

Chapter 2

Adaptation and Contraction Theory for the Synchronization of Complex Neural Networks

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Abstract In this chapter, we will present two different approaches to solve the problem of synchronizing networks of interacting dynamical systems. The former will be based on making the coupling between agents in the network adaptive and evolving so that synchronization can emerge asymptotically. The latter will be using recent results from contraction theory to give conditions on the node dynamics and the network topology that result into the desired synchronized motion. The theoretical results will be illustrated by means of some representative examples, including networks of neural oscillators.

2.1 Introduction

Synchronization and coordination of motion are key features of many biological and living systems. For example, circadian rhythms regulate the functions of cells in our bodies which are entrained to the day/night cycle via the sophisticated action of various gene regulatory networks, see, for example, [14]. More notably, synchronization has been proposed as a powerful mechanism to explain some of the patterns observed in the brain [1, 3, 19, 27]. For instance, as noted in [18], partial synchrony in cortical networks can be used to explain several brain oscillatory patterns such as the alpha and gamma EEG rhythms. Also, synchronization of brain waves has

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been proposed to explain pathologies such as epilepsy while coordinated neuronal evolution is required to generate patterns essential for locomotion, breathing, etc.

In all of these instances, sets of agents (genes, neurons etc.) communicate with each other over a complex web of interconnections exchanging information on their mutual state. Through this distributed communication process, agents successfully negotiate how to change their own behavior so that some synchronous evolution emerges. Recent studies in synchronization have clearly shown that such a process is, in general, completely decentralized with no central supervisor determining how individual agents should behave. This fascinating feature of synchronization makes it possible for it to be observed also in those cases where uncertainties and external disturbances can affect the agent behavior. Indeed, networks of agents have been shown to achieve synchronization in a plethora of different ways under all sort of different conditions, for example, [14, 16, 17, 19, 22, 23, 37].

Usually in the literature to model and investigate synchronization phenomena, a network of interacting dynamical agents is often considered. Specifically, let $\mathcal{G} = \{\mathcal{N}, \mathcal{E}\}$ be a graph defined by the set of nodes $\mathcal{N}$ and the set of edges $\mathcal{E}$, having cardinality N and M , respectively. The classical model that has been used to describe a complex network of N nodes is

$$\dot{x}_i = f(x_i, t) - \sigma \sum_{j=1}^N \ell_{ij} h(x_j), \quad i = 1, \dots, N. \quad (2.1)$$

Here, $x_i \in \mathbb{R}^n$ is the state of node i , $f : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is the vector field describing the dynamics of an isolated node, $\sigma \in \mathbb{R}$ is the coupling strength, $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the output function through which the network nodes are coupled, and ℓ_{ij} is the ij -th element of the Laplacian matrix $\mathcal{L}$ associated to the graph $\mathcal{G}$ describing the network topology.

Several simplifying assumptions are often made in the literature in order to study the problem. For example, it is typically assumed that all nodes share the same dynamics, that the network topology is time-invariant and that a unique coupling gain, σ , determines the strength of the coupling between neighboring nodes. Clearly, these assumptions are unrealistic when compared to the natural world. Indeed in Nature, the coupling strength between agents as well as the very structure of their interconnections vary in time so that the network can adapt and evolve in order to maintain a synchronous evolution. Moreover, often in biological networks evolution leads to systems and coupling laws whose structure facilitate the emergence of synchronization.

In this chapter, we discuss two approaches to analyze and induce synchronization in networks of dynamical agents with the aim of mimicking these features of the natural world. Specifically, we will present a strategy to adapt and evolve a given network of dynamical agents so as to make all agents synchronize onto a common evolution. Also, we will show that by using contraction theory, a tool from the theory of dynamical systems, it is possible to analyze and design the agent behavior and coupling so that the network spontaneously synchronizes. We will validate both strategies on a network of FitzHugh–Nagumo neurons showing that indeed they

are effective methodologies to guarantee their synchronization. This is, of course, only a first step in showing that adaptation and evolution can play an important role in determining the effectiveness of those synchronization processes commonly detected in natural systems and the brain.

The rest of the chapter is outlined as follows. We first present in Sect. 2.2 the basic mathematical tools used for deriving our results. In Sect. 2.3, we present two synchronization strategies based on the use of adaptive coupling strengths. In Sect. 2.4, we then turn our attention to the problem of tuning some properties of neurons so that synchronization spontaneously emerges, while in Sect. 2.4.1 we present the main tool used to this aim. Finally, in Sect. 2.5 and Sect. 2.6 we illustrate our theoretical results and validate them for networks of FitzHugh–Nagumo neurons. Conclusions are drawn in Sect. 2.8.

2.2 Preliminaries

We consider systems of ordinary differential equations, generally time-dependent:

$$\dot{x} = f(t, x) \quad (2.2)$$

defined for $t \in [0, \infty)$ and $x \in C$, where C is a subset of $\mathbb{R}^n$. It will be assumed that $f(t, x)$ is differentiable on x , and that $f(t, x)$, as well as the Jacobian of f with respect to x , denoted as $J(t, x) = \frac{\partial f}{\partial x}(t, x)$, are both continuous in (t, x) . In applications of the theory, it is often the case that C will be a closed set, for example given by non-negativity constraints on variables as well as linear equalities representing mass-conservation laws. For a nonopen set C , differentiability in x means that the vector field $f(t, \bullet)$ can be extended as a differentiable function to some open set which includes C , and the continuity hypotheses with respect to (t, x) hold on this open set.

The solution of Eq. 2.2 is in principle defined only on some interval $s \leq t < s + \varepsilon$, but we will assume that it is defined for all $t \geq s$. Conditions which guarantee such a property are often satisfied in biological applications, for example, whenever the set C is closed and bounded, or whenever the vector field f is bounded. (See Appendix C in [36] for more discussion, as well as [2] for a characterization of the forward completeness property.)

One of the main assumptions used in literature to prove asymptotic synchronization of network Eq. 2.1 is the so-called QUAD assumption, which can be defined as follows:

Definition 2.1 A function $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ is QUAD(Δ, ω) if and only if, for any $x, y \in \mathbb{R}^n, t \in \mathbb{R}^+$:

$$(x - y)^T [f(x, t) - f(y, t)] - (x - y)^T \Delta (x - y) \leq -\omega (x - y)^T (x - y), \quad (2.3)$$

where Δ is an $n \times n$ diagonal matrix and ω is a real positive scalar.

Table 2.1 Standard matrix measures used in this work

Vector norm, $ \cdot $	Induced matrix measure, $\mu(A)$
$ x _1 = \sum_{j=1}^m x_j $	$\mu_1(A) = \max_j (a_{jj} + \sum_{i \neq j} a_{ij})$
$ x _2 = (\sum_{j=1}^n x_j ^2)^{\frac{1}{2}}$	$\mu_2(A) = \max_i (\lambda_i \{ \frac{A+A^*}{2} \})$
$ x _\infty = \max_{1 \leq j \leq m} x_j $	$\mu_\infty(A) = \max_i (a_{ii} + \sum_{j \neq i} a_{ij})$

Another possibility is to make use of contraction theory, a powerful tool from the theory of dynamical systems aimed at assessing the stability between trajectories in the state space of a system or network of interest. The idea is to assess under what conditions all trajectories of a given network or system converge towards each other and hence onto some common asymptotic evolution. In order to properly define this concept, we recall (see, for instance, [24]) that, given a vector norm on Euclidean space ($|\bullet|$), with its induced matrix norm $\|A\|$, the associated *matrix measure* μ is defined as the directional derivative of the matrix norm, that is,

$$\mu(A) := \lim_{h \searrow 0} \frac{1}{h} (\|I + hA\| - 1).$$

For example, if $|\bullet|$ is the standard Euclidean 2-norm, then $\mu(A)$ is the maximum eigenvalue of the symmetric part of A . As we shall see, however, different norms will be useful for our applications. In particular, we will make use of the 1-norm and ∞ -norm, which are reported in Table 2.1.

We say that a system such as the one in Eq. 2.2 is (*infinitesimally*) *contracting* [21] on a convex set $C \subseteq \mathbb{R}^n$ if there exists some norm in C , with associated matrix measure μ such that, for some constant $c \in \mathbb{R} - \{0\}$,

$$\mu(J(x, t)) \leq -c^2, \quad \forall x \in C, \quad \forall t \geq 0. \quad (2.4)$$

The key theoretical result about contracting systems links infinitesimal and global contractivity, and is stated below. This result was presented in [21] and historically can be traced, under different technical assumptions, to for example, [15, 20, 26].

Theorem 2.1 *Suppose that C is a convex subset of $\mathbb{R}^n$ and that $f(t, x)$ is infinitesimally contracting with contraction rate c^2 . Then, for every two solutions $x(t)$ and $z(t)$ rooted in $\xi \in C$ and $\zeta \in C$ respectively, of Eq. 2.2, it holds that:*

$$|x(t) - z(t)| \leq e^{-c^2 t} |\xi - \zeta|, \quad \forall t \geq 0. \quad (2.5)$$

2.3 Adaptive Synchronization of Complex Networks

In the classical description of a complex network, see Eq. 2.1, the intensity of the coupling σ between nodes is often assumed to be constant and equal for all pairs of

interconnected nodes. Following [5–7], it is possible to consider a different coupling strategy where neighboring nodes negotiate the strength of their mutual coupling, effectively adapting its magnitude as a function of their mismatch. The network equation equipped with such a decentralized gain adaptation law can be written as

$$\dot{x}_i = f(x_i, t) + \sum_{j \in \mathcal{E}_i} \sigma_{ij}(t) (h(x_j) - h(x_i)), \quad (2.6)$$

$$\dot{\sigma}_{ij} = g(h(x_j) - h(x_i)), \quad (2.7)$$

where $\sigma_{ij}(t)$ is the coupling gain associated to the generic edge (i, j) between node i and node j and $g : \mathbb{R}^n \mapsto \mathbb{R}$ is some scalar function of the output error between neighboring nodes which is used to adapt and evolve the value of the gain itself.

Obviously, the effectiveness of the adaptation process is closely linked to the choice of the function g which is fundamental in determining the occurrence and performance of the synchronization process on the network. Here, we propose two possible alternative local strategies to guarantee the emergence of a synchronous evolution. The *vertex-based* strategy where the same adaptive gain is associated with all edges outgoing from a given node; and the *edge-based* strategy, where, instead, an adaptive gain is associated with every link in the network. We now briefly describe each of the two strategies.

- **Vertex-based strategy.** A first possible way of introducing gain adaptation is to associate an adaptive gain $\sigma_i(t)$ to every vertex i in the network. The gains associated to the outgoing links of each node are then updated comparing its output with the output of its neighbors, see [7]. Namely, the adaptation law is chosen as:

$$\dot{\sigma}_{ij}(t) = \dot{\sigma}_i(t) = \eta \left\| \sum_{j \in \mathcal{E}_i} (h(x_j) - h(x_i)) \right\|, \quad \mu > 0, (i, j) \in \mathcal{E}, \quad (2.8)$$

where η is some scalar parameter to be chosen as part of the design process, determining how sensitive the gain variation is to changes in the error between the node outputs. An alternative vertex-based strategy was independently proposed by Zhou and Kurths in [40] and is characterized by upper-bounding the maximal rate of change of the coupling gain.

- **Edge-based strategy.** Another possible decentralized approach to tune the coupling gains is to adaptively update each non-null element of the adjacency matrix. In other words, an adaptive gain σ_{ij} is associated with every $(i, j) \in \mathcal{E}$ [7]. We propose the following adaptation law:

$$\dot{\sigma}_{ij}(t) = \alpha \|h(x_j) - h(x_i)\|, \quad \alpha > 0, (i, j) \in \mathcal{E}. \quad (2.9)$$

Adopting this strategy, each couple of nodes negotiates the strength of their mutual coupling on the basis of the distance between their outputs.

The stability of these adaptive strategies was investigated in [6]. It was shown that, assuming $h(x) = x$, that is, all the nodes are linearly coupled through all their

state variables, a sufficient condition for asymptotic synchronization is that the vector field describing the node dynamics is QUAD(Δ, ω) with $\Delta - \omega I \leq 0$. In [5], the stability analysis was extended to a more general class of adaptive laws. Namely, the adaptive function belongs to one of the two following classes:

- Class 1: $g(e_{ij}) = \alpha \|e_{ij}\|^p$, with $e_{ij} = x_j - x_i$, $0 < p \leq 2$.
- Class 2: $g(e_{ij}) = \kappa(\|e_{ij}\|)$, where κ is a bounded differentiable class k function¹.

Theorem 2.2 *Let us assume that f is a continuous and differentiable vector field which is also QUAD(Δ, ω) with $\Delta - \omega I \leq 0$, and that $h(x) = x$. If the function g is selected from classes 1 or 2, then the network in Eq. 2.6, Eq. 2.7 asymptotically synchronizes onto a common evolution with*

$$\lim_{t \rightarrow \infty} (x_j - x_i) = 0, \quad \forall (i, j) \in \mathcal{E}, \quad (2.10)$$

and

$$\lim_{t \rightarrow \infty} \sigma_{ij}(t) = c_{ij} < +\infty, \quad \forall (i, j) \in \mathcal{E}. \quad (2.11)$$

(See [5] for the proof which is based on the use of appropriate Lyapunov functions to show that all nodes converge asymptotically towards the same synchronous evolution.)

This theorem ensures the asymptotic stability and the boundedness of the adaptive gains for any connected network, under the assumption that the vector field is QUAD. Nonetheless, in neural networks the coupling is typically on only one state variable (see, for instance, [25]), and, moreover, the vector field describe the node dynamics is QUAD, but $\Delta - \omega I$ is not negative semidefinite. Therefore, we consider the case in which the output function is $h(x) = \Gamma x$, with $\Gamma \geq 0$ being the *inner coupling matrix*, defining the subset of state variables through which the network nodes are coupled. Accordingly, we select the following adaptive function:

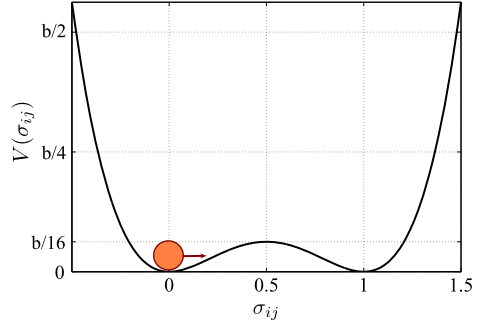
$$g(x_j - x_i) = (x_j - x_i)^T \Gamma (x_j - x_i). \quad (2.12)$$

With this choice, we can further extend the stability results as shown in [39].

Theorem 2.3 *Let us assume that $h(x) = \Gamma x$, with $\Gamma \geq 0$, and that f is a continuous and differentiable vector field which is also QUAD(Δ, ω) with $\Delta = H \Gamma$, where H is an arbitrary diagonal matrix. Then, the network in Eq. 2.6 asymptotically synchronizes under adaptive law in Eq. 2.12 and all the coupling gains converge to finite values, that is, Eq. 2.10 and Eq. 2.11 hold.*

¹A function $F : I \rightarrow \mathbb{R}$ is positive definite if $F(x) > 0$, $\forall x \in I$, $x \neq 0$ and $F(0) = 0$. A function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class k if it is continuous, positive definite and strictly increasing.

Fig. 2.1 Evolving networks. Bistable potential driving the evolution of each σ_{ij}



2.3.1 Evolving the Network Structure

In the adaptive scheme given in Eqs. 2.6, 2.7, each pair of nodes in the network is allowed to mutually negotiate the intensity of the coupling between themselves. Nonetheless, if at time 0 edge (i, j) is absent from the network, nodes i and j remain uncoupled for all time. In what follows, we describe the edge-snapping coupling mechanism presented in [8], where neighboring nodes are also allowed to activate or deactivate the link connecting them in such a way as for the network structure itself to evolve in time. This approach can be a very useful tool when the topology of the connections needs to be rearranged after edges/nodes failures while preserving the synchronous behavior. Specifically, we propose to model the gain evolution as a second order dynamical system:

$$\ddot{\sigma}_{ij} + \rho \dot{\sigma}_{ij} + \frac{d}{d\sigma_{ij}} V(\sigma_{ij}) = g(x_i, x_j). \quad (2.13)$$

Here, the coupling gain σ_{ij} can be viewed as a unitary mass in a double-well potential V subjected to an external force g , function of the states of the neighboring nodes i and j , while ρ represents the damping. We assume that g vanishes as the network nodes synchronize.

A possible choice of the potential V is the following:

$$V(\sigma_{ij}) = b\sigma_{ij}^2(\sigma_{ij} - 1)^2. \quad (2.14)$$

The shape of the potential is represented in Fig. 2.1, where b is a constant parameter regulating the depth of the wells. As can be observed from the figure, if the external force g dependent upon the distance between node i and node j is bounded and asymptotically vanishes, then the gain evolution determined by Eq. 2.13 must converge to one of its stable equilibria, that is, 0 (link is off) or 1 (link is on). Therefore, if synchronization is achieved, a topology emerges at steady-state, where only a subset of the possible edges of the network are activated.

A representative application of this adaptive approach to synchronization of a neural network is reported in Sect. 2.5.2

2.4 Inducing Synchronization via Design

An alternative strategy to achieve synchronization is to make sure that the properties of the node dynamics and the coupling functions are such that synchronization emerges from the network behavior. A powerful concept to assess this possibility is contraction theory as introduced in Sect. 2.2. Specifically, contraction theory can be effectively used to study synchronization by using a set of properties and results on contracting systems which are summarized briefly in what follows (all proofs can be found in [21, 33, 35]).

The key idea is to use the concept of *partial* contraction discussed in [38]. Basically, given a smooth nonlinear n -dimensional system of the form $\dot{x} = f(x, x, t)$, assume that the auxiliary system $\dot{y} = f(y, x, t)$ is contracting with respect to y . If a particular solution of the auxiliary y -system verifies a smooth specific property, then all trajectories of the original x -system verify this property exponentially. The original system is said to be *partially contracting*.

Indeed, the virtual y -system has two particular solutions, namely $y(t) = x(t)$ for all $t \geq 0$ and the particular solution with the specific property. Since all trajectories of the y -system converge exponentially to a single trajectory, this implies that $x(t)$ verifies the specific property exponentially.

We will now use the above basic results of nonlinear contraction analysis to study synchronization of complex networks by showing, first, that it is possible to effectively prove that a system (or virtual system) is contracting by means of an appropriate algorithmic procedure.

2.4.1 An Algorithm for Proving Contraction

As mentioned above, the key step in the use of contraction theory to achieve synchronization of networks is to prove that a given system of interest is contracting. In general, this can be a cumbersome task but, as recently shown in [34], the use of matrix measures and norms induced by non-Euclidean vector norms, (such as μ_1 , μ_∞ , $\|\cdot\|_1$, $\|\cdot\|_\infty$), can be effectively exploited to obtain alternative conditions to check for contraction of a dynamical system of interest, both in continuous-time and discrete-time.

In particular, we show now that by means of these measures and norms, it is possible to obtain an algorithmic procedure for checking if a system is contracting or for imposing such property. Formally, the conditions required by the algorithm for a system to be contracting are sufficient conditions. This means that, if the conditions are satisfied, then the system is contracting, while the vice-versa is not true.

The outcome of the algorithm is to provide a set of conditions on the elements of the system Jacobian, J , (and hence on the dynamics of $f(\cdot, \cdot)$) that can be used to prove contraction.

Here, we present an outline of the proposed algorithm for both continuous-time and discrete-time systems. The proof of the results on which the algorithm is based can be found in [34].

The first step of the algorithm is to differentiate the system of interest, in order to obtain the Jacobian matrix, $J := \frac{\partial f}{\partial x}$:

$$\begin{bmatrix} J_{1,1}(x, t) & J_{1,2}(x, t) & \dots & J_{1,n}(x, t) \\ J_{2,1}(x, t) & J_{2,2}(x, t) & \dots & J_{2,n}(x, t) \\ \dots & \dots & \dots & \dots \\ J_{n,1}(x, t) & J_{n,2}(x, t) & \dots & J_{n,n}(x, t) \end{bmatrix} \quad (2.15)$$

which is, in general, state/time dependent.

The next step of the algorithm is then to construct a directed graph from the system Jacobian. To this aim, we first derive an adjacency matrix from J , say $\mathcal{A}$, using the following rules:

1. initialize $\mathcal{A}$ so that $\mathcal{A}(i, j) = 0, \forall i, j$;
2. for all $i \neq j$, set $\mathcal{A}(i, j) = \mathcal{A}(j, i) = 1$ if either $J_{i,j}(x, t) \neq 0$, or $J_{j,i}(x, t) \neq 0$.

Such a matrix describes an undirected graph (see e.g. [13]), say $\mathcal{G}(\mathcal{A})$. The second step in the algorithm is then to associate directions to the edges of $\mathcal{G}(\mathcal{A})$ to obtain a directed graph, say $\mathcal{G}_d(\mathcal{A})$. This is done by computing the quantity

$$\alpha_{i,j}(x, t) = \frac{|J_{i,j}(x, t)|}{|J_{i,i}(x, t)|} (n - n_{0i} - 1). \quad (2.16)$$

In the above expressions n_{0i} is the number of zero elements on the i -th row of $\mathcal{A}$. (Note that if $J_{i,i}(x, t) = 0$ for some i , then, before computing (2.16), the system parameters/structure must be engineered so that $J_{i,i}(x, t) \neq 0$, for all i .)

The directions of the edges of $\mathcal{G}_d(\mathcal{A})$ are then obtained using the following simple rule:

the edge between node i and node j is directed from i to j if the quantity $\alpha_{i,j}(x, t) < 1$ while it is directed from j to i if $\alpha_{i,j}(x, t) \geq 1$.

Note that the quantities $\alpha_{i,j}(x, t)$ will be in general time-dependent, therefore the graph directions might be time-varying.

Once the directed graph $\mathcal{G}_d(\mathcal{A})$ has been constructed, contraction is then guaranteed under the following conditions:

1. uniform negativity of all the diagonal elements of the Jacobian, i.e. $J_{i,i}(x, t) < 0$ for all i ;
2. for all t , the directed graph $\mathcal{G}_d(\mathcal{A})$ does not contain loops of any length;
3. $\alpha_{ij}(x, t)\alpha_{ji}(x, t) < 1$.

Note that, when the above conditions are not satisfied, the algorithm can be used to impose contraction for the system of interest by:

1. using, if possible, a control input to impose the first condition of the algorithm for all the elements $J_{i,i}(x, t)$ that do not fulfill it;
2. *re-direct* (using an appropriate control input, or tuning system parameters) some edges of the graph $\mathcal{G}_d(\mathcal{A})$ in order to satisfy the *loopless* condition;

3. associate to each reverted edge (e.g., the edge between node i and node j) one of the following inequalities:

- $\alpha_{i,j}(x, t) \geq 1$, if the edge is reverted from j to i ;
- $\alpha_{i,j}(x, t) < 1$, if the edge is reverted from i to j .

2.4.2 Remarks

- Notice that the procedure presented above is based on the use of $\mu_\infty(\Theta J \Theta^{-1})$ and $\|\Theta J \Theta^{-1}\|_\infty$ for proving contraction. Other matrix measures and norms can also be used. In particular, for the continuous-time case, it is easy to prove that, using $\mu_1(\Theta J \Theta^{-1})$, yields the same algorithmic procedure applied on J^T . If this is the case, the resulting algorithm will follow the same logical steps presented above, with the only difference being the expression of $\alpha_{i,j}(x, t)$:

$$\alpha_{i,j}(x, t) := \frac{|J_{j,i}(x, t)|(m - c_{0i} - 1)}{|J_{i,i}(x, t)|}, \quad (2.17)$$

where c_{0i} denotes the number of zero elements of the i -th column of J .

- The procedure presented here can have a clear physical interpretation. This is the case, for example, of molecular systems, that is, systems composed by genes and proteins, where the state variables represent the concentrations of the species involved into the system. In this case, each term $\alpha_{i,j}(x, t)$ represent a *normalized production rate* between species i and species j and the resulting set of inequalities provided by the algorithm points towards a balance of some *flow-like* quantities in the system (see, e.g., [31] and [30]);
- The *robustness* of the conditions outlined above can be evaluated. Specifically, we consider two cases: (i) robustness with respect to additive white noise acting on the system; (ii) robustness with respect to parameter variations. In the former case, if the conditions of the algorithm are fulfilled by the dynamics of the noise-free system, then all of its trajectories globally converge to a unique trajectory, say $\tilde{x}(t)$. Since our conditions ensure contraction, we have that (see [29]) all trajectories of the noisy system globally converge towards a boundary layer around $\tilde{x}(t)$. In the latter case, instead, the results provided by the algorithm are robust with respect to all parameter variations that do not affect the topology of the graph $\mathcal{G}_d(\mathcal{A})$. Such properties can be particularly useful for control and synchronization of networks having non-identical nodes and for biochemical systems which are typically characterized by high noise intensity and parameter uncertainties;
- Note that our approach can also be used to check partial contraction of the virtual system describing a network of oscillators. In this case, the algorithm is applied on the Jacobian of such a system and can be used to analyze synchronization phenomena in networks of coupled dynamical systems as will be further illustrated in Sect. 2.6.

2.5 Adaptive Synchronization of FitzHugh–Nagumo Neurons

To illustrate the theoretical results presented above and show the viability of the techniques discussed so far for the study of synchronization in neural systems, we consider now networks of FitzHugh–Nagumo (FN) oscillators. Such oscillators were first presented in [11] and their model take the form:

$$\begin{aligned}\dot{v} &= c \left(v + w - \frac{1}{3}v^3 + u(t) \right), \\ \dot{w} &= -\frac{1}{c}(v - a + bw),\end{aligned}\tag{2.18}$$

where:

- v is the membrane potential;
- w is a recovery variable;
- $u(t)$ is the magnitude of the stimulus current.

Note that, equivalently, the model can be rewritten in the form $\dot{x} = f(x, t)$ where $x = (v, w)^T$ and $f(x) = (c(v + w - \frac{1}{3}v^3 + u(t)), -\frac{1}{c}(v - a + bw))^T$.

2.5.1 Properties of the Node Dynamics

We show here that networks of FitzHugh–Nagumo neurons can indeed be synchronized using a decentralized adaptive approach. As explained in Sect. 2.3, this is possible if the node dynamics satisfies the QUAD condition.

Computing the Jacobian J of the oscillator model Eq. 2.18, we have:

$$J(v) = \begin{bmatrix} c - v^2 & c \\ -\frac{1}{c} & -\frac{b}{c} \end{bmatrix}.\tag{2.19}$$

The symmetric part J_{sym} of J is:

$$J_{\text{sym}}(v) = \begin{bmatrix} c - v^2 & (c - \frac{1}{c})/2 \\ (c - \frac{1}{c})/2 & -\frac{b}{c} \end{bmatrix}.\tag{2.20}$$

From trivial matrix algebra, we have that

$$\lambda_{\max}(J_{\text{sym}}(v)) \leq \lambda_{\max}(J_{\text{sym}}(0)),\tag{2.21}$$

where

$$\lambda_{\max}(J_{\text{sym}}(0)) = \frac{-b + c^2 + \sqrt{1 + b^2 + 2(b-1)c^2 + 2c^4}}{2c} := \gamma.\tag{2.22}$$

Also, we can write:

$$J_{\text{sym}}(v) = J_{\text{sym}}^1 + J_{\text{sym}}^2(v), \quad (2.23)$$

where

$$J_{\text{sym}}^1 = \begin{bmatrix} -d & (c - \frac{1}{c})/2 \\ (c - \frac{1}{c})/2 & -\frac{b}{c} \end{bmatrix}, \quad (2.24)$$

$$J_{\text{sym}}^2(v) = \begin{bmatrix} d + c - v^2 & 0 \\ 0 & 0 \end{bmatrix} \quad (2.25)$$

and d is some arbitrary positive scalar.

Now, we are ready to state the following results.

Theorem 2.4 *The FitzHugh–Nagumo oscillator is QUAD(Δ, ω), with $\Delta - \omega I \geq \gamma I$.*

Proof Assuming that $u(t)$ is continuously differentiable, also $f(x, t)$ will be continuously differentiable. Therefore, following [9], there exists a $\tilde{\xi} \in [0, 1]$ such that

$$(x - y)^T [f(x, t) - f(y, t)] = (x - y)^T \left(\frac{\partial}{\partial x} f(y + \tilde{\xi}(x - y), t) \right) (x - y). \quad (2.26)$$

Thus, from Eqs. 2.21–2.22, we have:

$$(x - y)^T [f(x, t) - f(y, t)] = \gamma (x - y)^T (x - y). \quad (2.27)$$

Comparing Eq. 2.27 with Eq. 2.3, the thesis follows. $\square$

Theorem 2.5 *Let us assume that $\Delta = H\Gamma$, where H is an arbitrary diagonal matrix and*

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The FitzHugh–Nagumo oscillator is QUAD(Δ, ω) with $H_{11} > d + c$.

Proof Considering that the Jacobian depends only on the first state variable v , Eq. 2.26 can be equivalently rewritten as

$$(x - y)^T [f(x, t) - f(y, t)] = (x - y)^T J_{\text{sym}}(\tilde{v})(x - y), \quad (2.28)$$

where $\tilde{v}$ is the first component of the point $\tilde{x} = y + \tilde{\xi}(x - y)$. From Eq. 2.23, we have

$$(x - y)^T [f(x, t) - f(y, t)] = (x - y)^T J_{\text{sym}}^1(x - y) + (x - y)^T J_{\text{sym}}^2(\tilde{v})(x - y). \quad (2.29)$$

It is easy to show that, for any c , we can choose a $d > 0$ such that $\lambda_{\max}(J_{\text{sym}}^1) = \mu$. Now, we can write

$$\begin{aligned} (x - y)^T [f(x, t) - f(y, t)] \\ \leq \mu(x - y)^T (x - y) + (d + c - \tilde{v}^2)(v_x - v_y)^2 \end{aligned} \quad (2.30)$$

$$\leq \mu(x - y)^T (x - y) + (d + c)(v_x - v_y)^2. \quad (2.31)$$

Equation 2.31 can be rewritten as follows:

$$\begin{aligned} (x - y)^T [f(x, t) - f(y, t)] - (x - y)^T \begin{bmatrix} d + c & 0 \\ 0 & 0 \end{bmatrix} (x - y) \\ \leq \mu(x - y)^T (x - y). \end{aligned} \quad (2.32)$$

Comparing Eq. 2.32 with Eq. 2.3, and considering that $\Delta = H\Gamma$ proves the theorem. $\square$

2.5.2 Numerical Validation

In what follows, we consider a network of FitzHugh–Nagumo coupled through the adaptive scheme in Eqs. 2.6–2.7. In particular, we assume that the oscillators are coupled linearly on the first state variable v . Namely, $h(x) = \Gamma x$, with

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Moreover, according with Eq. 2.12, we choose:

$$\dot{\sigma}_{ij} = (v_i - v_j)^2, \quad (2.33)$$

so that neighboring oscillators adapt the strength of their coupling according to the distance between their membrane potentials.

It is easy to show that under these assumptions all the hypothesis of Theorem 2.3 are satisfied. In fact, from Theorem 2.5, the FitzHugh–Nagumo oscillator is QUAD with $\Delta = H\Gamma$.

As a numerical example, we consider a small world network of 500 nodes, with average degree equal to 8, and initial conditions taken randomly from a normal distribution.

As illustrated in Fig. 2.2, all the nodes' trajectories converge towards each other as expected with all coupling gains settling onto finite steady-state values.

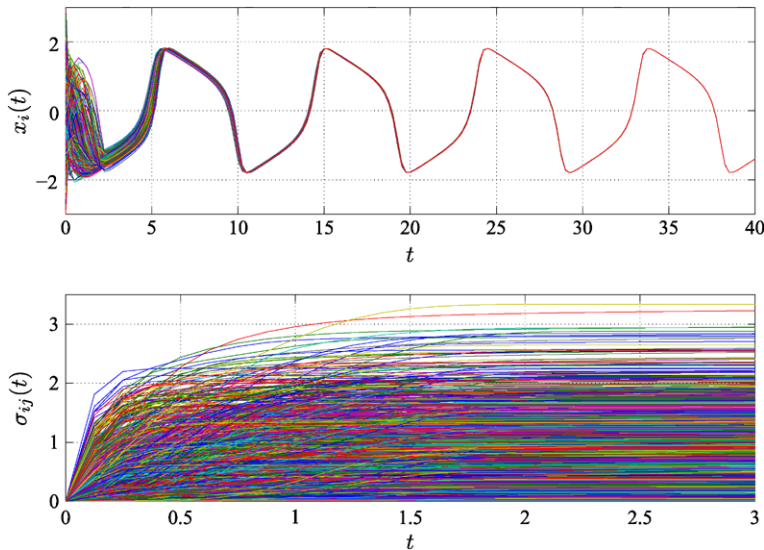


Fig. 2.2 Network of 500 FitzHugh–Nagumo oscillators coupled through the adaptive strategy Eq. 2.33. Evolution of the states (*top*) and of the adaptive gains (*bottom*)

2.5.2.1 Evolving the Network Topology

We consider now the case in which nodes can also decide whether to switch on or off a link between themselves according to the edge-snapping dynamics given by Eqs. 2.13, 2.14 as described in Sect. 2.3.1.

Specifically, we choose the external force g driving the gains' evolution as:

$$g(x_i, x_j) = (v_j - v_i)^2.$$

Moreover, we set $\sigma_{ij}(0) = 0$ so that, initially, all nodes are practically disconnected from each other. As we can see from Fig. 2.3, the gains then evolve converging towards 0 or 1 asymptotically so that a specific network topology emerges that guarantees synchronization.

2.6 Using Contraction

Now, we check if FitzHugh–Nagumo neurons can be synchronized via two alternative coupling strategies, diffusive coupling or local field potentials. In both cases, we will make use of contraction theory to assess the synchronizability of the network of interest and give conditions on the features of the node dynamics, the coupling and the network structure guaranteeing the emergence of a common asymptotic solution.

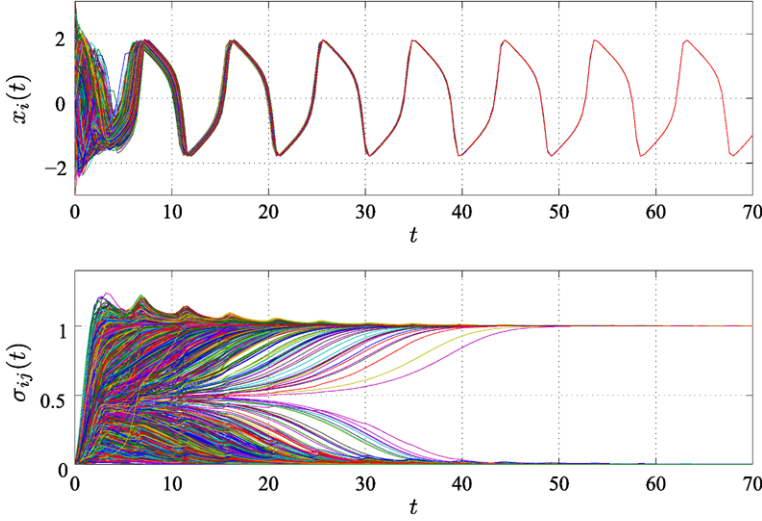


Fig. 2.3 Network of 500 FitzHugh–Nagumo oscillators coupled through the edge snapping strategy Eq. 2.13. Evolution of the states (*top*) and of the adaptive gains (*bottom*)

2.6.1 Node Dynamics

For the purpose of our analysis, it is convenient to define a new state variable, $w^* := \varepsilon w$, where ε is an arbitrary real number, and then write the system in the new coordinates (v, w^*) . We have:

$$\begin{aligned} \dot{v} &= c \left(v + \frac{w^*}{\varepsilon} - \frac{1}{3} v^3 + u(t) \right), \\ \dot{w}^* &= -\frac{\varepsilon}{c} \left(v - a + b \frac{w^*}{\varepsilon} \right). \end{aligned} \quad (2.34)$$

Differentiation of the above system yields the Jacobian matrix

$$J := \begin{bmatrix} c - v^2 & \frac{c}{\varepsilon} \\ -\frac{\varepsilon}{c} & -\frac{b}{c} \end{bmatrix}. \quad (2.35)$$

Notice that the element $J_{11}(x, t)$ is not uniformly negative definite as required by the algorithm presented in Sect. 2.4.1. Hence, the node dynamics has to be somehow engineered in order to make the system contracting. We now propose a simple way to achieve this task. We will show later that this modification will be useful to derive sufficient conditions for network synchronization.

Specifically, we modify the system by adding an extra function $h(v)$ to the dynamics of v in (2.34). This is a simple form of feedback whose specific form will be selected so as to fulfill the condition $J_{ii}(x, t) < 0$. Differentiation of the modified

dynamics yields:

$$J := \begin{bmatrix} c - v^2 + \frac{\partial h}{\partial v} & \frac{c}{\varepsilon} \\ -\frac{\varepsilon}{c} & -\frac{b}{c} \end{bmatrix}. \quad (2.36)$$

Therefore, in order to fulfill the condition $J_{11} < 0$, it immediately follows that a suitable choice for the function $h(v)$ is:

$$h(v) := -(c + \gamma_1)v \quad \text{for some positive scalar } \gamma_1.$$

Now, the Jacobian of the modified system is:

$$J := \begin{bmatrix} -\gamma_1 - v^2 & \frac{c}{\varepsilon} \\ -\frac{\varepsilon}{c} & -\frac{b}{c} \end{bmatrix}. \quad (2.37)$$

Thus, we have that both $J_{11}(x, t)$ and $J_{22}(x, t)$ are uniformly negative. Then, in order for the system to be contracting, we need to find conditions ensuring that no loops are present in $G_d(A)$. Since FN has a two dimensional phase space, it follows that $G_d(A)$ contains only two nodes. Thus, no loops are present in such a graph if:

$$\begin{aligned} \alpha_{12}(x, t) &< 1; \\ \alpha_{12}(x, t) &> 1, \end{aligned}$$

or viceversa, with α_{ij} defined as in Eq. 2.16.

In our case, we have:

$$\begin{aligned} \alpha_{12} &= \frac{|J_{12}(x, t)|}{|J_{11}(x, t)|} = \frac{c}{\varepsilon(\gamma_1 + v^2)}, \\ \alpha_{21} &= \frac{|J_{21}(x, t)|}{|J_{22}(x, t)|} = \frac{\varepsilon}{b}. \end{aligned}$$

Now,

$$\alpha_{12} \leq \frac{c}{\varepsilon\gamma_1}.$$

Thus, the graph conditions are satisfied if:

$$\begin{aligned} \frac{c}{\varepsilon\gamma_1} &< 1, \\ \frac{\varepsilon}{b} &\geq 1. \end{aligned}$$

Recall, now, that ε is an arbitrary real number. It then suffices to choose $\varepsilon > \max\{\frac{c}{\gamma_1}, b\}$ so that the above two conditions are satisfied. Finally, the last condition of the algorithm requires that

$$\alpha_{12}\alpha_{21} < 1.$$

Here, we have:

$$\alpha_{12}\alpha_{21} = \frac{c}{b\gamma_1}.$$

Thus, this condition is satisfied if:

$$\gamma_1 > \frac{c}{b}. \quad (2.38)$$

Therefore, an appropriate reengineering of the node dynamics and the use of the feedback input $h(v)$ can effectively make the FitzHugh–Nagumo oscillators contracting. In what follows, we will refer to such an input as a *contracting input* in accordance with what reported in [12].

Heuristically, from a physical viewpoint, the constraint represented by Eq. 2.38 can be interpreted as a lower bound on the strength of the feedback action introduced by the contracting input $h(v)$ on the node dynamics. This might be the principle behind the input to neurons in the brain provided by the small cortical circuit described in, for example, [4, 12], which may act as a frequency selector for many cortical pyramidal cells.

2.6.2 Synchronization Results

Now, we use the results obtained by means of the algorithm presented above to synchronize networks of N FN oscillators. We recall that the state variable of the i -th neuron is x_i , with $x_i := (v_i, w_i)^T$, while in this case we denote with $f(x_i)$ the dynamics of the i -th FN with the contracting input, that is,

$$f(x_i) := c \left(v + w - \frac{1}{3}v^3 + u(t) \right) + h(v_i) - \frac{1}{c}(v - a + bw),$$

where $h(v_i)$ is chosen as:

$$h(v_i) = -\delta(t)(c + \gamma_1)v_i$$

with $\delta(t) : \mathbb{R}^+ \rightarrow [0, 1]$ being some functions that determines when the contracting input is activated and $u(t)$ some periodic signal.

In what follows, we consider three different mechanisms for interconnecting different neurons. Specifically:

1. FN diffusively coupled;
2. FN coupled by means of field potentials;
3. FN with excitatory-only coupling.

2.6.2.1 Diffusive Coupling

In what follows, we will couple pairs of neighboring nodes only by means of the state variable v_i . Furthermore, we assume that the coupling function is linear. Specifically, let ϕ be any (coupling) function having diagonal Jacobian, with bounded nonnegative diagonal elements. The network considered here is then:

$$\dot{x}_i = f(x_i) + \sum_{j \in \mathcal{C}_i} (\phi(x_j) - \phi(x_i)), \quad (2.39)$$

where:

$$\phi(x) = \begin{pmatrix} Kx \\ 0 \end{pmatrix}.$$

In what follows, we make use of the following theorem, proved in [32], which generalize a previous result in [28].

Theorem 2.6 *Assume that in Eq. 2.39 the nodes dynamics are contracting. Then, all nodes dynamics globally exponentially converge towards each other, that is:*

$$|x_j(t) - x_i(t)| \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Furthermore, the rate of convergence towards the synchronous solution is the contraction rate of the network nodes.

In [32], an analogous of the above result is also given ensuring cluster synchronization. Clearly, such a choice for the coupling function satisfies the hypotheses required by Theorem 2.6.

Now, recall that the input $h(v)$ is a contracting input, with $\gamma_1 > c/b$ in the sense that it makes the system contracting. Thus, by means of Theorem 2.6, we have that when the nodes are diffusively coupled and the contracting input is activated, that is, $\delta(t) = 1$, then the nodes synchronize.

Figure 2.4 confirms this theoretical prediction. Specifically, we observe that all nodes start converging towards the same synchronous state from the moment the contracting input is activated at about $T \approx 2$.

2.6.2.2 Local Field Potentials

We now analyze the effects of a communication mechanism among neurons similar to the so-called quorum sensing in biological networks, for example, [32, 33]. In a neuronal context, a mechanism similar to that of quorum sensing may involve *local field potentials*, which may play an important role in the synchronization of clusters of neurons, [1, 10, 27, 37]. From a network dynamics viewpoint, the key characteristic of the quorum sensing-like mechanisms lies in the fact that communication between nodes (e.g., bacteria or neurons) occurs by means of a shared quantity. Furthermore, the production and degradation rates of such a quantity are affected by all

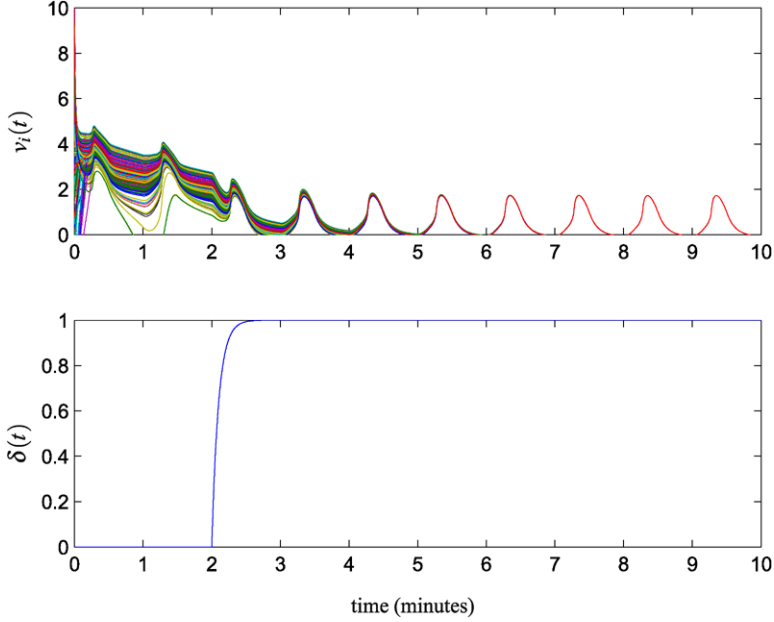


Fig. 2.4 Simulation of Eq. 2.39, with $N = 500$. Evolution of the oscillators' membrane potentials (*top panel*) and of the contracting input $\delta(t)$ (*bottom panel*). The system parameters are set as follows: $a = 0$, $b = 2$, $c = 2.5$, $K = 0.01$, $\gamma_1 = 1.5$ with $\delta(t) = [1 - e^{-10(t-2)}] \cdot H(t - 2)$, $H(t)$ being the Heaviside function

the nodes of the network. Therefore, a detailed model of such a mechanism needs to keep track of the temporal evolution of the shared quantity, resulting in an additional set of ordinary differential equations:

$$\begin{aligned}\dot{x}_i &= f(x_i, z, t), \\ \dot{z} &= g(z, x_1, \dots, x_N, t),\end{aligned}\tag{2.40}$$

where $z(t) \in \mathbb{R}^d$ denotes the medium dynamics.

We need the following result from [32].

Theorem 2.7 *All nodes trajectories of the network in Eq. 2.40 globally exponentially converge towards each other if the function $f(x, v_1(t), t)$ is contracting for any $v_1(t) \in \mathbb{R}^d$.*

We will consider the case where the coupling between the nodes and the common medium is diffusive. In this case, the mathematical model in Eq. 2.40 reduces to:

$$\begin{aligned}\dot{x}_i &= f(x_i, t) + \phi_1(z) - \phi_2(x_i), \quad i = 1, \dots, N, \\ \dot{z} &= g(z, t) + \sum_{i=1}^N [\phi_3(x_i) - \phi_4(z)].\end{aligned}\tag{2.41}$$

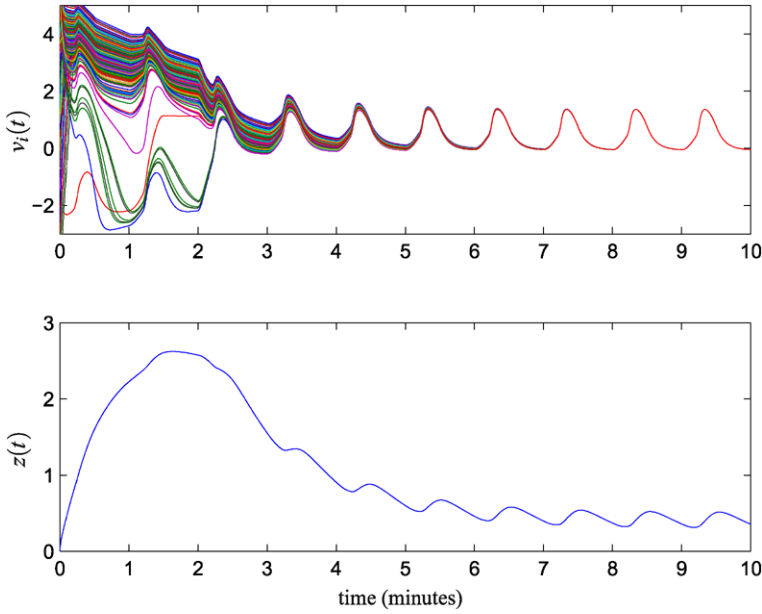


Fig. 2.5 Simulation of Eq. 2.41, with $N = 500$. System parameters are set as follows: $a = 0$, $b = 2$, $c = 2.5$, $K = 0.01$, $\gamma_1 = 1.5$

That is, the nodes and the common medium are coupled by means of the coupling functions $\phi_1 : \mathbb{R}^d \rightarrow \mathbb{R}^n$, $\phi_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\phi_3 : \mathbb{R}^n \rightarrow \mathbb{R}^d$, $\phi_4 : \mathbb{R}^d \rightarrow \mathbb{R}^d$. In our simulations, all the coupling functions are linear, that is,

$$\begin{aligned} \phi_1(z) &= \begin{pmatrix} K \\ 0 \end{pmatrix} z, & \phi_2(x_i) &= \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_i \\ w_i \end{pmatrix}, \\ \phi_3(x_i) &= \begin{pmatrix} K & 0 \end{pmatrix} \begin{pmatrix} v_i \\ w_i \end{pmatrix}, & \phi_4(z) &= Kz. \end{aligned}$$

In this case, Theorem 2.7 implies that synchronization is attained and $z(t)$ remains bounded if $f(x, t) - \phi_2(x)$ is contracting.

Recall that we want the neurons to be weakly coupled. In particular, if K is sufficiently small this part of the input cannot make the system contracting. The behavior drastically changes when the contracting input is applied.

Figure 2.5 shows that when the contracting input is applied, a transition to the synchronous behavior occurs.

2.6.2.3 Excitatory-Only Coupling

We now turn our attention to the analysis of synchronization of excitatory-only coupled neurons, which are described by:

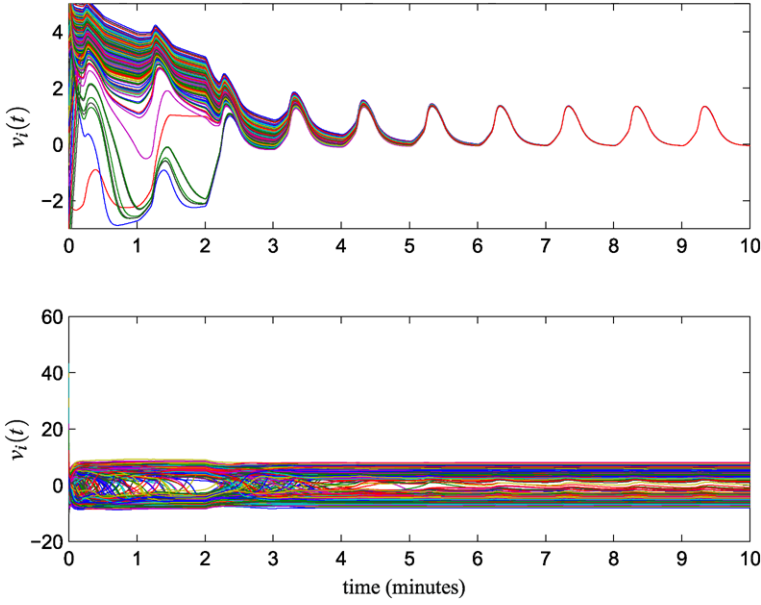


Fig. 2.6 Simulation of Eq. 2.41, with $N = 500$. System parameters are set as follows: $a = 0$, $b = 2$, $c = 2.5$, $\gamma_1 = 1.5$ $K = 0.01$ (top), $K = 5$ (bottom)

$$\dot{x}_i = f(x_i) + \sum_{j \in N_i} \phi(x_j), \quad (2.42)$$

where the function ϕ is chosen as

$$\phi(x_j) := \begin{pmatrix} \gamma_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_i \\ w_i \end{pmatrix}.$$

In [33], it has been proven that the above coupling function ensures synchronization if: (i) the contracting input is applied; (ii) γ_2 is *small* enough.

Figure 2.6 confirms the above prediction.

2.7 Materials & Methods

All the simulations have been performed using Matlab/Simulink, with variable step size ODE solver *ode45* with relative tolerance set to $1e-5$. Simulations for diffusive and excitatory-only coupling involved a directed small world network of $N = 500$ nodes.

2.8 Conclusions

In this chapter, we presented two alternative approaches to achieve synchronization of networks of interacting dynamical agents, one based on an adaptive coupling strategy, the other on the use of an appropriate contracting input. In the former case, we showed that, under the hypothesis that the node dynamics satisfies the so-called QUAD condition, it is possible for a set of agents to negotiate the strength of their mutual coupling so as to make a synchronous evolution emerge. The results obtained were applied to a representative network of FitzHugh–Nagumo neurons showing that indeed all neurons oscillate in synchrony with the strength of their couplings converging toward bounded constant values.

The alternative approach we took was that of considering contraction theory as a tool to engineer networks that synchronize. This can be seen as an effective design tool in all those cases where synchronization needs to be induced or suppressed by the appropriate engineering of the node dynamics and/or coupling structure and topology. After presenting the concept of contraction and briefly discussing its relevance for synchronization, we discussed a graphical procedure to check if a system or network of interest is contracting. This was then used to study the case of networks of FitzHugh–Nagumo oscillators, showing that, under the action of an *ad hoc* contracting input, the network can indeed synchronize. Three alternative coupling strategies were tested to show that synchronization can indeed be achieved in the presence of diffusive coupling, local mean field potentials or excitatory-only coupling.

The results presented show that synchronization is indeed a possible emergent phenomenon in current models of neuronal networks being used in the literature. Similar derivations can be applied to other neuronal models without major changes. The challenge now is to understand the exact role played by synchronization phenomena in the brain and the mechanism through which they emerge and are maintained even in the presence of disturbances and noise. In our view, the flexibility and robustness of these phenomena in the brain is probably due to the presence of the two features that were discussed in this chapter: adaptation of the network structure and coupling in time and specific properties of the node dynamics and interaction topology.

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