Preface

“F-friends,” said Fyodor Simeonovich ...
“But this is the Ben B-Betzalel’s p-problem.
C-Cagliostro has proved that it does not have a s-solution indeed.”
“We do know that it does not have a solution,” said Junta...
“We wish to know how to solve it.”
“You are somehow arguing oddly, C-Christo. . .
H-how to s-search for a s-solution, when it does not exist?
It’s a nonsense.
“I am sorry, Fyodor, but it’s you who are arguing strangely.
The nonsense is to search for a solution when it exists anyway.
The question is how to deal with a problem that does not have a solution.
This is a profoundly principled question ...”

A. Strugatsky and B. Strugatsky, “Monday Begins on Saturday”

At the International Mathematical Congress in Paris (1900), D. Hilbert put forth his famous 23 problems. In Hilbert’s opinion, these problems had to predefine the mainstream of mathematics in the twentieth century. By now, most of Hilbert’s problems have been solved successfully. However, despite the fact that many mathematical disciplines have arisen and new important problems were put forth in the twentieth century, Hilbert’s problems remain fundamental [3].

Among Hilbert’s problems, the 20th problem – “the general problem of boundary values” – takes its deserved place. This problem is formulated in the following way: “has not every regular variation problem a solution, provided certain assumptions regarding the given boundary conditions are satisfied (say that the functions concerned in these boundary conditions are continuous and have in sections one or more derivatives) and provided also if need be that the notion of a solution shall be suitably extended?” (see [25]).

The 20th problem is outstanding because D. Hilbert put it on extending the classical solution when there was neither the concept of completion of metric space nor the concept of normed space that serves as a basis of such a notion as “generalized solution of operator equations”. The idea of the generalized solution is quite simple: consider an operator equation $A(x) = y$, where A is a continuous operator (linear or nonlinear) from metric or Banach space E into F. Operator equations cover wide classes of differential equations (including boundary value problems), integral equations, integro-differential equations and more. In many situations, the operator equation $A(x) = y$ does not have a classical solution, since the right-hand side y does not belong to the range $R(A) \subset F$ of the operator A, but we can introduce a weaker topology in E, so that the completion $\widetilde{E}$ of E in this topology is a wider space: $E \subset \widetilde{E}$ and the operator A can be extended by continuity to $\widetilde{E}$, so that the right-hand side y belongs to the range $R(\widetilde{A})$ of the extended operator $\widetilde{A}$. Thus, the operator equation $\widetilde{A}(x) = y$ ($x \in \widetilde{E}$, $y \in F$, $\widetilde{A} : \widetilde{E} \to F$) has a classical solution $\widetilde{x} \in \widetilde{E}$ called a generalized solution of the original equation $A(x) = y$. This is exactly such an extension of the concept of solution about which D. Hilbert wrote.

The concept of generalized solution is closely related to the concept of a near-solution x_ε of the operator equation $A(x) = y$; this is such an element in E that $A(x_\varepsilon) = y_\varepsilon$ differs less than ε from y: $\rho(y, y_\varepsilon) < \varepsilon$. In some cases, x_ε may be considered as an approximate solution of the equation $A(x) = y$. If we put $\varepsilon = \varepsilon_n \to 0$ as $n \to \infty$ and consider a sequence of the near-solutions x_{ε_n}, then in $\widetilde{E}$ (but not in E!) the sequence x_{ε_n} converges to a generalized solution $\widetilde{x}$. In the case of linear operator A, the computation of the near-solution is reduced to the problem of computation of the approximate (or precise) solution of a system of linear algebraic equations. This is why we give so much attention to these issues and propose various methods for solving this problem.

Along with the investigation of generalized solutions, we study the so-called generalized extreme elements which are closely related to this concept. Let D be a region in a Banach or metric space E and a continuous functional $f(x)$ is defined on D. As a rule, the region D is non-compact in an infinite dimensional space, therefore the extreme element x^* from D, at which $f(x)$ attains its minimum or maximum value may not exist. Determination of a "generalized" extreme element resembles the construction of generalized solution. We introduce a weaker topology $\mathscr{T}_D$ on the set D, such that the completion $\widetilde{D}$ of D with respect to the topology $\mathscr{T}_D$ is a compact topological space, and the functional f may be extended on $\widetilde{D}$ by continuity, such that there is a classical extreme element x^* in $\widetilde{D}$. This element is considered as a generalized extreme element, since $x^* \notin D$. Note that the concept of a generalized extreme element may be defined in other ways. These ways are considered in the book as well.

By an *operator equation* we will always mean an equation where some known operator $\mathscr{L}$ from E into F acts on an unknown element u (a vector, sequence or function), where F may differ from E. The spaces E and F may be finite or infinite dimensional spaces, normed spaces (in particular, Banach), metric spaces, topological vector spaces, topological or differentiable manifold, and so on. In a general way, an operator equation has the following form

$$\mathscr{L}u = f,$$

where u is an unknown element in E, f is the known element in F, and $\mathscr{L}$ is the known operator which acts from E into F. The most important problems related to operator equations are the existence and uniqueness of a solution. The uniqueness of a solution is ensured by the condition of invertibility of the operator $\mathscr{L}$, that may be satisfied by the corresponding factorization of the space E (at least theoretically). It is clear that a solution of the equation $\mathscr{L}u = f$ exists iff the right-hand side f belong in the range $R(\mathscr{L})$ of the operator $\mathscr{L}$. Thus, if $f \in R(\mathscr{L})$ then the issue of the existence of a solution of the equation $\mathscr{L}u = f$ has, in principle, a positive answer. However, in many cases the right-hand side f does not belong to the set $R(\mathscr{L})$, so this equation does not have a solution in a classical sense. Nevertheless, from the practical point of view such equations may have "intuitive solutions", that must be defined correctly. The problem of construction of a generalized solution of the operator equation is closely related with the problem of introducing the "natural" notion of a generalized solution of the equation $\mathscr{L}u = f$ for all $f \in F$; in particular, when $f \in F \setminus R(\mathscr{L})$, and with the investigation of the properties of such generalized solutions. The point is that the description of a function set of $R(\mathscr{L})$ is extremely difficult. Therefore it is impossible to establish the criteria for the solvability of the equation $\mathscr{L}u = f$. We could say that it is possible to formulate the criterion of the solvability of the equation $\mathscr{L}u = f$ only in exceptional cases. For example, even in the simplest case of the investigation of the classical solvability of an ordinary differential equation $u'(t) = f(t)$ when $1 > t > 0$ and $u(0) = 0$, it is necessary to test the convergence of an integral (possibly improper integral)

$$\int_0^1 f(t)\,\mathrm{d}t.$$

However, as is well known, there are no general effective criteria for testing the convergence of improper integrals.

Consider one of the approaches to the formalization of such solutions. Suppose that in any ε-neighborhood f (in topological space F – in any neighborhood f) there exists such an element f_ε, that $\mathscr{L}u_\varepsilon = f_\varepsilon$ for some $u_\varepsilon \in E$. Then for small $\varepsilon > 0$ one could think that $f_\varepsilon \approx f$, since the distance $\rho(f_\varepsilon, f) < \varepsilon$, therefore the element u_ε can be accepted as a "generalized" solution of the operator equation $\mathscr{L}u = f$ (if topological space F is non-metrizable, then these reasonings must be slightly modified, but this is not a principal issue).

Consider the issue of the existence of classical and generalized solutions on concrete examples. Suppose that we want to obtain the best unbiased linear estimation x^* of an unknown mathematical expectation of a continuous random process $x(t)$ ($t \in [0,T]$) with a constant mathematical expectation and a correlation function $K(t,s)$. If we look for this estimation in the form

$$x^* = \int_0^T x(t)u(t)\,\mathrm{d}t,$$

then the problem is reduced to looking for the solution $u(t)$ of the integral equation

$$\int_0^T K(t,s)u(t)\,\mathrm{d}t = 1 \tag{P.1}$$

in the function class $L_2(0,T)$. In general case, the matter concerns the equation

$$\int_D K(t,s)u(t)\,\mathrm{d}t = f(s), \quad t \in \bar{D}. \tag{P.2}$$

However, solutions of such equations have the square integrability property very seldom (see example [23]). For example, it is shown in [36] that (P.1) never has a classical solution if a correlation function $K(\tau)$ of the stationary random process $x(t)$ has a spectral density. Nevertheless, it has the generalized solution. In some cases, the fact that the integral equation (P.1) does not have a solution in the class of square-integrable functions can be proved directly. For example, if a correlation function has the form $K(t,s) = \mathrm{e}^{-\beta|t-s|}$, which corresponds to a stationary Markov process when all probability distributions are normal, then it is impossible to construct a function $u(t)$ that sets the best unbiased estimation x^* of an unknown mathematical expectation. To prove this statement let us consider the integral equation

$$\int_0^T \mathrm{e}^{-\beta|t-s|}\,\mathrm{d}F(t) = \frac{2}{2+\beta T}.$$

It is easy to examine that this equation is satisfied by the following function of bounded variation

$$F(t) = \frac{\Theta(t) + \Theta(t-T) + \beta t}{2+\beta T},$$

where $\int_0^T \mathrm{d}F(t) = 1$ and $\Theta(t)$ is the Heaviside function:

$$\Theta(t) = \begin{cases} 0, \text{ if } t < 0 \\ 1, \text{ if } t \geq 0. \end{cases}$$

Hence, the expression

$$x^* = \frac{x(0) + x(T) + \beta \int_0^T x(t)\,\mathrm{d}t}{2+\beta T}$$

defines an unbiased estimation x^* having the least variance in the class of unbiased linear estimations (actually, this estimation is also the best in a much more wider estimation class [23]). Since the estimation x^* is unique and the formula for x^* contains Dirac delta-functions $\delta(t)$ and $\delta(t-T)$, that do not belong to $L_2(0,T)$, it is impossible to construct the function $u(t)$ from $L_2(0,T)$, that defines the estimation x^* and is a solution of (P.1). Therefore, (P.1) does not have the classical solution in $L_2(0,T)$. The issues related with the problem described above are listed in [96]: "The problems are: in which functional spaces should one look for the solution?

Is the solution unique? Is the solution of the equation also is the solution of the estimation problem? Does the solution depend continuously on the initial data, for example, on f and K? How can the solution be found analytically and numerically? What are the properties of the solutions? For example, what is the order of singularity? How can the properties of the integral operator be described, for example, in $L^2(D)$?"

Another possible example that requires the introduction of a generalized solution is the problem of optimal control of a system with a generalized external impact

$$\mathscr{L}u = f(h), \tag{P.3}$$

$$J(h) = \Phi(u(h),h) \to \min_h, \qquad h \in U, \tag{P.4}$$

where h is a control from an admissible set U, $\mathscr{L} : E \to F$ is some operator, J is a performance functional. To express this problem correctly it is necessary to ensure the solvability of (P.3) for all $h \in U$, i.e., it is necessary to ensure the inclusion $f(U) \subset R(\mathscr{L})$. However, generally it is very difficult to describe the range of f and $\mathscr{L}$; therefore, it is very hard to check the condition $f(U) \subset R(\mathscr{L})$. Moreover, often such an inclusion does not occur at all (in spite of the fact that a physical interpretation of the equation is natural and reasonable from the practical point of view). Thus, we must develop a theory of generalized solvability of (P.3) for an arbitrary right-hand side f from the set $f(U)$, or (much better) for all $f \in F$. In a general sense, (P.3) has a solution $u(h)$ for an arbitrary control $h \in U$. It is clear, that we must know peculiarities of these generalized solutions to prove some meaningful statements about the problem of the minimization of (P.4).

Now, problems of complex system control with singular impacts have a fundamental importance. For example, simulation of devices with laser and pulse impacts, correction of space vehicles movement, modelling of water transport in porous media with point sources and sinks are closely related with the equations with a singular right-hand side. The singularity of a control impact means that a control map f takes on a value in a space of generalized function. Traditionally, the natural range of the operator L does not contain generalized functions. So, lumped singularity in space and time bring us outside of the classical problem definitions. So, we face with the need to develop a theory of generalized solvability of (P.3).

The problem of construction of generalized solutions becomes the most important in the case of linear operator $\mathscr{L}$ (e.g., differential or integral) which acts between linear topological spaces E,F, in particular, between Banach or Hilbert spaces. Note that the "naturalness" of generalized solution means the conservation of the main properties of operator $\mathscr{L}$ (linearity, continuity, injectivity and so on) under extension on the class of generalized solutions. Thus, the offered problem fundamentally differs from various definitions of approximate solutions, pseudo-solutions, quasi-solution, and so on. [47, 107, 112].

The problems of construction of generalized solutions of equations with linear differential and integral operators are quite typical. They have been investigated successfully for a long time. For example, this problem for the classical operator of differentiation $\frac{\mathrm{d}}{\mathrm{d}t} : C^1([0,1]) \to L_2(0,1)$ may be solved by introducing of the

Sobolev generalized derivative and corresponding Sobolev spaces. In this sense, the theory of generalized functions may be considered as the first step in solving the posed problem.

The method of a priori inequations (e.g. [5, 39, 54]) is a very effective tool for the investigation of existence and uniqueness of solutions of various classical linear problems with generalized impacts. It was often used in the context of rigged Hilbert spaces. In [5], the theory of generalized solvability for equations with elliptic differential operators acting in the Sobolev discrete scale of Hilbert spaces was constructed. This theory is based on the concept of weak solutions (in the context of the theory of generalized functions). Berezanskii proved the theorems of a unique generalized solvability of elliptic operator equations for major problems of mathematical physics and investigated the smoothness of the generalized solutions. The theorems proved are the criteria of solvability (i.e., an operator determines a topological isomorphism). For example, the theorems of a unique solvability (in L_2 and other spaces) of the equations of mathematical physics of various types were proved in [44, 45]. Some criteria of solvability of parabolic equations are described in [2]. The issues of generalized solvability for pseudo-parabolic equations of order more than two were investigated in [58, 100], for pseudo-hyperbolic equations – in [73, 79, 85, 100], for Sobolev type system – in [59, 76, 100], for wave systems of fifth order – [60, 61, 64, 80, 82], and in many other papers (see also [62, 63]). Note that in these papers were used a priori inequalities in negative norms when a generalized solution belongs to Sobolev type spaces.

The generalized solvability of linear integral equations is closely related with Fredholm and Volterra; integral equations of the first kind [23] and [68, 87, 88, 92]. It must be stressed that in many above-mentioned papers the proofs of existence and uniqueness of a generalized solution are based on the classical idea of relations between direct and "adjoint" equations and the coercive inequality. Therefore, these theorems can be considered as the developing of classical results of S.G. Krein (e.g., see [39]).

There is one more important aspect of the theory of generalized solutions. It is related to the problem of optimal control (P.3), (P.4), rather than with only (P.3). As it is well known, there are problems of calculus of variations and optimal control which have no solutions in "traditional" sets of curves (in spaces of smooth functions). This problem was solved in classical papers on optimal control theory in the generalized statement. For example, the general plan of looking for generalized extreme curves is described in [116]. The plan involves the following activities: to densely embed the control space (and therefore an admissible set of controls) in a new topological space such that a functional in question is still sequentially continuous and an admissible set is sequentially compact. This idea naturally connects the optimal control problems with the Schwarz distributions spaces. We have to mention L. Young [118] among the authors who began to apply the ideas of the theory of generalized functions to the calculus of variations problems and the optimal control problems. From the Young's point of view, the spaces of curves with "traditional" topologies are poorly adaptable for the calculus of variations. More convenient are the topologies which induce so-called "generalized curves" (by Young) that are equivalent to the concepts of weak controls and gliding regimes. In the optimal control theory for ordinary and partial differential equations, Filippov and Gamkrelidze considered the weak solutions and analogous constructions [17, 22] ("gliding regimes"), Warga studied the "generalized curves" [115], McShane investigated the "generalized controls" [70, 71], Chouila-Houri considered "boundary controls" [10], and this list might be continued (see, e.g. books of A. Chikrii [9], M. Zgurovsky, V. Mel'nik [120], V. Kuntsevich [42], J.-L. Lions [51, 52], and B. Mordukhovich [72]).

These results naturally pose the general problem of looking for generalized extreme elements in various classes of functionals. These problems are interesting even in the simplest case when we look for an extremum of a continuous functional defined on a bounded set in a Banach space.

Thus, there are many papers, where existence and uniqueness of generalized solution of an operator equation or extremal problem solutions were investigated. Multiplicity and similarity of these papers suggest that there is a general approach to the construction of the concept of generalized solvability. The major elements of this approach are described in our book.

The book consists of the preface, eight chapters, divided into sections, and a bibliography. The numbering of definitions, lemmas, theorems, and so on, is continuous. Chapter 1 contains major definitions, concepts, and auxiliary facts used in the book. Chapter 2 is an introduction to the theory of generalized solutions of operator equations. It describes the simple schemes of generalized solutions for linear operator equations. In Chap. 3, we investigate the method of a priori estimates for generalized solutions. Chapter 4 describes some applications of the theory of generalized solvability of linear equations. Chapter 5 is devoted to numerical aspects of the theory. Chapter 6 describes the general topological method of construction of generalized solutions of linear operator equations. In Chap. 7, the issues of generalized solvability of nonlinear operator equations are considered. Chapter 8 is devoted to the generalized solvability of extreme problems.

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http://www.springer.com/978-1-4614-0618-1

Generalized Solutions of Operator Equations and Extreme Elements

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2012, XXII, 202 p., Hardcover
ISBN: 978-1-4614-0618-1