Chapter 2

Basic Examples

We will work our way through examples in this chapter, looking at representations and characters of some familiar finite groups. We focus on complex representations, but any algebraically closed field of characteristic zero (e.g., the algebraic closure $\overline{\mathbb{Q}}$ of the rationals) could be substituted for $\mathbb{C}$.

Recall that the character χ_ρ of a finite-dimensional representation ρ of a group G is the function on the group specified by

$$\chi_\rho(g) = \operatorname{Tr} \rho(g). \tag{2.1}$$

Characters are invariant under conjugation, and so χ_ρ takes a constant value $\chi_\rho(C)$ on any conjugacy class C. As we have seen before in (1.50),

$$\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)} \qquad \text{for all } g \in G, \tag{2.2}$$

for any complex representation ρ. We say that a character is *irreducible* if it is the character of an irreducible representation. A *complex character* is the character of a complex representation.

We denote by $\mathcal{R}_G$ a maximal set of inequivalent irreducible complex representations of G. Let $\mathcal{C}_G$ be the set of all conjugacy classes in G. If C is a conjugacy class, then we denote by C^{-1} the conjugacy class consisting of the inverses of the elements in C.

It will be useful to keep at hand some facts (proofs are given in Chap. 7) about complex representations of any finite group G: (a) there are only finitely many inequivalent irreducible complex representations of G and these are all finite-dimensional; (b) two finite-dimensional complex representations

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of G are equivalent if and only if they have the same character; (c) a complex representation of G is irreducible if and only if its character χ_ρ satisfies

$$\sum_{g\in G}|\chi_\rho(g)|^2 = \sum_{C\in\mathcal{C}_G}|C||\chi_\rho(C)|^2 = |G|; \tag{2.3}$$

and (d) the number of inequivalent irreducible complex representations of G is equal to the number of conjugacy classes in G.

In going through the examples in this chapter, we will sometimes pause to use or verify some standard properties of complex characters of a finite group G (again, proofs are given in Chap. 7). These properties are summarized in the orthogonality relations among complex characters:

$$\begin{aligned}\sum_{h\in G}\chi_\rho(gh)\chi_{\rho_1}(h^{-1}) &= |G|\chi_\rho(g)\delta_{\rho\,\rho_1},\\ \sum_{\rho\in\mathcal{R}_G}\chi_\rho(C')\chi_\rho(C^{-1}) &= \frac{|G|}{|C|}\delta_{C'\,C},\end{aligned} \tag{2.4}$$

where $\delta_{a\,b}$ is 1 if $a = b$ and is 0 otherwise, the relations above being valid for all $\rho, \rho_1 \in \mathcal{R}_G$, all conjugacy classes $C, C' \in \mathcal{C}_G$, and all elements $g \in G$. Specializing this to specific cases (such as $\rho = \rho_1$ or $g = e$), we have

$$\begin{aligned}\sum_{\rho\in\mathcal{R}_G}(\dim\rho)^2 &= |G|,\\ \sum_{\rho\in\mathcal{R}_G}\dim\rho\,\chi_\rho(g) &= 0 \qquad \text{if } g \neq e,\\ \sum_{g\in G}\chi_{\rho_1}(g)\chi_{\rho_2}(g^{-1}) &= |G|\delta_{\rho_1\,\rho_2}\dim\rho \qquad \text{for } \rho_1, \rho_2 \in \mathcal{R}_G.\end{aligned} \tag{2.5}$$

2.1 Cyclic Groups

Let us work out all irreducible representations of a cyclic group C_n containing n elements. Being cyclic, C_n contains a *generator* c, which is an element such that C_n consists exactly of the powers $c, c^2, ..., c^n$, where c^n is the identity e in the group. Figure **2.1** displays C_8 as eight equally spaced points around the unit circle in the complex plane.

Let ρ be a representation of C_n on a complex vector space $V \neq 0$. By Proposition 1.6, there is a basis of V relative to which the matrix of $\rho(c)$ is

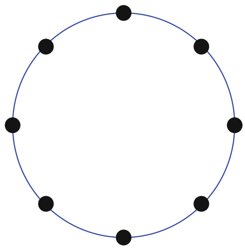

Fig. 2.1 The cyclic group C_8

diagonal, with each diagonal entry being an nth root of unity. If V is of finite dimension d, then

$$\text{matrix of } \rho(c) = \begin{bmatrix} \eta_1 & 0 & 0 & \dots & 0 \\ 0 & \eta_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \eta_d \end{bmatrix}.$$

Since c generates the full group C_n, the matrix for ρ is diagonal on all the elements c^j in C_n. Thus, V is a direct sum of one-dimensional subspaces, each of which provides a representation of C_n. Of course, any one-dimensional representation is automatically irreducible.

Let us summarize our observations:

Theorem 2.1 *Let C_n be a cyclic group of order $n \in \{1, 2, ...\}$. Every complex representation of C_n is a direct sum of irreducible representations. Each irreducible complex representation of C_n is one-dimensional, specified by the requirement that a generator element $c \in G$ act through multiplication by an nth root of unity. Each nth root of unity provides, in this way, an irreducible complex representation of C_n, and these representations are mutually inequivalent.*

Thus, there are exactly n inequivalent irreducible complex representations of C_n.

Everything we have done here applies for representations of C_n over a field containing n distinct roots of unity.

Let us now look at what happens when the field does not contain the requisite roots of unity. Consider, for instance, the representations of C_3 over the field $\mathbb{R}$ of real numbers. There are three geometrically apparent representations:

1. The one-dimensional ρ_1 representation that associates the identity operator (multiplication by 1) with every element of C_3;

2. The two-dimensional representation ρ_2^+ on $\mathbb{R}^2$ in which c is associated with rotation by $120°$;

3. The two-dimensional representation ρ_2^- on $\mathbb{R}^2$ in which c is associated with rotation by $-120°$.

These are clearly all irreducible. Moreover, any irreducible representation of C_3 on $\mathbb{R}^2$ is clearly either (2) or (3).

Now consider a general real vector space V on which C_3 has a representation ρ. Choose a basis B in V, and let $V_{\mathbb{C}}$ be the complex vector space with B as a basis (put another way, $V_{\mathbb{C}}$ is $\mathbb{C} \otimes_{\mathbb{R}} V$, viewed as a complex vector space). Then ρ gives, naturally, a representation of C_3 on $V_{\mathbb{C}}$. Then $V_{\mathbb{C}}$ is a direct sum of complex one-dimensional subspaces, each invariant under the action of C_3. Since a complex one-dimensional vector space is a real two-dimensional space, and we have already determined all two-dimensional real representations of C_3, we have finished classifying all real representations of C_3. Too fast, you say? Then proceed to Exercise 2.6.

Finite Abelian groups are products of cyclic groups. This could give the impression that there is nothing very interesting in the representations of such groups. But even a very simple representation can be of great use. For any prime p, the nonzero elements in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ form a group $\mathbb{Z}_p^*$ under multiplication. For any $a \in \mathbb{Z}_p^*$, define

$$\lambda_p(a) = a^{(p-1)/2},$$

this being 1 in the case $p = 2$. Since its square is $a^{p-1} = 1$, $\lambda_p(a)$ is necessarily ± 1. Clearly,

$$\lambda_p : \mathbb{Z}_p^* \to \{1, -1\}$$

is a group homomorphism, and hence gives a one-dimensional representation, which is the same as a one-dimensional character of $\mathbb{Z}_p^*$. The *Legendre symbol* $\left(\frac{a}{p}\right)$ is defined for any integer a by

$$\left(\frac{a}{p}\right) = \begin{cases} \lambda_p(a \bmod p) & \text{if } a \text{ is coprime to } p \\ 0 & \text{if } a \text{ is divisible by } p. \end{cases}$$

The celebrated law of quadratic reciprocity, conjectured by Euler and Legendre and proved first, and many times over, by Gauss, states that

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)/2}(-1)^{(q-1)/2},$$

if p and q are odd primes. For an extension of these ideas using the character theory of general finite groups, see the article by Duke and Hopkins [25].

2.2 Dihedral Groups

The dihedral group D_n, for n any positive integer, is a group of $2n$ elements generated by two elements c and r, where c has order n, r has order 2, and conjugation by r turns c into c^{-1}:

$$c^n = e, \qquad r^2 = e, \qquad rcr^{-1} = c^{-1}. \tag{2.6}$$

Geometrically, think of c as counterclockwise rotation in the plane by the angle $2\pi/n$ and r as reflection across a fixed line through the origin. The distinct elements of D_n are

$$e, c, c^2, \ldots, c^{n-1}, r, cr, c^2r, \ldots, c^{n-1}r.$$

This geometric view of D_n, illustrated in Fig. **2.2**, immediately yields a real two-dimensional representation: let c act on $\mathbb{R}^2$ through counterclockwise rotation by angle $2\pi/n$ and let r act through reflection across the x-axis. Relative to the standard basis of $\mathbb{R}^2$ these two linear maps have the following matrix forms:

$$\rho(c) = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix}, \qquad \rho(r) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It instructive to see what happens when we complexify and take this representation over to $\mathbb{C}^2$. Choose in $\mathbb{C}^2$ the basis given by eigenvectors of $\rho(c)$:

$$b_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad b_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

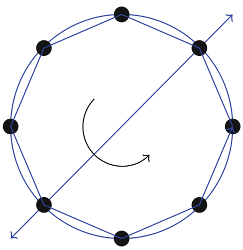

Fig. 2.2 The dihedral group D_4

Then

$$\rho_{\mathbb{C}}(c)b_1 = \eta b_1 \quad \text{and} \quad \rho_{\mathbb{C}}(c)b_2 = \eta^{-1}b_2,$$

where $\eta = \mathrm{e}^{2\pi\mathrm{i}/n}$, and

$$\rho_{\mathbb{C}}(r)b_1 = b_2 \quad \text{and} \quad \rho_{\mathbb{C}}(r)b_2 = b_1.$$

Thus, relative to the basis given by b_1 and b_2, the matrices of $\rho_{\mathbb{C}}(c)$ and $\rho_{\mathbb{C}}(r)$ are

$$\begin{bmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Switching our perspective from the standard basis to that given by b_1 and b_2 produces a two-dimensional complex representation ρ_1 on $\mathbb{C}^2$ given by

$$\rho_1(c) = \begin{bmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{bmatrix}, \qquad \rho_1(r) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.7}$$

Having been obtained by a change of basis, this representation is equivalent to the representation $\rho_{\mathbb{C}}$, which in turn is the complexification of the rotation–reflection real representation ρ of the dihedral group on $\mathbb{R}^2$.

More generally, we have the representation ρ_m specified by requiring

$$\rho_m(c) = \begin{bmatrix} \eta^m & 0 \\ 0 & \eta^{-m} \end{bmatrix}, \qquad \rho_m(r) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for any $m \in \mathbb{Z}$; of course, to avoid repetition, we may focus on $m \in \{1, 2, ..., n-1\}$. The values of ρ_m on all elements of D_n are given by

$$\rho_m(c^j) = \begin{bmatrix} \eta^{mj} & 0 \\ 0 & \eta^{-mj} \end{bmatrix}, \qquad \rho_m(c^j r) = \begin{bmatrix} 0 & \eta^{mj} \\ \eta^{-mj} & 0 \end{bmatrix}.$$

(Having written this, we notice that this representation makes sense over any field $\mathbb{F}$ containing nth roots of unity. However, we stick to the ground field $\mathbb{C}$, or at least $\mathbb{Q}$ with any primitive nth root of unity adjoined.)

Clearly, ρ_m repeats itself when m changes by multiples of n. Thus, we need only focus on $\rho_1, ..., \rho_{n-1}$.

Is ρ_m reducible? Yes if, and only if, there is a nonzero vector $v \in \mathbb{C}^2$ fixed by $\rho_m(r)$ and $\rho_m(c)$. Being fixed by $\rho_m(r)$ means that such a vector must be a multiple of $(1,1)$ in $\mathbb{C}^2$. But $\mathbb{C}(1,1)$ is also invariant under $\rho_m(c)$ if and only if η^m is equal to η^{-m}.

Thus, ρ_m for $m \in \{1, ..., n-1\}$ is irreducible if $n \neq 2m$ and is reducible if $n = 2m$.

Are we counting things too many times? Indeed, the representations ρ_m are not all inequivalent. Interchanging the two axes converts ρ_m into $\rho_{-m} = \rho_{n-m}$. Thus, we can narrow our focus to ρ_m for $1 \leq m < n/2$.

We have now identified $n/2 - 1$ irreducible two-dimensional complex representations if n is even, and $(n-1)/2$ irreducible two-dimensional complex representations if n is odd.

The character χ_m of ρ_m is obtained by taking the trace of ρ_m on the elements of the group D_n:

$$\chi_m(c^j) = \eta^{mj} + \eta^{-mj}, \qquad \chi_m(c^j r) = 0.$$

Now consider a one-dimensional complex representation θ of D_n. First, from $\theta(r)^2 = 1$, we see that $\theta(r) = \pm 1$. If we apply θ to the relation that rcr^{-1} equals c^{-1}, it follows that $\theta(c)$ must also be ± 1. But then, from $c^n = e$, it follows that $\theta(c)$ can be -1 only if n is even. Thus, we have the one-dimensional representations specified by

$$\begin{aligned} \theta_{+,\pm}(c) = 1, \quad \theta_{+,\pm}(r) = \pm 1 \qquad &\text{if } n \text{ is even or odd}, \\ \theta_{-,\pm}(c) = -1, \quad \theta_{-,\pm}(r) = \pm 1 \qquad &\text{if } n \text{ is even}. \end{aligned} \tag{2.8}$$

This gives us four one-dimensional complex representations if n is even, and two if n is odd. (Indeed, the reasoning here works for any ground field.)

Thus, for n is even we have identified a total of $3 + n/2$ irreducible representations, and for n is odd we have identified $(n+3)/2$ irreducible representations.

As noted in the first equation in (2.5), the sum $\sum_{\chi \in \mathcal{R}_G} d_\chi^2$ over all distinct complex irreducible characters of a finite group G is the total number of elements in G. In this case the sum should be $2n$. Working out the sum over all the irreducible characters χ we have determined, we obtain

$$\begin{aligned} \left(\frac{n}{2}-1\right)2^2+4 &= 2n \qquad \text{for even } n; \\ \left(\frac{n-1}{2}\right)2^2+2 &= 2n \qquad \text{for odd } n. \end{aligned} \tag{2.9}$$

Thus, our list of irreducible complex representations contains all irreducible complex representations, up to equivalence.

Our next objective is to work out all complex characters of D_n. Since characters are constant on conjugacy classes, let us first determine the conjugacy classes in D_n.

Since rcr^{-1} is c^{-1}, it follows that

$$r(c^j r)r^{-1} = c^{-j}r = c^{n-j}r.$$

This already indicates that the conjugacy class structure is different for n is even and n is odd. In fact, notice that conjugating $c^j r$ by c results in increasing j by 2:

$$c(c^j r)c^{-1} = c^{j+1}cr = c^{j+2}r.$$

If n is even, the conjugacy classes are:

$$\begin{aligned} \{e\}, \{c, c^{n-1}\}, \{c^2, c^{n-2}\}, ..., \{c^{n/2-1}, c^{n/2+1}\}, \{c^{n/2}\}, \\ \{r, c^2 r, ..., c^{n-2}r\}, \{cr, c^3 r, ..., c^{n-1}r\}. \end{aligned} \tag{2.10}$$

Note that there are $3+n/2$ conjugacy classes, and this exactly matches the number of inequivalent irreducible complex representations obtained earlier.

To see how this plays out in practice, let us look at D_4. Our analysis shows that there are five conjugacy classes:

$$\{e\}, \{c, c^3\}, \{c^2\}, \{r, c^2 r\}, \{cr, c^3 r\}.$$

There are four one-dimensional complex representations $\theta_{\pm,\pm}$, and one irreducible two-dimensional complex representation ρ_1 specified through

$$\rho_1(c) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad \rho_1(r) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Table **2.1** contains the *character table* of D_4, listing the values of the irreducible complex characters of D_4 on the various conjugacy classes. The latter

Table 2.1 Complex irreducible characters of D_4

	1	2	1	2	2
	e	c	c^2	r	cr
$\theta_{+,+}$	1	1	1	1	1
$\theta_{+,-}$	1	1	1	-1	-1
$\theta_{-,+}$	1	-1	1	1	-1
$\theta_{-,-}$	1	-1	1	-1	1
χ_1	2	0	-2	0	0

Table 2.2 Complex irreducible characters of $D_3 = S_3$

	1	2	3
	e	c	r
$\theta_{+,+}$	1	1	1
$\theta_{+,-}$	1	1	-1
χ_1	2	-1	0

are displayed in a row (second from top), each conjugacy class identified by an element it contains; above each conjugacy class we have listed the number of elements it contains. Each row in the main body of the table displays the values of a character on the conjugacy classes.

The case for odd n proceeds similarly. Take, for instance, $n = 3$. The group D_3 is generated by elements c and r subject to the relations

$$c^3 = e, \quad r^2 = e, \qquad rcr^{-1} = c^{-1}.$$

The conjugacy classes are

$$\{e\}, \{c, c^2\}, \{r, cr, c^2r\}$$

The irreducible complex representations are $\theta_{+,+}$, $\theta_{+,-}$, ρ_1. Their values are displayed in Table **2.2**, where the first row displays the number of elements in the conjugacy classes listed (by choice of an element) in the second row. The dimensions of the representations can be read off from the first column in the main body of the table. Observe that the sum of the squares of the dimensions of the representations of S_3 listed in the table is

$$1^2 + 1^2 + 2^2 = 6,$$

which is exactly the number of elements in D_3. This verifies the first property listed earlier in (2.5).

Table 2.3 Conjugacy classes in S_4

Number of elements	1	6	8	6	3
Conjugacy class	ι	(12)	(123)	(1234)	(12)(34)

2.3 The Symmetric Group S_4

The symmetric group S_3 is isomorphic to the dihedral group D_3, and we have already determined the irreducible representations of D_3 over the complex numbers. Let us turn now to the symmetric group S_4, which is the group of permutations of $\{1, 2, 3, 4\}$. Geometrically, this is the group of rotational symmetries of a cube.

Two elements of S_4 are conjugate if and only if they have the same cycle structure; thus, for instance, (134) and (213) are conjugate, and these are not conjugate to (12)(34). The following elements belong to all the distinct conjugacy classes:

$$\iota, \qquad (12), \qquad (123), \qquad (1234), \qquad (12)(34),$$

where ι is the identity permutation. The conjugacy classes, each identified by one element they contain, are listed with the number of elements in each conjugacy class in Table 2.3.

There are two one-dimensional complex representations of S_4 we are familiar with: the trivial one, associating 1 with every element of S_4, and the signature representation ϵ whose value is $+1$ on even permutations and -1 on odd ones.

We also have seen a three-dimensional irreducible complex representation of S_4; recall the representation R of S_4 on $\mathbb{C}^4$ given by permutation of coordinates:

$$(x_1, x_2, x_3, x_4) \mapsto (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(4)})$$

Equivalently,

$$R(\sigma)e_j = e_{\sigma(j)} \qquad \text{for } j \in \{1, 2, 3, 4\},$$

where $e_1, ..., e_4$ are the standard basis vectors of $\mathbb{C}^4$. The three-dimensional subspace

$$E_0 = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$$

Table 2.4 The characters χ_R and χ_0 on conjugacy classes

Conjugacy class	ι	(12)	(123)	(1234)	(12)(34)
χ_R	4	2	1	0	0
χ_0	3	1	0	-1	-1
χ_1	3	-1	0	1	-1

is mapped into itself by the action of R, and the restriction to E_0 gives an irreducible representation R_0 of S_4. In fact,

$$\mathbb{C}^4 = E_0 \oplus \mathbb{C}(1,1,1,1)$$

decomposes the space $\mathbb{C}^4$ into complementary invariant, irreducible subspaces. The subspace $\mathbb{C}(1,1,1,1)$ carries the trivial representation (all elements act through the identity map). Examining the effect of the group elements on the standard basis vectors, we can work out the character of R. For instance, $R((12))$ interchanges e_1 and e_2, and leaves e_3 and e_4 fixed, and so its matrix is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the trace is

$$\chi_R((12)) = 2.$$

Subtracting the trivial character, which is 1 on all elements of S_4, we obtain the character χ_0 of the representation R_0. All this is displayed in the first three rows in Table **2.4**.

We can create another three-dimensional complex representation R_1 by tensoring R_0 with the signature ϵ:

$$R_1 = R_0 \otimes \epsilon.$$

The character χ_1 of R_1 is then written down by taking products, and is displayed in the fourth row in Table **2.4**.

Since R_0 is irreducible and R_1 acts by a simple ± 1 scaling of R_0, it is clear that R_1 is also irreducible. Thus, we now have two one-dimensional complex representations and two three-dimensional complex irreducible representations. The sum of the squares of the dimensions is

$$1^2 + 1^2 + 3^2 + 3^2 = 20.$$

From the first relation in (2.5) we know that the sum of the squares of the dimensions of all the inequivalent irreducible complex representations is $|S_4| = 24$. Thus, looking at the equation

$$24 = 1^2 + 1^2 + 3^2 + 3^2 + ?^2,$$

we see that we are missing a two-dimensional irreducible complex representation R_2. Leaving the entries for this blank, we have Table **2.5**.

Table 2.5 Character table for S_4 with a missing row

	1	6	8	6	3
	ι	(12)	(123)	(1234)	(12)(34)
Trivial	1	1	1	1	1
ϵ	1	-1	1	-1	1
χ_0	3	1	0	-1	-1
χ_1	3	-1	0	1	-1
χ_2	2	?	?	?	?

As an illustration of the power of character theory, let us work out the character χ_2 of this "missing" representation R_2, without even bothering to search for the representation itself. Recall from (2.5) the relation

$$\sum_{\rho} (\dim \rho)\, \chi_\rho(\sigma) = 0, \qquad \text{if } \sigma \neq \iota,$$

Table 2.6 Character table for S_4

	1	6	8	6	3
	ι	(12)	(123)	(1234)	(12)(34)
Trivial	1	1	1	1	1
ϵ	1	-1	1	-1	1
χ_0	3	1	0	-1	-1
χ_1	3	-1	0	1	-1
χ_2	2	0	-1	0	2

where the sum runs over a maximal set of inequivalent irreducible complex representations of S_4 and σ is any element of S_4. This means that *the vector formed by the first column* in the main body of the table (i.e., the column for the conjugacy class $\{\iota\}$) *is orthogonal to the vectors* formed by the columns *for the other conjugacy classes*. Using this we can work out the entries missing from the character table. For instance, taking $\sigma = (12)$, we have

$$2\chi_2((12)) + 3 * \underbrace{(-1)}_{\chi_1((12))} + 3 * 1 + 1 * (-1) + 1 * 1 = 0,$$

which yields

$$\chi_2((12)) = 0.$$

For $\sigma = (123)$, we have

$$2\chi_2((123)) + 3 * \underbrace{0}_{\chi_1((123))} + 3 * 0 + 1 * 1 + 1 * 1 = 0,$$

which produces

$$\chi_2((123)) = -1.$$

Filling in the entire last row of the character table in this way produces the Table **2.6**.

Just to be sure that the indirectly detected character χ_2 is irreducible, let us run the check given in (2.3) for irreducible complex characters: the sum of the quantities $|C||\chi_2(C)|^2$ over all the conjugacy classes C should be 24. Indeed, we have

$$\sum_C |C||\chi_2(C)|^2 = 1*2^2 + 6*0^2 + 8*(-1)^2 + 6*0^2 + 3*2^2 = 24 = |S_4|,$$

a pleasant proof of the power of the theory and tools promised to be developed in the chapters ahead.

2.4 Quaternionic Units

Before moving on to general theory in the next chapter, let us look at another example which produces a little surprise. The unit quaternions

$$1, -1, i, -i, j, -j, k, -k$$

form a group Q under multiplication. We can take

$$-1, i, j, k$$

as generators, with the relations

$$(-1)^2 = 1,\ i^2 = j^2 = k^2 = -1,\ ij = k.$$

The conjugacy classes are

$$\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}.$$

We can spot the one-dimensional representations as follows. Since

$$ijij = k^2 = -1 = i^2 = j^2,$$

the value of any one-dimensional representation τ on -1 must be 1 because

$$\tau(-1) = \tau(ijij) = \tau(i)\tau(j)\tau(i)\tau(j) = \tau(i^2j^2) = \tau(1) = 1, \tag{2.11}$$

and then the values on i and j must each be ± 1. (For another formulation of this argument, see Exercise 4.6.) A little thought shows that

Table 2.7 Character table for Q, missing the last row

	1	2	1	2	2
	1	i	-1	j	k
$\chi_{+1,+1}$	1	1	1	1	1
$\chi_{+1,-1}$	1	1	1	-1	-1
$\chi_{-1,+1}$	1	-1	1	1	-1
$\chi_{-1,-1}$	1	-1	1	-1	1
χ_2	2	?	?	?	?

Table 2.8 Character table for Q

	1	2	1	2	2
	1	i	-1	j	k
$\chi_{+,+}$	1	1	1	1	1
$\chi_{+,-}$	1	1	1	-1	-1
$\chi_{-,+}$	1	-1	1	1	-1
$\chi_{-,-}$	1	-1	1	-1	1
χ_2	2	0	-2	0	0

$\big(\tau(i), \tau(j)\big)$ could be taken to be any of the four possible values $(\pm 1, \pm 1)$ and this would specify a one-dimensional representation τ. Thus, there are four one-dimensional representations. Given that Q contains eight elements, writing this as a sum of squares of dimensions of irreducible complex representations, we have

$$8 = 1^2 + 1^2 + 1^2 + 1^2 + ?^2$$

Clearly, what we are missing is an irreducible complex representation of dimension 2. The incomplete character table is displayed in Table **2.7**.

Remarkably, everything here, with the potential exception of the missing last row, is identical to the information in Table **2.1** for the dihedral group D_4. Then, since the last row is entirely determined by the information available, the entire character table for Q must be identical to that of D_4. Thus the complete character table for Q is as shown in Table **2.8**.

A guess at this stage would be that Q must be isomorphic to D_4, a guess bolstered by the observation that certainly the conjugacy classes look much the same, in terms of the number of elements at least. But this guess is shown to be invalid upon second thought: the dihedral group D_4 has four elements r, cr, c^2r, and c^3r each of order 2, whereas the only element of order 2 in Q is -1. So we have an interesting observation here: *two nonisomorphic groups can have identical character tables*!

2.5 Afterthoughts: Geometric Groups

In closing this chapter, let us note some important classes of finite groups, although we will not explore their representations specifically.

The group Q of special quaternions we studied in Sect. 2.4 is a particular case of a more general setting. Let V be a finite-dimensional real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. There is then the *Clifford algebra* $C_{\mathrm{real},d}$, which is an associative algebra over $\mathbb{R}$, with a unit element 1, whose elements are linear combinations of formal products $v_1 ... v_m$ (with this being 1 if $m = 0$), linear in each $v_i \in V$, with the requirement that

$$vw + wv = -2\langle v, w \rangle 1 \qquad \text{for all } v, w \in V.$$

If e_1, ..., e_d form an orthonormal basis of V, then the products $\pm e_{i_1} \ldots e_{i_k}$, for $k \in \{0, ..., d\}$, form a group Q_d under the multiplication operation of the algebra $C_{\mathrm{real},d}$. When $d = 2$, we write $i = e_1$, $j = e_2$, and $k = e_1 e_2$, and obtain $Q_2 = \{1, -1, i, -i, j, -j, k, -k\}$, the quaternionic group.

In chemistry one studies *crystallographic groups*, which are finite subgroups of the group of Euclidean motions in $\mathbb{R}^3$. *Reflection groups* are groups generated by reflections in Euclidean spaces. Let V be a finite-dimensional real vector space with an inner product $\langle \cdot, \cdot \rangle$. If w is a unit vector in V, then the reflection r_w across the hyperplane

$$w^{\perp} = \{v \in \mathbb{R}^n : \langle v, w \rangle = 0\}$$

takes w to $-w$ and holds all vectors in the "mirror" $w^{\perp}$ fixed; thus,

$$r_w(v) = v - 2\langle v, w \rangle w \qquad \text{for all } v \in V. \tag{2.12}$$

If r_1 and r_2 are reflections across planes $w_1^{\perp}$ and $w_2^{\perp}$, where w_1 and w_2 are unit vectors in V with angle $\theta = \cos^{-1}\langle w_1, w_2 \rangle \in [0, \pi]$ between them, then, geometrically,

$$\begin{aligned} r_1^2 &= r_2^2 = I, \\ r_1 r_2 &= r_2 r_1 \quad \text{if } \langle w_1, w_2 \rangle = 0, \\ r_1 r_2 &= \text{rotation by angle } 2\theta \text{ in the } w_1\text{–}w_2 \text{ plane}. \end{aligned} \tag{2.13}$$

An abstract *Coxeter group* is a group generated by a family of elements r_i of order 2, with the restriction that certain pair products $r_i r_j$ also have finite order. Of course, for such a group to be finite, every pair product $r_i r_j$ needs to have finite order. An important class of finite Coxeter groups is formed by the *Weyl groups* that arise in the study of Lie algebras. Consider a very special type of Weyl group: the group generated by reflections across the hyperplanes $(e_j - e_k)^{\perp}$, where e_1, ..., e_n form the standard basis of $\mathbb{R}^n$, and j and k are distinct elements running over $[n]$. We can recognize this as essentially the symmetric group S_n, realized geometrically through the faithful representation R in (1.3). From this point of view, S_n can be viewed as being generated by elements r_1, ..., r_{n-1}, with r_i standing for the transposition $(i\, i+1)$, satisfying the relations

$$\begin{aligned} r_j^2 = \iota \qquad &\text{for all } j \in [n-1], \\ r_j r_{j+1} r_j = r_{j+1} r_j r_{j+1} \qquad &\text{for all } j \in [n-2], \\ r_j r_k = r_j r_k \qquad &\text{for all } j, k \in [n-1] \text{ with } |j-k| \geq 2, \end{aligned} \tag{2.14}$$

where ι is the identity element. It would seem to be more natural to write the second equation as $(r_j r_{j+1})^3 = \iota$, which would be equivalent provided each r_j^2 is ι. However, just holding on to the second and third equations generates another important class of groups, the *braid groups* B_n, where B_n is generated abstractly by elements $r_1, \ldots, r_{n-1}$ subject to just the second and third conditions in (2.14). Thus, there is a natural surjection $B_n \to S_n$ mapping r_i to $(i\, i+1)$ for each $i \in [n-1]$.

If $\mathbb{F}$ is a subfield of a field $\mathbb{F}_1$, such that $\dim_{\mathbb{F}} \mathbb{F}_1 < \infty$, then the set of all automorphisms σ of the field $\mathbb{F}_1$ for which $\sigma(c) = c$ for all $c \in \mathbb{F}$ is a finite group under composition. This is the *Galois group* of $\mathbb{F}_1$ over $\mathbb{F}$; the classical case is where $\mathbb{F}_1$ is defined by adjoining to $\mathbb{F}$ roots of polynomial equations over $\mathbb{F}$. Morally related to these ideas are fundamental groups of surfaces; an instance of this, the fundamental group of a compact oriented surface of genus g, is the group with $2g$ generators $a_1, b_1, \ldots, a_g, b_g$ satisfying the constraint

$$a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = e. \tag{2.15}$$

Such equations, with a_i and b_j represented in more concrete groups, have come up in two- and three-dimensional gauge theories. Far earlier, in his first major work developing character theory, Frobenius [29] studied the number of solutions of equations of this and related types, with each a_i and b_j represented in some finite group. In Sect. 7.9 we will study the Frobenius formula for counting the number of solutions of the equation

$$s_1 \ldots s_m = e$$

for $s_1, \ldots, s_m$ running over specified conjugacy classes in a finite group G. In the case $G = S_n$, restricting the s_i to run over transpositions, a result of Hurwitz relates this number to counting n-sheeted Riemann surfaces with m branch points (see Curtis [14] for related history).

Exercises

2.1. Work out the character table for D_5.

2.2. Consider the subgroup of S_4 given by

$$V_4 = \{\iota, (12)(34), (13)(24), (14)(23)\}.$$

Being a union of conjugacy classes, V_4 is a normal subgroup of S_4. Now view S_3 as the subgroup of S_4 consisting of the permutations that fix 4. Thus, $V_4 \cap S_3 = \{\iota\}$. Show that the mapping

$$S_3 \to S_4/V_4 : \sigma \mapsto \sigma V_4$$

is an isomorphism. Obtain an explicit form of a two-dimensional irreducible complex representation of S_4 for which the character is χ_2 as given in Table **2.6**.

2.3. In S_3 there is the cyclic group C_3 generated by (123), which is a normal subgroup. The quotient $S_3/C_3 \simeq S_2$ is a two-element group. Work out the one-dimensional representation of S_3 that arises from this by the method in Exercise 2.2.

2.4. Construct a two-dimensional irreducible representation of S_3, over any field $\mathbb{F}$ in which $3 \neq 0$, using matrices that have integer entries.

2.5. The alternating group A_4 consists of all even permutations in S_4. It is generated by the elements

$$c = (123), \quad x = (12)(34), \quad y = (13)(24), \quad z = (14)(23)$$

satisfying the relations

$$cxc^{-1} = z, \quad cyc^{-1} = x, \quad czc^{-1} = y, \quad c^3 = \iota, \quad xy = yx = z.$$

Table 2.9 Character table for A_4

	1	3	4	4
	ι	(12)(34)	(123)	(132)
ψ_0	1	1	1	1
ψ_1	1	1	ω	ω^2
ψ_2	1	1	ω^2	ω
χ_1	?	?	?	?

(a) Show that the conjugacy classes are

$$\{\iota\},\ \{x, y, z\},\ \{c, cx, cy, cz\},\ \{c^2, c^2x, c^2y, c^2z\}.$$

Note that c and c^2 are in different conjugacy classes in A_4, even though in S_4 they are conjugate.

(b) Show that the group A_4 generated by all commutators $aba^{-1}b^{-1}$ is $V_4 = \{\iota, x, y, z\}$, which is just the set of commutators in A_4.

(c) Check that there is an isomorphism given by

$$C_3 \mapsto A_4/V_4 : c \mapsto cV_4.$$

(d) Obtain three one-dimensional representations of A_4.

(e) The group $A_4 \subset S_4$ acts by permutation of coordinates on $\mathbb{C}^4$ and preserves the three-dimensional subspace $E_0 = \{(x_1, ..., x_4) : x_1 + \cdots + x_4 = 0\}$. Work out the character χ_3 of this representation of A_4.

(f) Work out the full character table for A_4, by filling in the last row in Table **2.9**.

2.6. Let V be a real vector space and $T : V \to V$ be a linear mapping with $T^m = I$ for some positive integer m. Choose a basis B of V and let $V_{\mathbb{C}}$

be the complex vector space with basis B. Define the *conjugation* map $C : V_{\mathbb{C}} \to V_{\mathbb{C}} : v \mapsto \overline{v}$ by

$$C\left(\sum_{b \in B} v_b b\right) = \sum_{b \in B} \overline{v_b} b,$$

where each $v_b \in \mathbb{C}$, and on the right we just have the ordinary complex conjugates $\overline{v}_b$. Show that

$$x = \tfrac{1}{2}(v + Cv) \text{ and } y = -\tfrac{i}{2}(v - Cv)$$

are in V for every $v \in V_{\mathbb{C}}$. If $v \in V_{\mathbb{C}}$ is an eigenvector of T, show that T maps the subspace $\mathbb{R}x + \mathbb{R}y$ of V spanned by x and y into itself.

2.7. Work out an irreducible representation of the group

$$Q = \{1, -1, i, -i, j, -j, k, -1\}$$

of unit quaternions on $\mathbb{C}^2$, by associating suitable 2×2 matrices with the elements of Q.

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