The Geometry of Extremal Elements in a Lie Algebra

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Abstract Let L be a simple finite-dimensional Lie algebra over an algebraically closed field of characteristic distinct from 2 and from 3. Then L contains an extremal element, that is, an element x such that $[x, [x, L]]$ is contained in the linear span of x in L. Suppose that L contains no sandwich, that is, no element x such that $[x, [x, L]] = 0$. Then, up to very few exceptions in characteristic 5, the Lie algebra L is generated by extremal elements and we can construct a building of irreducible and spherical type on the set of extremal elements of L. Therefore, by Tits' classification of such buildings, L is determined by a known shadow space of a building. This gives a geometric alternative to the classical classification of finite-dimensional simple Lie algebras over the complex numbers and of classical finite-dimensional simple modular Lie algebras over algebraically closed fields of characteristic ≥ 5. This paper surveys developments pertaining to this kind of approach to classical Lie algebras.

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1 Extremal Elements in Lie Algebras

Let k be a field and consider a Lie algebra L over k. For each $z \in L$ we denote by ad_z as usual left multiplication by z, so $\mathrm{ad}_z(x) = [z, x]$ for $x \in L$.

If the characteristic of k is distinct from 2, a nonzero element x of L is called *extremal* if $[x, [x, L]]$ is contained in the linear span of x in L. Then, by linearity

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of $[x,[x,y]]$ in y, an element x of L is extremal if and only if there is a linear functional $g_x : L \to k$ such that, for all $y \in L$,

$$[x,[x,y]] = 2g_x(y)x. \tag{1}$$

Note that $g_x(y) = 0$ if x and y commute.

The two identities of the following lemma were first obtained by Premet through polarization (see [7, 14]).

Lemma 1.1. *Suppose the characteristic of k is not even and let $x \in L$ be extremal. Then, for all $y, z \in L$,*

$$[[x,y],[x,z]] = g_x([y,z])x + g_x(z)[x,y] - g_x(y)[x,z], \tag{2}$$

$$[x,[y,[x,z]]] = g_x([y,z])x - g_x(z)[x,y] - g_x(y)[x,z]. \tag{3}$$

Observe that the number 2 occurring in (1) has disappeared. Accordingly, these identities cannot be derived from the above definition of extremality if the characteristic of k is even. In order to include the proper notion of extremality for even characteristic, we need to take these identities along.

Definition 1.2. A nonzero element x of L is called *extremal* if there is a linear functional $g_x : L \to k$ such that

(i) $[x,[x,y]] = 2g_x(y)x$ for each $y \in L$;
(ii) $[[x,y],[x,z]] = g_x([y,z])x + g_x(z)[x,y] - g_x(y)[x,z]$ for all $y, z \in L$.

The set of extremal elements of L is denoted $\mathrm{E}(L)$, or just E if no confusion is imminent. Likewise, a projective point kx with $x \in L \setminus \{0\}$ is called *extremal* if x is extremal. The set of extremal points is denoted $\mathcal{E}(L)$ or just $\mathcal{E}$.

In view of (i), extremal elements are nilpotent of order 3. If $x \in \mathrm{E}$ is such that Definition 1.2 is satisfied with $g_x = 0$, then we call x a *sandwich*.

By definition, each extremal element satisfies the first Premet identity (2). An extremal element is easily shown to satisfy the second Premet identity (3), and so Lemma 1.1 holds for all characteristics. According to the above discussion, requirement (ii) is not needed if the characteristic of k is not even. In characteristic 2, formulas (i) and (ii) do not uniquely define the linear functional g_x, but the leeway is removed by insisting that $g_x = 0$ whenever x is a sandwich; see [10, Lemma 16] for details. We will adopt this convention. A sandwich $x \in \mathrm{E}$ is nilpotent of order 2 and, due to (3), satisfies $\mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x = 0$ for each $y \in L$, which explains to some extent the name.

The fact that extremal elements are special nilpotent elements is exhibited by the direct relation with automorphisms. For $x \in L$ and $t \in k$ define the map $\exp(x,t) : L \to L$ by

$$\exp(x,t)y = y + t[x,y] + t^2 g_x(y)x. \tag{4}$$

Lemma 1.3. *Let $x \in L$ be extremal with functional g_x. Then the map $t \mapsto \exp(x,t)$ is an injective homomorphism of groups $k^+ \to Aut(L)$, where k^+ stands for the additive group of k.*

Here are some elementary facts on extremal elements. Proofs can be found in [10].

Proposition 1.4. *Let $x, y \in \mathrm{E}$ and $z \in L$.*

(i) *The subalgebra $\langle \mathrm{E} \rangle$ of L generated by* E *is linearly spanned by* E.
(ii) *On $\langle \mathrm{E} \rangle$ there is a unique symmetric bilinear form g with value $g_x(y)$ at (x, y).*
(iii) *For all $a, b, c \in \langle \mathrm{E} \rangle$, we have* $g([a,b],c) = g(a,[b,c])$. *In particular, the form g of (ii) is associative.*
(iv) *Assume that L is generated by* E. *Then the radical of g, that is, $Rad(g) = \{x \in L \mid g(x, L) = 0\}$, is a nilpotent ideal of L containing all sandwiches of L.*

A consequence of the last assertion is that sandwiches in Lie algebras generated by extremal elements disappear in the transition to the quotient by $\mathrm{Rad}(g)$. Moreover, if $x \in \mathrm{E}(L)$ is not a sandwich, then $x + \mathrm{Rad}(g)$ is an extremal element of $L/\mathrm{Rad}(g)$ and is not a sandwich. So, going over from L to $L/\mathrm{Rad}(g)$, we retain all extremal elements that are not a sandwich and remove all sandwiches. For a characterization of simple Lie algebras generated by extremal elements that are not sandwiches, the assertion allows us to assume that they have no sandwiches.

Example 1.5. Let V be a vector space over k. Then each member X of $\mathfrak{gl}(V)$, the Lie algebra of all endomorphisms of V, of rank 1 with $X^2 = 0$ is extremal. The linear 1-space of X determines the incident pair (x, H) of the point that is its image $x = XV$ and the hyperplane $H = \ker X$ that is its kernel.

Suppose that V has finite dimension n. Then the subalgebra $\langle \mathrm{E} \rangle$ coincides with $\mathfrak{sl}(V)$, the matrices of trace 0 in $\mathfrak{gl}(V)$. If the characteristic of k is a prime dividing n, then the identity belongs to $\mathrm{Rad}(g)$. The subalgebra $\mathfrak{n}(V)$ of all upper triangular matrices with respect to a chosen basis of V all of whose main diagonal entries are zero, is easily seen to be generated by $\mathrm{E}(\mathfrak{gl}(V)) \cap \mathfrak{n}(V)$ and to coincide with its own g-radical. If $n = 3$, we refer to this example as the *Heisenberg algebra* and denote it by $\mathfrak{h}$. It is a 3-dimensional Lie algebra with one-dimensional center $Z(\mathfrak{h})$ such that $[\mathfrak{h}, \mathfrak{h}] = Z(\mathfrak{h})$.

Example 1.6. Let k be a field of characteristic 3 and let V be the vector space k^3. We will consider the 7-dimensional Lie algebra $L = \mathfrak{sl}(V)/Z$, where $Z = Z(\mathfrak{sl}(V))$ is the (one-dimensional) center of $\mathfrak{sl}(V)$ spanned by the identity matrix. Besides the images mod Z of the rank 1 matrices X in $\mathfrak{sl}(V)$ with $X^2 = 0$, the images mod Z of the rank 2 matrices Y in $\mathfrak{sl}(V)$ with $Y^2 \neq 0$ and $Y^3 = 0$ also belong to $\mathrm{E}(L)$. The group generated by all $\exp(x,t)$ for $t \in k$ and $x \in \mathrm{E}(L)$ consists of the k-rational points of the split algebraic group of type G_2. The Lie algebra of this algebraic group is 14-dimensional and contains L as an ideal.

Example 1.7. Let V be a vector space over k supplied with a symplectic form f. Let $\mathfrak{sp}(V, f)$ be the Lie algebra of all endomorphisms of V preserving f (so $X \in$

$\mathfrak{sp}(V, f)$ if and only if $X \in \mathfrak{gl}(V)$ and $f(Xa, b) + f(a, Xb) = 0$ for all $a, b \in V$). Then each infinitesimal symplectic transvection, that is, endomorphism of $\mathfrak{sl}(V)$ of the form $x \mapsto f(x, a)a$ for some $a \in V$, is extremal in $\mathfrak{sp}(V, f)$. If $n = 2$ and f is non-degenerate, then $\mathfrak{sp}(V, f) = \mathfrak{sl}(V)$. If $n = 3$ and f has rank 2, then $\mathfrak{sp}(V, f)$ is a semi-direct product of the ideal $\mathrm{Rad}(g)$, which is isomorphic to $\mathfrak{h}$ (of Example 1.5), and $\mathfrak{sl}(k^2)$.

Example 1.8. Let V be a vector space over k supplied with a quadratic form κ. Let $\mathfrak{o}(V, \kappa)$ be the Lie algebra of all endomorphisms of V preserving κ. Denote by f the associated bilinear form: $f(a, b) = \kappa(a + b) - \kappa(a) - \kappa(b)$ for all $a, b \in V$. Then $X \in \mathfrak{o}(V, \kappa)$ if and only if $X \in \mathfrak{gl}(V)$ and $f(Xa, a) = 0$ for all $a \in V$. Each infinitesimal Siegel transvection, that is, endomorphism of $\mathfrak{sl}(V)$ of the form $x \mapsto f(x, a)b + f(x, b)a$ for some $a, b \in V$ with $\kappa(a) = f(a, b) = \kappa(b) = 0$, is extremal in $\mathfrak{o}(V, \kappa)$.

There is a unitary variant of this example, which we denote $\mathfrak{u}(V, \kappa)$, where κ is a unitary form on V.

Example 1.9. Consider an arbitrary k-split connected reductive linear algebraic group G, with reductive rank n and semisimple rank ℓ. Fix a maximal split torus T of G and let Φ be the root system with respect to T, let X be the character group of T, and Y its dual so there is a bilinear pairing $\langle \cdot, \cdot \rangle \to \mathbb{Z}$. Fix a basis $e_1, \ldots, e_n$ of X and a dual basis $f_1, \ldots, f_n$ of Y (so $\langle e_i, f_j \rangle = \delta_{ij}$). There is a coroot system $\Phi^\star$ in Y with a bijective correspondence $\alpha \mapsto \alpha^\star$ between Φ and $\Phi^\star$ such that $\langle \alpha, \alpha^\star \rangle = 2$ for each $\alpha \in \Phi$. Let L be the Lie algebra of G and H the Lie subalgebra of T. Then L has basis elements e_α for $\alpha \in \Phi$ and $h_i \in H$ for $i = 1, \ldots, n$ with structure constants:

$$[h_i, h_j] = 0, \tag{5}$$

$$[e_\alpha, h_i] = \langle \alpha, f_i \rangle\, e_\alpha, \tag{6}$$

$$[e_{-\alpha}, e_\alpha] = \sum_{i=1}^{n} \langle e_i, \alpha^\star \rangle h_i, \tag{7}$$

$$[e_\alpha, e_\beta] = \begin{cases} N_{\alpha\beta}\, e_{\alpha+\beta} & \text{for } \alpha + \beta \in \Phi, \\ 0 & \text{for } \alpha + \beta \notin \Phi,\ \beta \neq -\alpha, \end{cases} \tag{8}$$

for certain integral constants $N_{\alpha\beta}$ (see [6, 28]). These integers are interpreted as elements of the field k. Such a basis of L is called a *Chevalley basis*. Now fix a long root $\alpha \in \Phi$. Then $e_\alpha \in \mathrm{E}(L)$. For k of characteristic distinct from 2, this can be checked as follows: as α is long, $2\alpha + \beta \in \Phi$ is a root only if $\beta = -\alpha$, so

$$[e_\alpha, [e_\alpha, L]] \subseteq [e_\alpha, [e_\alpha, H]] + \sum_{\beta \in \Phi} [e_\alpha, [e_\alpha, e_\beta]] \subseteq \sum_{\beta \in \Phi : 2\alpha + \beta \in \Phi} k e_{2\alpha+\beta} \subseteq k e_\alpha.$$

Now G acts on L via the adjoint representation and the full G-orbit of ke_α is contained in $\mathcal{E}(L)$. In most cases, this is all of $\mathcal{E}(L)$. This example generalizes Examples 1.5, 1.7, and 1.8. If G is not semisimple, then $\langle E(L)\rangle$ is properly contained in L. If G is semisimple and the characteristic of k is not bad for G, then $\langle E(L)\rangle = L$ (see [13]).

There are five essentially different positions two extremal elements can be in with respect to each other.

Definition 1.10. For $i \in \{-2,-1,0,1,2\}$, define relations E_i on E as follows.

i	$x\mathrm{E}_i y$	Name
-2	$kx = ky$	Identical
-1	$[x,y]=0, kx+ky \subseteq \mathrm{E}\cup\{0\}, kx \neq ky$	Strongly commuting
0	$[x,y]=0, kx+ky \not\subseteq \mathrm{E}\cup\{0\}$	Polar
1	$[x,y]\neq 0, g(x,y)=0$	Special
2	$g(x,y)\neq 0$	Hyperbolic

The name refers to the pair (x, y) in the individual cases. As the relation E_i on E does not depend on the choice of vector in the 1-space spanned by x or y, the names are also valid for the projective pairs (kx, ky) from $\mathcal{E}$. The corresponding projective relations are denoted $\mathcal{E}_i$.

Clearly, the relations E_i are symmetric and partition $\mathrm{E}\times\mathrm{E}$, and the $\mathcal{E}_i$ partition $\mathcal{E}\times\mathcal{E}$.

The following cases of few extremal generators are discussed in [14]; see also [18].

Lemma 1.11. *Let $x, y, z \in$ E.*

(i) The subalgebra of L generated by x and y is at most 3-dimensional. It is commutative if $x\mathrm{E}_i y$ with $i \leq 0$, it is isomorphic to the Heisenberg algebra $\mathfrak{h}$ if $x\mathrm{E}_1 y$, and it is isomorphic to $\mathfrak{sl}(k^2)$ if $x\mathrm{E}_2 y$.

(ii) The subalgebra $\langle x, y, z\rangle$ is at most 8-dimensional. It is nilpotent, isomorphic to (a possibly twisted form of) $\mathfrak{sl}(k^3)$ (like $\mathfrak{u}(\mathbb{R}^3,\kappa)$ where κ is a non-compact unitary form), or an extension of $\mathfrak{sl}(k^2)$ by a nilpotent ideal.

An application in [14] of a beautiful result of Zelmanov and Kostrikin [34] gives that the dimension of a Lie algebra generated by a finite number of extremal elements is finite. Besides, there is always a nilpotent Lie algebra of maximal dimension $d(N)$ for any given finite number N of extremal generators. The above lemma shows that $d(2) = 3$ and $d(3) = 8$ are attained by simple Lie algebras. It is also known that $d(4) = 28$ and $d(5) = 537$ if k has characteristic 0. The former case is not only realized by a nilpotent Lie algebra but also by the 28-dimensional Lie algebra $\mathfrak{o}(k^8,\kappa)$ of Example 1.8, where κ is a quadratic form on k^8 of Witt index 4. The Lie algebras of types E_6, E_7 and E_8 are generated by five extremal elements. This leads to an intriguing question.

Problem 1.12. Is there a 'generic' Lie algebra of maximal possible dimension 537 in the 5 generator case?

The notion of genericity is made precise by Draisma and in't panhuis in [17]. For any given finite graph Γ, they construct an algebraic variety X over k whose k-points parametrize Lie algebras over k generated by a set of extremal elements in bijective correspondence with the vertex set of the graph, with prescribed commutation relations for pairs corresponding to nonedges of Γ. If Γ is a connected, simply laced Dynkin diagram of finite or affine type, then they prove that X is an affine space, and that all points in an open dense subset of X parametrize Lie algebras isomorphic to a single fixed Lie algebra. If Γ is of affine type, then this fixed Lie algebra is the split finite-dimensional simple Lie algebra corresponding to the associated finite-type Dynkin diagram. This gives a new construction of these Lie algebras, in which they come together with interesting degenerations, corresponding to points outside the open dense subset. For graphs Γ on five vertices, the maximal dimensions of Lie algebras realizing Γ have been investigated by Roozemond [26]. In [19], the classical Lie algebras are characterized by series of graphs Γ. The easiest series is the path of length n (Coxeter diagram A_n), which corresponds to $\mathfrak{sp}(k^n, f)$, where f is a symplectic form on k^n of maximal rank. So, if n is even, then f is non-degenerate and the classical Lie algebra is of type $\mathrm{C}_{n/2}$ and if n is odd, it is a semi-direct product of $\mathrm{Rad}(g)$ and the classical Lie algebra of type $\mathrm{C}_{(n-1)/2}$. If $n = 3$, we recover the semidirect product of $\mathfrak{h}$ and $\mathfrak{sl}(k^2)$ described in Example 1.7.

2 Geometry from Extremal Elements

Throughout this section L will be a Lie algebra over the field k. In [22, 23] Premet proved that extremal elements occur in many simple Lie algebras.

Theorem 2.1. *If k is algebraically closed of characteristic distinct from 2 and 3 and L is finite-dimensional and simple, then L has an extremal element.*

Using powerful methods, Benkart [1, Theorem 3.2] showed that if a simple Lie algebra over an algebraically closed field of characteristic $p \geq 7$ or $p = 0$ contains a nilpotent element of order at most $p - 1$ and no sandwiches, then it is of classical type. In this section, we survey an attempt to replace this proof by a geometric argument with recourse to Tits' theory of buildings [32]. For the classification of simple modular Lie algebras in characteristic at least 5, the reader is referred to [2, 9, 24, 29].

Our approach will consist of three steps, two of which have been completed. First, in this section, we discuss how one can produce a geometry from L. The geometry will have point set $\mathcal{E}(L)$ and its line set $\mathcal{F}$ will be the projective lines in L fully contained in $\mathcal{E}(L)$. This geometry satisfies properties summarized by the term root filtration space (see Definition 3.1).

Then, in a second step, root filtration spaces are characterized as certain (root) shadow spaces of spherical buildings.

The third and final step will be to show that, given a root filtration space $(\mathcal{E}, \mathcal{F})$ satisfying some mild conditions, then, up to isomorphism, there is at most one simple Lie algebra whose root filtration space is isomorphic to $(\mathcal{E}, \mathcal{F})$. There is an abundance of examples of root filtration spaces that do not arise from Lie algebras.

We now return to the first step. The point set of our geometry will be $\mathcal{E}$. By Theorem 2.1, this set is not empty if k is algebraically closed. This is a good starting point in view of the fact that, at least for k of characteristic distinct from 2, 3, and 5, a self-contained proof by Tange is available in [30]. But we need $\mathcal{E}$ to be more than a singleton.

Example 2.2. Let k be a field of characteristic $p = 5$ and take $W_{1,1}(5)$ to be the vector space over k with basis $z^i \partial_z$, for $i = 0, \ldots, 4$. The Lie bracket is defined on two of these elements by

$$[z^i \partial_z, z^j \partial_z] := (j - i) z^{i+j-1} \partial_z,$$

with the convention that $z^i = 0$ whenever $i \notin \{0, \ldots, 4\}$. The Lie bracket extends bilinearly to a multiplication on $W_{1,1}(5)$ which turns it into a Lie algebra of dimension 5 over k.

Now $x = -z^2 \partial_z$ is extremal in $W_{1,1}(5)$. Together with $y = \partial_z$ and $h = 2z\partial_z$ it forms an $\mathfrak{sl}_2$-triple in $W_{1,1}(5)$. But $[y, [y, 2z^4 \partial_z]] = x$, so y is not extremal in $W_{1,1}(5)$. In fact, no nilpotent element generating a subalgebra isomorphic to $\mathfrak{sl}(k^2)$ with x is extremal in $W_{1,1}(5)$. So, although the Jacobson-Morozov theorem (see [20, page 98]) holds, it does not provide a second extremal element. It readily follows that $W_{1,1}(5)$ is not generated by $\mathrm{E}(W_{1,1}(5))$.

The failure of Jacobson–Morozov elements to be extremal is exceptional. A relatively succinct proof in [12] shows that L generally has more than one extremal point.

Theorem 2.3. *Let the characteristic of k be distinct from 2 and 3, and let L be simple. Suppose that L contains an extremal element that is not a sandwich. Then either k has characteristic 5 and L is isomorphic to $W_{1,1}(5)$ or L is linearly spanned by extremal elements.*

Note that the field k need not be algebraically closed. If no pair $(x, y) \in \mathrm{E} \times \mathrm{E}$ would lie in E_2, then, in view of Lemma 1.11(i), the Lie algebra L would be nilpotent. So, in simple Lie algebras containing extremal elements, E_2 will be non-empty. In the presence of a hyperbolic pair of extremal elements, the following structural result from [10] is very useful.

Proposition 2.4. *Suppose that $x, y \in E$ satisfy $g_x(y) = 1$ and write $U = \{u \in L \mid g_x(u) = g_y(u) = g_x([y, u]) = 0\}$.*

(i) There is a $\mathbb{Z}$-grading

$$L = L_{-2}(x, y) + L_{-1}(x, y) + L_0(x, y) + L_1(x, y) + L_2(x, y), \qquad (9)$$

with $L_{-2}(x,y) = kx$, $L_2(x,y) = ky$, $L_0(x,y) = N_L(kx) \cap N_L(ky)$, $L_{-1}(x,y) = [x,U]$, *and* $L_1(x,y) = [y,U]$. *Furthermore,* ad_x *induces a linear isomorphism* $L_1(x,y) \to L_{-1}(x,y)$ *with inverse* $-\mathrm{ad}_y$.

(ii) There is a filtration

$$L_{\leq -2}(x) \subseteq L_{\leq -1}(x) \subseteq L_{\leq 0}(x) \subseteq L_{\leq 1}(x) \subseteq L_{\leq 2}(x) = L, \qquad (10)$$

where $L_{\leq i}(x) = \sum_{j=-2}^{i} L_j(x,y)$. *Moreover,* $L_{\leq 1}(x) = \{z \in L \mid g_x(z) = 0\}$, $L_{\leq 0}(x) = N_L(kx)$, *and* $L_{\leq -1}(x) = kx + [x, L_{\leq 1}(x)]$. *In particular, the subspaces* $L_{\leq i}(x)$ *are independent of the choice of* y.

For each $i \in \{-2,-1,0,1,2\}$, the subspace $L_i(x,y)$ is contained in the i-eigenspace of $\mathrm{ad}_{[x,y]}$. If the characteristic is distinct from 2, 3, 5, these eigenspaces coincide with the subspaces of the grading of (i). This illustrates why it is difficult to obtain the result for characteristic 5 and why there is no result for characteristics 2 and 3.

Recall that $\mathcal{F}$ is the set of projective lines of L that lie entirely in $\mathcal{E}$. This implies that two points are collinear if and only if they strongly commute. Being a subspace of the projective space on L, the line space $(\mathcal{E},\mathcal{F})$ is partial linear. It satisfies a host of useful properties. For notions not explained here, see [8] or the beginning of the next section.

Theorem 2.5. *Let* L *be generated by* $\mathrm{E}(L)$ *and suppose* L *has no sandwiches. Consider the space* $(\mathcal{E},\mathcal{F})$ *together with the symmetric relations* $\mathcal{E}_i$ $(i = -2, ..., 2)$ *on* $\mathcal{E}$. *It satisfies the following properties, where we write* $\mathcal{E}_{\leq i}$ *for* $\cup_{j \leq i} \mathcal{E}_j$.

(A) The relation $\mathcal{E}_{-2}$ *is equality on* $\mathcal{E}$.
(B) The relation $\mathcal{E}_{-1}$ *is collinearity of distinct points of* $\mathcal{E}$.
(C) There is a map $\mathcal{E}_1 \to \mathcal{E}$, *denoted by* $(u,v) \mapsto [u,v]$ *such that, if* $(u,v) \in \mathcal{E}_1$ *and* $x \in \mathcal{E}_i(u) \cap \mathcal{E}_j(v)$, *then* $[u,v] \in \mathcal{E}_{\leq i+j}(x)$.
(D) For each $(x,y) \in \mathcal{E}_2$, *we have* $\mathcal{E}_{\leq 0}(x) \cap \mathcal{E}_{\leq -1}(y) = \emptyset$.
(E) For each $x \in \mathcal{E}$, *the subsets* $\mathcal{E}_{\leq -1}(x)$ *and* $\mathcal{E}_{\leq 0}(x)$ *are subspaces of* $(\mathcal{E},\mathcal{F})$.
(F) For each $x \in \mathcal{E}$, *the subset* $\mathcal{E}_{\leq 1}(x)$ *is a hyperplane of* $(\mathcal{E},\mathcal{F})$.

This is the geometric structure that we will take as a starting point of our geometric characterization attempt for classical Lie algebras.

3 Root Filtration Spaces

We introduce root filtration spaces and derive some of their properties. We begin with some notation for relations on a set $\mathcal{E}$. Let $x \in \mathcal{E}$. For a relation $\mathcal{X}$ on $\mathcal{E}$, we denote by $\mathcal{X}(x)$ the set of all elements $y \in \mathcal{E}$ with $(x,y) \in \mathcal{X}$. If, in addition, $y,z \in \mathcal{E}$ and $Y \subseteq \mathcal{E}$, we write $\mathcal{X}(x,y)$ for $\mathcal{X}(x) \cap \mathcal{X}(y)$, $\mathcal{X}(x,y,z)$ for $\mathcal{X}(x) \cap \mathcal{X}(y) \cap \mathcal{X}(z)$, and $\mathcal{X}(Y)$ for $\bigcap_{y \in Y} \mathcal{X}(y)$, etc.

A *point-line space* (or just *space*) is a pair $(\mathcal{E}, \mathcal{F})$ consisting of a set $\mathcal{E}$ (of points) and a collection $\mathcal{F}$ of subsets of $\mathcal{E}$ of size at least 2 (whose members are called lines). A space is called a *gamma space* if, for each point p and each line l not on p, the set of points on l collinear with p is either empty, a singleton, or all of l. It is called a *partial linear space* if every pair of distinct points is on at most one line, and a *linear space* if every pair of distinct points is on precisely one line. A *subspace* of $(\mathcal{E}, \mathcal{F})$ is a subset of $\mathcal{E}$ containing each line that has at least two points in common with it. The *rank* of a linear space is one less than the length of a maximal chain of proper subspaces; if there is no such finite chain, the rank is said to be ∞. A *singular subspace* of a space is a subspace in which any two points are collinear. The *singular rank* of a partial linear space is the supremum of all ranks of maximal singular subspaces.

We say that a subspace of a point-line space is a *hyperplane* if every line has a non-empty intersection with it. Thus, the whole point set is a hyperplane.

The definitions of *polar space*, *nondegeneracy* of a polar space, and *rank* of a polar space, are as in [8].

Definition 3.1. Let $(\mathcal{E}, \mathcal{F})$ be a partial linear space. For $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$, a quintuple of disjoint symmetric relations partitioning $\mathcal{E}\times\mathcal{E}$, we call $(\mathcal{E}, \mathcal{F})$ a *root filtration space with filtration* $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$ if the properties (A)–(F) of Theorem 2.5 are satisfied.

If, in addition, $(\mathcal{E}, \mathcal{F})$ satisfies the following two conditions, it is called a *non-degenerate* root filtration space.

(G) For each $x \in \mathcal{E}$ the set $\mathcal{E}_2(x)$ is not empty.
(H) The graph $(\mathcal{E}, \mathcal{E}_{-1})$ is connected.

Thus, by definition, Theorem 2.5 can be restated as in the first part of the following result of [10].

Theorem 3.2. *Suppose that L is a Lie algebra over k generated by* E. *Let $\mathcal{F}$ be the set of projective lines all of whose points belong to $\mathcal{E}$. If L does not contain sandwiches then $(\mathcal{E}, \mathcal{F})$ is a root filtration space with filtration $(\mathcal{E}_i)_{-2\leq i\leq 2}$ as defined in Definition 3.1. Furthermore, let $\mathcal{B}_i$ $(i \in I)$ be the connected components of the graph $(\mathcal{E}, \mathcal{E}_2)$. Then each $\mathcal{B}_i$ is either a non-degenerate root filtration space or a root filtration space with an empty set of lines. Furthermore, L is the direct sum of the Lie subalgebras $\langle\mathcal{B}_i\rangle$ generated by $\mathcal{B}_i$ for $i \in I$.*

The second part of the theorem shows that the decomposition of L into direct summands can be recovered from the geometry.

Condition (C) gives a kind of filtration around each extremal point x. According to Lemma 3.3(ii) below, $[u, v]$ is the unique point in $\mathcal{E}_{\leq -1}(u) \cap \mathcal{E}_{\leq -1}(v)$, so the map $[\cdot, \cdot]$ is uniquely determined by the relations $(\mathcal{E}_i)_{-2\leq i\leq 2}$.

Condition (D) is referred to as the *triangle condition on x, y, z* by Timmesfeld [31]. It can be replaced by the statement that, for each $(x, y) \in \mathcal{E}_2$, we have $\mathcal{E}_{\leq i}(x)\cap \mathcal{E}_{\leq j}(y) = \emptyset$ whenever $i + j < 0$.

Condition (E) can be replaced by the statement that $\mathcal{E}_{\leq i}(x)$ is a subspace of $(\mathcal{E}, \mathcal{F})$ for each i (see Lemma 3.3(i) below).

We will use some of the terminology introduced for Lie algebras in Definition 1.10, calling a pair $(x, z) \in \mathcal{E}_i$ *hyperbolic* if $i = 2$, *special* if $i = 1$, *polar* if $i = 0$, *collinear* if $i = -1$ (so collinearity is only used for distinct points). In addition we say (x, z) are *commuting* (notation $[x, z] = 0$) if $i \leq 0$. The following properties of root filtration spaces are derived in [11].

Lemma 3.3. *In a root filtration space $(\mathcal{E}, \mathcal{F})$, the following properties hold.*

(i) For each $i \in \{-2, \dots, 2\}$ and each $x \in \mathcal{E}$, the subset $\mathcal{E}_{\leq i}(x)$ is a subspace of $(\mathcal{E}, \mathcal{F})$.
(ii) If $(u, v) \in \mathcal{E}_1$, then $[u, v]$ is the unique common neighbor of both u and v in the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ of $(\mathcal{E}, \mathcal{F})$.
(iii) If $(u, v) \in \mathcal{E}_1$, then $\mathcal{E}_0(u) \cap \mathcal{E}_2(v) \subseteq \mathcal{E}_1([u, v])$.
(iv) If $(x, y) \in \mathcal{E}_0$ and $z \in \mathcal{E}_{-1}(y)$, then either $z \in \mathcal{E}_{\leq 0}(x)$, or $z \in \mathcal{E}_1(x)$ and $\mathcal{E}_{-1}(x, y, z) = \{[x, z]\}$.
(v) If (x, q) and (u, z) belong to $\mathcal{E}_1$ whereas $u = [x, q]$ and $q = [u, z]$, then $(x, z) \in \mathcal{E}_2$.
(vi) If P is a pentagon in the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ (that is, the induced subgraph is a pentagon), then each distinct non-collinear pair of points of P is polar.
(vii) If $(u, v) \in \mathcal{E}_1$, then $\mathcal{E}_{-1}(u) \cap \mathcal{E}_0([u, v]) \subseteq \mathcal{E}_1(v)$.
(viii) Let $y \in \mathcal{E}$ and $l \in \mathcal{F}$ be such that $y \in \mathcal{E}_0(l)$. Then $\mathcal{E}_{\leq -1}(y, l)$ is a non-empty singular subspace of $(\mathcal{E}, \mathcal{F})$.

Here are some examples of root filtration spaces.

Example 3.4. Every linear space is a trivial example of a root filtration space with $\mathcal{E}_i = \emptyset$ for $i \geq 0$.

Every space without lines is a trivial example of a root filtration space with $\mathcal{E}_i = \emptyset$ for $-2 < i < 2$ and $\mathcal{E}_2$ the relation of being distinct. Even if we keep $\mathcal{E}_1 = \mathcal{E}_{-1} = \emptyset$ and allow for the relation of being distinct to be partitioned in $\mathcal{E}_0$ and $\mathcal{E}_2$, the result is a root filtration space. For example, if $(\mathcal{E}, \mathcal{L})$ is a polar space, taking $\mathcal{E}_2$ to be the non-collinearity relation, $\mathcal{E}_{-2}$ the identity relation, and $\mathcal{E}_0$ the complement in $\mathcal{E} \times \mathcal{E}$ of $\mathcal{E}_{-2} \cup \mathcal{E}_2$, we obtain a root filtration space $(\mathcal{E}, \emptyset)$ with $\mathcal{E}_{-1} = \mathcal{E}_1 = \emptyset$. Nondegeneracy of this polar space $(\mathcal{E}, \mathcal{L})$ is equivalent to condition (G). This makes clear why some nondegeneracy conditions like (G) and (H) are needed.

Example 3.5. Every generalized hexagon $(\mathcal{E}, \mathcal{F})$ with $\mathcal{E}_i$ for $i = 1, 2$ the set of points at mutual distance $i + 1$, and $[x, y]$ the unique point collinear with both x and y for $(x, y) \in \mathcal{E}_1$, is a non-degenerate root filtration space with $\mathcal{E}_0 = \emptyset$. Conversely, if $(\mathcal{E}, \mathcal{F})$ is a non-degenerate root filtration space with $\mathcal{E}_0 = \emptyset$, then it is a generalized hexagon.

Example 3.6. Let $\mathbb{P}$ and $\mathbb{H}$ be projective spaces in duality: $\mathbb{H}$ is a collection of hyperplanes forming a subspace of the dual of $\mathbb{P}$ annihilating $\mathbb{P}$. The latter means that the intersection of all hyperplanes of $\mathbb{H}$ is empty. If $\mathbb{P}$ has finite rank, this condition forces $\mathbb{H}$ to be the dual of $\mathbb{P}$. Take $\mathcal{E}$ to be the set of incident pairs from $\mathbb{P} \times \mathbb{H}$. The set $\mathcal{F}$ of lines is built up of two kinds: those consisting of all (x, H) with hyperplane $H \in \mathbb{H}$ fixed and x running through the points of a line of $\mathbb{P}$ inside H, and those consisting of all (x, H) with point x fixed and H running through the hyperplanes in $\mathbb{H}$ containing a fixed codimension 2 subspace of $\mathbb{P}$ containing x. So, $((x, H), (y, K)) \in \mathcal{E}_{-1}$ iff $x = y$ or $H = K$ (but not both). Then $(\mathcal{E}, \mathcal{F})$ is a root filtration space with $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$ defined as follows: $(x, H) \in \mathcal{E}_0((y, K))$ iff $x \in K$ and $y \in H$ and $x \neq y$ and $H \neq K$, $(x, H) \in \mathcal{E}_1((y, K))$ iff $x \in K$ (in which case $[(x, H), (y, K)] = (x, K))$ or $y \in H$ (in which case $[(x, H), (y, K)] = (y, H))$ but not both, and $(x, H) \in \mathcal{E}_2((y, K))$ iff $x \notin K$ and $y \notin H$. We denote this root filtration space by $\mathcal{E}(\mathbb{P}, \mathbb{H})$.

Example 3.7. Let $(\mathcal{P}, \mathcal{E})$ be a non-degenerate polar space. Then the Grassmann space $(\mathcal{E}, \mathcal{F})$ on $(\mathcal{P}, \mathcal{E})$, where $\mathcal{F}$ consists of the pencils of lines on a point in a singular plane, is a root filtration space with $l \in \mathcal{E}_{-1}(m)$ iff l and m span a singular plane, $l \in \mathcal{E}_0(m)$ iff either l and m span a singular subspace not contained in a plane, or l and m intersect but do not span a singular plane, $l \in \mathcal{E}_1(m)$ iff there is a unique line n such that both the span of l and n and the span of n and m are singular planes (in which case $n = [l, m]$, and the span of l and m is not a singular subspace), and $\mathcal{E}_2$ is the complement of $\mathcal{E}_{\leq 1}$ in $\mathcal{E} \times \mathcal{E}$. The same construction for a projective space instead of a polar space leads to a root filtration space with $\mathcal{E}_1 = \mathcal{E}_2 = \emptyset$.

Let $\mathbb{P}$ and $\mathbb{H}$ be as in Example 3.6. Consider the space $(\mathcal{P}, \mathcal{L})$ whose point set $\mathcal{P}$ is the disjoint union of $\mathbb{P}$ and $\mathbb{H}$ and whose line set $\mathcal{L}$ is the union of the line set of $\mathbb{P}$, the line set of $\mathbb{H}$ and the set of all unordered pairs $\{x, H\}$ with $x \in \mathbb{P}$ and $H \in \mathbb{H}$ such that $x \in H$. This is a non-degenerate polar space, called the *dualized projective space* of $\mathbb{P}$ and $\mathbb{H}$. The root filtration space $\mathcal{E}(\mathbb{P}, \mathbb{H})$ defined in Example 3.6 is a subspace of the Grassmann space on the dualized projective space $(\mathcal{P}, \mathcal{L})$.

Example 3.8. Suppose that $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ and $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ are root filtration spaces. Let $\mathcal{E}$ be the disjoint union of $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ and let $\mathcal{F}$ be the disjoint union of $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$. Then $(\mathcal{E}, \mathcal{F})$ is a root filtration space with filtration $\mathcal{E}_0 = \mathcal{E}_0^{(1)} \cup \mathcal{E}_0^{(2)} \cup (\mathcal{E}^{(1)} \times \mathcal{E}^{(2)}) \cup (\mathcal{E}^{(2)} \times \mathcal{E}^{(1)})$ and $\mathcal{E}_i = \mathcal{E}_i^{(1)} \cup \mathcal{E}_i^{(2)}$ for $i \neq 0$.

Under mild conditions, an arbitrary root filtration space can be deconstructed in the vein of Example 3.8.

Lemma 3.9. *If $(\mathcal{E}, \mathcal{F})$ is a root filtration space satisfying (G), then $(\mathcal{E}, \mathcal{F})$ is the disjoint union of connected subspaces $\mathcal{B}_i$ such that $\mathcal{B}_i \times \mathcal{B}_j \subseteq \mathcal{E}_0$ whenever $i \neq j$ unless $\mathcal{B}_i \times \mathcal{B}_j \subseteq \mathcal{E}_2$, in which case $\mathcal{B}_i$ and $\mathcal{B}_j$ are singletons. Moreover, if $x, y \in \mathcal{B}_i$ for some i and $(x, y) \in \mathcal{E}_0$ then $\mathcal{E}_{-1}(x, y) \neq \emptyset$.*

Remark 3.10. The relations $(\mathcal{E}_i)_{-2 \leq i \leq 2}$ and the map $[\cdot, \cdot] : \mathcal{E}_1 \to \mathcal{E}$ of a non-degenerate root filtration space $(\mathcal{E}, \mathcal{F})$ are fully determined by the space $(\mathcal{E}, \mathcal{F})$ itself. For, $\mathcal{E}_{-1}$, $\mathcal{E}_0 \cup \mathcal{E}_1$, and $\mathcal{E}_2$ are the relations of having distance 1, 2, and 3 in the collinearity graph of $(\mathcal{E}, \mathcal{F})$, and, for x, y in $\mathcal{E}$ at mutual distance 2, we have $x \in \mathcal{E}_1(y)$ if and only if x and y have a unique common neighbor (which coincides with $[x, y]$). Therefore, we will often not mention the filtration explicitly when introducing a non-degenerate root filtration space.

Each spherical building whose Coxeter diagram comes from a Dynkin diagram corresponds to a root filtration space. This holds in particular for all thick spherical buildings. To clarify this, we need the following definitions. Recall that a Dynkin diagram is the Coxeter diagram of a Weyl group whose bonds with an even label greater than 2 are directed.

Definition 3.11. Let Y_n be an irreducible Dynkin diagram of rank $n > 1$. We number its nodes with $1, \ldots, n$ as in Bourbaki [4]. Denote by X_n the corresponding Coxeter diagram (obtained by removing the arrow of a multiple bond). To avoid confusion between the Dynkin diagram and the Coxeter diagram, we shall write $X_n = (\mathrm{B}|\mathrm{C})_n$ for the Coxeter diagram corresponding to both B_n and C_n.

Let $\widetilde{Y}_n$ be the extended (or affine) Dynkin diagram of Y_n. Its nodes are those of Y_n and an additional node, numbered 0. By J we denote the subset of $\{1, \ldots, n\}$ consisting of all nodes of Y_n adjacent to 0. Then $J = \{1, n\}$ if $M = \mathrm{A}_n$, and $J = \{j\}$, where $j = 2$ if $Y_n = \mathrm{B}_n$, $j = 1$ if $Y_n = \mathrm{C}_n$, $j = 2$ if $Y_n = \mathrm{D}_n$ or E_6, $j = 1$ if $Y_n = \mathrm{E}_7$, $j = 8$ if $Y_n = \mathrm{E}_8$, $j = 1$ if $Y_n = \mathrm{F}_4$, $j = 2$ if $Y_n = \mathrm{G}_2$. We shall call J the *root nodes* or, if appropriate, j the *root node* of Y_n.

Let $\mathcal{C}$ be a building of type X_n. Following [33], we view it as a chamber system over $R = \{1, \ldots, n\}$. Let J be an arbitrary subset of R. The J*-shadow* of a chamber c of $\mathcal{C}$ is the $(R \setminus J)$-cell containing c. For $j \in J$, we define a j*-line* to be the union of all $(R \setminus J)$-cells containing a chamber from a given j-panel. The pair $(\mathcal{E}, \mathcal{F})$ consisting of the set $\mathcal{E}$ of all J-shadows and the set $\mathcal{F}$ of all j-lines, for $j \in J$, is called the *shadow space* for $\mathcal{C}$ of type $X_{n,J}$. If $J = \{j\}$, we also write $X_{n,j}$ instead of $X_{n,J}$. If J is the set of root nodes of a corresponding Dynkin diagram Y_n, we call $(\mathcal{E}, \mathcal{F})$ the *root shadow space* of $\mathcal{C}$ with respect to Y_n. If X_n has multiple bonds, there are two choices for Y_n, whence two root shadow spaces of $\mathcal{C}$.

Example 3.12. Let $(\mathcal{E}, \mathcal{F})$ be the root shadow space of a spherical building of irreducible Dynkin type Y_n with $n \geq 3$. If $Y_n = \mathrm{C}_n$, then $(\mathcal{E}, \mathcal{F})$ is a non-degenerate polar space. If $Y_n \neq \mathrm{C}_n$, then $(\mathcal{E}, \mathcal{F})$ is a non-degenerate root filtration space.

These results are proved in [10], by means of the following facts, where J is the set of root nodes.

– The various mutual geometric positions two J-cells can be in (in case of a sufficiently transitive automorphism group, such as the examples coming from algebraic groups, this means the orbits of the group on pairs of J-cells) are parameterized by left and right J-reduced elements of the corresponding Coxeter group of type X_n.
– In turn, these reduced elements are parameterized by the inner products between pairs of long roots of the associated root system. The five possible values are the familiar integers $-2, -1, 0, 1, 2$.

Root shadow spaces of buildings of rank at least 3 have been discussed. Those of rank 1 are non-degenerate polar spaces of rank 1 and those of irreducible Dynkin type of rank 2 are either generalized hexagons (cases A_2 and G_2) or generalized quadrangles (cases, B_2 and C_2). Therefore, every root shadow space of a building of irreducible spherical type is a non-degenerate root filtration space or a non-degenerate polar space.

Example 3.5 represents the root shadow space of a building of type G_2, its dual corresponds to the other interpretation of G_2 as a Dynkin diagram. Example 3.6 represents a generalization to arbitrary rank of type A_n, Example 3.7 represents both Dynkin types B_n and D_n, and the polar spaces of Example 3.4 are of Dynkin type C_n.

Timmesfeld's non-degenerate sets of abstract root subgroups are in fact non-degenerate root filtration spaces. To explain this observation, we begin by recalling the notion of abstract root subgroups as appearing in [31, Definition (1.1), Chap. II] (the wording is adjusted to our setting). If A and B are subgroups of a given group G, then $[A, B]$ stands for the subgroup of G generated by all commutators $[a, b] := a^{-1}b^{-1}ab = a^{-1}a^b$ with $a \in A$ and $b \in B$. Similarly for $[a, B]$ and $[A, b]$.

Definition 3.13. Let G be a group. A set $\mathcal{E}$ of abelian non-trivial subgroups of G is called a set of *abstract root subgroups* of G if it satisfies the following two conditions.

(I) $G = \langle \mathcal{E} \rangle$ and $\mathcal{E}^g \subseteq \mathcal{E}$ for each $g \in G$.
(II) For each pair $a, b \in \mathcal{E}$ one of the following cases occurs, where $X = \langle a, b \rangle$:

- (-2) $a = b$, and so $X = a = b$.
- (-1) $[a, b] = 1$, $a \neq b$, and $X \setminus \{1\}$ is partitioned by $c \setminus \{1\}$ for $c \in \mathcal{E}$ with $c \leq X$. Here, we call the *line ab* the set of elements $c \in \mathcal{E}$ with $c \leq ab$. By $\mathcal{F}$ we denote the set of lines.
- (0) $[a, b] = 1$ and $X \setminus \{1\}$ is not partitioned by $c \setminus \{1\}$ for $c \in \mathcal{E}$ with $c \leq X$.
- (1) $[a, b]$ belongs to $\mathcal{E}$ and coincides with $[a_0, b]$ and with $[a, b_0]$ for every nontrivial $a_0 \in a$ and $b_0 \in b$; this subgroup is nontrivial and is contained in $Z(X)$, the center of X.
- (2) For each nontrivial $a_0 \in a$ there exists a nontrivial $b_0 \in b$ such that $a^{b_0} = b^{a_0}$; and similarly with a and b interchanged.

Case (2) above is described as 'X is a rank one group with unipotent subgroups a and b'. Chapter I of [31] is concerned with the structure of such groups. The subgroups a and b as in (II)(2) are X-conjugate and their X-conjugacy class is called a *hyperbolic line*. Typical examples of X are the groups (P)SL$(2, k)$ for a (skewfield) k, in which case the hyperbolic line corresponds to the points of the projective line on which X acts 2-transitively. When viewed as the rational points over a field k of an algebraic group, its Lie algebra corresponds to $\mathfrak{sl}(k^2)$ (up to central isogeny), and a and b to a hyperbolic pair of points therein.

Case (1) is the so-called special case; typical examples are extra-special p-groups of order p^3, often suggestively denoted by p^{1+2}. When viewed as the $(\mathbb{Z}/p\mathbb{Z})$-rational points of the upper triangular group in SL(k^3) for k a field of characteristic p, its Lie algebra is the Heisenberg Lie algebra $\mathfrak{h}$ over k of Example 1.5.

For each $j \in \{-2, -1, 0, 1, 2\}$ we write $\mathcal{E}_j$ to denote the relation on $\mathcal{E}$ expressing that a, b are in case (j). So $(a, b) \in \mathcal{E}_{-2}$ is equivalent to $a = b$, and $a \in \mathcal{E}_{-1}(b)$ means that a and b belong to a line in $\mathcal{F}$. Notice that $\mathcal{E}_{\leq 0}(x)$ is the set of subgroups in $\mathcal{E}$ commuting with x.

The following result shows that the images of the maps in Lemma 1.3 are abstract root subgroups of the algebraic group $\mathrm{Aut}(L)$.

Theorem 3.14. *Suppose that L is a Lie algebra over k generated by extremal elements and containing no sandwiches. Then the set of subgroups $\{\exp(x,t) \mid t \in k\}$ of $Aut(L)$ for $x \in E$ is a set of abstract root subgroups of the subgroup of $Aut(L)$ which they generate.*

Example 3.15. Let G be a group. In [31, Definition (1.1)], nondegeneracy of a set $\mathcal{E}$ of abstract root subgroups in G is defined by the condition that $\mathcal{E}_{\leq 0}$, $\mathcal{E}_1$, $\mathcal{E}_2$ be non-empty. Suppose that $\mathcal{E}$ is a non-degenerate set of abstract root subgroups of G such that $(\mathcal{E}, \mathcal{E}_2)$ is connected and $\mathcal{E}_{-1}$ is nonempty. If G has no solvable normal subgroup, then $(\mathcal{E}, \mathcal{F})$, for $\mathcal{F}$ the set of lines of $\mathcal{E}$, is a non-degenerate root filtration space with thick lines. In [31, Sect. II.4], Timmesfeld showed that, when G is generated by a set of abstract root groups, the solvable radical of G is nilpotent of class at most two.

Observe that, for $(a,b) \in \mathcal{E}_1$, we have $[a,b] \in \mathcal{E}$, so we have a map $[\cdot,\cdot] : \mathcal{E}_1 \to \mathcal{E}$ as required in the definition of root filtration space.

For the proof (in [10]) that the set of abstract root subgroups of G gives rise to a root filtration space, some properties from [31] are needed that occur at an early stage.

4 Parapolar Spaces

This section is concerned with a converse of Example 3.12, which states that each non-degenerate root filtration space is the root shadow space of a building of spherical type. The result is proved in [11].

Theorem 4.1. *Let $(\mathcal{E}, \mathcal{F})$ be a non-degenerate root filtration space. If the singular rank of $(\mathcal{E}, \mathcal{F})$ is finite, then $(\mathcal{E}, \mathcal{F})$ is isomorphic to a shadow space of type $\mathrm{A}_{n,\{1,n\}}$ $(n \geq 2)$, $(\mathrm{B}|\mathrm{C})_{n,2}$ $(n \geq 3)$, $\mathrm{D}_{n,2}$ $(n \geq 4)$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, $\mathrm{F}_{4,1}$, or $\mathrm{G}_{2,2}$.*

The only root shadow spaces missing from this result are polar spaces, that is, the shadow spaces of type $(\mathrm{B}|\mathrm{C})_{n,1}$. A characterization of polar spaces in terms of degenerate root filtration spaces is given by Cuypers [15] who extended earlier results of Timmesfeld [31]. See Sect. 5, where it is also described how Cuypers' result can be used in the characterization of classical simple Lie algebras with degenerate root filtration spaces.

We briefly describe the proof of Theorem 4.1. In order to reconstruct the building from a non-degenerate root filtration space, the elements of the building of types other than point and line need to be found. An initial step is to show that the space is a parapolar space. We recall some definitions from [8].

Definition 4.2. A *parapolar space* is a connected partial linear gamma space possessing a collection of geodesically closed subspaces, called *symplecta* (singular: *symplecton*), isomorphic to non-degenerate polar spaces of rank at least 2, with the properties that each line is contained in a symplecton and that each pair of distinct non-collinear points having at least 2 common neighbors is contained in a unique symplecton. (This definition comes from [11] and is a slight improvement of the one in [8], where the existence of symplecta was only required for quadrangles.) If all symplecta are polar spaces of rank k (respectively, of rank at least k) the space is said to have *polar rank* k (respectively, polar rank at least k).

A very useful condition that separates the parapolar spaces of our interest from others is the following condition, which originates from [9].

(K) For each point x and each symplecton S the set of points of S collinear with x is either empty or contains a line.

Note that the set of points inside a symplecton that are collinear with a point outside of it form a singular subspace, so (K) excludes the possibility of a singleton.

Recall the root filtration space $\mathcal{E}(\mathbb{P}, \mathbb{H})$ of Example 3.6. In [11] the following result is established.

Theorem 4.3. *Let $(\mathcal{E}, \mathcal{F})$ be a non-degenerate root filtration space.*

(i) If some line in $\mathcal{F}$ is contained in a unique maximal singular subspace, then one of the following cases prevails.

I. The singular rank is 1 *and $(\mathcal{E}, \mathcal{F})$ is a root shadow space of type* $\mathrm{G}_{2,1}$.

II. The singular rank is at least 2*, there is a point that belongs to at least 3 maximal singular subspaces, and $(\mathcal{E}, \mathcal{F})$ is a root shadow space of type* $(\mathrm{B}|\mathrm{C})_{3,2}$.

III. The singular rank is at least 2*, on each point there are precisely 2 maximal singular subspaces, and $(\mathcal{E}, \mathcal{F})$ is isomorphic to $\mathcal{E}(\mathcal{M}, \mathcal{N})$ for a certain pair $(\mathcal{M}, \mathcal{N})$ of projective spaces in duality.*

(ii) If some line in $\mathcal{F}$ is contained in more than one maximal singular subspace, then $\mathcal{E}_1 \neq \emptyset$ and $(\mathcal{E}, \mathcal{F})$ is a parapolar space satisfying condition (K) *whose polar rank is at least* 3.

We briefly discuss the proof. For $(\mathcal{E}, \mathcal{F})$ as in the hypotheses, the set of common neighbors of a polar pair (x, y) is a non-degenerate polar space. In case (ii) this polar space has rank at least 2 and the required properties can be derived by use of Lemma 3.3.

In case (i) it is of rank 1; in other words, the subspace $\mathcal{E}_{-1}(x, y)$ of $\mathcal{E}$ has no lines. It is relatively easy to prove that each line l is in a unique maximal singular subspace $M(l)$. If the singular rank of $(\mathcal{E}, \mathcal{F})$ equals 1, then it can be shown that $\mathrm{E}_2 = \emptyset$, and it follows that $(\mathcal{E}, \mathcal{F})$ is a generalized hexagon, whence case I.

If there is a point in $\mathcal{E}$ that is in at least 3 maximal singular subspaces, then $(\mathcal{E}, \mathcal{F})$ has singular rank 2, and the symplecta of the resulting space can be shown to support the structure of a polar space (in which a line is the collection of symplecta on a line of $(\mathcal{E}, \mathcal{F})$). This leads to case II.

The case remains where each point in $\mathcal{E}$ is in exactly two maximal singular subspaces. Then the collection of maximal singular subspaces of $(\mathcal{E}, \mathcal{F})$ can be partitioned into two classes $\mathcal{M}$ and $\mathcal{N}$, such that each of $\mathcal{M}$ and $\mathcal{N}$ carries the structure of a projective space whose lines are the members of the class containing exactly one point of a line in $\mathcal{F}$. Moreover, the space $\mathcal{N}$ can be viewed as a subspace of the dual of $\mathcal{M}$ annihilating $\mathcal{M}$, and, with this identification, $(\mathcal{E}, \mathcal{F})$ is isomorphic to $\mathcal{E}(\mathcal{M}, \mathcal{N})$; hence, case III.

This ends the discussion of the proof of Theorem 4.3. We proceed with the brief overview of the proof of Theorem 4.1. In case (ii) of Theorem 4.3, the hypotheses of the Kasikova–Shult theorem of [17] hold. Below we state this theorem, which deals with point-line spaces of arbitrary rank. In order to cover the case where the polar rank is big, the following condition is needed.

(L) If all symplecta have rank at least 4, then the singular rank of $(\mathcal{E}, \mathcal{F})$ is finite.

The main result of [17, 27] reads as follows.

Theorem 4.4. *Each parapolar space of polar rank at least* 3 *satisfying (K) and (L), and containing a special pair is a shadow space of type* $(\mathrm{B}|\mathrm{C})_{n,2}$ $(n \geq 4)$*,* $\mathrm{D}_{n,2}$ $(n \geq 4)$*,* $\mathrm{E}_{6,2}$*,* $\mathrm{E}_{7,1}$*,* $\mathrm{E}_{8,8}$*, or* $\mathrm{F}_{4,1}$.

In [27], Shult shows that, if $(\mathcal{E}, \mathcal{F})$ is a parapolar space of polar rank at least 3 in which (K) holds, then, for each singular plane π and each line l meeting π at a point x and not lying in a singular subspace containing π, either one or all lines m on x in π have the property that $l \cup m$ belongs to a symplecton. In an earlier version of Theorem 4.4, this property was stated as a condition.

Theorem 4.4 is proved by reconstructing the elements of missing types and invoking [33] to recognize the building. The reconstruction is carried out by deriving conditions needed in earlier theorems characterizing buildings by means of point-line spaces, like [9].

Theorem 4.1 follows from Theorems 4.3 and 4.4. For, the singular rank of $(\mathcal{E}, \mathcal{F})$ is finite, so case III of Theorem 4.3 leads to $(\mathcal{E}, \mathcal{F})$ being a shadow space of type $\mathrm{A}_{n,\{1,n\}}$.

5 Degenerate Root Filtration Spaces

In this section, we are concerned with root filtration spaces that are polar spaces. In particular, we review how Lie algebras generated by extremal elements in which no strongly commuting pairs occur, have a root filtration space coming from a polar space, as described in Example 3.4.

At the root filtration space level, we have a useful result by Cuypers [15], which generalizes an abstract root group version by Timmesfeld [31]. In order to state it, we need the notion of transversal coclique.

Definition 5.1. A *dual affine plane* is a point-line space obtained from a projective plane by removing a point and all lines containing that point. A *transversal coclique* in a dual affine plane is the set of points forming such a removed line together with the removed point.

Theorem 5.2. *Let* $(\mathcal{E}, \mathcal{H})$ *be a point-line space and let* $\sim$ *denote being collinear and distinct. Write* $\perp$ *for its complement. Suppose* $(\mathcal{E}, \mathcal{H})$ *satisfies the following six conditions.*

(i) $(\mathcal{E}, \mathcal{H})$ *is a connected partial linear space that is not linear.*
(ii) Each line contains at least four points.
(iii) The subspace of $(\mathcal{E}, \mathcal{H})$ *generated by any triple of points* x, y, z *with* $x \sim y \sim z \perp x$ *is a dual affine plane.*
(iv) If a point is not collinear with two points of a transversal coclique in a subspace isomorphic to a dual affine plane, then it is not collinear with any point of that coclique.
(v) If x, y *are points with* $x^{\perp} \subseteq y^{\perp}$*, then* $x = y$.
(vi) $(\mathcal{E}, \perp)$ *is connected.*

Then $(\mathcal{E}, \perp)$ *is the collinearity graph of a non-degenerate polar space.*

The following characterization of the geometry on $\mathcal{E}(L)$ can be found in the PhD thesis [18] of in't panhuis.

Theorem 5.3. *Let* L *be a Lie algebra over a field* k *of size at least* 3 *generated by its extremal elements. Assume the following two conditions hold.*

(i) The sets $\mathcal{E}_{-1}$ *and* $\mathcal{E}_1$ *are empty.*
(ii) The set $\mathcal{E}_0$ *is nonempty and the graph* $(\mathcal{E}, \mathcal{E}_2)$ *is connected.*

Then $(\mathcal{E}, \mathcal{E}_0)$ *is the collinearity graph of a non-degenerate polar space.*

This result suffices for finding the full polar space as lines of a non-degenerate polar space are of the form $\{x, y\}^{\perp\perp}$ for distinct collinear points x and y. The proof of this theorem is based on Cuypers' Theorem 5.2. In the application to L, the members of the line set $\mathcal{H}$ are *hyperbolic lines*, that is, the set of extremal points of a subalgebra of L generated by two extremal elements forming a hyperbolic pair. The relation $\perp$ is just $\mathcal{E}_{-2} \cup \mathcal{E}_0$.

The remaining case, where k has two elements, is harder to characterize in terms of root filtration spaces. It is more easily described when hyperbolic lines are added, as the following result of [18] shows. The theory of abstract root groups was an initiative to view the beautiful results on 3-transpositions by Fischer as a special case of a more general set-up in which the original setting would be the case where a parameterizing field k would have two elements. In this respect, and with the root filtration spaces as geometric generalizations of abstract root groups, it is fitting that Fischer spaces occur as the root filtration spaces for $k = \mathbb{Z}/2\mathbb{Z}$.

Definition 5.4. A *Fischer space* is a partial linear space in which each plane is isomorphic to a dual affine plane of order two or to an affine plane of order three.

Theorem 5.5. *Let L be a Lie algebra over $\mathbb{Z}/2\mathbb{Z}$ generated by extremal elements whose root filtration space satisfies conditions (i) and (ii) of Theorem 5.3. Let $\mathcal{H}$ be the collection of hyperbolic lines of L. Then $(\mathcal{E}, \mathcal{H})$ is a connected Fischer space.*

6 Conclusion

Although there is work in progress, the third step in the characterization of classical Lie algebras by means of root filtration spaces has not yet been finalized and so it is formulated here as a question.

Problem 6.1. Under which conditions (e.g. nondegeneracy) is it true that, for a given root filtration space $(\mathcal{E}, \mathcal{F})$, there is, up to isomorphism, at most one simple Lie algebra whose root filtration space is isomorphic to $(\mathcal{E}, \mathcal{F})$?

This question is more general than the problem posed at the beginning of Sect. 2. If 'classical type' of a simple Lie algebra is interpreted as being the Lie algebra of an algebraic group or a central quotient thereof, Benkart's results are covered by the following proposition.

Proposition 6.2. *Let L be a simple Lie algebra generated by extremal elements over an algebraically closed field k. Then $G = \langle \exp(x,t) \mid x \in \mathrm{E}(L),\ t \in k \rangle$ is an algebraic subgroup of $\mathrm{Aut}(L)$ whose Lie algebra $\mathrm{Lie}(G)$ contains an ideal isomorphic to L.*

Proof. As the subgroups U_x for $x \in \mathrm{E}$ are closed connected subgroups of the algebraic group $\mathrm{Aut}(L)$ (cf. Lemma 1.3), the subgroup G of $\mathrm{Aut}(L)$ generated by all of them is also closed in $\mathrm{Aut}(L)$ (see [3, Proposition 2.2]). The Lie algebra $\mathrm{Lie}(G)$ can be viewed as a Lie subalgebra of $\mathrm{Der}(L)$, the subalgebra of all $X \in \mathfrak{gl}(L)$ such that $X[a,b] = [Xa,b] + [a,Xb]$ whenever $a, b \in L$. If $x \in \mathrm{E}$, the derivative of the embedding of U_x in $\mathrm{Aut}(L)$ has image $k\,\mathrm{ad}_x$, so ad_L is a subalgebra of $\mathrm{Lie}(G)$. It is even an ideal of $\mathrm{Lie}(G)$ as it is an ideal of $\mathrm{Der}(L)$, which contains $\mathrm{Lie}(G)$. As L is simple, it is isomorphic to ad_L. We conclude that L is isomorphic to the ideal ad_L of $\mathrm{Lie}(G)$, as required.

The Lie algebras L and $\mathrm{Lie}(G)$ need not coincide as follows from Example 6.3. If k has characteristic 0, all derivations of L are known to be inner (cf. [20, Theorem 6]), and so $\mathrm{Lie}(G)$, being contained in $\mathrm{Der}(\mathrm{ad}_L)$, coincides with ad_L; therefore, L is the Lie algebra of G if k has characteristic 0. This statement is even true if L is not simple provided its center is trivial (and the characteristic is still 0), as it is isomorphic then to the subalgebra ad_L of $\mathfrak{gl}(L)$, which is generated by the algebraic Lie subalgebras $k\,\mathrm{ad}_x$ for $x \in \mathrm{E}(L)$, so [3, Corollary 7.7] applies.

Example 6.3. In Example 1.6, we have seen an exceptional case (with k of characteristic 3) in which the dimension of L is as low as half the dimension of $\mathrm{Lie}(G)$. Take p to be a prime greater than 3 and let k be algebraically closed of characteristic p. Then the analogous construction yields a simple Lie algebra that still deviates from the Lie algebra of the corresponding group although the dimension gap is smaller: put $V = k^p$ and let $L = \mathfrak{sl}(V)/Z$, where $Z = Z(\mathfrak{sl}(V))$ is the (one-dimensional) center of $\mathfrak{sl}(V)$ spanned by the identity matrix. Then L is simple of dimension $p^2 - 2$. So, if H is a connected algebraic subgroup of $\mathrm{GL}(L)$ with $\mathrm{Lie}(H) = \mathrm{ad}_L$, then it must be simple of dimension $p^2 - 2$. But no simple connected algebraic group has such a dimension (as $p > 4$), so L is not even the Lie algebra of an algebraic group.

In Example 3.4, we have seen that there is little structure on root filtration spaces with $\mathcal{E}_{-1} = \mathcal{E}_1 = \emptyset$, and that polar spaces are among them. There is an abundance of examples of root filtration spaces that do not arise from a Lie algebra. For instance, let k be algebraically closed of characteristic distinct from two and let V be a vector space over k with a non-degenerate quadratic form κ. Define $(\mathcal{E}, \mathcal{F})$ as the polar space of the quadric determined by κ in the projective space on V. If the dimension of V is equal to 3, then the exterior product map leads to a Lie algebra structure on V isomorphic to $\mathfrak{o}(V, \kappa)$. For higher dimensions, no Lie algebra structure (nor any other antisymmetric product) on V exists that is invariant under the orthogonal group on V with respect to κ. But the embedding of $(\mathcal{E}, \mathcal{F})$ in the projective space on V is universal, and so there is no Lie algebra whose root filtration space is isomorphic to $(\mathcal{E}, \mathcal{F})$.

The Lie algebra $\mathfrak{o}(V, \kappa)$ is generated by extremal elements, but, if the dimension is higher than 3, the corresponding root filtration space is not isomorphic to $(\mathcal{E}, \mathcal{F})$. The root filtration space of $\mathfrak{o}(V, \kappa)$ is the Grassmannian of lines on this polar space.

Similarly, not every root filtration space comes from a set of abstract root groups. Another way to understand what is going on, is to consider a thick building of an irreducible spherical type having multiple bonds. Then there are two root shadow spaces of this building, according to two distinct interpretations of the Coxeter diagram as a Dynkin diagram. Except for 'bad characteristics', only one of these choices will lead to a correspondence with long root subgroups, and hence to abstract root subgroups. The number of positions two short root groups can be in is usually larger than five.

In the case where the field has size 2 and the root filtration space of the Lie algebra over $\mathbb{Z}/2\mathbb{Z}$ is degenerate, the hyperbolic lines lead to a Fischer space (see Theorem 5.5). Conversely, there is a universal way to construct a Lie algebra from the geometry such that the points of the geometry map to extremal points of the Lie algebra, which runs as follows. Given a Fischer space $(\mathcal{E}, \mathcal{H})$, let A be the vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis $\mathcal{E}$. Define the multiplication $*$ on A as the one determined by

$$x * y = \begin{cases} x + y + z & \text{if } \{x, y, z\} \in \mathcal{H} \\ 0 & \text{otherwise} \end{cases}.$$

Following [16], we consider the symplectic form f on A satisfying $f(x, y) = 1$ if x and y are distinct and collinear, and $f(x, y) = 0$ otherwise, for $x, y \in \mathcal{E}$. Then

f is associative, in the sense that $f(x, y * z) = f(x * y, z)$ for all $x, y, z \in A$, so $\mathrm{Rad}(f)$ is an ideal of A.

Now the quotient algebra $A/\mathrm{Rad}(f)$ is a Lie algebra if and only if each subset of $\mathcal{E}$ that is an affine plane of $(\mathcal{E}, \mathcal{H})$ belongs to $\mathrm{Rad}(f)$. This Lie algebra $A/\mathrm{Rad}(f)$ may collapse, so that again examples of Fischer spaces without corresponding Lie algebras occur. The Fischer space related to the biggest Fischer group F_{24} is an example where this happens. Also, two nonisomorphic Fischer spaces may lead to isomorphic Lie algebras. In one of these instances, the Lie algebra $A/\mathrm{Rad}(f)$ of the Fischer space related to Fischer's group F_{22} is isomorphic to the Lie algebra over $\mathbb{Z}/2\mathbb{Z}$ of twisted type ${}^2\mathrm{E}_6$, which leads to a geometric proof of the existence of an embedding of F_{22} in the twisted Chevalley group ${}^2\mathrm{E}_6(2)$. Many more details are to be found in [16, 18].

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