inverse operator $u = \mathscr{L}^{-1}(f)$, which acts from the vector space $R(\mathscr{L})$ into E. Denote by T the topology induced in $R(\mathscr{L}) \subset F$ by the norm of the Banach space F, and denote by $(R(\mathscr{L}), T)$, $(E, \sigma(E, R(\mathscr{L}^*)))$ the vector spaces $R(\mathscr{L})$ and E endowed with the topologies T and $\sigma(E, R(\mathscr{L}^*))$, respectively. Let us prove that the inverse operator $B = \mathscr{L}^{-1}$ is a continuous linear operator, which acts from the normed space $(R(\mathscr{L}), T)$ into the Hausdorff topological vector space $(E, \sigma(E, R(\mathscr{L}^*)))$. Since the set

$$W(l_1, \dots, l_n; \varepsilon) = \{u : u \in E,\ l_1(u) < \varepsilon, \dots, l_n(u) < \varepsilon\},$$

where $\varepsilon \in \mathbb{R}$, $l_i \in R(\mathscr{L}^*)$, $i \in \{1, 2, \dots, n\}$, form a fundamental system of neighborhoods of zero in $(E, \sigma(E, R(\mathscr{L}^*)))$, it is enough to prove that the following preimages $B^{-1}[W(l_1, \dots, l_n; \varepsilon)] = \mathscr{L}[W(l_1, \dots, l_n; \varepsilon)]$ are neighborhoods of zero in $(R(\mathscr{L}), T)$.

Indeed,

$$\begin{aligned}\mathscr{L}[W(l_1,\dots,l_n;\varepsilon)] &= \{\mathscr{L}u : l_1(u) < \varepsilon,\dots,l_n(u) < \varepsilon\}\\ &= \{\mathscr{L}u : \varphi_1\big(\mathscr{L}u\big) < \varepsilon,\dots,\varphi_n\big(\mathscr{L}u\big) < \varepsilon\},\end{aligned}$$

where $l_i(u) = (\mathscr{L}^*\varphi_i)(u) = \varphi_i(\mathscr{L}u)$, $\varphi_i \in D(\mathscr{L}^*)$, $i \in \{1,2\dots,n\}$. Therefore,

$$\begin{aligned}\mathscr{L}[W(l_1,\dots,l_n;\varepsilon)] &= \{f \in R(\mathscr{L}) \,:\, \varphi_1(f) < \varepsilon,\dots,\varphi_n(f) < \varepsilon\}\\ &= W_{R(\mathscr{L})}(\varphi_1,\dots,\varphi_n;\varepsilon),\end{aligned}$$

where $W_{R(\mathscr{L})}(\varphi_1,\dots,\varphi_n;\varepsilon)$ is a neighborhood that belongs to a fundamental system of neighborhoods of zero in the vector space $R(\mathscr{L})$ endowed with the topology $\sigma(R(\mathscr{L}),D(\mathscr{L}^*))$. Since the normed topology T is stronger than the weak topology $\sigma(R(\mathscr{L}),D(\mathscr{L}^*))$, then the set $W_{R(\mathscr{L})}(\varphi_1,\dots,\varphi_n;\varepsilon)$ is a neighborhood of zero with respect to the topology T. Thus, the operator $B = \mathscr{L}^{-1} : (R(\mathscr{L}),T) \to (E,\sigma(E,R(\mathscr{L}^*)))$ is continuous.

Since the space $\widetilde{E}$ is complete, F,E are the Hausdorff topological vector spaces, and every continuous linear map B of the space $(R(\mathscr{L}),T)$ into E is uniquely extendable to a continuous linear map $\widetilde{B}$ from F into $\widetilde{E}$ [7], then the sequence $\{u_n = \mathscr{L}^{-1}(f_n) = \widetilde{B}(f_n)\}$ converges to some element $\widetilde{u} \in \widetilde{E}$, which is a limit element of the near-solution $\{u_n = \mathscr{L}^{-1}(f_n)\}$. As it was shown above, in this case $\widetilde{u}$ is a weak generalized solution of (2.1). Thus, the existence of a weak generalized solution of (2.1) is proved.

2.6 Relation Between Weak and Strong Solutions of a Linear Operator Equation

Let us establish the relation between the solvability in sense of Definitions 2.1 and 2.3.

Theorem 2.3. *The space $\bar{E}$ is densely embedded into the space $\widetilde{E}$.*

Proof. Let some network $\{u_\alpha\}_{\alpha\in\mathscr{A}}, u_\alpha \in E$ converges to 0 with respect to the topology of the space $\bar{E}$. Then $\mathscr{L}u_\alpha \to 0$ in F, hence $\varphi(\mathscr{L}u_\alpha) \to 0$ for any $\varphi \in F^*$. Thus, $l(u_\alpha) \to 0$ for all $l \in R(\mathscr{L}^*)$. Therefore, the topology $\widetilde{E}$ is weaker than the topology $\bar{E}$. It remained only to prove that if $u_\alpha \to u$ with respect to the topology of the space $\bar{E}$ and $u_\alpha \to 0$ with respect to the topology $\widetilde{E}$, then $u = 0$ (condition π)). Taking into account the fact that $u_\alpha \to u$ is a convergent sequence, we have that

$$l(u_\alpha) = \mathscr{L}^*\varphi(u_\alpha) = \varphi(\mathscr{L}u_\alpha) \to \varphi(\bar{\mathscr{L}}u)$$

for all $l \in R(\mathscr{L}^*)$. In addition, the fact that $u_\alpha \to 0$ implies that $l(u_\alpha) \to 0$ for all $l \in R(\mathscr{L}^*)$ also. Thus, we have that $\varphi(\bar{\mathscr{L}}u) = 0$ for any $\varphi \in D(\mathscr{L}^*)$. Since the set

$D(\widetilde{\mathscr{L}^*})$ is total and the operator $\bar{\mathscr{L}}$ is injective, then $u = 0$. Thus, the embedding $\bar{E} \subset \widetilde{E}$ is proved.

The fact that the embedding is dense follows from the fact that the spaces $\bar{E} \subset \widetilde{E}$ are obtained as a result of completing of the set $D(\mathscr{L})$, i.e. $D(\mathscr{L})$ is a dense set both in $\bar{E}$ and $\widetilde{E}$. □

Theorem 2.4. *Definitions 2.1 and 2.3 are equivalent.*

Proof. Let $u \in \bar{E}$ be a strong generalized solution of the equation $\mathscr{L}u = f$. Taking into account the fact that the set $R(\mathscr{L})$ is dense in F, we have that there exists such a sequence $f_n \in R(\mathscr{L})$ that converges to f, or, in other words, there exists such an element $u_n \in D(\mathscr{L})$, that $u_n \to u$ in $\bar{E}$. By virtue of Theorem 2.3 the elements $u \in \bar{E}$ belongs to the space $\widetilde{E}$, and in addition $u_n \to u$ in $\widetilde{E}$. Now, it is easy to see that, from one hand, for all $l = \mathscr{L}^*\varphi \in R(\mathscr{L})$

$$l(u_n) = \mathscr{L}^*\varphi(u_n) = \varphi(\mathscr{L}u_n) \to \varphi(f),$$

and, from the other hand, – $l(u_n) \to l(u)$ as $n \to \infty$. Thus, u – is a weak generalized solution.

Let us prove that the solution $u \in \widetilde{E}$ in the sense of Definition 2.3 is a solution in the sense of Definition 2.1 (and vice versa). Indeed, there exists a solution $u^* \in \bar{E}$ of the equation $\mathscr{L}u = f$. It is clear that $\mathscr{L}^*\varphi(u) = \varphi(f) = \varphi(\bar{\mathscr{L}}u^*)$ for all $\varphi \in D(\mathscr{L}^*)$. Hence $u = Ou^*$, where O is an operator of embedding of the space $\bar{E}$ into the space $\widetilde{E}$. □

Finally, let us point out that the concept of a generalized solution of the operator equation $\mathscr{L}u = y$ is very different from various concepts u^* of such equations (for example, from the concept of a quasi-solution introduce by V. K. Ivanov), which are described in [47] and [112], as far as $\bar{\mathscr{L}}\bar{u} = y$ for the generalized solution $\bar{u}$ always, where $\bar{\mathscr{L}}$ is a natural extension of the operator $\mathscr{L}$, whereas the equality $\bar{\mathscr{L}}u^* = y$ for the generalized solutions u^* holds not always.

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Klyushin, D.A.; Lyashko, S.I.; Nomirovskii, D.A.; Petunin,
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