

Chapter 2

Regularity of Sets

2.1 Motivations

We present here two examples of motivations of the study of regularity concepts for sets and their importance in applications.

2.1.1 Calculus Rules

Calculus rules of the Clarke tangent cones and the convexified (Clarke) normal cones of sets is one of the mathematical domains where the regularity of sets in some sense plays a crucial role to obtain exact formulas. For instance, let X be a Banach space, S_1, S_2 be two subsets in X with $\bar{x} \in S_1 \cap S_2$ and S_2 be epi-Lipschitz¹ around \bar{x} . Then

$$T^C(S_1; \bar{x}) \cap T^C(S_2; \bar{x}) \subset T^C(S_1 \cap S_2; \bar{x}), \quad (*)$$

whenever the qualification condition (QC)

$$T^C(S_1; \bar{x}) \cap \text{int} T^C(S_2; \bar{x}) \neq \emptyset,$$

is satisfied. The corresponding normal formula that holds under (QC) and some other hypothesis is

$$N^C(S_1 \cap S_2; \bar{x}) \subset N^C(S_1; \bar{x}) + N^C(S_2; \bar{x}). \quad (**)$$

¹A closed nonempty subset S of X is said to be *epi-Lipschitz* around $\bar{x} \in S$ if it can be represented near \bar{x} as the epigraph of a Lipschitz function.

In [88], Clarke showed that formulas $(*)$ and $(**)$ become equalities if both sets S_1 and S_2 are *tangentially regular* at \bar{x} , that is, the Clarke tangent cone $T^C(S; \bar{x})$ coincides with the Bouligand tangent cone $K(S; \bar{x})$. So, we will obtain under (QC) and the tangential regularity of both sets at \bar{x}

$$T^C(S_1 \cap S_2; \bar{x}) = T^C(S_1; \bar{x}) \cap T^C(S_2; \bar{x})$$

and

$$N^C(S_1 \cap S_2; \bar{x}) = N^C(S_1; \bar{x}) + N^C(S_2; \bar{x}).$$

There are many other calculus rules for tangent and normal cones that need the hypothesis of tangential regularity to become equalities, for examples: sets with constraint structure $S = A \cap F^{-1}(D)$, product subsets $S = \prod_{i=1}^m S_i, \dots$, etc.

2.1.2 Differential Inclusions

Another type of problem which is an important area of applications of our main results in these chapters is the differential inclusion problems (see for details Chaps. 5 and 6), more precisely, the first and second order sweeping process problems. Recall that the first order sweeping process problem was introduced in the 1970s by Moreau [207–210] and extensively studied by himself and other authors (Castaing [73, 75, 78, 81, 82], Valadier [255–257], and their students). Let \mathbf{H} be a Hilbert space, $T > 0$ be some real positive number, and $C : [0, T] \rightrightarrows \mathbf{H}$ be a set-valued mapping taking nonempty closed values in \mathbf{H} . The first order sweeping process consists in solving the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N^C(C(t); x(t)), & \text{a.e. } t \in [0, T], \\ x(t) \in C(t), & \forall t \in [0, T], \\ x(0) = x_0 \in C(0). \end{cases} \quad (SP)$$

Under the convexity assumption on $C(t)$ and other natural hypothesis, Moreau proved existence and uniqueness of a solution to (SP) . A natural question, which many authors attacked, is whether similar results can be obtained if $C(t)$ in (SP) is not assumed to be convex. The most important regularity hypothesis assumed on $C(t)$ in the setting of Hilbert space, in order to obtain existence results of (SP) , are: uniform prox-regularity in the sense of [229] (or equivalently proximal smoothness in the sense of [89]) (see [42, 58, 248]), epi-Lipschitz property (see [92, 255–257]), and Φ -convexity (see [92]). We will see with more details in Chap. 5 the important role of the uniform prox-regularity in such problems. The second order sweeping process was firstly studied by Castaing [80] with convex-valued set-valued mappings and later by the author and other investigators in some regular cases (see for instance [43, 54, 64] and the references therein). It consists in solving the following abstract second order differential inclusion:

$$\begin{cases} \ddot{x}(t) \in -N^C(C(x(t)); \dot{x}(t)), & \text{a.e. } t \in [0, T], \\ \dot{x}(t) \in C(x(t)), & \forall t \in [0, T], \\ x(0) = x_0 \in \mathbf{H} \text{ and } \dot{x}(0) \in C(x(0)), \end{cases} \quad (SOSP)$$

where C is a set-valued mapping defined from \mathbf{H} to itself. This problem (SOSP) and many of its variants will be the main purpose of Chap. 6 when the convexity of $C(t)$ is excluded and the uniform prox-regularity is employed.

2.2 Tangential Regularity of Sets

There are several ways to define general concepts of regularity of a subset S at some point $\bar{x} \in S$ in Nonsmooth Analysis Theory. One of them is the tangential regularity introduced by Clarke in [88], that is, the Clarke tangent cone $T^C(S; \bar{x})$ coincides with the Bouligand tangent cone $K(S; \bar{x})$ (Note that one always has $T^C(S; \bar{x}) \subset K(S; \bar{x})$). Let us recall the definition of these two classical tangent cones when the space X is assumed to be topological vector space not necessarily normed. In all this section X will be a topological vector space.

Definition 2.1. Let S be a nonempty closed subset of X and $\bar{x} \in S$.

- (i) The Bouligand tangent cone $K(S; \bar{x})$ to S at \bar{x} is the set of all $h \in X$ such that for every neighborhood H of h in X and for every $\varepsilon > 0$, there exists $t \in (0, \varepsilon)$ such that

$$(\bar{x} + tH) \cap S \neq \emptyset.$$

- (ii) The Clarke tangent cone $T^C(S; \bar{x})$ to S at \bar{x} is the set of all $h \in X$ such that for every neighborhood H of h in X there exist a neighborhood U of \bar{x} in X and a real number $\varepsilon > 0$ such that

$$(x + tH) \cap S \neq \emptyset \quad \text{for all } x \in U \cap S \text{ and } t \in (0, \varepsilon).$$

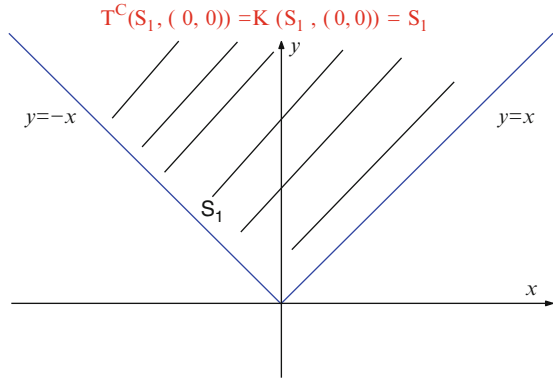
Exercise 2.1.

1. Assume that X is a normed vector space. Prove that the definitions of the Clarke tangent cone and the Bouligand tangent cone given above coincide with the ones given in Chap. 1 in Definition 1.5 in terms of the directional derivatives of the distance function d_S .
2. Show that the Parts (1) and (2) in Exercise 1.9 are still true even when X is a topological vector space.

There is a large class of subsets for which the inclusion $T^C(S; \bar{x}) \subset K(S; \bar{x})$ has the equality form. Following Clarke [88], we get with the following definition.

Definition 2.2. We will say that S is *tangentially regular* at \bar{x} provided that $T^C(S; \bar{x}) = K(S; \bar{x})$.

Fig. 2.1 Tangential regularity of convex sets



We give in the following example some regular and irregular subsets.

Example 2.1.

1. Any closed convex subset is tangentially regular at each of its points. This follows directly from the Part (2) in the previous exercise. For example, let $S_1 = \{(x,y) \in \mathbf{R}^2 : y \geq |x|\}$ and $\bar{x} = (0,0)$ (see Fig. 2.1). This subset is convex and hence it is tangentially regular at \bar{x} with

$$T^C(S_1; \bar{x}) = K(S_1; \bar{x}) = T^{\text{conv}}(S_1; \bar{x}) = \text{cl}[\mathbf{R}_+(S_1 - \bar{x})] = \text{cl}[\mathbf{R}_+(S_1)] = S_1.$$

The last equality follows from the fact that S_1 is a closed cone.

2. Let $S_2 = \{(x,y) \in \mathbf{R}^n : f(x) = 0\}$ with $f \in C^1$. This subset is tangentially regular at each point $\bar{x} \in S$ satisfying $\nabla f(\bar{x}) \neq 0$ and one has

$$T^C(S_2; \bar{x}) = K(S_2; \bar{x}) = \{v \in \mathbf{R}^n : \nabla f(\bar{x})v = 0\}.$$

More generally, the constraint set $S = F^{-1}(D) = \{x \in X : F(x) \in D\}$ is tangentially regular at each point $\bar{x} \in S$, whenever F is a C^1 mapping, D is tangentially regular at $F(\bar{x})$, and under some natural conditions (such as Robinson qualification condition see for instance [35, 233]) and in this case one has

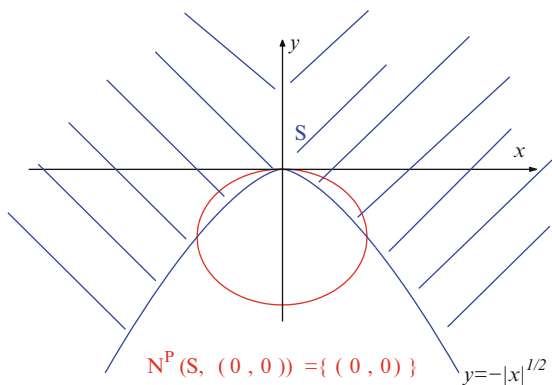
$$R(S; \bar{x}) = \nabla F(\bar{x})^{-1} R(D; F(\bar{x}))$$

where $R(S; u) = T^C(S; u) = K(S; u)$. Note that in the general case when D is not tangentially regular one only has the following inclusions

$$K(S; \bar{x}) \subset \nabla F(\bar{x})^{-1} K(D; F(\bar{x})) \quad \text{and} \quad \nabla F(\bar{x})^{-1} T^C(D; F(\bar{x})) \subset T^C(S; \bar{x}).$$

3. Let S_3 be the closure of the complement of the set S_1 (see Fig. 1.2) and $\bar{x} = (0,0)$. Show that $T^C(S; \bar{x}) = -S_1$ and $K(S; \bar{x}) = S_3$? and hence S_3 is not tangentially regular at \bar{x} .

Fig. 2.2 A set which is (FNR) but not (PNR)



4. Let $S_4 = \{(x, y) \in \mathbf{R}^2 : \|(x-1, y)\| \leq 1\} \cup \{(x, y) \in \mathbf{R}^2 : \|(x+1, y)\| \leq 1\}$ and $\bar{x} = (0, 0)$ (see Fig. 1.3). Show that $T^C(S_4; \bar{x}) = \{(0, 0)\}$ and $K(S_4; \bar{x}) = \mathbf{R}^2$? and hence one gets that S_4 is not tangentially regular at \bar{x} .

2.3 Fréchet and Proximal Normal Regularity of Sets

Another natural concept of regularity of a subset S at some point $\bar{x} \in S$, that needs to be considered is the normal regularity. This means that the convexified (Clarke) normal cone $N^C(S; \bar{x})$ of S at \bar{x} coincides with a prescribed normal cone $N^\#(S; \bar{x})$ of S at \bar{x} . We state here the case of the Fréchet or the proximal normal cones.

Definition 2.3. Let S be a nonempty closed subset of X and let $\bar{x} \in S$. We will say that S is *Fréchet normally* (resp. *proximal normally*) *regular* at \bar{x} if one has $\hat{N}(S; \bar{x}) = N^C(S; \bar{x})$ (resp. $N^P(S; \bar{x}) = N^C(S; \bar{x})$).

Remark 2.1. As one always has $N^P(S; \bar{x}) \subset \hat{N}(S; \bar{x})$, one sees that the proximal normal regularity (PNR) always implies the Fréchet normal regularity (FNR). The converse is not true. Indeed, we take $S = \{(x, y) \in \mathbf{R}^2 : y \geq -|x|^\alpha\}$, with $1 < \alpha < 2$ (for instance $\alpha = \frac{3}{2}$) and $\bar{x} = (0, 0)$. One has $N^P(S; \bar{x}) = \{(0, 0)\}$, because no ball whose interior fails intersect S can have \bar{x} on its boundary. While all other normal cones coincide and are equal to $\{(0, -r) : r \in \mathbf{R}_+\}$ (Fig. 2.2).

2.4 Scalar Regularity of Sets

One more but not less natural notion of regularity for a subset S at $\bar{x} \in S$ is the *scalar regularity*, which means the regularity of an associated scalar function with the subset S . The scalar function which will be used here is the associated distance

function d_S . Note that d_S is always globally 1-Lipschitz function on X . So, we will need some preliminaries and some regularity concepts for locally Lipschitz functions.

Remark 2.2. Note that a general study about various notions of regularity of lower semicontinuous (non necessarily locally Lipschitz) functions will be given in Chap. 3. Because of the use of the distance function here, we restrict our attention on the following preliminaries in the case of locally Lipschitz functions.

Let $f : X \rightarrow \mathbf{R}$ be a locally Lipschitz function and $\bar{x} \in X$. Like in the case of subsets, there are many ways to define general concepts of regularity for functions in Nonsmooth Analysis Theory. The well known is the directional regularity defined as follows.

Definition 2.4. Following Clarke [88], we will say that f is *directionally regular* at \bar{x} if one has

$$f^0(\bar{x}; \cdot) = f^-(\bar{x}; \cdot)$$

where $f^0(\bar{x}; \cdot)$ and $f^-(\bar{x}; \cdot)$ are, respectively, the generalized directional derivative and the lower Dini directional derivative of f at \bar{x} .

Example 2.2.

1. Any convex continuous function f is directionally regular and

$$f^0(\bar{x}; v) = f^-(\bar{x}; v) = f'(\bar{x}; v).$$

2. Any continuously differentiable function f (i.e. $f \in C^1$) is directionally regular and one has

$$f^0(\bar{x}; v) = f^-(\bar{x}; v) = \nabla f(\bar{x})(v).$$

3. Let $f : X \rightarrow \mathbf{R}$ be defined by $f(x) = -\|x\|$ and $\bar{x} = 0$. Then for all $v \in X$ one has $f^0(\bar{x}; v) = \|v\|$ and $f^-(\bar{x}; v) = -\|v\|$. Hence, f is not directionally regular at \bar{x} .
4. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be defined by $f(x) = x^2 \sin(\frac{1}{x})$ and $\bar{x} = 0$. This function is differentiable at \bar{x} and its derivative $\nabla f(\bar{x}) = 0$, but it is not directionally regular at \bar{x} . Indeed, for all $v \in X$ one has $f^0(\bar{x}; v) = \|v\|$ and $f^-(\bar{x}; v) = 0$.

Remark 2.3. The Part (4) in the last example shows that the differentiability of f at \bar{x} is not sufficient for f to be directionally regular at \bar{x} .

Another natural concept of regularity for functions is the *subdifferential regularity*. This means that the generalized gradient (Clarke subdifferential) $\partial^C f(\bar{x})$ of f at \bar{x} coincides with a prescribed subdifferential $\partial^\# f(\bar{x})$ (for example: $\partial^\# = \widehat{\partial}$ (for Fréchet), $\partial^\# = \partial^P$ (for proximal), $\partial^\# = \partial$ (for basic),...) of f at \bar{x} . We restrict our attention here only on the Fréchet and the proximal cases. The following proposition

states a very important relationship between the subdifferential concept and the normal cone concept which will be used in our study. It cannot be proved for a general subdifferential and a general normal cone. Each case has been proved separately and we refer the reader to different works, in each one we can find the proof of a particular case (see [88, 91, 140, 180, 196, 198]).

Proposition 2.1. *Let $(\partial^\#; N^\#) \in \{(\partial^P; N^P), (\widehat{\partial}; \widehat{N}), (\partial^C; N^C), (\partial; N)\}$. Then one has*

$$\partial^\# f(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in N^\#(\text{epi } f; (\bar{x}, f(\bar{x})))\}. \quad (2.1)$$

Now we are in position to introduce the second type of scalar regularity.

Definition 2.5. We will say that f is *Fréchet* (resp. *proximal*) *subdifferentially regular* at \bar{x} provided that $\widehat{\partial}f(\bar{x})$ (resp. $\partial^P f(\bar{x})$) coincides with $\partial^C f(\bar{x})$.

Remark 2.4. Obviously, one has proximal subdifferential regularity implies Fréchet subdifferential regularity. The converse does not hold. Indeed, take $f(x) = -(|x|)^{\frac{3}{2}}$, $f \in C^1$. One has $\widehat{\partial}f(0) = \partial^C f(0) = \{0\}$, while $\partial^P f(0) = \emptyset$ (because $N^P(\text{epi } f; (0, 0)) = \{(0, 0)\}$).

Conclusion. We have considered three types of regularity for locally Lipschitz functions. Consequently, we will obtain three additional types of regularity for subsets by taking $f = d_S$ and then we will say that

- S is (DR) at $\bar{x} \in S$ if and only if d_S is directionally regular at \bar{x} .
- S is (FSR) at $\bar{x} \in S$ if and only if d_S is Fréchet subdifferentially regular at \bar{x} ;
- S is (PSR) at $\bar{x} \in S$ if and only if d_S is proximally subdifferentially regular at \bar{x} .

Now, we summarize all what we have considered as a concept of regularity for a subset S at a point $\bar{x} \in S$:

- (i) Tangential regularity (TR), i.e., $T^C(S, \bar{x}) = K(S, \bar{x})$;
- (ii) Proximal Normal Regularity (PNR), i.e., $N^P(S, \bar{x}) = N^C(S, \bar{x})$;
- (iii) Fréchet Normal Regularity (FNR), i.e., $\widehat{N}(S, \bar{x}) = N^C(S, \bar{x})$;
- (iv) Directional Regularity (DR), i.e., $d_S^0(\bar{x}; \cdot) = d_S^-(\bar{x}; \cdot)$;
- (v) Proximal Subdifferential Regularity (PSR), i.e., $\partial^P d_S(\bar{x}) = \partial^C d_S(\bar{x})$;
- (vi) Fréchet Subdifferential Regularity (FSR), i.e., $\widehat{\partial} d_S(\bar{x}) = \partial^C d_S(\bar{x})$.

Our main goal in the sequel of this Chapter is to study the relationships between all of these notions of regularity. We will proceed as follows:

1. (TR) \iff (DR)? (can be seen as a scalarization of (TR));
2. (FNR) \iff (FSR)? (can be seen as a scalarization of (FNR));
3. (PNR) \iff (PSR)? (can be seen as a scalarization of (PNR));
4. (TR) \iff (FNR)? (We can see this equivalence as a bridge between the primal notion of regularity (TR) and the dual notion of regularity (FNR)).

2.5 Scalarization of Tangential Regularity: [(TR) \Leftrightarrow (DR)?]

In this section, we prove the following main theorem due to Burke et al. [71].

Theorem 2.1. *Let S be a closed nonempty subset of a normed vector space X and let $\bar{x} \in S$.*

1. *If d_S is directionally regular at $\bar{x} \in S$, then S is tangentially regular (TR) at $\bar{x} \in S$;*
2. *If, in addition, $\dim X < +\infty$, then the converse holds, i.e., (TR) \Longleftrightarrow (DR).*

Proof.

1. By Definition 1.5 in Chap. 1 and the directional regularity of d_S at \bar{x} we have

$$K(S; \bar{x}) = \{v \in X : d_S^-(\bar{x}, v) = 0\} = \{v \in X : d_S^0(\bar{x}, v) = 0\} = T^C(S; \bar{x})$$

and hence S is tangentially regular at \bar{x} .

2. Assume that $\dim X < +\infty$, and S is tangentially regular at \bar{x} . In order to make clear the idea of the proof of this Part (2), we omit the following facts:

Fact 1.

$$d_S^0(\bar{x}, v) \leq d_{T^C(S; \bar{x})}(v), \quad \text{for all } v \in X.$$

Fact 2.

$$d_S^-(\bar{x}, v) = d_{K(S; \bar{x})}(v), \quad \text{for all } v \in X.$$

We will give the proof of these two facts after completing the proof of the Part (2).

Then one has for all $v \in X$

$$\begin{aligned} d_S^0(\bar{x}, v) &\leq d_{T^C(S; \bar{x})}(v) && \text{(by Fact 1)} \\ &= d_{K(S; \bar{x})}(v) && \text{(by (TR))} \\ &= d_S^-(\bar{x}, v). && \text{(by Fact 2)} \end{aligned}$$

Hence, $d_S^0(\bar{x}, v) \leq d_S^-(\bar{x}, v)$ and as the reverse inequality is always true, one gets the equality, i.e., the directional regularity of d_S at \bar{x} . This completes the proof of the Part (2). \square

Now let us prove Fact 1 and Fact 2.

Proof of Fact 1. Fix any $v \in X$ and any $\varepsilon > 0$. There exists (by the definition of the infimum) some $\bar{v} \in T^C(S; \bar{x})$ such that

$$\|v - \bar{v}\| \leq d_{T^C(S; \bar{x})}(v) + \varepsilon. \quad (2.2)$$

Consider a sequence (t_n, x_n) in $(0, +\infty) \times S$ converging to $(0, \bar{x})$ satisfying

$$d_S^0(\bar{x}; v) = \lim_n t_n^{-1} d_S(x_n + t_n v) = \lim_n d_{t_n^{-1}(S - x_n)}(v). \quad (2.3)$$

By the sequential characterization of the Clarke tangent cone $T^C(S; \bar{x})$, there exists a sequence $v_n \rightarrow \bar{v}$ such that $x_n + t_n v_n \in S$, i.e., $v_n \in t_n^{-1}(S - x_n)$, for all n . Thus,

$$d_{t_n^{-1}(S-x_n)}(v) \leq d_{t_n^{-1}(S-x_n)}(v_n) + \|v - v_n\| \leq \|v - v_n\|.$$

By letting $n \rightarrow +\infty$, one gets by (2.2) and (2.3)

$$d_S^0(\bar{x}; v) \leq \|v - \bar{v}\| \leq d_{T^C(S; \bar{x})}(v) + \varepsilon.$$

This completes the proof of Fact 1. \square

Proof of Fact 2. With the same method as in the proof of Fact 1, we can prove the inequality

$$d_S^-(\bar{x}; v) \leq d_{K(S; \bar{x})}(v), \text{ for all } v \in X.$$

So, we proceed now to prove the reverse inequality, i.e.,

$$d_{K(S; \bar{x})}(v) \leq d_S^-(\bar{x}; v), \text{ for all } v \in X. \quad (2.4)$$

Fix any $v \in X$. Let us consider a sequence of real positive numbers $t_n \downarrow 0$ such that

$$d_S^-(\bar{x}; v) = \lim_n t_n^{-1} d_S(\bar{x} + t_n v) = \lim_n d_{t_n^{-1}(S - \bar{x})}(v). \quad (2.5)$$

For each $n \in \mathbf{N}$, we choose $v_n \in t_n^{-1}(S - \bar{x})$ with

$$\|v - v_n\| \leq d_{t_n^{-1}(S - \bar{x})}(v) + t_n.$$

Then, by (2.5) one gets

$$\lim_n \|v - v_n\| \leq d_S^-(\bar{x}; v). \quad (2.6)$$

This ensures that the sequence $(v_n)_{n \in \mathbf{N}}$ is bounded. Hence, as $\dim X < +\infty$, some subsequence converges to some vector $\bar{v} \in X$. Consequently, the sequential characterization of $K(S; \bar{x})$ and the choice of v_n ensures that \bar{v} must lie in $K(S; \bar{x})$. It follows then, by (2.6) that

$$d_{K(S; \bar{x})}(v) \leq \|v - \bar{v}\| \leq d_S^-(\bar{x}; v).$$

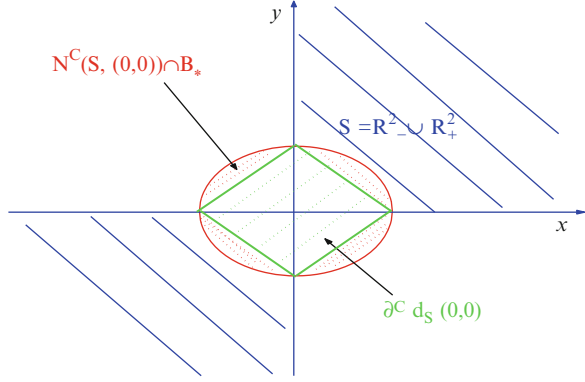
This completes the proof of (2.4) and hence the proof of Fact 2 is finished. \square

Remark 2.5. Note that Fact 1 and its corresponding inequality for $d_S^-(\bar{x}; \cdot)$ and $d_{K(S; \bar{x})}(\cdot)$ are true for any closed nonempty subset S in any normed vector space X , without the hypothesis $\dim X < \infty$.

It is well known, in convex analysis theory, that the bridge formula between the normal cone $N^{\text{conv}}(S; \bar{x})$ to a nonempty closed subset S at $\bar{x} \in S$ and the subdifferential $\partial^{\text{conv}} d_S(\bar{x})$ of its distance function d_S at \bar{x} is the following:

$$\partial^{\text{conv}} d_S(\bar{x}) = N^{\text{conv}}(S; \bar{x}) \cap \mathbf{B}_*. \quad (2.7)$$

Fig. 2.3 A set with a strict inclusion in (2.8)
(Example 2.3)



In the nonconvex case, this formula may fail for the convexified (Clarke) normal cone $N^C(S; \bar{x})$ and the generalized gradient $\partial^C d_S(\bar{x})$ and one only has the direct inclusion, i.e.,

$$\partial^C d_S(\bar{x}) \subset N^C(S; \bar{x}) \cap \mathbf{B}_*. \quad (2.8)$$

The proof of this inclusion will be given later, see the proof of Theorem 2.2.

We recall the following example, given in [71], proving this fact.

Example 2.3. Let $S = \mathbf{R}^2_- \cup \mathbf{R}^2_+$, $X = \mathbf{R}^2$ endowed with the Euclidean norm and let $\bar{x} = (0, 0)$. It is not hard to prove that $N^C(S; \bar{x}) = \mathbf{R}^2$ and hence

$$N^C(S; \bar{x}) \cap \mathbf{B}_* = \{(\xi_1, \xi_2) : \xi_1^2 + \xi_2^2 \leq 1\}$$

and one can also check that $d^0(\bar{x}; (v_1, v_2)) = \max\{|v_1|, |v_2|\}$, which ensures that

$$\partial^C d_S(\bar{x}) = \{(\xi_1, \xi_2) : |\xi_1| + |\xi_2| \leq 1\}.$$

This shows that $\partial^C d_S(\bar{x})$ is strictly included in $N^C(S; \bar{x}) \cap \mathbf{B}_*$ (see Fig. 2.3).

In the following theorem, we show that the inclusion (2.8) becomes equality whenever S is tangentially regular at \bar{x} and $\dim X < \infty$.

Theorem 2.2. *Let S be a closed subset of a finite dimensional vector space X and let $\bar{x} \in S$. Assume that S is tangentially regular at \bar{x} . Then one has*

1. $d_S^0(\bar{x}, v) = d_{T^C(S; \bar{x})}(v)$, for all $v \in X$ and
2. $\partial^C d_S(\bar{x}) = N^C(S; \bar{x}) \cap \mathbf{B}_*$.

Proof.

1. Assume that S is tangentially regular at \bar{x} . Then by Theorem 2.1, the function d_S is directionally regular at \bar{x} (because $\dim X < \infty$), i.e., $d_S^0(\bar{x}, v) = d_S^-(\bar{x}, v)$, for all $v \in X$. Therefore, by Theorem 2.1 once again and by the tangential regularity of S at \bar{x} one gets for all $v \in X$

$$d_S^0(\bar{x}, v) = d_S^-(\bar{x}, v) = d_{K(S; \bar{x})}(v) = d_{T^C(S; \bar{x})}(v).$$

2. First, we prove that the direct inclusion, $\partial^C d_S(\bar{x}) \subset N^C(S; \bar{x}) \cap \mathbf{B}_*$, holds in every normed vector space X (without the finite dimensional assumption of X and the tangential regularity of S at \bar{x}).

Fix any $x^* \in \partial^C d_S(\bar{x})$. Then, by the relations (1.12) and (1.13) and by the fact that d_S is 1-Lipschitz, one gets

$$\langle x^*, v \rangle \leq d_S^0(\bar{x}, v) \leq \|v\|, \text{ for all } v \in X. \quad (2.9)$$

This ensures that $\|x^*\| \leq 1$. Now, we wish to show that $x^* \in N^C(S; \bar{x})$. By Fact 1 in Theorem 2.1 and the relation (1.12) one gets

$$\langle x^*, v \rangle \leq d_S^0(\bar{x}, v) \leq d_{T^C(S; \bar{x})}(v), \text{ for all } v \in X$$

and hence $\langle x^*, v \rangle \leq 0$ for all $v \in T^C(S; \bar{x})$. This ensures that $x^* \in (T^C(S; \bar{x}))^0 = N^C(S; \bar{x})$.

Now, we use the hypothesis $\dim X < \infty$ and the tangential regularity of S at \bar{x} to prove the reverse inclusion. Fix any $x^* \in N^C(S; \bar{x}) = (T^C(S; \bar{x}))^0$ with $\|x^*\| \leq 1$. Then for any $v \in T^C(S; \bar{x})$ one has $\langle x^*, v \rangle \leq 0$. Thus, the function $h: X \rightarrow \mathbf{R}$ defined by $h(v) = -\langle x^*, v \rangle$ is 1-Lipschitz (because $\|x^*\| \leq 1$) and satisfies $h(v) \geq h(0) = 0$ for all $v \in T^C(S; \bar{x})$. Using the exact penalization in the Part (3) of Exercise 1.5, one obtains

$$0 \leq h(v) + d_{T^C(S; \bar{x})}(v), \text{ for all } v \in X$$

and hence by the Part (1) of the theorem one gets

$$\langle x^*, v \rangle \leq d_{T^C(S; \bar{x})}(v) = d_S^0(\bar{x}, v), \text{ for all } v \in X.$$

This ensures that $x^* \in \partial^C d_S(\bar{x})$ and hence the proof of the theorem is complete. \square

Another interesting relationship between the normal cone concept and the subdifferential concept of the distance function, which will be used hereafter, is the following

$$N^\#(S; x) = \text{cl}_{w^*}(\mathbf{R}_+ \partial^\# d_S(x)), \quad (2.10)$$

for $(N^\#, \partial^\#) \in \{(\widehat{N}, \widehat{\partial}), (N^C, \partial^C)\}$.

Exercise 2.2. Prove that the relation (2.10) is always true for $(N^\#, \partial^\#) = (N^C, \partial^C)$, whenever X is a real normed space.

Observe that the relation (2.10) is weaker than the equality

$$\partial^\# d_S(x) = N^\#(S; x) \cap \mathbf{B}_*, \quad (2.11)$$

i.e., (2.11) \Rightarrow (2.10). The reverse is not true in general as Exercise 2.2 and Example 2.3 prove it.

2.6 Scalarization of Fréchet Normal Regularity: [(FNR) \Leftrightarrow (FSR)]?

We begin this subsection by presenting a simple and different proof given by Bounkhel and Thibault [61] of a result due to Ioffe [140] and Kruger [166] (It can also be found in [167]). It proves that the inclusion (2.8) becomes equality when we replace the convexified (Clarke) normal cone and the generalized gradient (Clarke subdifferential) by their corresponding in the Fréchet case.

Theorem 2.3. *Let X be a normed vector space, S be a nonempty closed subset of X and x be a point in S . Then*

$$\widehat{\partial}d_S(x) = \widehat{N}(S; x) \cap \mathbf{B}_*.$$

Proof. We begin by proving the inclusion

$$\widehat{\partial}d_S(x) \subset \widehat{N}(S; x) \cap \mathbf{B}_*. \quad (2.12)$$

Fix $x^* \in \widehat{\partial}d_S(x)$. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x' \in x + \delta\mathbf{B}$

$$\langle x^*, x' - x \rangle \leq d_S(x') - d_S(x) + \varepsilon \|x' - x\|.$$

Hence

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x' - x\|, \quad \text{for all } x' \in S \cap (x + \delta\mathbf{B}),$$

which ensures that $x^* \in \widehat{N}(S; x)$. Then as one always has $\widehat{\partial}d_S(x) \subset \partial^C d_S(x) \subset \mathbf{B}_*$, the inclusion (2.12) is proved.

Now, we prove the reverse inclusion, i.e.

$$\widehat{N}(S; x) \cap \mathbf{B}_* \subset \widehat{\partial}d_S(x).$$

Let $x^* \in \widehat{N}(S; x)$ with $\|x^*\| \leq 1$ and let $\varepsilon > 0$. Fix $0 < \varepsilon' < \frac{\varepsilon}{2}$. Then, there exists $\delta > 0$ such that, for all $x' \in S \cap (x + \delta\mathbf{B})$

$$\langle x^*, x' - x \rangle \leq \varepsilon' \|x' - x\|. \quad (2.13)$$

Fix $r > 0$ such that $0 < 2r < \delta$. Since the function $x' \mapsto \langle x^*, x - x' \rangle + \varepsilon' \|x - x'\|$ is L -Lipschitz with $L = \|x^*\| + \varepsilon'$, then using (2.13) and the exact penalization in the Part (3) of Exercise 1.5, we obtain for all $x' \in X$

$$\begin{aligned} \langle x^*, x' - x \rangle &\leq \varepsilon' \|x' - x\| + (\|x^*\| + \varepsilon') d_{S \cap (x + \delta\mathbf{B})}(x') \\ &\leq \varepsilon' \|x' - x\| + d_{S \cap (x + \delta\mathbf{B})}(x') + \varepsilon' d_{S \cap (x + \delta\mathbf{B})}(x') \end{aligned}$$

$$\begin{aligned} &\leq \varepsilon' \|x' - x\| + d_{S \cap (x + \delta \mathbf{B})}(x') + \varepsilon' \|x' - x\| \\ &\leq 2\varepsilon' \|x' - x\| + d_{S \cap (x + \delta \mathbf{B})}(x'). \end{aligned}$$

On the other hand, we can easily check that for all $x' \in x + r\mathbf{B}$

$$d_{S \cap (x + 2r\mathbf{B})}(x') = d_S(x').$$

Thus, for all $\varepsilon > 0$, there exists $r > 0$ such that for all $x' \in x + r\mathbf{B}$

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x' - x\| + d_S(x') - d_S(x).$$

Therefore, $x^* \in \widehat{\partial} d_S(x)$ and hence the proof is finished. \square

Now, we can state the first result on the relationship between the Fréchet normal regularity (FNR) of a set and the Fréchet subdifferential regularity (FSR) of its distance function.

Theorem 2.4. *Let X be any Banach space and let S be a nonempty closed subset of X with $x \in S$. Suppose that S is Fréchet normally regular at x . Then, d_S is Fréchet subdifferentially regular at x . If, in addition, X is reflexive, then one has the equivalence.*

Proof. Assume that S is Fréchet normally regular at x , i.e., $N^C(S; x) = \widehat{N}(S; x)$. Then, by the relations (2.12) and (2.8) one has

$$\partial^C d_S(x) \subset N^C(S; x) \cap \mathbf{B}_* = \widehat{N}(S; x) \cap \mathbf{B}_* = \widehat{\partial} d_S(x),$$

which ensures the Fréchet subdifferential regularity of d_S at x .

Let us prove the reverse implication under the additional hypotheses of the theorem. Assume that X is reflexive and d_S is Fréchet subdifferentially regular at x . The definition of Fréchet subdifferential regularity ensures that $\widehat{\partial} d_S(x) = \partial^C d_S(x)$ and hence by Exercise 2.2 and Theorem 2.3 one gets

$$N^C(S; x) = \text{cl}_{w^*}(\mathbf{R}_+ \widehat{\partial} d_S(x)) = \text{cl}_{w^*}(\widehat{N}(S; x)).$$

By Part (3) in Proposition 1.8, the set $\widehat{N}(S; x)$ is strongly closed convex in X^* and hence it is weak star closed in X^* since X is reflexive. Thus, $N^C(S; x) = \text{cl}_{w^*}(\widehat{N}(S; x)) = \widehat{N}(S; x)$, which ensures the Fréchet normal regularity of S at x . The proof of the theorem is then complete. \square

Remark 2.6. The equivalence in the previous theorem is still true for Asplund–Banach spaces which are more general than reflexive spaces, if we assume that the subset S is compactly epi-Lipschitz at x . For more details on this result and the definition of compactly epi-Lipschitz property we refer the reader to [61] and the references therein.

Corollary 2.1. *Let S be a nonempty closed subset of \mathbf{R}^N and let $x \in S$. Then the following assertions are equivalent:*

- (i) S is Fréchet normally regular at x ;
- (ii) d_S is Fréchet subdifferentially regular at x .

Consider now another concept of normal regularity introduced by Mordukhovich [192] in the finite dimensional setting and used latter by Mordukhovich and Shao [195–198] in Asplund–Banach spaces. Its definition is not in the same way like Definition 2.3. A subset S of an Asplund–Banach space X is said to be *Mordukhovich regular* (called *normally regular* in Mordukhovich [192]) at $x \in S$ provided that $N(S; x) = \widehat{N}(S; x)$. In the following theorem, we prove that Fréchet normal regularity and Mordukhovich regularity are equivalent in reflexive Banach spaces (see also [241] for the finite dimensional setting). We need the following relationship between the convexified (Clarke) normal cone and the basic normal cone. For its proof we refer the reader, for instance, to [89, 192]:

$$N^C(S; \bar{x}) = \text{cl}_{w^*} \text{co} [N(S; \bar{x})]. \quad (2.14)$$

Theorem 2.5. *Let S be a nonempty closed subset of a reflexive Banach space X with $x \in S$. Then the set S is Fréchet normally regular at x if and only if it is Mordukhovich regular at x .*

Proof. If S is Fréchet normally regular at x , then

$$N^C(S; x) = \widehat{N}(S; x) \subset N(S; x) \subset N^C(S; x)$$

and hence $\widehat{N}(S; x) = N(S; x)$, that is, S is Mordukhovich regular at x .

Assume now that S is Mordukhovich regular at x , i.e.,

$$\widehat{N}(S; x) = N(S; x). \quad (2.15)$$

By (2.14) and the convexity of the Fréchet normal cone we get

$$N^C(S; x) = \text{cl}_{w^*} \text{co} [N(S; x)] = \text{cl}_{w^*} [\widehat{N}(S; x)]. \quad (2.16)$$

As $\widehat{N}(S; x)$ is strongly closed (see Part (3) in Proposition 1.8) and convex, it is a weak star closed convex set (since X is reflexive). So, the assumption (2.15) and the equality (2.16) ensure that $N^C(S; x) = \widehat{N}(S; x)$, i.e., S is Fréchet normally regular at x . \square

In a similar way, the concept of *Mordukhovich regularity* of a function $f : X \rightarrow \mathbf{R}$ can be defined as $\widehat{\partial}f(x) = \partial f(x)$ and the equivalence between the Mordukhovich regularity of the function d_S and its Fréchet subdifferential regularity can be established in reflexive Banach spaces. So, the arguments used in the proofs of the two above theorems give the following result.

Theorem 2.6. *Let X be a reflexive Banach space and S be a closed subset of X with $x \in S$. Then the following assertions are equivalent:*

- (i) S is Mordukhovich regular at x ;
- (ii) d_S is Mordukhovich regular at x ;
- (iii) d_S is Fréchet subdifferentially regular at x .

2.7 Scalarization of Proximal Normal Regularity: [(PNR) \Leftrightarrow (PSR)]?

In this section, we will assume that X is a *reflexive Banach space*.

We have already seen that Fréchet normal regularity is not equivalent to proximal normal regularity. So, the present section is devoted to study some properties of proximal normal regularity, essentially we will give conditions under which this normal regularity can be characterized in terms of the distance function.

We establish first the following result on the relationship between the proximal normal cone and the proximal subdifferential of the distance function, which is the corresponding formula of (2.7) in the proximal case. It is due to Bounkhel and Thibault [61].

Theorem 2.7. *Let S be a nonempty closed subset of X and $x \in S$. Then*

$$\partial^P d_S(x) = N^P(S; x) \cap \mathbf{B}_*.$$

Proof. We begin by proving the inclusion

$$\partial^P d_S(x) \subset N^P(S; x) \cap \mathbf{B}_*.$$

Let $x^* \in \partial^P d_S(x)$. Then there exist $\sigma > 0$ and $\delta > 0$ such that for all $x' \in x + \delta \mathbf{B}$

$$\langle x^*, x' - x \rangle \leq \sigma \|x' - x\|^2 + d_S(x') - d_S(x) = \sigma \|x' - x\|^2 + d_S(x')$$

and hence for all $x' \in S \cap (x + \delta \mathbf{B})$

$$\langle x^*, x' - x \rangle \leq \sigma \|x' - x\|^2$$

which ensures that $x^* \in N^P(S; x)$. Then, as one always has $\partial^P d_S(x) \subset \partial^C d_S(x) \subset \mathbf{B}_*$, then $x^* \in N^P(S; x) \cap \mathbf{B}_*$.

Now, we show the reverse inclusion

$$N^P(S; x) \cap \mathbf{B}_* \subset \partial^P d_S(x).$$

Fix $x^* \in N^P(S; x)$ with $\|x^*\| \leq 1$. Then there exist $\sigma > 0$ and $\delta > 0$ such that

$$\langle x^*, x' - x \rangle \leq \sigma \|x' - x\|^2 \quad \text{for all } x' \in S \cap (x + \delta \mathbf{B}). \quad (2.17)$$

Fix now $\gamma = \min\{1, \frac{\delta}{3}\}$ and fix also any z in $x + \gamma \mathbf{B}$ and choose y_z in S such that

$$\|y_z - z\| \leq d_S(z) + \|z - x\|^2. \quad (2.18)$$

Then $y_z \in x + \delta \mathbf{B}$, because (by (2.18)) $\|y_z - x\| \leq \|y_z - z\| + \|z - x\| \leq 3\|z - x\| \leq 3\gamma \leq \delta$, and hence

$$\begin{aligned} \langle x^*, z - x \rangle &= \langle x^*, y_z - x \rangle + \langle x^*, z - y_z \rangle \\ &\leq \sigma \|y_z - x\|^2 + \|y_z - z\| && \text{(by (2.17))} \\ &\leq 9\sigma \|z - x\|^2 + d_S(z) + \|z - x\|^2 && \text{(by (2.18))} \\ &\leq d_S(z) - d_S(x) + (9\sigma + 1)\|z - x\|^2. \end{aligned}$$

This ensures that $x^* \in \partial^P d_S(x)$ and hence the proof is finished. \square

Remark 2.7. Note that the idea of the proof of Theorem 2.7 can also be used to give another and different proof for Theorem 2.3. Note also that the proof shows that Theorem 2.7 holds for any normed vector space.

Theorem 2.8. *Let S be a nonempty closed subset of X and $x \in S$. Then S is proximally normally regular at x if and only if the function d_S is proximally subdifferentially regular at x .*

Proof.

1. Suppose that S is proximally normally regular at x , that is $N^P(S; x) = N^C(S; x)$. Then, by Theorem 2.7 one has

$$\partial^C d_S(x) \subset N^C(S; x) \cap \mathbf{B}_* = N^P(S; x) \cap \mathbf{B}_* = \partial^P d_S(x) \subset \partial^C d_S(x)$$

which ensures the proximal subdifferential regularity of d_S at x .

2. Now, we assume that d_S is proximally subdifferentially regular (PSR) at x . Then the definition of the Fréchet subdifferential and the definition of (PSR) ensure that

$$\partial^P d_S(x) = \widehat{\partial} d_S(x) = \partial^C d_S(x).$$

So by Theorems 2.3 and 2.7 one has $N^P(S; x) = \widehat{N}(S; x)$. Moreover, as X is reflexive and $\widehat{N}(S; x)$ is convex and strongly closed in X^* (see Part (3) in Proposition 1.8), $\widehat{N}(S; x)$ is weak star closed and hence so is $N^P(S; x)$. Thus, the relation (2.10) yields

$$N^C(S; x) = \text{cl}_{w^*}(\mathbf{R}_+ \partial^C d_S(x))$$

$$\begin{aligned}
&= \text{cl}_{w^*}(\mathbf{R}_+ \partial^P d_S(x)) \\
&= \text{cl}_{w^*}(N^P(S; x)) \\
&= N^P(S; x).
\end{aligned}$$

This completes the proof. \square

Corollary 2.2. *Let S be a nonempty closed subset of \mathbf{R}^N and let $x \in S$. Then S is proximal normally regular at x if and only if d_S is proximally subdifferentially regular at x .*

Remark 2.8. Note that in finite dimensional spaces (see Theorem 2.12 below), tangential regularity and Fréchet normal regularity are equivalent. Consequently, according to that equivalence, Corollaries 2.1 and 2.2, one concludes that even in finite dimensional spaces, tangential regularity and proximal normal regularity are not equivalent.

2.8 Weak Tangential Regularity of Sets

Another natural notion of regularity, in infinite dimensional setting, is the weak tangential regularity, that is, the Clarke tangent cone $T^C(S; \bar{x})$ coincides with the weak contingent cone $K^w(S; \bar{x})$, where

$K^w(S; \bar{x})$ is the set of all $v \in X$ for which there exist a sequence of positive numbers $t_k \rightarrow 0$ and a sequence $v_k \xrightarrow{w} v$ such that $x + t_k v_k \in S$ for all k , where \xrightarrow{w} means the weak convergence.

Note that one always has $T^C(S; x) \subset K(S; x) \subset K^w(S; x)$.

We have proved in the previous section that in infinite dimensional case, Fréchet normal regularity is not equivalent to tangential regularity. One of our interests in this section is to prove (see Theorem 2.13) that Fréchet normal regularity can be characterized as weak tangential regularity, whenever the space X is reflexive. First, we prove the following theorem. Its proof is in the line of the proof of Fact 1 in Theorem 2.1.

Theorem 2.9. *Let S be a nonempty closed subset of a reflexive Banach space X and let $x \in S$. Then*

$$d_S^-(x; v) \geq d(K^w(S; x); v), \quad \text{for all } v \in X.$$

Proof. Fix any $v \in X$. Let (t_k) be a sequence of positive numbers converging to zero such that

$$d_S^-(x; v) = \lim_{k \rightarrow +\infty} t_k^{-1} [d_S(x + t_k v) - d_S(x)].$$

For each integer k choose $w_k \in t_k^{-1}(S - x)$ such that

$$d_{t_k^{-1}(S-x)}(v) \geq \|v - w_k\| - t_k.$$

Then,

$$\begin{aligned} d_S^-(x; v) &= \lim_{k \rightarrow +\infty} t_k^{-1} d_S(x + t_k v) \\ &= \lim_{k \rightarrow +\infty} d_{t_k^{-1}(S-x)}(v) \\ &\geq \limsup_{k \rightarrow +\infty} \|v - w_k\|. \end{aligned}$$

Since $d_S^-(x; v) \leq \|v\|$ (since d_S is 1-Lipschitz), the sequence (w_k) is bounded, and hence some subsequence converges weakly to $w \in K^w(S; x)$. Hence,

$$\begin{aligned} d_S^-(x; v) &\geq \limsup_{k \rightarrow +\infty} \|v - w_k\| \\ &\geq \liminf_{k \rightarrow +\infty} \|v - w_k\| \\ &\geq \|v - w\| \\ &\geq d(K^w(S; x); v), \end{aligned}$$

which completes the proof. \square

In the following theorem we establish a relationship between the weak tangential regularity of a subset and the directional regularity of its distance function.

Theorem 2.10. *Let S be a nonempty closed subset of a reflexive Banach space and let $x \in S$. Assume that S is weakly tangentially regular at x . Then d_S is directionally regular at x .*

Proof. Assume that S is weakly tangentially regular at x , that is $K^w(S; x) = T^C(S; x)$. Then by Facts 1 and 2 in Theorem 2.1 and by Theorem 2.8 we obtain for all $v \in X$

$$\begin{aligned} d_S^-(x; v) &\leq d_S^0(x; v) \\ &\leq d(T^C(S; x); v) \\ &= d(K^w(S; x); v) \\ &\leq d_S^-(x; v). \end{aligned}$$

This finishes the proof. \square

Now, we give the relationship between the tangential regularity of a nonempty closed subset $S \subset X$ at $x \in S$ and another type of normal regularity. First we need the definition of the lower Dini directional derivative for l.s.c functions (not necessarily locally Lipschitz continuous). Let $f : X \rightarrow \mathbf{R} \cup \{+\infty\}$ and $\bar{x} \in \text{dom } f$. The lower Dini directional derivative of f at \bar{x} is given by

$$f^-(\bar{x}; v) = \liminf_{\substack{v' \rightarrow v \\ t \downarrow 0}} t^{-1} f(\bar{x} + tv') - f(\bar{x}). \quad (2.19)$$

Using the same idea as in Sect. 1.3.2, we define the Dini subdifferential of f at \bar{x} , i.e.

$$\partial^- f(\bar{x}) = \{x^* \in X^* : \langle x^*, v \rangle \leq f^-(\bar{x}; v), \text{ for all } v \in X\}.$$

Now, we define the new type of normal regularity of a closed set S at any point $\bar{x} \in S$ as follows: We will say that S is *Dini normally regular* provided that $N^C(S; x) = N^-(S; x)$, where $N^-(S; x)$ is the Dini normal cone to S at x defined as the Dini subdifferential of the indicator function ψ_S (note that ψ_S is l.s.c whenever S is closed and it is not Lipschitz) of S at x (see Chap. 3 for more details on the Dini subdifferential for lower semicontinuous functions). Note that one always has the inclusion $N^-(S; x) \subset N^C(S; x)$.

Theorem 2.11. *Let X be a normed vector space and let S be a nonempty closed subset of X with $x \in S$. Then S is tangentially regular (TR) at x if and only if it is Dini normally regular (DNR) at x .*

Proof. First, we prove the following relationship between the lower Dini derivative of the indicator function and the Bouligand tangent cone

$$\psi_S^-(x; v) = \psi_{K(S; x)}(v), \text{ for all } v \in X. \quad (2.20)$$

Assume that $v \notin K(S; x)$, then by Part (2) in Proposition 1.6 one has $x + t_n v_n \notin S$ for all $t_n \downarrow 0$ and all $v_n \rightarrow v$. Then

$$\psi_S^-(x; v) = \liminf_{\substack{v' \rightarrow v \\ t \downarrow 0}} t^{-1} \psi_S(x + tv') = +\infty = \psi_{K(S; x)}(v).$$

Assume now that $v \in K(S; x)$. Then by Part (2) in Proposition 1.6 once again there exists $t_n \downarrow 0$ and $v_n \rightarrow v$ such that $x + t_n v_n \in S$ for all n . By the definition recalled above of the lower Dini derivative one has

$$0 \leq \psi_S^-(x; v) = \liminf_{\substack{v' \rightarrow v \\ t \downarrow 0}} t^{-1} \psi_S(x + tv') \leq \liminf_{n \rightarrow +\infty} t_n^{-1} \psi_S(x + t_n v_n) = 0 = \psi_{K(S; x)}(v)$$

and hence in both cases one has $\psi_S^-(x; v) = \psi_{K(S; x)}(v)$.

Now, we assume that S is tangentially regular at x and we prove that it is Dini normally regular at the same point x . By (2.20) and the definition of the Dini normal cone one gets

$$\begin{aligned} N^-(S; x) &= \{x^* \in X^* : \langle x^*, v \rangle \leq \psi_S^-(x; v) = \psi_{K(S; x)}(v), \text{ for all } v \in X\} \\ &= \{x^* \in X^* : \langle x^*, v \rangle \leq 0, \text{ for all } v \in K(S; x)\} = K^0(S; x), \end{aligned}$$

where $K^0(S; x)$ denotes the negative polar of $K(S; x)$. Therefore, by the tangential regularity of S at x one obtains

$$N^-(S; x) = K^0(S; x) = T^0(S; x) = N^C(S; x).$$

For the converse, we assume that S is Dini normally regular at x , that is $N^-(S; x) = N^C(S; x)$ and we show that S is tangential regularity at x . Then,

$$K(S; x) \subset (N^-(S; x))^0 = (N^C(S; x))^0 = T^C(S; x)$$

and hence $K(S; x) = T^C(S; x)$ (because the reverse inclusion always holds). \square

Now, we show that the Fréchet normal regularity of a nonempty closed subset $S \subset X$ at x implies the tangential regularity of S at x . If, in addition, X is assumed to be a finite dimensional space, then we have the equivalence.

Theorem 2.12. *Let X be a Banach space and let S be a nonempty closed subset of X with $x \in S$. Assume that S is Fréchet normally regular at x . Then, S is tangentially regular at x . If, in addition, X is a finite dimensional space, then one has the equivalence.*

Proof.

1. Assume that S is Fréchet normally regular at x , i.e., $N^C(S; x) = \widehat{N}(S; x)$. As one always has $\widehat{N}(S; x) \subset N^-(S; x) \subset N^C(S; x)$, the Dini normal regularity of S at x is ensured. Consequently, by Theorem 2.11, S is tangentially regular at x .
2. Now, we assume that X is a finite dimensional space and S is tangentially regular at x , i.e., $T^C(S; x) = K(S; x)$. Let $x^* \in N^C(S; x) = (T^C(S; x))^0$. Then

$$\langle x^*, v \rangle \leq 0 \quad \text{for all } v \in T^C(S; x) = K(S; x).$$

Consider a sequence (x_k) in S that converges to x with $x_k \neq x$ and such that

$$\limsup_{x' \rightarrow S_x} \left\langle x^*, \frac{x' - x}{\|x' - x\|} \right\rangle = \lim_{k \rightarrow +\infty} \left\langle x^*, \frac{x_k - x}{\|x_k - x\|} \right\rangle.$$

Extracting a subsequence if necessary we may suppose that

$$\frac{x_k - x}{\|x_k - x\|} \rightarrow v \in K(S; x) = T^C(S; x).$$

Therefore, $\langle x^*, v \rangle \leq 0$ and hence

$$\limsup_{x' \rightarrow x} \left\langle x^*, \frac{x' - x}{\|x' - x\|} \right\rangle \leq 0$$

that is, $x^* \in \widehat{N}(S; x)$. So, $N^C(S; x) = \widehat{N}(S; x)$ and hence the proof is complete. \square

Theorem 2.12, Corollary 2.1, and Theorem 2.6 give the following result. Note that the equivalence between (i), (ii), and (iii) of the corollary has been established (in a different way) in Corollary 6.29 of the book of Rockafellar and Wets [241].

Corollary 2.3. *Let S be a nonempty closed subset of \mathbf{R}^N and let $x \in S$. Then the following assertions are equivalent:*

- (i) S is Fréchet normally regular at x ;
- (ii) S is Mordukhovich regular at x ;
- (iii) S is tangentially regular at x ;
- (iv) d_S is Fréchet subdifferentially regular at x ;
- (v) d_S is directionally regular at x .

We have showed in the previous corollary that the equivalence between tangential regularity and Fréchet normal regularity is ensured whenever X is a finite dimensional space. We will see via a set constructed in Borwein and Fabian [36] that for any infinite dimensional space, this equivalence does not hold. For this purpose, we recall the following result due to Borwein and Strojwas (Proposition 3.1 in [37] see also Kruger [167, 175]) and we will characterize Fréchet normal regularity as weak tangential regularity in reflexive Banach spaces.

Proposition 2.2. *Let X be a Banach space and let S be a nonempty closed subset of X with $x \in S$. Then*

$$\widehat{N}(S; x) \subset (K^w(S; x))^0.$$

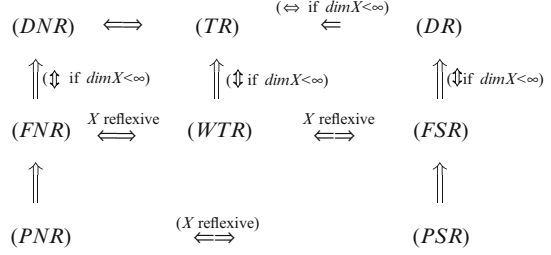
If, furthermore, X is a reflexive Banach space, then equality holds in the inclusion above.

Theorem 2.13. *Let X be a Banach space and let S be a nonempty closed subset of X with $x \in S$. Then,*

- (i) S is weakly tangentially regular at x whenever it is Fréchet normally regular at x ;
- (ii) if, in addition, X is reflexive, then (i) becomes an equivalence.

Proof. 1. Assume that S is Fréchet normally regular at x , i.e., $\widehat{N}(S; x) = N^C(S; x) = (T^C(S; x))^0$. This ensures by the previous proposition that $(T^C(S; x))^0 \subset (K^w(S; x))^0$. Therefore, as $T^C(S; x)$ is a closed convex cone, we obtain $K^w(S; x) \subset T^C(S; x)$ and hence $K^w(S; x) = T^C(S; x)$ since the reverse inclusion always holds. So, S is weakly tangentially regular at x .

Fig. 2.4 Relationships between various concepts of regularity of sets



2. Now, assume that X is reflexive and S is weakly tangentially regular at x i.e. $T^C(S; x) = K^w(S; x)$. Then, one has $N^C(S; x) = (T^C(S; x))^0 = (K^w(S; x))^0$. On the other hand, we have (by Proposition 2.2) $\widehat{N}(S; x) = (K^w(S; x))^0$, and hence

$$\widehat{N}(S; x) = N^C(S; x)$$

which ensures the Fréchet normal regularity of S at x . \square

In Theorem 2.13, we have proved that when X is a reflexive Banach space one has (1) “ $\widehat{N}(S; x) = N^C(S; x)$ ” \Longleftrightarrow (2) “ S is w -tangentially regular at x .” Moreover, we have proved in Theorem 2.10 that (2) \implies (3) “ d_S is directionally regular at x .” So (1) \implies (3). On the other hand, in [36] Borwein and Fabian proved that we can find in any infinite dimensional Banach space a nonempty closed subset S of X and a point x in S for which (4) “ S is tangentially regular at x ” and d_S is not directionally regular at x . Consequently, in infinite dimensional Banach spaces (4) $\not\implies$ (3). This ensures that, for *infinite dimensional reflexive* Banach spaces, there is *no equivalence between tangential regularity and Fréchet normal regularity*.

Remark 2.9.

- In the infinite dimensional case (even if X is assumed to be a reflexive Banach space), the inclusion $\widehat{N}(S; x) \subset N^-(S; x)$ is strict for the subset S constructed in Borwein and Fabian [36]. Indeed, according to the arguments above $\widehat{N}(S; x)$ is strictly included in $N^C(S; x)$. On the other hand, as S is tangentially regular at x , Theorem 2.11 ensures that S is normally Dini regular at x , i.e., $N^C(S; x) = N^-(S; x)$. Therefore, $\widehat{N}(S; x)$ is strictly included in $N^-(S; x)$ and in *infinite dimensional spaces Fréchet normal regularity and Dini normal regularity are not equivalent*.
- The same subset S in [36] also shows (with the help of Theorem 2.11) that in *infinite dimensional spaces the Dini normal regularity of a set is not equivalent to the Dini subdifferential regularity of the distance function associated with this set*.

Conclusion. In the following diagram, we summarize all the relationships between all the various concepts of regularity of sets considered in this chapter (Fig. 2.4). Let X be a normed vector space, S be a nonempty closed subset of X , and x be some point in S . Then one has

However, there are important classes of sets (in finite and infinite dimensional spaces) for which all these types of regularity hold. It is obviously the case for convex sets. Another classes that appeared very recently are the one of prox-regular sets introduced by Poliquin and Rockafellar [229] and the one of proximally smooth sets introduced by Clarke et al. [89]. The next section is devoted to study this important class.

2.9 Uniform Prox-Regularity of Sets

First, we begin by recalling that, for a given $r \in (0, +\infty]$, a subset S is uniformly r -prox-regular (see [230]) (or equivalently r -proximally smooth see [89]) if and only if every nonzero proximal normal to S can be realized by an r -ball. This means that for all $\bar{x} \in S$ and all $0 \neq \xi \in N^P(S; \bar{x})$ one has

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r} \|x - \bar{x}\|^2$$

for all $x \in S$. We make the convention $\frac{1}{r} = 0$ for $r = +\infty$ and we will just say in the sequel that S is uniformly r -prox-regular. Recall that for $r = +\infty$, the uniform r -prox-regularity of S is equivalent to the convexity of S . The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel of the book. For the proof of these results we refer the reader to [89, 230]. We use the notation $\text{proj}_S(x)$ instead of $\text{Proj}_S(x)$ whenever this set has a unique point.

Proposition 2.3. *Let S be a nonempty closed subset in \mathbf{H} and let $r > 0$. If the subset S is uniformly r -prox-regular, then the following hold:*

- (i) *For all $x \in \mathbf{H}$ with $d_S(x) < r$, $\text{proj}_S(x)$ exists;*
- (ii) *For every $r' \in (0, r)$, the enlarged subset $S(r') := \{x \in \mathbf{H} : d_S(x) \leq r'\}$ is uniformly $(r - r')$ -prox-regular;*
- (iii) *The generalized gradient and the proximal subdifferential of d_S coincide at all points $x \in \mathbf{H}$ with $d_S(x) < r$.*

The following proposition shows that in the inequality above characterizing the uniform prox-regularity one may use the proximal subdifferential of the distance function in place of the proximal normal cone. For a given subset S in \mathbf{H} and a given $r > 0$ we will set

$$(P_r) \quad \left\{ \begin{array}{l} \text{for all } \bar{x} \in S \text{ and all } 0 \neq \xi \in N^P(S; \bar{x}) \text{ one has} \\ \left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r} \|x - \bar{x}\|^2 \quad \text{for all } x \in S \end{array} \right.$$

and

$$(P'_r) \quad \begin{cases} \text{for all } \bar{x} \in S \text{ and all } \xi \in \partial^P d_S(\bar{x}) \text{ one has} \\ \langle \xi, x - \bar{x} \rangle \leq \frac{1}{2r} \|x - \bar{x}\|^2 \quad \text{for all } x \in S. \end{cases}$$

Proposition 2.4. *Let S be a nonempty closed subset in \mathbf{H} and let $r > 0$. Then $(P_r) \Leftrightarrow (P'_r)$.*

Proof. $((P_r) \Rightarrow (P'_r))$. Assume that S satisfies (P_r) . The property (P'_r) obviously holds for $\xi = 0$. Let $\bar{x} \in S$ and $0 \neq \xi \in \partial^P d_S(\bar{x}) \subset N^P(S; \bar{x})$. Then by (P_r) one has for all $x \in S$

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r} \|x - \bar{x}\|^2$$

and hence

$$\langle \xi, x - \bar{x} \rangle \leq \frac{\|\xi\|}{2r} \|x - \bar{x}\|^2 \leq \frac{1}{2r} \|x - \bar{x}\|^2$$

because $\|\xi\| \leq 1$. The property (P'_r) then holds.

$((P'_r) \Rightarrow (P_r))$. Assume now that S satisfies (P'_r) . Let $\bar{x} \in S$ and $0 \neq \xi \in N^P(S; \bar{x})$. Then by Theorem 2.7 one has $\frac{\xi}{\|\xi\|} \in \partial^P d_S(\bar{x})$ and hence one gets (by (P'_r))

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r} \|x - \bar{x}\|^2$$

for all $x \in S$. This completes the proof of the second implication and so the proof of the proposition is finished. \square

The following lemma is proved in Sect. 2.11 in the context of a general normed vector space. It will be used in the proof of the next theorem. For the convenience of the reader, we show how the Hilbert norm allows us to give another simple proof. The reader will also note that the arguments work for any Kadec norm of a reflexive Banach space (see, e.g., [106–108] for the definition and properties of Kadec norms).

Lemma 2.1. *Let S be a nonempty closed subset in \mathbf{H} and let $r > 0$. Then for all $x \notin S(r)$ one has*

$$d_{S(r)}(x) = d_S(x) - r. \quad (2.21)$$

Proof. As the set $\{x \notin S(r) : \text{Proj}_S(x) \neq \emptyset\}$ is dense in $X \setminus S(r)$ by [178], and as the functions d_S and $d_{S(r)}$ are continuous, it is enough to prove (2.21) only for points $x \notin S(r)$ satisfying $\text{Proj}_S(x) \neq \emptyset$. Fix any such point x and fix also p in S such that $d_S(x) = \|x - p\|$. Set

$$u := p + \left(\frac{r}{\|x - p\|} \right) (x - p). \quad (2.22)$$

We observe that u is in $S(r)$ because (2.22) and the relation $p \in S$ ensure $d_S(u) \leq \|u - p\| = r$.

Let us prove now that $u \in \text{Proj}_{S(r)}(x)$. Consider any $y \in S(r)$, that is, $d_S(y) \leq r$, and fix any positive number ε . We may choose some $y_\varepsilon \in S$ satisfying

$$\|y - y_\varepsilon\| \leq d_S(y) + \varepsilon \leq r + \varepsilon.$$

Consequently

$$\|y - x\| \geq \|y_\varepsilon - x\| - \|y_\varepsilon - y\| \geq \|x - p\| - r - \varepsilon = \|x - u\| - \varepsilon$$

and this yields $d_{S(r)}(x) \geq \|x - u\| - \varepsilon$. As this holds for all $\varepsilon > 0$, we have $d_{S(r)}(x) \geq \|x - u\|$ and hence $d_{S(r)}(x) = \|x - u\|$ because u is in $S(r)$ as observed above. Writing by (2.22)

$$d_{S(r)}(x) = \|x - u\| = \|x - p\| - r = d_S(x),$$

the proof of the lemma is finished. \square

We establish now the main result of this section from which some new characterizations of uniformly r -prox-regular sets will be derived. Here, the point where the proximal subdifferential of d_S is considered is not required to stay in S contrarily to Proposition 2.4.

Theorem 2.14. *Let S be a nonempty closed subset in \mathbf{H} and let $r > 0$. Assume that S is uniformly r -prox-regular. Then the following holds:*

$$(P_r'') \left\{ \begin{array}{l} \text{for all } x \in \mathbf{H}, \text{ with } d_S(x) < r, \text{ and all } \xi \in \partial^P d_S(x) \text{ one has} \\ \langle \xi, x' - x \rangle \leq \frac{8}{r - d_S(x)} \|x' - x\|^2 + d_S(x') - d_S(x), \\ \text{for all } x' \in \mathbf{H} \text{ with } d_S(x') \leq r. \end{array} \right.$$

Proof.

Step 1. Firstly, we prove a stronger property for $x \in S$, more precisely we prove the following:

$$(P_r''') \left\{ \begin{array}{l} \text{for all } x \in S \text{ and all } \xi \in \partial^P d_S(x) \text{ one has} \\ \langle \xi, x' - x \rangle \leq \frac{2}{r} \|x' - x\|^2 + d_S(x'), \\ \text{for all } x' \in \mathbf{H} \text{ with } d_S(x') < r. \end{array} \right.$$

Fix any $x \in S$ and any $\xi \in \partial^P d_S(\bar{x})$. Fix also any $z \in \mathbf{H}$ satisfying $d_S(z) < r$. As S is uniformly r -prox-regular one can find some $y_z \in \text{Proj}_S(z) \neq \emptyset$, that is, y_z is in S and

$$\|z - y_z\| = d_S(z). \quad (2.23)$$

Then,

$$\|y_z - x\| \leq \|y_z - z\| + \|z - x\| \leq 2\|z - x\|$$

and hence by (P'_r) and the inequality $\|\xi\| \leq 1$, and also by the equality (2.23) one gets

$$\begin{aligned} \langle \xi, z - x \rangle &= \langle \xi, y_z - x \rangle + \langle \xi, z - y_z \rangle \\ &\leq \frac{1}{2r} \|y_z - x\|^2 + \|\xi\| \|y_z - z\| \\ &\leq \frac{2}{r} \|z - x\|^2 + d_S(z) - d_S(x). \end{aligned}$$

This completes the proof of (P'''_r) .

Step 2. Note that (see Part (ii) in Proposition 2.3) for every $0 < r' < r$ the enlarged set $S(r')$ is uniformly $(r - r')$ -prox-regular. Further, for any $u' \in \mathbf{H}$ it can be seen that the inequality $d_{S(r')}(u') < r - r'$ holds if and only if $d_S(u') < r$. Indeed, if we suppose that $d_{S(r')}(u') < r - r'$, then there exists some z in \mathbf{H} with $d_S(z) \leq r'$ and $\|u' - z\| < r - r'$, and hence

$$d_S(u') \leq d_S(z) + \|u' - z\| < r.$$

Suppose now that $d_S(u') < r$. In the case $u' \in S(r')$, we can write $d_{S(r')}(u') = 0 < r - r'$. In the other case $u' \notin S(r')$, we have by Lemma 2.1

$$d_{S(r')}(u') = d_S(u') - r' < r - r'.$$

The equivalence then holds, and hence the property $(P'''_{(r-r')})$ may be written as

$$(P'''_{(r-r')}) \quad \left\{ \begin{array}{l} \text{for all } u \in S(r'), \text{ and all } \zeta \in \partial^P d_{S(r')}(u) \text{ one has} \\ \langle \zeta, u' - u \rangle \leq \frac{2}{(r - r')} \|u' - u\|^2 + d_{S(r')}(u'), \\ \text{for all } u' \in \mathbf{H} \text{ with } d_S(u') < r. \end{array} \right.$$

Now, fix any $x \in \mathbf{H}$ with $d_S(x) < r$ and any $\xi \in \partial^P d_S(x)$. We distinguish two cases:

Case 1. If $x \in S$, then by (P'''_r) one obtains for all $x' \in \mathbf{H}$ with $d_S(x') < r$

$$\langle \xi, x' - x \rangle \leq \frac{2}{r} \|x' - x\|^2 + d_S(x') - d_S(x). \quad (2.24)$$

Case 2. If $x \notin S$, we put $r' := d_S(x) > 0$ in this case. Firstly, one observes that $\xi \in \partial^P d_{S(r')}(x)$. Indeed, one knows by Theorems 4.1 and 4.3 in Bounkhel and Thibault [61] (see also Theorem 3.2 in [89] for the equality in the following relation) that

$$\partial^P d_S(x) = N^P(S(r'), x) \cap \{\xi : \|\xi\| = 1\} \subset \partial^P d_{S(r')}(x)$$

and hence as ξ is fixed in $\partial^P d_S(x)$, one then gets $\xi \in \partial^P d_{S(r')}(x)$. Applying $(P'''_{(r-r')})$ in the form obtained above one gets for any $x' \in \mathbf{H}$ with $d_S(x') < r$

$$\langle \xi, x' - x \rangle \leq \frac{2}{r - r'} \|x' - x\|^2 + d_{S(r')}(x').$$

Consequently, for any $x' \in \mathbf{H}$ satisfying $d_S(x') < r$ and $x' \notin S(r')$ (that is, $r' < d_S(x') < r$) one gets according to Lemma 2.1

$$\langle \xi, x' - x \rangle \leq \frac{2}{r - r'} \|x' - x\|^2 + d_S(x') - d_S(x). \quad (2.25)$$

Now fix any $x' \in \mathbf{H}$ satisfying $d_S(x') < r$ and $x' \in S(r')$. We begin by noting that $(P'''_{(r-r')})$ ensures that the inequality

$$\langle \xi, y - x \rangle \leq \frac{2}{r - r'} \|y - x\|^2. \quad (2.26)$$

holds for all $y \in \mathbf{H}$ with $d_S(y) \leq r'$. Choose now, according to $\|\xi\| = 1$, some $u \in \mathbf{H}$ with $\|u\| = 1$ and such that $\langle \xi, u \rangle = 1$. Put $t := d_S(x) - d_S(x') \geq 0$. Then $x' + tu \in S(r')$, because $d_S(x' + tu) \leq d_S(x') + t = d_S(x) = r'$. Therefore, (2.26) allows us to write

$$\langle \xi, x' - x \rangle = \langle \xi, x' + tu - x \rangle - \langle \xi, tu \rangle \leq \frac{2}{r - r'} \|x' + tu - x\|^2 - t. \quad (2.27)$$

Observing that

$$\|x' + tu - x\| \leq \|x' - x\| + t \leq 2\|x' - x\|$$

we deduce from (2.27)

$$\langle \xi, x' - x \rangle \leq \frac{8}{r - r'} \|x' - x\|^2 + d_S(x') - d_S(x).$$

It then follows from (2.24), (2.25) and the last inequality that one has for all $x \in \mathbf{H}$ with $d_S(x) < r$ and all $\xi \in \partial^P d_S(x)$

$$\langle \xi, x' - x \rangle \leq \frac{8}{r - r'} \|x' - x\|^2 + d_S(x') - d_S(x) \quad \text{for all } x' \in \mathbf{H} \text{ with } d_S(x') < r.$$

Taking the continuity of both members of that inequality with respect to x' into account, we may replace the requirement $d_S(x') < r$ by $d_S(x) \leq r$. The proof of the theorem is then complete. \square

The following corollary of the above theorem adds some further characterizations of uniformly prox-regular sets to the lists in Clarke et al. [89] and Poliquin et al. [230].

Corollary 2.4. *Let S be a nonempty closed subset of \mathbf{H} and let $r > 0$. Then, the following assertions are equivalent:*

- (a) S is uniformly r -prox-regular;
- (b) the property (P_r'') holds for the proximal subdifferential of d_S ;
- (c) the property (P_r'') holds for the Fréchet subdifferential of d_S ;
- (d) the property (P_r'') holds for the basic (limiting) subdifferential of d_S ;
- (e) the property (P_r'') holds for the generalized gradient (Clarke subdifferential) of d_S .

Proof. The implication (a) \Rightarrow (b) follows from Theorem 2.14 and (b) \Rightarrow (c) holds because any $\xi \in \widehat{\partial}d_S(x)$ is the weak limit of a sequence $(\xi_n)_n$ such that $\xi_n \in \partial^P d_S(x_n)$ and $(x_n)_n$ converges to x . In the same way, the implication (c) implies (d) is true. The implication (d) \Rightarrow (e) can be seen easily as a consequence of the definition of (P_r'') and of the formula characterizing the generalized gradient of a Lipschitz function as the closed convex hull of its basic (limiting) subdifferential. So, it remains to see (e) \Rightarrow (a). We know that $\partial^C d_S(x)$ is nonempty at any x (see [88]). Supposing that (e) holds. This property tells us that any generalized subgradient is a proximal subgradient. Therefore, for any $x \in \mathbf{H}$ with $d_S(x) < r$ we have $\partial^P d_S(x) \neq \emptyset$. The implication is thus a consequence of corollary 4.3 in [230] (see also Theorem 4.1 in [89]). \square

Observe that the assertion (e) in the corollary entails that the generalized gradient (Clarke subdifferential) and the proximal subdifferential (and hence also the Fréchet subdifferentials) of d_S coincide at all points $x \in \mathbf{H}$ with $d_S(x) < r$ provided that S is uniformly r -prox-regular. In fact, it is easily seen that this equality property of these subdifferentials characterizes the uniformly r -prox-regular sets.

2.10 Arc-Wise Essential Tangential Regularity

This section is devoted to study a different type of regularity for closed sets in Banach spaces. This concept has been introduced by Borwein and Moors in [34] in \mathbf{R}^n . Let us start with the following definition.

Definition 2.6. A closed subset C of a Banach space X is *arc-wise essentially tangentially regular* in X and we will write $C \in \mathcal{A}_{\text{TR}}(X)$, if for each $x \in \mathcal{S}_e((0, 1), X)$, the set

$$\{t \in (0, 1) : x(t) \in C \text{ and } x'(t) \in K(C; x(t)) \setminus T^C(C; x(t))\}$$

has null measure, where $\mathcal{S}_e((0, 1), X)$ is the class of all locally Lipschitz mappings $x : (0, 1) \rightarrow X$ which are strictly differentiable almost everywhere on $(0, 1)$. The sets

$K(C; x)$ and $T^C(C; x)$ denote the contingent cone and the Clarke tangent cone of C at x respectively (see Chap. 1).

Remark 2.10. As one always has $K(C; x) = T^C(C; x) = X$, for each $x \in \text{int} C$ (the topological interior of C), we can take x only in $bd C$ (the boundary of C), in Definition 2.6, that is, C is arc-wise essentially tangentially regular if and only if for each $x \in \mathcal{S}_e((0, 1), X)$ one has

$$\mu(\{t \in (0, 1) : x(t) \in bd C \text{ and } x'(t) \in K(C; x(t)) \setminus T^C(C; x(t))\}) = 0.$$

We proceed now to establish a characterization of the class $\mathcal{A}_{\text{TR}}(X)$.

Proposition 2.5. *Let C be a nonempty closed subset of X . Then the set C is arc-wise essentially tangentially regular if and only if for each $x \in \mathcal{S}_e((0, 1), X)$, both sets*

$$\{t \in (0, 1) : x(t) \in C \text{ and } x'(t) \in K(C; x(t)) \setminus T^C(C; x(t))\}$$

and

$$\{t \in (0, 1) : x(t) \in C \text{ and } -x'(t) \in K(C; x(t)) \setminus T^C(C; x(t))\}$$

have null measure.

Proof. “ \Rightarrow ” Assume that C is arc-wise essentially tangentially regular, that is for each $z \in \mathcal{S}_e((0, 1), X)$, one has

$$\mu(\{t \in (0, 1) : z(t) \in C \text{ and } z'(t) \in K(C; z(t)) \setminus T^C(C; z(t))\}) = 0.$$

Fix any $x \in \mathcal{S}_e((0, 1), X)$ and put

$$E_x := \{t \in (0, 1) : x(t) \in C \text{ and } x'(t) \in K(C; x(t)) \setminus T^C(C; x(t))\}$$

and

$$\tilde{E}_x := \{t \in (0, 1) : x(t) \in C \text{ and } -x'(t) \in K(C; x(t)) \setminus T^C(C; x(t))\}.$$

We will show that $\mu(E_x \cup \tilde{E}_x) = 0$. Since by hypothesis, we have $\mu(E_x) = 0$, it suffices to show that $\mu(\tilde{E}_x) = 0$. To this end, define $h : (0, 1) \rightarrow (0, 1)$ by $h(s) = 1 - s$ and $y : (0, 1) \rightarrow X$ by $y(s) = (x \circ h)(s) = x(1 - s)$. Clearly, $y \in \mathcal{S}_e((0, 1), X)$ and everywhere $y'(s) = -x'(h(s))$. For $s := 1 - t$ one has

$$\begin{aligned} \tilde{E}_x &= \{h(s) \in (0, 1) : x(h(s)) \in C \text{ and } -x'(h(s)) \in K(C; x(h(s))) \setminus T^C(C; x(h(s)))\} \\ &= \{h(s) \in (0, 1) : y(s) \in C \text{ and } y'(s) \in K(C; y(s)) \setminus T^C(C; y(s))\} \\ &= h(E_y), \end{aligned}$$

where $E_y := \{t \in (0, 1) : y(t) \in C \text{ and } y'(t) \in K(C; y(t)) \setminus T^C(C; y(t))\}$. Since $y \in \mathcal{S}_e((0, 1), X)$, the set E_y has null measure and hence $\mu(\tilde{E}_x) = 0$. Therefore, the proof of “ \Rightarrow ” is finished.

“ \Leftarrow ” This implication is obvious. \square

Before proving the main theorem of this section which can be seen as a scalarization of arc-wise essential tangential regularity, we need the following different type of regularity for locally Lipschitz functions.

Definition 2.7. Let f be a locally Lipschitz function on a nonempty open subset Ω of X . We will say that f is arc-wise essentially strictly differentiable on Ω and we will write $f \in \mathcal{A}_{s,d}(\Omega)$ if for each $x \in \mathcal{AC}((0, 1), \Omega)$, the set $\{t \in (0, 1) : f \text{ is not strictly differentiable at } x(t) \text{ in the direction } x'(t)\}$ is Lebesgue-null in \mathbf{R} . Here, $\mathcal{AC}((0, 1), \Omega)$ denotes the class of all absolutely continuous mappings defined from $(0, 1)$ to Ω .

Also, we recall that a locally Lipschitz function f from X into \mathbf{R} is directionally regular at x if $f'(x; v)$ exists for all $v \in X$ and $f^0(x; v) = f'(x; v)$ (See Chap. 1 for this definition and Chap. 3 for more details on directionally regular functions not necessarily locally Lipschitz).

Theorem 2.15. Let C be a nonempty closed subset of a Banach space X .

- (i) C is arc-wise essentially tangentially regular whenever the associated distance function d_C is arc-wise essentially tangentially regular.
- (ii) If the norm $\|\cdot\|_X$ is uniformly Gâteaux differentiable, then C is arc-wise essentially tangentially regular if and only if d_C is arc-wise essentially strictly differentiable.

Proof.

1. Assume that $d_C \in \mathcal{A}_{s,d}(X, \mathbf{R})$, i.e., for each $x \in \mathcal{S}_e((0, 1), X)$, the set

$$A := \{t \in (0, 1) : d_C \text{ is not s.d. at } x(t) \text{ in the direction } x'(t)\}$$

is a Haar-null set. We will show that C is arc-wise essentially tangentially regular, i.e., $\mu(B) = 0$ where $B := \{t \in (0, 1) : x(t) \in C \text{ and } x'(t) \in K(C; x(t)) \setminus T^C(C; x(t))\}$. It is enough to prove that $B \subset A$. Let $t_0 \notin A$. Then d_C is s.d. at $x(t_0)$ in the direction $x'(t_0)$ and hence $d'_C(x(t_0); x'(t_0)) = d^-_C(x(t_0); x'(t_0)) = d^0_C(x(t_0); x'(t_0)) = 0$. So $x'(t_0) \in T^C(C; x(t_0))$ and hence $t_0 \notin B$. Consequently, each $t_0 \notin A$ does not lie in B . This completes the proof of the inclusion $B \subset A$.

2. Assume now that $\|\cdot\|_X$ is a uniformly Gâteaux differentiable norm and assume that C is arc-wise essentially tangentially regular. Then, for each fixed x in $\mathcal{S}_e((0, 1), X)$ by Proposition 2.5 we have

$$\mu(B_x) = 0,$$

where

$$\begin{aligned} B_x &= B_x^1 \cup B_x^2, \\ B_x^1 &:= \{t \in (0, 1) : x(t) \in C, \text{ and } x'(t) \in K(C; x(t)) \setminus T^C(C; x(t))\} \text{ and} \\ B_x^2 &:= \{t \in (0, 1) : x(t) \in C \text{ and } -x'(t) \in K(C; x(t)) \setminus T^C(C; x(t))\}. \end{aligned}$$

Put

$$A := \{t \in (0, 1) : d_C \text{ is not s.d. at } x(t) \text{ in the dir. } x'(t)\}.$$

It is not difficult to check that

$$A = \{t \in (0, 1) : x(t) \in bd C, d_C \text{ is not s.d. at } x(t) \text{ in the dir. } x'(t)\}.$$

Indeed, if $t \in (0, 1)$ with $x(t) \in (X \setminus C) \cup \text{int} C$ and d_C is not s.d. at $x(t)$ in the direction $x'(t)$, then $(-d_C)$ is not s.d. at $x(t)$ in the direction $x'(t)$ and so $(-d_C)$ is not directionally regular at $x(t)$ in the direction $x'(t)$, which is impossible, because $x(t) \in (X \setminus C) \cup \text{int} C$, and Theorem 8 in [31]. Put now $D_{x'} := \{t \in (0, 1) : x'(t) \text{ exists}\}$ hence

$$\mu(A \setminus D_{x'}) = 0 \quad (2.28)$$

and put also $I := I_r \cup I_l$ with I_r (resp. I_l) denotes the set of all isolated points in $A \cap D_{x'}$ relatively to the right topology (resp. the left topology). It is not difficult to check that I is countable and hence $\mu(I) = 0$. Fix $t_0 \in (A \cap D_{x'}) \setminus I$. Then there exist two sequences of real positive numbers $(\lambda_n)_n$ and $(\varepsilon_n)_n$ converging to zero such that for n sufficiently large $t_0 + \lambda_n$ and $t_0 - \varepsilon_n$ lie in $(A \cap D_{x'}) \setminus I$ and hence $x(t_0 + \lambda_n)$ and $x(t_0 - \varepsilon_n)$ lie in $bd C$, for n sufficiently large.

Put

$$v_n := \lambda_n^{-1} [x(t_0 + \lambda_n) - x(t_0)] \quad \text{and} \quad w_n := \varepsilon_n^{-1} [x(t_0 - \varepsilon_n) - x(t_0)].$$

Clearly, $v_n \rightarrow x'(t_0)$ and $w_n \rightarrow -x'(t_0)$ and for n sufficiently large $x(t_0) + \lambda_n v_n$ and $x(t_0) + \varepsilon_n w_n$ lie in $bd C$. It follows by the sequential characterization of the contingent cone given Proposition 1.6, that $x'(t_0)$ and $-x'(t_0)$ lie in $K(C; x(t_0))$. Now, we distinguish two cases. Firstly, if $x'(t_0) \in K(C; x(t_0)) \setminus T^C(C; x(t_0))$, then $t_0 \in B_x$. Secondly, if $x'(t_0) \in T^C(C; x(t_0))$, then $-x'(t_0) \in K(C; x(t_0)) \setminus T^C(C; x(t_0))$ (because, if $-x'(t_0) \in T^C(C; x(t_0))$, we would have

$$d_C^0(x(t_0); x'(t_0)) = -d_C^0(x(t_0); -x'(t_0)) = 0,$$

so d_C would be s.d. at $x(t_0)$ in the direction $x'(t_0)$, which would contradict that $t_0 \in A$). Hence $t_0 \in B_x$. Thus $(D_{x'} \cap A) \setminus I \subset B_x$ and hence

$$\mu((D_{x'} \cap A) \setminus I) = 0. \quad (2.29)$$

Finally, according to (2.28) and (2.29), we obtain $\mu(A) = 0$. This ensures that $d_C \in \mathcal{A}_{s,d}(X, \mathbf{R})$ and hence the proof is finished. \square

Remark 2.11. As observed by Borwein and Moors [34] all sets that are directionally tangentially regular except on a countable set are arc-wise essentially tangentially regular. Thus, all closed convex sets and smooth manifolds are arc-wise essentially tangentially regular.

2.11 More on the Regularity of Sets

In Sects. 2.5–2.7, we have scalarized some geometric notions of regularity of sets such as (TR), (FNR), and (PNR), via the distance function whenever the point is in the set. In this section we are interested in the following natural question: Given a closed nonempty set S and a point $\bar{x} \notin S$, is it possible to characterize the regularity of d_S at \bar{x} , which is well defined, in terms of some geometric notion regularity of the set S , or in other words is it possible to “geometrize” the regularity of d_S at points outside the set S . We restrict our study in this section to two concepts of regularity. We recall the following result from [71] which will be used in the present section.

Theorem 2.16. *Let S be a nonempty closed subset of a normed vector space X and let $x \notin S$ with d_S directionally regular at x and $\text{Proj}_S(x) \neq \emptyset$. Then*

$$\partial^C d_S(x) = N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}.$$

2.11.1 Fréchet Case

In this subsection, we will show that for every subset S of a reflexive Banach space X and every $x \notin S$, with $\text{Proj}_S(x) \neq \emptyset$, the Fréchet subdifferential regularity of d_S at x implies the Fréchet normal regularity of the set $S(r)$ at x , where $r = d_S(x)$. In addition, if d_S is directionally regular at x , we have the equivalence.

In this subsection, we will put $r = d_S(x)$ for some point $x \notin S$. We begin this case with the following lemma which will be used in the proof of the next theorem. The lemma will be also used in the proximal case.

Lemma 2.2. *Let X be a normed vector space, S be a nonempty closed subset of X . Then for all $x' \notin S(r)$ one has*

$$d_{S(r)}(x') = d_S(x') - r.$$

Proof. Fix any $x' \notin S(r)$. Consider any $y \in S(r)$, that is, $d_S(y) \leq r$, and consider also any $\varepsilon > 0$. We may choose some $y_\varepsilon \in S$ satisfying

$$\|y - y_\varepsilon\| \leq d_S(y) + \varepsilon \leq r + \varepsilon.$$

Consequently

$$\|y - x'\| \geq \|y_\varepsilon - x'\| - \|y_\varepsilon - y\| \geq d_S(x') - \|y_\varepsilon - y\| \geq d_S(x') - r - \varepsilon.$$

As the inequality $\|y - x'\| \geq d_S(x') - r - \varepsilon$ holds for all $y \in S(r)$ and all $\varepsilon > 0$ we deduce

$$d_{S(r)}(x') \geq d_S(x') - r.$$

Let us prove the reverse inequality. Fix any $y \in S$ and consider the real-valued function h defined on $[0, +\infty)$ by $h(s) := d_S(sx' + (1-s)y)$. Observing that $h(0) = 0$ (because $y \in S$) and $h(1) > r$ (because $x' \notin S(r)$), we may apply the classical intermediate value theorem to get some $s_0 \in (0, 1)$ such that $h(s_0) = r$. Putting $z := s_0x' + (1-s_0)y$, we have $d_S(z) = r$ and

$$\|x' - y\| = \|x' - z\| + \|z - y\|.$$

Therefore, because $y \in S$ we obtain

$$\|x' - y\| \geq \|x' - z\| + d_S(z) = \|x' - z\| + r$$

and as $z \in S(r)$, it follows that

$$\|x' - y\| \geq d_{S(r)}(x') + r.$$

This yields the inequality $d_S(x') \geq d_{S(r)}(x') + r$ that completes the proof of the lemma. \square

Now we establish the following result on the relationship between the Fréchet subdifferential of the distance function d_S at a point $x \notin S$ and the Fréchet normal cone of $S(r)$ at x . This result has been stated in [167] but the proof therein seems to need some further arguments. In the proof below, we use the previous lemma and strong ideas in the proof of Proposition 2.16 in Kruger [167].

Theorem 2.17. *Let X be a Banach space, S be a nonempty closed subset of X , and let $x \notin S$. Then,*

$$\widehat{\partial}d_S(x) \subset \widehat{N}(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}. \quad (2.30)$$

If, furthermore, X is a reflexive Banach space and $\|\cdot\|$ denotes a Kadec equivalent norm on X , then equality holds in (2.30).

Proof. We begin by showing (2.30). Fix x^* in $\widehat{\partial}d_S(x)$ and fix also $\varepsilon > 0$. By definition there exists $\delta > 0$ such that

$$\langle x^*, x' - x \rangle \leq d_S(x') - d_S(x) + \varepsilon \|x' - x\|, \quad \text{for all } x' \in x + \delta \mathbf{B}.$$

As $d_S(x') - d_S(x) \leq 0$ for all $x' \in S(r)$, one obtains

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x' - x\|$$

for all $x' \in (x + \delta \mathbf{B}) \cap S(r)$, which ensures that $x^* \in \widehat{N}(S(r); x)$.

Now, we show that $\|x^*\| = 1$. Fix $\varepsilon > 0$. As $x^* \in \widehat{\partial} d_S(x)$, there exists $\delta > 0$ such that for all $x' \in x + \delta \mathbf{B}$

$$\langle x^*, x' - x \rangle \leq d_S(x') - d_S(x) + \varepsilon \|x' - x\|. \quad (2.31)$$

Fix now, $\alpha := \min\{1, \varepsilon, \frac{\delta}{1+d_S(x)}\}$ and choose x_α in S such that

$$\|x - x_\alpha\| \leq d_S(x) + \alpha^2.$$

Put $x' := x + \alpha(x_\alpha - x)$. Since $\|x' - x\| \leq \alpha\|x - x_\alpha\| \leq \alpha d_S(x) + \alpha^2 \leq \alpha(1 + d_S(x)) \leq \delta$, one gets (by (2.31))

$$\begin{aligned} \langle x^*, x' - x \rangle &\leq \|x' - x_\alpha\| - \|x - x_\alpha\| + \alpha^2 + \varepsilon \alpha \|x - x_\alpha\| \\ &= (1 - \alpha)\|x - x_\alpha\| - \|x - x_\alpha\| + \alpha^2 + \varepsilon \alpha \|x - x_\alpha\| \\ &= -\alpha\|x - x_\alpha\| + \alpha^2 + \varepsilon \alpha \|x - x_\alpha\|. \end{aligned}$$

Thus,

$$\begin{aligned} \langle x^*, x_\alpha - x \rangle &\leq -\|x - x_\alpha\| + \alpha + \varepsilon \|x - x_\alpha\| \\ &\leq -\|x - x_\alpha\| + \varepsilon(1 + \|x - x_\alpha\|), \end{aligned}$$

and hence

$$\frac{\langle x^*, x - x_\alpha \rangle}{\|x - x_\alpha\|} \geq 1 - \varepsilon \left(1 + \frac{1}{\|x - x_\alpha\|}\right) \geq 1 - \varepsilon \left(1 + \frac{1}{d_S(x)}\right).$$

This ensures that $\|x^*\| \geq 1$. Thus, as one always has $\widehat{\partial} d_S(x) \subset \mathbf{B}_*$, then $\|x^*\| \leq 1$ and hence $\|x^*\| = 1$. This completes the proof of (2.30).

Now, we assume that X is a reflexive Banach space and that the norm $\|\cdot\|$ of X is Kadec. Fix x^* in $\widehat{N}(S(r); x)$, with $\|x^*\| = 1$ and fix $\varepsilon > 0$. On the one hand, observe first that $x^* \in \widehat{\partial} d_{S(r)}(x)$ by Theorem 2.3. So, there exists $\delta_1 > 0$ such that for all $x' \in x + \delta_1 \mathbf{B}$

$$\langle x^*, x' - x \rangle \leq d_{S(r)}(x') - d_{S(r)}(x) + \varepsilon \|x' - x\|.$$

By Lemma 3.2 one gets for any $x' \in (x + \delta_1 \mathbf{B}) \setminus S(r)$

$$\langle x^*, x' - x \rangle \leq d_S(x') - d_S(x) + \varepsilon \|x' - x\|. \quad (2.32)$$

On the other hand, as $x^* \in \widehat{N}(S(r); x)$, there exists $\delta_2 > 0$ such that for all $x' \in (x + \delta_2 \mathbf{B}) \cap S(r)$

$$\langle x^*, x' - x \rangle \leq \frac{\varepsilon}{2} \|x' - x\|. \quad (2.33)$$

Since $\|x^*\| = 1$, we can choose $u \in X$, with $\|u\| = 1$, such that $\langle x^*, u \rangle = 1$.

Fix now $\delta_3 \in (0, \delta_2/2)$ and $x' \in (x + \delta_3 \mathbf{B}) \cap S(r)$ and put $t_{x'} := d_S(x) - d_S(x') \geq 0$.

Then, $x' + t_{x'}u \in S(r) \cap (x + \delta_2 \mathbf{B})$ because

$$d_S(x' + t_{x'}u) \leq d_S(x') + t_{x'} = d_S(x) = r$$

and

$$\|x' + t_{x'}u - x\| \leq \|x' - x\| + t_{x'} \leq 2\|x' - x\| \leq 2\delta_3 \leq \delta_2.$$

By (2.33) one gets

$$\langle x^*, x' + t_{x'}u - x \rangle \leq \frac{\varepsilon}{2} \|x' + t_{x'}u - x\| \leq \varepsilon \|x' - x\|,$$

and hence

$$\begin{aligned} \langle x^*, x' - x \rangle &= \langle x^*, x' + t_{x'}u - x \rangle - \langle x^*, t_{x'}u \rangle \\ &\leq \varepsilon \|x' - x\| - t_{x'} \\ &\leq \varepsilon \|x' - x\| + d_S(x') - d_S(x). \end{aligned} \quad (2.34)$$

According to (2.32) and (2.34), one obtains that for all $x' \in x + \delta \mathbf{B}$ with $\delta := \min\{\delta_1, \delta_3\}$ one has

$$\langle x^*, x' - x \rangle \leq d_S(x') - d_S(x) + \varepsilon \|x' - x\|.$$

So $x^* \in \widehat{\partial} d_S(x)$ and hence the proof is complete. \square

In order to establish the result on the Fréchet normal regularity of $S(r)$, we need to recall the following result of Borwein and Giles [31].

Theorem 2.18. *Let X be a reflexive Banach space, S be a nonempty closed subset of X and $x \notin S$. Let $\|\cdot\|$ denote a Kadec equivalent norm on X . If $\widehat{\partial} d_S(x) \neq \emptyset$, then $\text{Proj}_S(x) \neq \emptyset$.*

Now we are in position to state and prove the main result of this case.

Theorem 2.19. *Let X be a reflexive Banach space, S be a nonempty closed subset of X , and $x \notin S$. Let $\|\cdot\|$ denote a Kadec equivalent norm on X .*

- (i) *If the function d_S is Fréchet subdifferentially regular at x , then $S(r)$ is Fréchet normally regular at x . Further, $\text{Proj}_S(x) \neq \emptyset$ and d_S is directionally regular at x .*
- (ii) *Conversely, if $S(r)$ is Fréchet normally regular at x , d_S is directionally regular at x , and $\text{Proj}_S(x) \neq \emptyset$, then d_S is Fréchet subdifferentially regular at x .*

Proof. (i) Assume that d_S is Fréchet subdifferentially regular at x , i.e. $\widehat{\partial}d_S(x) = \partial^C d_S(x)$. As one always has $\partial^C d_S(x) \neq \emptyset$, one has $\text{Proj}_S(x) \neq \emptyset$ by Theorem 2.18. Furthermore, d_S is directionally regular at x . Thus, by Theorem 2.16, we obtain

$$\partial^C d_S(x) = N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}. \quad (2.35)$$

Now, fix any $x^* \in N^C(S(r); x)$ with $x^* \neq 0$. By (2.35) and Theorem 3.2, one has

$$\frac{x^*}{\|x^*\|} \in \partial^C d_S(x) = \widehat{\partial}d_S(x) \subset \widehat{N}(S(r); x) \cap \{u^* \in X^* : \|u^*\| = 1\},$$

which ensures that $x^* \in \widehat{N}(S(r); x)$ and hence $N^C(S(r); x) \subset \widehat{N}(S(r); x)$. Since the reverse inclusion always holds, the proof of (i) is complete.

(ii) Now, we assume $\widehat{N}(S(r); x) = N^C(S(r); x)$ as well as the other hypothesis in (ii) of the statement of the theorem. Thus, by Theorem 3.2 once again one has

$$\begin{aligned} \widehat{\partial}d_S(x) &= \widehat{N}(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\} \\ &= N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}. \end{aligned}$$

Since d_S is directionally regular at x and $\text{Proj}_S(x) \neq \emptyset$, one also has by Theorem 2.16

$$\partial^C d_S(x) = N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}.$$

Consequently, $\partial^C d_S(x) = \widehat{\partial}d_S(x)$ and hence the proof is complete. \square

Remark 2.12. As X is reflexive, the Mordukhovich regularity of d_S at x could be used in the previous theorem in place of Fréchet subdifferential regularity.

2.11.2 Proximal Case

In this subsection, we will assume that X is a *reflexive Banach space*. We will also assume that the norm $\|\cdot\|$ on X is *Kadec* and we will put $r = d_S(x)$ for some point $x \notin S$.

We have already seen that Fréchet normal regularity is not equivalent to proximal normal regularity. So, the present subsection is devoted to study similar properties of proximal normal regularity as in the previous subsection, essentially we will give conditions under which the proximal normal regularity can be characterized in terms of the distance function at points outside the set.

The following result has been established by Clarke, Stern, and Wolenski (see [89]) in the context of Hilbert spaces. Their proof is strongly based on the scalar product of the Hilbert space. Here we prove it in the general reflexive Banach space setting with a method in the line of the proof of Theorem 3.2.

Theorem 2.20. *Let S be a nonempty closed subset of X and $x \notin S$. Then,*

$$\partial^P d_S(x) = N^P(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}.$$

Proof. We begin by showing the inclusion

$$\partial^P d_S(x) \subset N^P(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}. \quad (2.36)$$

Fix $x^* \in \partial^P d_S(x)$. Then there exist $\sigma > 0$ and $\delta > 0$ such that for all $x' \in x + \delta \mathbf{B}$

$$\langle x^*, x' - x \rangle \leq d_S(x') - d_S(x) + \sigma \|x' - x\|^2.$$

As $d_S(x') - d_S(x) \leq 0$ for all $x' \in S(r)$, one gets

$$\langle x^*, x' - x \rangle \leq \sigma \|x' - x\|^2,$$

for all $x' \in (x + \delta \mathbf{B}) \cap S(r)$, which ensures that $x^* \in \widehat{N}(S(r); x)$. On the other hand, as in the proof of Theorem 3.2, we can check that $\|x^*\| = 1$. So, the proof of the inclusion (2.36) is complete.

Return now to the proof of the reverse inclusion

$$N^P(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\} \subset \partial^P d_S(x).$$

Fix $x^* \in N^P(S(r); x)$ with $\|x^*\| = 1$. By Theorem 2.7, one has $x^* \in \partial^P d_{S(r)}(x)$ and hence there exist $\sigma_1 > 0$ and $\delta_1 > 0$ such that for all $x' \in x + \delta_1 \mathbf{B}$

$$\langle x^*, x' - x \rangle \leq d_{S(r)}(x') - d_{S(r)}(x) + \sigma_1 \|x' - x\|^2.$$

By Lemma 3.2, one gets

$$\langle x^*, x' - x \rangle \leq d_S(x') - d_S(x) + \sigma_1 \|x' - x\|^2, \quad (2.37)$$

for any $x' \in (x + \delta \mathbf{B}) \setminus S(r)$. On the other hand, as $x^* \in N^P(S(r); x)$, there exist $\sigma_2 > 0$ and $\delta_2 > 0$ such that for all $x' \in (x + \delta_2 \mathbf{B}) \cap S(r)$

$$\langle x^*, x' - x \rangle \leq \sigma_2 \|x' - x\|^2. \quad (2.38)$$

Since $\|x^*\| = 1$, we can choose $u \in X$, with $\|u\| = 1$, such that $\langle x^*, u \rangle = 1$.

Fix now $\delta := \min\{\delta_1, \delta_2/2\}$ and $x' \in (x + \delta \mathbf{B}) \cap S(r)$ and put $t_{x'} := d_S(x) - d_S(x') \geq 0$. Then $(x' + t_{x'} u) \in (x + \delta_2 \mathbf{B}) \cap S(r)$ (as in the proof of Theorem 3.2) and hence as $\|x' + t_{x'} u - x\| \leq 2\|x' - x\|$ one gets by (2.38)

$$\begin{aligned} \langle x^*, x' - x \rangle &= \langle x^*, x' + t_{x'} u - x \rangle - \langle x^*, t_{x'} u \rangle \\ &\leq \sigma_2 \|x' + t_{x'} u - x\|^2 - t_{x'} \\ &\leq 4\sigma_2 \|x' - x\|^2 + d_S(x') - d_S(x). \end{aligned} \quad (2.39)$$

According to (2.37) and (2.39), one obtains that for all $x' \in x + \delta \mathbf{B}$ one has

$$\langle x^*, x' - x \rangle \leq d_S(x') - d_S(x) + \sigma \|x' - x\|^2,$$

where $\sigma := \max\{\sigma_1, 4\sigma_2\}$. So $x^* \in \partial^P d_S(x)$ and hence the proof is finished. \square

Theorem 2.21. *Let S be a nonempty closed subset of X , $x \notin S$.*

1. *If the function d_S is proximally subdifferentially regular at x , then $S(r)$ is proximally normally regular at x . Further, $\text{Proj}_S(x) \neq \emptyset$ and d_S is directionally regular at x .*
2. *Conversely, if $S(r)$ is proximally normally regular at x , d_S is directionally regular at x and $\text{Proj}_S(x) \neq \emptyset$, then d_S is proximally subdifferentially regular at x .*

Proof.

1. Assume that d_S is proximally subdifferentially regular at x , i.e.,

$$\partial^P d_S(x) = \partial^C d_S(x). \quad (2.40)$$

As $\partial^C d_S(x)$ is always nonempty, then $\partial^P d_S(x) \neq \emptyset$. Thus, by Theorem 2.18, one has $\text{Proj}_S(x) \neq \emptyset$. On the other hand, by (2.40) we have the directional regularity of d_S at x . Therefore, by Theorem 2.16, the following equality holds

$$\partial^C d_S(x) = N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}.$$

Thus (by (2.40) and Theorem 2.20),

$$N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\} = N^P(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}.$$

Consequently,

$$N^P(S(r); x) = N^C(S(r); x),$$

which ensures that $S(r)$ is proximally normally regular at x .

2. Now, we assume that $N^P(S(r); x) = N^C(S(r); x)$. Thus, by Theorem 2.20 once again one has

$$\begin{aligned} \partial^P d_S(x) &= N^P(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\} \\ &= N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}. \end{aligned} \quad (2.41)$$

On the other hand, since d_S is directionally regular at x and $\text{Proj}_S(x) \neq \emptyset$, one has by Theorem 2.16

$$\partial^C d_S(x) = N^C(S(r); x) \cap \{x^* \in X^* : \|x^*\| = 1\}. \quad (2.42)$$

Thus, by (2.41) and (2.42), we conclude that $\partial^C d_S(x) = \partial^P d_S(x)$ and hence the proof is complete. \square

2.12 Commentary to Chap. 2

In this chapter, we present various concepts of regularity for sets and the possible relationships between them.

The results in Sect. 2.5 are due to Burke et al. [71]. Most results presented in Sects. 2.6–2.8 and 2.11 are taken from Bounkhel and Thibault [61]. Section 2.9 is devoted to the presentation and study of the very important concept of uniform prox-regularity in Hilbert spaces. The results stated in that section are proved in Bounkhel and Thibault [58]. Concerning this concept and its extensions to Banach spaces, we refer the reader to the following list of recent papers [22–24] and to the recent survey [96]. We have to point out, that a recent and very interesting application of this concept to a real life phenomena is given in [186–189]. This application concerns the modeling of evacuation situations. The role of uniform prox-regularity in this application appears to be crucial. Indeed, the set of configurations in their model cannot be convex at all and it has been proved in [187–189] to be uniformly prox-regular.

In Sect. 2.10, we present a concept of regularity of sets introduced in finite dimensional settings in Borwein and Moors [33] and extended in Bounkhel [39] to Banach spaces. The results presented here are taken from [39]. The extension of this concept to set-valued mappings is introduced and studied recently in Bounkhel [41].

For interested readers on more regularity concepts for nonsmooth sets, the following list of references can be consulted [10, 22–24, 29, 36–39, 41, 44, 55, 61, 63, 71, 72, 88–91, 93, 94, 100–102, 104–106, 120, 133, 136, 137, 149, 160, 178, 190, 192, 193, 226–230, 235, 241, 258].



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