

Affine Gindikin–Karpelevich Formula via Uhlenbeck Spaces

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Abstract We prove a version of the Gindikin–Karpelevich formula for untwisted affine Kac–Moody groups over a local field of positive characteristic. The proof is geometric and it is based on the results of [Braverman, Finkelberg, and Gaitsgory, Uhlenbeck spaces via affine Lie algebras, *Progr. Math.*, 244, 17–135, 2006] about intersection cohomology of certain Uhlenbeck-type moduli spaces (in fact, our proof is conditioned upon the assumption that the results of [Braverman, Finkelberg, and Gaitsgory, Uhlenbeck spaces via affine Lie algebras, *Progr. Math.*, 244, 17–135, 2006] are valid in positive characteristic; we believe that generalizing [Braverman, Finkelberg, and Gaitsgory, Uhlenbeck spaces via affine Lie algebras, *Progr. Math.*, 244, 17–135, 2006] to the case of positive characteristic should be essentially straightforward but we have not checked the details). In particular, we give a geometric explanation of certain combinatorial differences between finite-dimensional and affine case (observed earlier by Macdonald and Cherednik), which here manifest themselves by the fact that the affine Gindikin–Karpelevich formula has an additional term compared to the finite-dimensional case. Very roughly speaking, that additional term is related to the fact that the loop group of an affine Kac–Moody group (which should be thought of as some kind of “double loop group”) does not behave well from algebro-geometric point of view; however, it

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has a better behaved version, which has something to do with algebraic surfaces. A uniform (i.e. valid for all local fields) and unconditional (but not geometric) proof of the affine Gindikin–Karpelevich formula is going to appear in [Braverman, Kazhdan, and Patnaik, The Iwahori-Hecke algebra for an affine Kac-Moody group (in preparation)].

Dedicated to S. Patterson on the occasion of his 60th birthday.

1 The Problem

1.1 Classical Gindikin–Karpelevich Formula

Let $\mathcal{K}$ be a non-archimedean local field with ring of integers $\mathcal{O}$ and let G be a split semi-simple group over $\mathcal{O}$. The classical Gindikin–Karpelevich formula describes explicitly how a certain intertwining operator acts on the spherical vector in a principal series representation of $G(\mathcal{K})$.¹ In more explicit terms, it can be formulated as follows.

Let us choose a Borel subgroup B of G and an opposite Borel subgroup B_- ; let U, U_- be their unipotent radicals. In addition, let Λ denote the coroot lattice of G , $R_+ \subset \Lambda$ – the set of positive coroots, Λ_+ – the subsemigroup of Λ generated by R_+ . Thus any $\gamma \in \Lambda_+$ can be written as $\sum a_i \alpha_i$, where α_i are the simple roots. We shall denote by $|\gamma|$ the sum of all the a_i .

Set now $\text{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$. Then it is known that $\mathcal{U}(\mathcal{K})$ -orbits on Gr are in one-to-one correspondence with elements of Λ (this correspondence will be reviewed in Sect. 2); for any $\mu \in \Lambda$, we shall denote by S^μ the corresponding orbit. The same thing is true for $U_-(\mathcal{K})$ -orbits. For each $\gamma \in \Lambda$, we shall denote by T^γ the corresponding orbit. It is well known that $T^\gamma \cap S^\mu$ is non-empty iff $\mu - \gamma \in \Lambda_+$ and in that case the above intersection is finite. The Gindikin–Karpelevich formula allows one to compute the number of points in $T^{-\gamma} \cap S^0$ for $\gamma \in \Lambda_+$ (it is easy to see that the above intersection is naturally isomorphic to $T^{-\gamma+\mu} \cap S^\mu$ for any $\mu \in \Lambda$). The answer is most easily stated in terms of the corresponding generating function:

Theorem 1. (*Gindikin–Karpelevich formula*)

$$\sum_{\gamma \in \Lambda_+} \#(T^{-\gamma} \cap S^0) q^{-|\gamma|} e^{-\gamma} = \prod_{\alpha \in R_+} \frac{1 - q^{-1} e^{-\alpha}}{1 - e^{-\alpha}}.$$

¹More precisely, the Gindikin–Karpelevich formula answers the analogous question for real groups; its analog for p -adic groups (usually also referred to as Gindikin–Karpelevich formula) is proved e.g., in Chap. 4 of [6].

1.2 Formulation of the Problem in the General Case

Let now G be a split symmetrizable Kac–Moody group functor in the sense of [8] and let $\mathfrak{g}$ be the corresponding Lie algebra. We also let $\widehat{G}$ denote the corresponding “formal” version of G (cf. page 198 in [8]). The notations $\Lambda, \Lambda_+, R_+, \text{Gr}_G, S^\mu, T^\gamma$ make sense for $\widehat{G}$ without any changes (cf. Sect. 2 for more detail).

Conjecture 1. For any $\gamma \in \Lambda_+$, the intersection $T^{-\gamma} \cap S^0$ is finite.

This conjecture will be proved in [2] when G is of affine type. In this paper, we are going to prove the following result:

Theorem 2. Assume that $\mathcal{K} = k((t))$ where k is finite. Then Conjecture 1 holds.

So now (at least when $\mathcal{K}$ is as above) we can ask the following

Question: Compute the generating function²

$$I_{\mathfrak{g}}(q) = \sum_{\gamma \in \Lambda_+} \#(T^{-\gamma} \cap S^0) q^{-|\gamma|} e^{-\gamma}.$$

One possible motivation for the above question is as follows: when G is finite-dimensional, Langlands [6] has observed that the usual Gindikin–Karpelevich formula (more precisely, some generalization of it) is responsible for the fact that the constant term of Eisenstein series induced from a parabolic subgroup of G is related to some automorphic L -function. Thus, we expect that generalizing the Gindikin–Karpelevich formula to general Kac–Moody group will eventually become useful for studying Eisenstein series for those groups. This will be pursued in further publications.

We do not know the answer for general G . In the case when G is finite-dimensional, the answer is given by Theorem 1. In this paper we are going to reprove that formula by geometric means and give a generalization to the case when G is untwisted affine.

1.3 The Affine Case

Let us now assume that $\mathfrak{g} = \mathfrak{g}'_{\text{aff}}$, where $\mathfrak{g}'$ is a simple finite-dimensional Lie algebra. The Dynkin diagram of $\mathfrak{g}$ has a canonical (“affine”) vertex and we let $\mathfrak{p}$ be the corresponding maximal parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}^\vee$ denote the Langlands dual algebra and let $\mathfrak{p}^\vee$ be the corresponding dual parabolic. We denote by $\mathfrak{n}(\mathfrak{p}^\vee)$ its (pro)nilpotent radical.

Let (e, h, f) be a principal $sl(2)$ -triple in $(\mathfrak{g}')^\vee$. Since the Levi subalgebra of $\mathfrak{p}^\vee$ is $\mathbb{C} \oplus \mathfrak{g}' \oplus \mathbb{C}$ (where the first multiple is central in $\mathfrak{g}^\vee$ and the second is responsible

²The reason that we use the notation $I_{\mathfrak{g}}$ rather than I_G is that it is clear that this generating function depends only on $\mathfrak{g}$ and not on G .

for the “loop rotation”), this triple acts on $\mathfrak{n}(\mathfrak{p}^\vee)$ and we let $\mathscr{W} = (\mathfrak{n}(\mathfrak{p}^\vee))^f$ (the centralizer of f in $\mathfrak{n}(\mathfrak{p}^\vee)$). We are going to regard $\mathscr{W}$ as a complex (with zero differential) and with grading coming from the action of h (thus, $\mathscr{W}$ is negatively graded). In addition, $\mathscr{W}$ is endowed with an action of $\mathbb{G}_m$, coming from the loop rotation in $\mathfrak{g}^\vee$. In the case when $\mathfrak{g}'$ is simply laced we have $(\mathfrak{g}')^\vee \simeq \mathfrak{g}'$ and $\mathfrak{n}(\mathfrak{p}^\vee) = t \cdot \mathfrak{g}'[t]$ (i.e., $\mathfrak{g}'$ -valued polynomials, which vanish at 0). Hence, $\mathscr{W} = t \cdot (\mathfrak{g}')^f[t]$ and the above $\mathbb{G}_m$ -action just acts by rotating t . Let $d_1, \dots, d_r$ be the exponents of $\mathfrak{g}'$ (here $r = \text{rank}(\mathfrak{g}')$). Then $(\mathfrak{g}')^f$ has a basis $(x_1, \dots, x_r)$, where each x_i is placed in the degree $-2d_i$. We let Fr act on $\mathscr{W}$ by requiring that it acts by $q^{i/2}$ on elements of degree i . Also for any $n \in \mathbb{Z}$, let $\mathscr{W}(n)$ be the same graded vector space but with Frobenius action multiplied by q^{-n} .

Consider now $\text{Sym}^*(\mathscr{W})$. We can again consider it as a complex concentrated in degrees ≤ 0 endowed with an action of Fr and $\mathbb{G}_m$. For each $n \in \mathbb{Z}$, we let $\text{Sym}^*(\mathscr{W})_n$ be the part of $\text{Sym}^*(\mathscr{W})$ on which $\mathbb{G}_m$ acts by the character $z \mapsto z^n$. This is a finite-dimensional complex with zero differential, concentrated in degrees ≤ 0 and endowed with an action of Fr .

We are now ready to formulate the main result. Let δ denote the minimal positive imaginary coroot of $\mathfrak{g}$. Set

$$\Delta_{\mathscr{W}}(z) = \sum_{n=0}^{\infty} \text{Tr}(\text{Fr}, \text{Sym}^*(\mathscr{W})_n) z^n.$$

In particular, when $\mathfrak{g}'$ is simply laced we have

$$\Delta(z) = \prod_{i=1}^r \prod_{j=0}^{\infty} (1 - q^{-d_i} z^j)^{-1}.$$

Theorem 3. (*Affine Gindikin–Karpelevich formula*)

Assume that the results of [1] are valid over k and let $\mathscr{K} = k((t))$. Then

$$I_{\mathfrak{g}}(q) = \frac{\Delta_{\mathscr{W}}(e^{-\delta})}{\Delta_{\mathscr{W}(1)}(e^{-\delta})} \prod_{\alpha \in R_+} \left(\frac{1 - q^{-1} e^{-\alpha}}{1 - e^{-\alpha}} \right)^{m_{\alpha}}.$$

Here, m_{α} denotes the multiplicity of the coroot α .

Remark. Although formally the paper [1] is written under the assumption that $\text{char } k = 0$, we believe that adapting all the constructions of [1] to the case $\text{char } k = p$ should be more or less straightforward. We plan to discuss it in a separate publication.

Let us make two remarks about the above formula: first, we see that it is very similar to the finite-dimensional case (of course in that case $m_{\alpha} = 1$ for any α) with the exception of a “correction term” (which is equal to $\frac{\Delta_{\mathscr{W}}(e^{-\delta})}{\Delta_{\mathscr{W}(1)}(e^{-\delta})}$). Roughly speaking, this correction term has to do with imaginary coroots of $\mathfrak{g}$. The second

remark is that the same correction term appeared in the work of Macdonald [7] from purely combinatorial point of view (cf. also [3] for a more detailed study). The main purpose of this note is to explain how the term $\frac{\Delta_{\mathcal{W}}(e^{-\delta})}{\Delta_{\mathcal{W}(1)}(e^{-\delta})}$ appears naturally from geometric point of view (very roughly speaking it is related to the fact that affine Kac–Moody groups over a local field of positive characteristic can be studied using various moduli spaces of bundles on an algebraic surface). The relation between the present work and the constructions of [3] and [7] will be discussed in [2].

2 Interpretation via Maps from $\mathbb{P}^1$ to $\mathcal{B}$

2.1 Generalities on Kac–Moody Groups

In what follows all schemes will be considered over a field k which at some point will be assumed to be finite. Our main reference for Kac–Moody groups is [8]. Assume that we are given a symmetrizable Kac–Moody root data and we denote by G (resp. $\widehat{G}$) the corresponding minimal (resp. formal) Kac–Moody group functor (cf. [8], page 198); we have the natural embedding $G \hookrightarrow \widehat{G}$. We also let W denote the corresponding Weyl group and we let $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ be the corresponding length function.

The group G is endowed with closed subgroup functors $U \subset B, U_- \subset B_-$ such that the quotients B/U and B_-/U_- are naturally isomorphic to the Cartan group H of G ; also H is isomorphic to the intersection $B \cap B_-$. Moreover, both U_- and B_- are still closed as subgroup functors of $\widehat{G}$. On the other hand, B and U are not closed in $\widehat{G}$ and we denote by $\widehat{B}$ and $\widehat{U}$ their closures.

The quotient G/B has a natural structure of an ind-scheme which is ind-proper; the same is true for the quotient $\widehat{G}/\widehat{B}$ and the natural map $G/B \rightarrow \widehat{G}/\widehat{B}$ is an isomorphism. This quotient is often called *the thin flag variety of G* . Similarly, one can consider the quotient $\mathcal{B} = \widehat{G}/B_-$; it is called *the thick flag variety of G or Kashiwara flag scheme*. As is suggested by the latter name, $\mathcal{B}$ has a natural scheme structure. The orbits of B on $\mathcal{B}$ are in one-to-one correspondence with the elements of the Weyl group W ; for each $w \in W$, we denote by $\mathcal{B}_w$ the corresponding orbit. The codimension of $\mathcal{B}_w$ is $\ell(w)$; in particular, $\mathcal{B}_e$ is open. There is a unique H -invariant point $y_0 \in \mathcal{B}_e$. The complement to $\mathcal{B}_e$ is a divisor in $\mathcal{B}$ whose components are in one-to-one correspondence with the simple roots of G .

In what follows Λ will denote the coroot lattice of G , $R_+ \subset \Lambda$ – the set of positive coroots, Λ_+ – the subsemigroup of Λ generated by R_+ . Thus, $\gamma \in \Lambda_+$ can be written as $\sum a_i \alpha_i$ where α_i are the simple coroots. We shall denote by $|\gamma|$ the sum of all the a_i .

In what follows we shall assume that G is “simply connected,” which means that Λ is equal to the full cocharacter lattice of H .

2.2 Some Further Notations

For any variety X and any $\gamma \in \Lambda_+$ we shall denote by $\text{Sym}^\gamma X$ the variety parametrizing all unordered collections $(x_1, \gamma_1), \dots, (x_n, \gamma_n)$, where $x_j \in X, \gamma_j \in \Lambda_+$ such that $\sum \gamma_j = \gamma$.

Assume that k is finite and let $\mathcal{S}$ be a complex of ℓ -adic sheaves on a variety X over k . We set

$$\chi_k(\mathcal{S}) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(\text{Fr}, H^i(\bar{X}, \mathcal{S})),$$

where $\bar{X} = X \times_{\text{Spec } k} \text{Spec } \bar{k}$.

We shall denote by $(\bar{\mathbb{Q}}_l)_X$ the constant sheaf with fiber $\bar{\mathbb{Q}}_l$. According to the Grothendieck–Lefschetz fixed point formula, we have

$$\chi_k((\bar{\mathbb{Q}}_l)_X) = \#X(k).$$

2.3 Semi-Infinite Orbits

As in the introduction, we set $\mathcal{K} = k((t))$, $\mathcal{O} = k[[t]]$. We let $\text{Gr} = \widehat{G}(\mathcal{K})/\widehat{G}(\mathcal{O})$, which we are just going to consider as a set with no structure. Each $\lambda \in \Lambda$ is a homomorphism $\mathbb{G}_m \rightarrow H$; in particular, it defines a homomorphism $\mathcal{K}^* \rightarrow H(\mathcal{K})$. We shall denote the image of t under the latter homomorphism by t^λ . Abusing the notation, we shall denote its image in Gr by the same symbol. Set

$$S^\lambda = \widehat{U}(\mathcal{K}) \cdot t^\lambda \subset \text{Gr}; \quad T^\lambda = U_-(\mathcal{K}) \cdot t^\lambda \subset \text{Gr}.$$

Lemma 1. *Gr is equal to the disjoint union of all the S^λ .*

Proof. This follows from the *Iwasawa decomposition* for G of [5]; we include a different proof for completeness. Since $\Lambda \simeq \widehat{U}(\mathcal{K}) \backslash \widehat{B}(\mathcal{K}) / \widehat{B}(\mathcal{O})$, the statement of the lemma is equivalent to the assertion that the natural map $\widehat{B}(\mathcal{K}) / \widehat{B}(\mathcal{O}) \rightarrow \widehat{G}(\mathcal{K}) / \widehat{G}(\mathcal{O})$ is an isomorphism; in other words, we need to show that $\widehat{B}(\mathcal{K})$ acts transitively on Gr . But this is equivalent to saying that $\widehat{G}(\mathcal{O})$ acts transitively on $\widehat{G}(\mathcal{K}) / \widehat{B}(\mathcal{K})$, which means that the natural map $\widehat{G}(\mathcal{O}) / \widehat{B}(\mathcal{O}) \rightarrow \widehat{G}(\mathcal{K}) / \widehat{B}(\mathcal{K})$ is an isomorphism. However, the left-hand side is $(\widehat{G}/\widehat{B})(\mathcal{O})$ and the right-hand side is $(\widehat{G}/\widehat{B})(\mathcal{K})$ and the assertion follows from the fact that the ind-scheme $\widehat{G}/\widehat{B}$ satisfies the valuative criterion of properness.

The statement of the lemma is definitely false if we use T^μ 's instead of S^λ 's since the scheme $\widehat{G}/B_-$ does not satisfy the valuative criterion of properness. Let us say that an element $g(t) \in \widehat{G}(\mathcal{K})$ is good if its projection to $\mathcal{B}(\mathcal{K}) = B_-(\mathcal{K}) \backslash \widehat{G}(\mathcal{K})$

comes from a point of $\mathcal{B}(\mathcal{O})$. Since $\mathcal{B}(\mathcal{O}) = B_-(\mathcal{O}) \backslash \widehat{G}(\mathcal{O})$, it follows that the set of good elements of $\widehat{G}(\mathcal{K})$ is just equal to $B_-(\mathcal{K}) \cdot G(\mathcal{O})$, which immediately proves the following result:

Lemma 2. *The preimage of $\bigcup_{\gamma \in \Lambda} T^\gamma$ in $\widehat{G}(\mathcal{K})$ is equal to the set of good elements of $\widehat{G}(\mathcal{K})$.*

2.4 Spaces of Maps

Recall that the Picard group of $\mathcal{B}$ can be naturally identified with $\Lambda^\vee$ (the dual lattice to Λ). Thus for any map $f : \mathbb{P}^1 \rightarrow \mathcal{B}$, we can talk about the degree of f as an element $\gamma \in \Lambda$. The space of such maps is non-empty iff $\gamma \in \Lambda_+$. We say that a map $f : \mathbb{P}^1 \rightarrow \mathcal{B}$ is *based* if $f(\infty) = y_0$. Let $\mathcal{M}^\gamma$ be the space of based maps $f : \mathbb{P}^1 \rightarrow \mathcal{B}$ of degree γ . It is shown in the Appendix to [1] that this is a smooth scheme of finite type over k of dimension $2|\gamma|$. We have a natural (“factorization”) map $\pi^\gamma : \mathcal{M}^\gamma \rightarrow \text{Sym}^\gamma \mathbb{A}^1$, which is related to how the image of a map $\mathbb{P}^1 \rightarrow \mathcal{B}$ intersects the complement to $\mathcal{B}_e$. In particular, if we set

$$\mathcal{F}^\gamma = (\pi^\gamma)^{-1}(\gamma \cdot 0),$$

then $\mathcal{F}^\gamma$ consists of all the based maps $f : \mathbb{P}^1 \rightarrow \mathcal{B}$ of degree γ such that $f(x) \in \mathcal{B}_e$ for any $x \neq 0$.

Theorem 4. *There is a natural identification $\mathcal{F}^\gamma(k) \simeq T^{-\gamma} \cap S^0$.*

Since $\mathcal{F}^\gamma$ is a scheme of finite type over k , it follows that $\mathcal{F}^\gamma(k)$ is finite and thus Theorem 4 implies Theorem 2.

The proof of Theorem 4 is essentially a repetition of a similar proof in the finite-dimensional case, which we include here for completeness.

Proof. First of all, let us construct an embedding of the union of all the $\mathcal{F}^\gamma(k)$ into $S^0 = \widehat{U}(\mathcal{K})/\widehat{U}(\mathcal{O})$. Indeed, an element of $\bigcup_{\gamma \in \Lambda_+} \mathcal{F}^\gamma$ is uniquely determined

by its restriction to $\mathbb{G}_m \subset \mathbb{P}^1$; this restriction is a map $f : \mathbb{G}_m \rightarrow \mathcal{B}_e$ such that $\lim_{x \rightarrow \infty} f(x) = y_0$. We may identify $\mathcal{B}_e$ with $\widehat{U}$ (by acting on y_0). Thus, we get

$$\bigcup_{\gamma \in \Lambda_+} \mathcal{F}^\gamma \subset \{u : \mathbb{P}^1 \setminus \{0\} \rightarrow \widehat{U} \mid u(\infty) = e\}. \quad (1)$$

We have a natural map from the set of k -points of the right-hand side of (1) to $\widehat{U}(\mathcal{K})$; this map sends every u as above to its restriction to the formal punctured neighbourhood of 0. We claim that after projecting $\widehat{U}(\mathcal{K})$ to $S^0 = \widehat{U}(\mathcal{K})/\widehat{U}(\mathcal{O})$, this map becomes an isomorphism. Recall that $\widehat{U}$ is a group-scheme, which can be written as a projective limit of finite-dimensional unipotent group-schemes U_i ;

moreover, each U_i has a filtration by normal subgroups with successive quotients isomorphic to $\mathbb{G}_a$. Hence, it is enough to prove that the above map is an isomorphism when $U = \mathbb{G}_a$. In this case, we just need to check that any element of the quotient $\mathbf{k}((t))/\mathbf{k}[[t]]$ has unique lift to a polynomial $u(t) \in \mathbf{k}[t, t^{-1}]$ such that $u(\infty) = 0$, which is obvious.

Now Lemma 2 implies that a map $u(t)$ as above extends to a map $\mathbb{P}^1 \rightarrow \mathcal{B}$ if and only if the corresponding element of S^0 lies in the intersection with some $T^{-\gamma}$.

It remains to show that $\mathcal{F}^\gamma(\mathbf{k})$ is exactly equal to $S^0 \cap T^{-\gamma}$ as a subset of S^0 . Let $\Lambda^\vee$ be the weight lattice of G and let $\Lambda_+^\vee$ denote the set of dominant weights of G . For each $\lambda^\vee \in \Lambda_+^\vee$, we can consider the Weyl module $L(\lambda^\vee)$, defined over $\mathbb{Z}$; in particular, $L(\lambda^\vee)(\mathcal{K})$ and $L(\lambda^\vee)(\mathcal{O})$ make sense. By the definition $L(\lambda^\vee)$ is the module of global sections of a line bundle $\mathcal{L}(\lambda^\vee)$ on $\mathcal{B}$. Moreover, we have a weight decomposition

$$L(\lambda^\vee) = \bigoplus_{\mu^\vee \in \Lambda^\vee} L(\lambda^\vee)_{\mu^\vee},$$

where each $L(\lambda^\vee)_{\mu^\vee}$ is a finitely generated free $\mathbb{Z}$ -module and $L(\lambda^\vee)_{\lambda^\vee} := l_{\lambda^\vee}$ has rank one. Geometrically, $l_{\lambda^\vee}$ is the fiber of $\mathcal{L}(\lambda^\vee)$ at y_0 and the corresponding projection map from $L(\lambda^\vee) = \Gamma(\mathcal{B}, \mathcal{L}(\lambda^\vee))$ to $l_{\lambda^\vee}$ is the restriction to y_0 .

Let $\eta_{\lambda^\vee}$ denote the projection of $L(\lambda^\vee)$ to $l_{\lambda^\vee}$. This map is U_- -equivariant (where U_- acts trivially on $l_{\lambda^\vee}$).

Lemma 3. *The projection of a good element $g \in G(\mathcal{K})$ lies in $T^\vee$ (for some $\mathbf{v} \in \Lambda$) if and only if for any $\lambda^\vee \in \Lambda^\vee$ we have:*

$$\eta_{\lambda^\vee}(g(L(\lambda^\vee)(\mathcal{O}))) \subset t^{\langle \mathbf{v}, \lambda^\vee \rangle} l_{\lambda^\vee}(\mathcal{O}); \quad \eta_{\lambda^\vee}(g(L(\lambda^\vee)(\mathcal{O}))) \not\subset t^{\langle \mathbf{v}, \lambda^\vee \rangle - 1} l_{\lambda^\vee}(\mathcal{O}). \quad (2)$$

Proof. First of all, we claim that if the projection of g lies in $T^\vee$ then the above condition is satisfied. Indeed, it is clearly satisfied by $t^\vee$; moreover, (2) is clearly invariant under left multiplication by $U_-(\mathcal{K})$ and under right multiplication by $G(\mathcal{O})$. Hence any $g \in U_-(\mathcal{K}) \cdot t^\vee \cdot G(\mathcal{O})$ satisfies (2).

On the other hand, assume that a good element $g \in G(\mathcal{K})$ satisfies (2). Since g lies in $U_-(\mathcal{K}) \cdot t^{\mathbf{v}'} \cdot G(\mathcal{O})$ for some $\mathbf{v}'$, it follows that g satisfies (2) when $\mathbf{v}$ is replaced by $\mathbf{v}'$. However, it is clear that this is possible only if $\mathbf{v} = \mathbf{v}'$.

It is clear that in (2) one can replace $g(L(\lambda^\vee)(\mathcal{O}))$ with $g(L(\lambda^\vee)(\mathbf{k}))$ (since the latter generates the former as an $\mathcal{O}$ -module).

Let now f be an element of $\mathcal{F}^\gamma$. Then $f^* \mathcal{L}(\lambda^\vee)$ is isomorphic to the line bundle $\mathcal{L}(\langle \gamma, \lambda^\vee \rangle)$ on $\mathbb{P}^1$. On the other hand, the bundle $\mathcal{L}(\lambda^\vee)$ is trivialized on $\mathcal{B}_e$ by means of the action of U ; more precisely, the restriction of $\mathcal{L}(\lambda^\vee)$ is canonically identified with the trivial bundle with fiber $l_{\lambda^\vee}$. Let now $s \in L(\lambda^\vee)(\mathbf{k})$; we are going to think of it as a section of $L(\lambda^\vee)$ on $\mathcal{B}$. In particular, it gives rise to a function $\tilde{s}: \mathcal{B}_e \rightarrow l_{\lambda^\vee}$. Let also $u(t)$ be the element of $U(\mathcal{K})$, corresponding to f . Then $\eta_{\lambda^\vee}(u(t)(s))$ can be described as follows: we consider the composition $\tilde{s} \circ f$ and restrict it to the formal neighbourhood of $0 \in \mathbb{P}^1$ (we get an element of $l_{\lambda^\vee}(\mathcal{K})$).

On the other hand, since $f \in \mathcal{F}^\gamma$, it follows that $f^* \mathcal{L}(\lambda^\vee)$ is trivialized away from 0 and any section of it can be thought of as a function $\mathbb{P}^1 \setminus \{0\}$ with pole of order $\leq \langle \gamma, \lambda^\vee \rangle$ at 0. Hence, $\tilde{s} \circ f$ has pole of order $\leq \langle \gamma, \lambda^\vee \rangle$ at 0.

To finish the proof it is enough to show that for some s the function $\tilde{s} \circ f$ has pole of order exactly $\langle \gamma, \lambda^\vee \rangle$ at 0 (indeed if $f \in T^{-\gamma'}$ for some $\gamma' \in \Lambda$, then by (2) $\tilde{s} \circ f$ has pole of order $\leq \langle \gamma', \lambda^\vee \rangle$ at 0 and for some s , it has pole of order exactly $\langle \gamma', \lambda^\vee \rangle$, which implies that $\gamma = \gamma'$). To prove this, let us note that since $\mathcal{L}(\lambda^\vee)$ is generated by global sections, the line bundle $f^* \mathcal{L}(\lambda^\vee)$ is generated by sections of the form $f^* s$, where s is a global section of $\mathcal{L}(\lambda^\vee)$. This implies that for any $\bar{s} \in \Gamma(\mathbb{P}^1, f^* \mathcal{L}(\lambda^\vee))$ there exists a section $s \in \Gamma(\mathcal{B}, \mathcal{L}(\lambda^\vee))$ such that the ratio $\bar{s}/s$ is a rational function on $\mathbb{P}^1$, which is invertible at 0. Taking $\bar{s}$ such that its pole with respect to the above trivialization of $f^* \mathcal{L}(\lambda^\vee)$ is exactly equal to $\langle \gamma', \lambda^\vee \rangle$ and taking s as above, we see that the pole of $f^* s$ with respect to the above trivialization of $f^* \mathcal{L}(\lambda^\vee)$ is exactly equal to $\langle \gamma', \lambda^\vee \rangle$.

3 Proof of Theorem 1 via Quasi-Maps

3.1 Quasi-Maps

We shall denote by $\mathcal{DM}^\gamma$ the space of based *quasi-maps* $\mathbb{P}^1 \rightarrow \mathcal{B}$. According to [4], we have the stratification

$$\mathcal{DM}^\gamma = \bigcup_{\gamma' \leq \gamma} \mathcal{M}^{\gamma'} \times \mathrm{Sym}^{\gamma-\gamma'} \mathbb{A}^1.$$

The factorization morphism π_0^γ extends to the similar morphism $\bar{\pi}^\gamma: \mathcal{DM}^\gamma \rightarrow \mathrm{Sym}^\gamma$ and we set $\overline{\mathcal{F}}^\gamma = (\bar{\pi}^\gamma)^{-1}(0)$. Thus, we have

$$\overline{\mathcal{F}}^\gamma = \bigcup_{\gamma' \leq \gamma} \mathcal{F}^{\gamma'}. \quad (3)$$

There is a natural section $i^\gamma: \mathrm{Sym}^\gamma \mathbb{A}^1 \rightarrow \mathcal{DM}^\gamma$. According to [4], we have

- Theorem 5.** 1. The restriction of $\mathrm{IC}_{\mathcal{DM}^\gamma}$ to $\mathcal{F}^{\gamma'}$ is isomorphic to $(\mathbb{Q}_l)_{\mathcal{F}^{\gamma'}}[2](1)^{\otimes |\gamma'|} \otimes \mathrm{Sym}^*(\mathfrak{n}_+^\vee[2](1))_{\gamma-\gamma'}$.
 2. There exists a $\mathbb{G}_m$ -action on $\mathcal{DM}^\gamma$, which contracts it to the image of i^γ . In particular, it contracts $\overline{\mathcal{F}}^\gamma$ to one point (corresponding to $\gamma' = 0$ in (3)).
 3. Let s_γ denote the embedding of $\gamma \cdot 0$ into $\mathrm{Sym}^\gamma \mathbb{A}^1$. Then

$$s_{\gamma!}^* \mathrm{IC}_{\mathcal{DM}^\gamma} = \mathrm{Sym}^*(\mathfrak{n}_+)_{\gamma}$$

(here the right hand is a vector space concentrated in cohomological degree 0 and with trivial action of Fr).

The assertion (2) implies that $\pi_!^\gamma \mathrm{IC}_{\mathcal{M}^\gamma} = i_!^\gamma \mathrm{IC}_{\mathcal{M}^\gamma}$ and hence

$$H_c^*(\overline{\mathcal{F}}, \mathrm{IC}_{\mathcal{M}^\gamma}|_{\overline{\mathcal{F}}^\gamma}) = s_\gamma^* \pi_!^\gamma \mathrm{IC}_{\mathcal{M}^\gamma} = s_\gamma^* i_!^\gamma \mathrm{IC}_{\mathcal{M}^\gamma} = \mathrm{Sym}^*(\mathfrak{n}_+)_{\gamma}.$$

Thus, setting, $\mathcal{S}^\gamma = \mathrm{IC}_{\mathcal{M}^\gamma}|_{\overline{\mathcal{F}}^\gamma}$ we get

$$\sum_{\gamma \in \Lambda_+} \chi_k(\mathcal{S}^\gamma) e^{-\gamma} = \prod_{\alpha \in R_+} \frac{1}{1 - e^{-\alpha}}. \quad (4)$$

On the other hand, according to (1) we have

$$\chi_k(\mathcal{S}^\gamma) = \sum_{\gamma' \leq \gamma} (\#\mathcal{F}^{\gamma'}) q^{-|\gamma'|} \mathrm{Tr}(\mathrm{Fr}, \mathrm{Sym}^*(\mathfrak{n}_+^\vee[2](1))_{\gamma-\gamma'}),$$

which implies that

$$\sum_{\gamma \in \Lambda_+} \chi_k(\mathcal{S}^\gamma) e^{-\gamma} = \frac{\sum_{\gamma \in \Lambda_+} \#\mathcal{F}^\gamma(k) q^{-|\gamma|} e^{-\gamma}}{\prod_{\alpha \in R_+} 1 - q^{-1} e^{-\alpha}} = \frac{I_{\mathfrak{g}}(q)}{\prod_{\alpha \in R_+} 1 - q^{-1} e^{-\alpha}}. \quad (5)$$

Hence,

$$I_{\mathfrak{g}}(q) = \prod_{\alpha \in R_+} \frac{1 - q^{-1} e^{-\alpha}}{1 - e^{-\alpha}}.$$

4 Proof of Theorem 3

4.1 Flag Uhlenbeck Spaces

We now assume that $G = (G')_{\mathrm{aff}}$ where G' is some semi-simple simply connected group. We want to follow the pattern of Sect. 3. Let $\gamma \in \Lambda_+$. As is discussed in [1], the corresponding space of quasi-maps behaves badly when G is replaced by G_{aff} . However, in this case one can use the corresponding *flag Uhlenbeck space* $\mathcal{U}^\gamma$. In fact, as was mentioned in the Introduction, in [1] only the case of k of characteristic 0 is considered. In what follows we are going to assume that the results of *loc. cit.* are valid also in positive characteristic.

The flag Uhlenbeck space $\mathcal{U}^\gamma$ has properties similar to the space of quasi-maps $\mathcal{M}^\gamma$ considered in the previous section. Namely, we have:

- $\mathcal{U}^\gamma$ is an affine variety of dimension $2|\gamma|$, which contains $\mathcal{M}^\gamma$ as a dense open subset.
- There is a factorization map $\pi^\gamma: \mathcal{U}^\gamma \rightarrow \mathrm{Sym}^\gamma \mathbb{A}^1$; it has a section $i_\gamma: \mathrm{Sym}^\gamma \mathbb{A}^1 \rightarrow \mathcal{U}^\gamma$.
- $\mathcal{U}^\gamma$ is endowed with a $\mathbb{G}_m$ -action, which contracts $\mathcal{U}^\gamma$ to the image of i_γ .

These properties are identical to the corresponding properties of $\mathcal{Q}\mathcal{M}^\gamma$ from the previous section. The next (stratification) property, however, is different (and it is in fact responsible for the additional term in Theorem 3). Namely, let δ denote the minimal positive imaginary coroot of G'_{aff} . Then we have

d. There exists a stratification

$$\mathcal{U}^\gamma = \bigcup_{\gamma' \in \Lambda_+, n \in \mathbb{Z}, \gamma - \gamma' - n\delta \in \Lambda_+} (\mathcal{M}^{\gamma - \gamma' - n\delta} \times \text{Sym}^{\gamma'}(\mathbb{A}^1) \times \text{Sym}^n(\mathbb{G}_m \times \mathbb{A}^1)). \quad (6)$$

In particular, if we now set $\overline{\mathcal{F}}^\gamma = (\pi^\gamma)^{-1}(\gamma \cdot 0)$, we get

$$\overline{\mathcal{F}}^\gamma = \left(\bigcup_{\gamma' \in \Lambda_+, n \in \mathbb{Z}, \gamma - \gamma' - n\delta \in \Lambda_+} \mathcal{F}^{\gamma'} \right) \times \text{Sym}^n(\mathbb{G}_m). \quad (7)$$

4.2 Description of the IC-Sheaf

In [1], we describe the IC-sheaf of $\mathcal{U}^\gamma$. To formulate the answer, we need to introduce some notation. Let $\mathcal{P}(n)$ denote the set of partitions of n . In other words, any $P \in \mathcal{P}(n)$ is an unordered sequence $n_1, \dots, n_k \in \mathbb{Z}_{>0}$ such that $\sum n_i = n$. We set $|P| = k$. For a variety X and any $P \in \mathcal{P}(n)$, we denote by $\text{Sym}^P(X)$ the locally closed subset of $\text{Sym}^n(X)$ consisting of all formal sums $\sum n_i x_i$ where $x_i \in X$ and $x_i \neq x_j$ for $i \neq j$. The dimension of $\text{Sym}^P(X)$ is $|P| \cdot \dim X$. Let also

$$\text{Sym}^*(W[2](1))_P = \bigotimes_{i=1}^k \text{Sym}^*(\mathcal{W}[2](1))_{n_i}.$$

Theorem 6. *The restriction of $\text{IC}_{\mathcal{U}^\gamma}$ to $\mathcal{M}^{\gamma - \gamma' - n\delta} \times \text{Sym}^{\gamma'}(\mathbb{A}^1) \times \text{Sym}^P(\mathbb{G}_m \times \mathbb{A}^1)$ is isomorphic to constant sheaf on that scheme tensored with*

$$\text{Sym}^*(\mathfrak{n}_+)_{\gamma'} \otimes \text{Sym}^*(\mathcal{W})_P[2|\gamma - \gamma' - n\delta|](|\gamma - \gamma' - n\delta|).$$

Corollary 1. *The restriction of $\text{IC}_{\mathcal{U}^\gamma}$ to $\overline{\mathcal{F}}^{\gamma - \gamma' - n\delta} \times \text{Sym}^P(\mathbb{G}_m)$ is isomorphic to the constant sheaf tensored with*

$$\text{Sym}^*(\mathfrak{n}_+^\vee)_{\gamma'} \otimes \text{Sym}^*(\mathcal{W})_P[2|\gamma - \gamma' - n\delta|](|\gamma - \gamma' - n\delta|).$$

Let now $\mathcal{S}^\gamma$ denote the restriction of $\text{IC}_{\mathcal{U}^\gamma}$ to $\overline{\mathcal{F}}^\gamma$. Then as in (4) we get

$$\sum_{\gamma \in \Lambda_+} \chi_k(\mathcal{S}^\gamma) e^{-\gamma} = \prod_{\alpha \in R_+} \frac{1}{(1 - e^{-\alpha})^{m_\alpha}}. \quad (8)$$

On the other hand, arguing as in (5) we get that

$$\sum_{\gamma \in \Lambda_+} \chi_k(\mathcal{S}^\gamma) e^{-\gamma} = A(q) \frac{I_{\mathfrak{g}}(q)}{\prod_{\alpha \in R_+} (1 - q^{-1} e^{-\alpha})^{m_\alpha}}, \quad (9)$$

where

$$A(q) = \sum_{n=0}^{\infty} \sum_{P \in \mathcal{P}(n)} \text{Tr}(\text{Fr}, H_c^*(\text{Sym}^P(\mathbb{G}_m), \overline{\mathbb{Q}}_l) \otimes \text{Sym}^*(\mathcal{W}[2](1))_P) e^{-n\delta}.$$

This implies that

$$I_{\mathfrak{g}}(q) = A(q) \prod_{\alpha \in R_+} \left(\frac{1 - q^{-1} e^{-\alpha}}{1 - e^{-\alpha}} \right)^{m_\alpha}.$$

It remains to compute $A(q)$. However, it is clear that

$$A(q) = \sum_{n=0}^{\infty} \text{Tr}(\text{Fr}, \text{Sym}^n(H_c^*(\mathbb{G}_m)) \otimes \mathcal{W}[2](1)) e^{-n\delta} = \frac{\Delta_{\mathcal{W}}(e^{-\delta})}{\Delta_{\mathcal{W}(1)}(e^{-\delta})}. \quad (10)$$

This is true since $H_c^i(\mathbb{G}_m) = 0$ unless $i = 1, 2$, and we have

$$H_c^1(\mathbb{G}_m) = \overline{\mathbb{Q}}_l, \quad H_c^2(\mathbb{G}_m) = \overline{\mathbb{Q}}_l(-1),$$

and thus if we ignore the cohomological $\mathbb{Z}$ -grading, but only remember the corresponding $\mathbb{Z}_2$ -grading, then we just have

$$\text{Sym}^*(H_c^*(\mathbb{G}_m) \otimes \mathcal{W}[2](1)) = \text{Sym}^*(\mathcal{W}) \otimes \Lambda^*(\mathcal{W}(1)),$$

whose character is exactly the right-hand side of (10).

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References

1. A. Braverman, M. Finkelberg and D. Gaitsgory, *Uhlenbeck spaces via affine Lie algebras*, The unity of mathematics, 17–135, Progr. Math., **244**, Birkhäuser Boston, Boston, MA, 2006.
2. A. Braverman, D. Kazhdan and M. Patnaik, *The Iwahori-Hecke algebra for an affine Kac-Moody group*, in preparation.
3. I. Cherednik and X. Ma, *A new take on spherical, Whittaker and Bessel functions*, arXiv:0904.4324.

4. B. Feigin, M. Finkelberg, A. Kuznetsov, I. Mirkovic, *Semi-infinite flags. II. Local and global intersection cohomology of quasimaps' spaces. Differential topology, infinite-dimensional Lie algebras, and applications*, 113–148, Amer. Math. Soc. Transl. Ser. 2, 194, Amer. Math. Soc., Providence, RI, 1999.
5. S. Gaussent and G. Rousseau, *Kac-Moody groups, hovels and Littelmann paths*, Annales de l'Institut Fourier, **58** no. 7 (2008), p. 2605–2657
6. R. P. Langlands, *Euler products*, A James K. Whittemore Lecture in Mathematics given at Yale University, 1967. Yale Mathematical Monographs, **1**. Yale University Press, New Haven, Conn.-London, 1971.
7. I. G. Macdonald, *A formal identity for affine root systems. Lie groups and symmetric spaces*, 195–211, Amer. Math. Soc. Transl. Ser. 2, **210**, Amer. Math. Soc., Providence, RI, 2003.
8. J. Tits, *Groups and group functors attached to Kac-Moody data*, Workshop Bonn 1984 (Bonn, 1984), 193–223, Lecture Notes in Math., **1111**, Springer, Berlin, 1985

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