

Growth Behavior and Zero Distribution of Maximally Convergent Rational Approximants

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Abstract Given a compact set E in $\mathbb{C}$ and a function f holomorphic on E , we investigate the distribution of zeros of rational uniform approximants $\{r_{n,m_n}\}$ with numerator degree $\leq n$ and denominator degree $\leq m_n$, where $m_n = o(n/\log n)$ as $n \rightarrow \infty$. We obtain a Jentzsch–Szegő type result, i.e., the zero distribution converges weakly to the equilibrium distribution of the maximal Green domain $E_{\rho(f)}$ of meromorphy of f if f has a singularity of multivalued character on the boundary $\partial E_{\rho(f)}$. Further, we show that any singular point of f on the boundary $\partial E_{\rho(f)}$, that is not a pole, is a limit point of zeros of the sequence $\{r_{n,m_n}\}$.

Let $f(z) = \sum_{v=0}^{\infty} a_v z^v$ be a power series with radius of convergence 1; set

$$s_n(z) := \sum_{v=0}^n a_v z^v, \quad n = 0, 1, 2, \dots$$

The classical theorem of Jentzsch concerns the limiting behavior of the zeros of the partial sums s_n as $n \rightarrow \infty$. Jentzsch [9] proved that each point of the unit circle is a limit point of zeros of $s_n(z)$, $n = 1, 2, \dots$. Later, Szegő [13] showed that there is an infinite sequence $\Lambda \subset \mathbb{N}$ such that the zeros of s_n , $n \in \Lambda$, are asymptotically uniformly distributed in the sense of Weyl.

Since the radius of convergence equals 1,

$$\limsup_{n \rightarrow \infty} |a_n|^{1/n} = 1.$$

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In the proof of Szegő's theorem, it is shown that the asymptotic uniform distribution for the zeros in the sense of Weyl holds for any sequence Λ that satisfies

$$\lim_{n \in \Lambda, n \rightarrow \infty} |a_n|^{1/n} = \limsup_{n \rightarrow \infty} |a_n|^{1/n} = 1.$$

Throughout this paper, we shall consider compact sets $E \subset \mathbb{C}$ whose complements $\Omega = \overline{\mathbb{C}} \setminus E$ with respect to the extended complex plane $\overline{\mathbb{C}}$ are connected. A set E is called *regular* if there exists a Green function $G(z) = G_E(z, \infty)$ on Ω with pole at ∞ , and $G(z) \rightarrow 0$ as $z \rightarrow \partial\Omega$. We define the Green domains E_ρ by

$$E_\rho := \{z \in \Omega : G(z) < \log \rho\} \cup E, \quad \rho > 1. \quad (1)$$

Since Ω is regular, the *equilibrium distribution* μ_E of E exists, as well as the equilibrium distribution μ_ρ for every $\bar{E}_\rho$, $\rho > 1$ (cf. [12] or [14]).

Given an open set $B \subset \mathbb{C}$, $\mathcal{A}(B)$ represents the class of functions f that are *holomorphic* (analytic and single-valued) in the set B . Further, the function f is *meromorphic* in B ($f \in \mathcal{M}(B)$) if in any closed subdomain of B , f has no more than a finite number of poles. We say that a function f is holomorphic (meromorphic) on the compact set E if f is holomorphic (meromorphic) in some open set U containing E and write $f \in \mathcal{A}(E)$ ($f \in \mathcal{M}(E)$). Further, $f \in \mathcal{M}_m(B)$ (f is m -meromorphic in B) iff f is meromorphic with no more than m poles in B . As usual, poles are counted with respect to their multiplicities. Finally, $\mathcal{R}_{n,m}$ is the collection of all rational functions $\{p/q : \deg p \leq n, \deg q \leq m, q \not\equiv 0\}$.

For $f \in C(E)$, we introduce the *radius of holomorphy* $\rho_0(f)$, resp. the *radius* $\rho_m(f)$ of m -meromorphy, $m \in \mathbb{N}_0$ as follows:

$$\rho_0(f) := \begin{cases} 1, & \text{if } f \notin \mathcal{A}(E), \\ \sup\{\rho > 1 : f \in \mathcal{A}(E_\rho)\}, & \text{otherwise} \end{cases}$$

and, respectively,

$$\rho_m(f) := \begin{cases} 1, & \text{if } f \notin \mathcal{A}(E), \\ \sup\{\rho > 1 : f \in \mathcal{M}_m(E_\rho)\}, & \text{otherwise.} \end{cases}$$

Given a pair (n, m) , $n, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let $R_{n,m}$ denote a best uniform approximation to $f \in C(E)$ in the class $\mathcal{R}_{n,m}$; that is,

$$e_{n,m}(f) := \inf_{r \in \mathcal{R}_{n,m}} \|f - r\|_E = \|f - R_{n,m}\|_E \quad (2)$$

and $\|\cdot\|_E$ denotes the max-norm (uniform norm of Chebyshev) on E . In the following, we assume that

$$R_{n,m} = P_{n,m}/Q_{n,m},$$

where the polynomials $P_{n,m}$ and $Q_{n,m}$ do not have a common factor. The starting point of our considerations is the following theorem.

Theorem 1 ([6]). *Let E be a regular compact set in $\mathbb{C}$ with connected complement and $m \in \mathbb{N}_0$ be fixed. Suppose that $f \in \mathcal{A}(E^o) \cap C(E)$ and $\rho_m(f) < \infty$. Then*

$$\limsup_{n \rightarrow \infty} e_{n,m}^{1/n} = 1/\rho_m(f) \quad (3)$$

and $f \in \mathcal{M}_m(E_{\rho_m(f)})$.

As known, Mergelyan's theorem says that $\lim_{n \rightarrow \infty} e_{n,m}(f) = 0$, where $m \in \mathbb{N}_0$ is fixed. Theorem 1 reveals the relation of the rate of decay to the analyticity of f .

Concerning the distribution of the zeros of $R_{n,m}(f)$, let us denote by $v_n = v(P_{n,m})$ the *normalized zero counting measure* of $P_{n,m}(f)$, i.e.,

$$v_n(B) = \frac{\# \text{ zeros of } P_{n,m} \text{ in } B}{\deg P_{n,m}}, \quad B \subset \mathbb{C}.$$

The next theorem characterizes the zero distribution of the sequence $\{R_{n,m}\}$ as $n \rightarrow \infty$, where m is fixed.

Theorem 2. *Under the conditions of Theorem 1, suppose that f is not identically 0 on any component of E . Suppose also that $\rho_m(f) < \infty$. Then there exists a subsequence $\Lambda \subset \mathbb{N}$ such that the normalized zero counting measures v_n of the numerators of $R_{n,m}$ converge weakly to the equilibrium distribution $\mu_{\rho_m(f)}$ of $\bar{E}_{\rho_m(f)}$.*

Theorem 2 follows from results of [1, 2]. In an independent way, it was proved in [10].

For our further purposes we use the concept of *convergence in m_1 -measure*. Let e be a set in $\mathbb{C}$. We set

$$m_1(e) := \inf \left(\sum_v |U_v| \right),$$

where the infimum is taken over all countable coverings $\{U_v\}$ by disks U_v , where $|U_v|$ denotes the radius of the disk U_v .

Let D be a domain in $\mathbb{C}$ and φ a function defined in D with values in $\overline{\mathbb{C}}$. A sequence of functions $\{\varphi_n\}$, meromorphic in D , is said to converge to a function φ with respect to the m_1 -measure *inside* D if for every $\varepsilon > 0$ and any compact set $K \subset D$ we have

$$m_1\{z \in K : |(\varphi - \varphi_n)(z)| \geq \varepsilon\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The sequence $\{\varphi_n\}$ is said to converge to φ *m_1 -almost locally uniformly inside D* if for any compact set $K \subset D$ and any $\varepsilon > 0$ there exists a set $K_\varepsilon \subset K$ such that $m_1(K \setminus K_\varepsilon) < \varepsilon$ and the sequence $\{\varphi_n\}$ converges uniformly to φ on K_ε . Hence, the m_1 -almost local uniform convergence inside D implies m_1 -convergence inside D [7].

We recall the basic properties of the convergence in m_1 -measure [7]. Suppose that $\{\varphi_n\}$ converges in m_1 -measure to φ inside the domain D , then:

- (a) If $\{\varphi_n\} \in \mathcal{A}(D)$, then $\{\varphi_n\}$ converges locally uniformly in D . Thus, the limit function is analytic in D .
- (b) If the functions φ_n are m -meromorphic in D , where $m \geq 0$ is a fixed integer, then the limit function φ is m_1 -equivalent to a function which is also m -meromorphic in D . Hence, if φ has a pole of order τ at a point $a \in D$, then at least τ poles of φ_n tend to a .
- (c) If $\varphi \in \mathcal{M}_m(D)$, then all functions φ_n (for n large enough) have at least m poles in D .

Let us return to Theorem 1. If $m \in \mathbb{N}$ is fixed and $\rho_m(f) > 1$, then it was shown in [6] that the best Chebyshev approximants $R_{n,m}$ converge m_1 -almost locally uniformly inside the domain $E_{\rho_m(f)}$ as $n \rightarrow \infty$. Denote by $\tilde{f}$ the limit function. By (b), $\tilde{f}$ is m_1 -equivalent to a function in $\mathcal{M}_m(E_{\rho_m(f)})$. Furthermore, Gončar proved a quantitative characteristic of the convergence of the sequence $\{R_{n,m}\}$ inside $E_{\rho_m(f)}$; namely, for every compact set $K \subset E_{\rho_m(f)}$ and $\varepsilon > 0$ there exists an open set Ω_ε with $m_1(\Omega_\varepsilon) < \varepsilon$ such that

$$\limsup_{n \rightarrow \infty} \|f - R_{n,m}\|_{K \setminus \Omega_\varepsilon}^{1/n} \leq \frac{\exp(\max_K G_E(z, \infty))}{\rho_m(f)}$$

([6], p. 153, Remark 1).

We consider here the question about the rate of convergence and the distribution of the zeros in the case of rational functions if the degrees of the denominators are not bounded.

For $f \in C(E)$, let us introduce the *radius of meromorphy* $\rho(f)$ of f , that is

$$\rho(f) := \sup_{m \geq 0} \rho_m(f).$$

Our new results are concerned with a sequence $\{m_n\}_{n=1}^\infty$, $m_n \in \mathbb{N}$, $n \in \mathbb{N}$, such that

$$m_n \rightarrow \infty \text{ and } m_n = o(n/\log n) \text{ as } n \rightarrow \infty. \quad (4)$$

Assume that $f \in \mathcal{A}(E)$ and $\rho(f) < \infty$. Then for sequences as in (4), Walsh's theorem ([15], p. 378, Theorem 3) implies that

$$\limsup_{n \rightarrow \infty} \|f - R_{n,m_n}\|_E^{1/n} \leq \frac{1}{\rho(f)}. \quad (5)$$

In Theorem 3, we state that the uniform geometric convergence of a sequence $\{r_{n,m_n}\}_{n \in \mathbb{N}}$ to f on E implies the geometric convergence of this sequence to a continuous function m_1 -almost locally uniformly inside some domain E_τ , $\tau > 1$.

Theorem 3 ([5]). Let E be compact in $\mathbb{C}$ with regular, connected complement $\Omega = \overline{\mathbb{C}} \setminus E$, $\{m_n\}_{n=1}^\infty$ a sequence in $\mathbb{N}_0$ with $m_n = o(n/\log n)$ as $n \rightarrow \infty$, $\{r_{n,m_n}\}_{n \in \mathbb{N}}$ a sequence of rational functions, $r_{n,m_n} \in \mathcal{R}_{n,m_n}$, such that for $f \in \mathcal{M}(E)$

$$\limsup_{n \rightarrow \infty} \|f - r_{n,m_n}\|_{\partial E}^{1/n} \leq \frac{1}{\tau} < 1. \quad (6)$$

Then there exists an extension $\tilde{f}$ of f to E_τ with the following property: For any $\varepsilon > 0$ there exists a subset $\Omega(\varepsilon) \subset \mathbb{C}$ with $m_1(\Omega(\varepsilon)) < \varepsilon$ such that $\tilde{f}$ is a continuous function on $E_\tau \setminus \Omega(\varepsilon)$ and

$$\limsup_{n \rightarrow \infty} \|\tilde{f} - r_{n,m_n}\|_{E_\sigma \setminus \Omega(\varepsilon)}^{1/n} \leq \frac{\sigma}{\tau} \quad (7)$$

for any σ with $1 < \sigma < \tau$, and $\{r_{n,m_n}\}_{n \in \mathbb{N}}$ converges m_1 -almost uniformly to $\tilde{f}$ inside E_τ .

Remark 1. Under the conditions of Theorem 1, in particular m is fixed, the function f can be continued to $f \in \mathcal{M}_m(E_{\rho_m(f)})$. In this case, $\tilde{f}$ of Theorem 3 is m_1 -equivalent to f in $E_{\rho_m(f)}$. But under the conditions of Theorem 3, the sequence $\{m_n\}_{n=1}^\infty$ need not be bounded. If f can be continued to $f \in \mathcal{M}(E_\rho)$, $1 < \rho$, then it is an open problem whether the continuous extension $\tilde{f}$ of f on $E_\rho \setminus \Omega(\varepsilon)$ is m_1 -equivalent to f on $(E_\tau \cap E_\rho) \setminus \Omega(\varepsilon)$, where $\varepsilon > 0$ is arbitrarily small and $m_1(\Omega(\varepsilon)) < \varepsilon$.

For obtaining results about the distribution of the zeros of the approximants r_{n,m_n} , we will assume that the continuous extension $\tilde{f}$ of Theorem 3 for $\tau \leq \rho(f)$ coincides with the meromorphic continuation of f into $E_{\rho(f)}$.

Definition 1. Let $f \in \mathcal{M}(E)$ and $\rho(f) < \infty$. A sequence $\{r_n\}_{n \in \mathbb{N}}$ with $r_n \in \mathcal{R}_{n,n}$ is called m_1 -maximally convergent to f on E if

$$\limsup_{n \rightarrow \infty} \|f - r_n\|_{\partial E}^{1/n} \leq \frac{1}{\rho(f)} \quad (A)$$

and for any $\varepsilon > 0$ there exists a set $\Omega(\varepsilon) \subset \mathbb{C}$ with $m_1(\Omega(\varepsilon)) < \varepsilon$ such that

$$\limsup_{n \rightarrow \infty} \|f - r_n\|_{E_\sigma \setminus \Omega(\varepsilon)}^{1/n} \leq \frac{\sigma}{\rho(f)} \quad (B)$$

for all σ , $1 < \sigma < \rho(f)$.

Remark 2. As an example for such m_1 -maximal convergent sequence one can take best real rational approximants of $f \in C(E)$, where $E = [-1, 1]$ and f is real-valued [3]. Another example is classical Padé approximation to a function f meromorphic in a neighborhood of 0 ([11] and [7]).

In our results for the distribution of the zeros of the rational approximants, a special class of meromorphic functions will play an essential role.

Definition 2. Let $f \in \mathcal{M}(E)$ with $\rho(f) < \infty$. A point $z_0 \in \partial E_{\rho(f)}$ is called a *singularity of multivalued character* of the function f if there exists a neighborhood U of z_0 such that f can be continued to any point of $U \setminus \{z_0\}$ and f is locally holomorphic, but not single-valued.

Examples of singularities of multivalued character are branch points.

The next theorem provides for functions with a multivalued singularity an exact estimate of the rate of approximation.

Theorem 4 ([5]). Let E be a compact, connected set in $\mathbb{C}$ with regular and connected complement. Moreover, let $\{m_n\}_{n=1}^\infty$ be a subsequence of $\mathbb{N}_0$ satisfying (4). Suppose that the sequence of rational functions $\{r_{n,m_n}\}_{n \in \mathbb{N}}$, $r_{n,m_n} \in \mathcal{R}_{n,m_n}$ is m_1 -maximally convergent to $f \in \mathcal{M}(E)$ on E with $\rho(f) < \infty$. If there exists a singularity of multivalued character of f on $\partial E_{\rho(f)}$, then

$$\limsup_{n \rightarrow \infty} \|f - r_{n,m_n}\|_{\partial E}^{1/n} = \frac{1}{\rho(f)}. \quad (8)$$

Theorem 4 and the results of [8] are fundamental for establishing an analogue of Szegő's theorem for the case being considered.

Theorem 5 ([5]). Under the conditions of Theorem 4, the normalized zero counting measures ν_n of r_{n,m_n} converge weakly to the equilibrium distribution of $\overline{E}_{\rho(f)}$, at least for a subsequence $\Lambda \subset \mathbb{N}$ as $n \rightarrow \infty$ with $n \in \Lambda$.

We are now interested how a singularity on the boundary $\partial E_{\rho(f)}$ of $E_{\rho(f)}$ impacts the behavior of the zeros of a m_1 -maximally convergent sequence $\{r_{n,m_n}\}$ as $n \rightarrow \infty$. If among the singularities on $\partial E_{\rho(f)}$ there is at least one of multivalued character, then by Theorem 5 each point on $\partial E_{\rho(f)}$ is a limit point of zeros of $\{r_{n,m_n}\}$ as $n \rightarrow \infty$. Furthermore, there is a sequence Λ such that the associated zero distributions ν_n converge weakly to the equilibrium measure of $\overline{E}_{\rho(f)}$ as $n \in \Lambda$, $n \rightarrow \infty$. It is of interest to study the case where there is no singular point of multivalued character on $\partial E_{\rho(f)}$.

Given a domain B and a function $g \in \mathcal{M}(B)$, we introduce the notation $Z(g, B)$; that is the number of zeros of g in B . Analogously, $P(g, B)$ denotes the number of poles of g in B . Recall that poles and zeros are counted with their multiplicities.

Concerning the limit distribution of the zeros and poles of a m_1 -maximally convergent sequence of rational function, we have the following result:

Theorem 6 ([4]). Let E be a regular compact set and let $f \in \mathcal{M}(E)$. Suppose that $\rho(f) < \infty$ and assume that the point $z_0 \in \partial E_{\rho(f)}$ is a singularity of the function f , but neither a pole nor a removable singularity. Suppose that the sequence of

rational functions $\{r_n\}$, $r_n \in \mathcal{R}_{n,n}$, is m_1 -maximally convergent to f on E . Then for any neighborhood U of z_0 the following statements hold:

- (a1) If $P(r_n, U) = o(n)$ as $n \rightarrow \infty$, then $\limsup_{n \rightarrow \infty} Z(r_n, U) = \infty$.
- (a2) If $P(r_n, U) = O(1)$ as $n \rightarrow \infty$, then $\limsup_{n \rightarrow \infty} Z(r_n, U)/n > 0$.
- (b1) If $Z(r_n, U) = o(n)$ as $n \rightarrow \infty$, then $\limsup_{n \rightarrow \infty} P(r_n, U) = \infty$.
- (b2) If $Z(r_n, U) = O(1)$ as $n \rightarrow \infty$, then $\limsup_{n \rightarrow \infty} P(r_n, U)/n > 0$.

In the proof of Theorem 6, we essentially use that $f \in \mathcal{M}(E)$, as well as the m_1 -maximal convergence of the sequence $\{r_n\}$ to the function f . Theorem 6 applies to Padé approximants and to rational approximations to a real valued function on a bounded real interval, when the degrees of the denominators satisfy the condition (4) (see [5] and [4]).

Let $a \in \mathbb{C}$, $B \subset \mathbb{C}$, r a rational function, and let $a(r, B)$ denote the number of a -values of r in B , i.e.,

$$a(r, B) := \#\{z \in B : r(z) = a\}.$$

Corollary 1. *Let E and f be as in Theorem 6. Suppose that $z_0 \in \partial E_{\rho(f)}$ is a singular point of f which is neither a pole nor a removable singularity. Suppose that $\{m_n\}_{n \in \mathbb{N}}$ with $m_n = o(n)$ as $n \rightarrow \infty$, and let $\{r_{n, m_n}\}_{n \in \mathbb{N}}$ be m_1 -maximally convergent to f on E . Then, for every neighborhood U of z_0*

$$\limsup_{n \rightarrow \infty} a(r_{n, m_n}, U) = \infty, \quad a \in \mathbb{C}.$$

Proof. Fix $a \in \mathbb{C}$. The proof follows from statement (a1) in Theorem 6, applied to the functions $r_{n, m_n} - a$ and $f - a$, after taking into account that $m_n = o(n)$ as $n \rightarrow \infty$. $\square$

Recalling the classical theorem of Picard concerning the behavior of a holomorphic function in a neighborhood of an essential singularity, we can summarize Corollary 1 by saying that a m_1 -maximally convergent sequence $\{r_{n, m_n}\}$ with $\{m_n\}$ as in (4) has an asymptotic essential singularity at each singularity of the curve of meromorphy which is neither a pole nor a removable singularity.

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