Chapter 2
Nonlinear Discrete Dynamical Systems

In this chapter, the basic concepts of nonlinear discrete systems will be presented. The Local and global theory of stability and bifurcation for nonlinear discrete systems will be discussed. The stability switching and bifurcation on specific eigenvectors of the linearized system at fixed points under specific period will be presented. The higher singularity and stability for nonlinear discrete systems on the specific eigenvectors will be developed. A few special cases in the lower dimensional maps will be presented for a better understanding of the generalized theory. The route to chaos will be discussed briefly, and the intermittency phenomena relative to specific bifurcations will be presented. The normalization group theory for 2-D discrete systems will be presented via Duffing discrete systems.

2.1 Discrete Dynamical Systems

Definition 2.1 For $\Omega_\alpha \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^m$ with $\alpha \in \mathbb{Z}$, consider a vector function $\mathbf{f}_\alpha : \Omega_\alpha \times \Lambda \to \Omega_\alpha$ which is C^r $(r \geq 1)$-continuous, and there is a discrete (or difference) equation in the form of

$$\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha) \text{ for } \mathbf{x}_k, \mathbf{x}_{k+1} \in \Omega_\alpha, k \in \mathbb{Z} \text{ and } \mathbf{p}_\alpha \in \Lambda. \tag{2.1}$$

With an initial condition of $\mathbf{x}_k = \mathbf{x}_0$, the solution of Eq. (2.1) is given by

$$\begin{gathered}\mathbf{x}_k = \underbrace{\mathbf{f}_\alpha(\mathbf{f}_\alpha(\cdots(\mathbf{f}_\alpha}_{k}(\mathbf{x}_0, \mathbf{p}_\alpha)))) \\ \text{for } \mathbf{x}_k \in \Omega_\alpha, k \in \mathbb{Z} \text{ and } \mathbf{p} \in \Lambda.\end{gathered} \tag{2.2}$$

(i) The difference equation with the initial condition is called a *discrete dynamical system*.
(ii) The vector function $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ is called a *discrete vector field* on domain Ω_α.
(iii) The solution $\mathbf{x}_k$ for each $k \in \mathbb{Z}$ is called a *flow* of discrete dynamical system.

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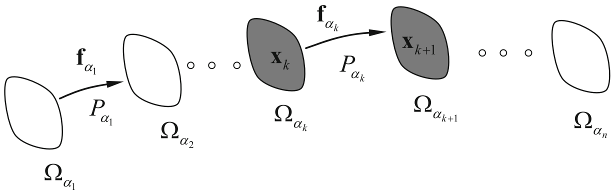

Fig. 2.1 Maps and vector functions on each sub-domain for discrete dynamical system

(iv) The solution $\mathbf{x}_k$ for all $k \in \mathbb{Z}$ on domain Ω_α is called the trajectory, phase curve or orbit of discrete dynamical system, which is defined as

$$\Gamma = \left\{\mathbf{x}_k | \mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha) \text{ for } k \in \mathbb{Z} \text{ and } \mathbf{p}_\alpha \in \Lambda\right\} \subseteq \cup_\alpha \Omega_\alpha. \tag{2.3}$$

(v) The discrete dynamical system is called *a uniform discrete system* if

$$\mathbf{x}_{k+1} = \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha) = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \text{ for } k \in \mathbb{Z} \text{ and } \mathbf{x}_k \in \Omega_\alpha \tag{2.4}$$

Otherwise, this discrete dynamical system is called a *non-uniform discrete system*.

Definition 2.2 For the discrete dynamical system in Eq. (2.1), the relation between state $\mathbf{x}_k$ and state $\mathbf{x}_{k+1}$ $(k \in \mathbb{Z})$ is called a discrete map if

$$P_\alpha : \mathbf{x}_k \xrightarrow{\mathbf{f}_\alpha} \mathbf{x}_{k+1} \text{ and } \mathbf{x}_{k+1} = P_\alpha \mathbf{x}_k \tag{2.5}$$

with the following properties:

$$P_{(k,n)} : \mathbf{x}_k \xrightarrow{\mathbf{f}_{\alpha_1}, \mathbf{f}_{\alpha_2}, \cdots, \mathbf{f}_{\alpha_n}} \mathbf{x}_{k+n} \text{ and } \mathbf{x}_{k+n} = P_{\alpha_n} \circ P_{\alpha_{n-1}} \circ \cdots \circ P_{\alpha_1} \mathbf{x}_k \tag{2.6}$$

where

$$P_{(k;n)} = P_{\alpha_n} \circ P_{\alpha_{n-1}} \circ \cdots \circ P_{\alpha_1}. \tag{2.7}$$

If $P_{\alpha_n} = P_{\alpha_{n-1}} = \cdots = P_{\alpha_1} = P_\alpha$, then

$$P_{(\alpha;n)} \equiv P_\alpha^{(n)} = P_\alpha \circ P_\alpha \circ \cdots \circ P_\alpha \tag{2.8}$$

with

$$P_\alpha^{(n)} = P_\alpha \circ P_\alpha^{(n-1)} \text{ and } P_\alpha^{(0)} = \mathbf{I}. \tag{2.9}$$

The total map with n-different submaps is shown in Fig. 2.1. The map P_{α_k} with the relation function $\mathbf{f}_{\alpha_k}$ $(\alpha_k \in \mathbb{Z})$ is given by Eq. (2.5). The total map $P_{(k,n)}$ is given in Eq. (2.7). The domains $\Omega_{\alpha_k}(\alpha_k \in \mathbb{Z})$ can fully overlap each other or can be completely separated without any intersection.

Definition 2.3 For a vector function in $\mathbf{f}_\alpha \in \mathscr{R}^n, \mathbf{f}_\alpha : \mathscr{R}^n \to \mathscr{R}^n$. The operator norm of $\mathbf{f}_\alpha$ is defined by

$$||\mathbf{f}_\alpha|| = \sum\nolimits_{i=1}^{n} |f_{\alpha(i)}(\mathbf{x}_k, \mathbf{p}_\alpha)|. \tag{2.10}$$

For an $n \times n$ matrix $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha) = \mathbf{A}_\alpha \mathbf{x}_k$ and $\mathbf{A}_\alpha = (a_{ij})_{n\times n}$, the corresponding norm is defined by

$$||\mathbf{A}_\alpha|| = \sum_{i,j=1}^{n} |a_{ij}|. \tag{2.11}$$

Definition 2.4 For $\Omega_\alpha \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^m$ with $\alpha \in \mathbb{Z}$, the vector function $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ with $\mathbf{f}_\alpha : \Omega_\alpha \times \Lambda \to \mathscr{R}^n$ is differentiable at $\mathbf{x}_k \in \Omega_\alpha$ if

$$\left.\frac{\partial \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)}{\partial \mathbf{x}_k}\right|_{(\mathbf{x}_k, \mathbf{p})} = \lim_{\Delta \mathbf{x}_k \to \mathbf{0}} \frac{\mathbf{f}_\alpha(\mathbf{x}_k + \Delta \mathbf{x}_k, \mathbf{p}_\alpha) - \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)}{\Delta \mathbf{x}_k}. \tag{2.12}$$

$\partial \mathbf{f}_\alpha / \partial \mathbf{x}_k$ is called the spatial derivative of $\mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)$ at $\mathbf{x}_k$, and the derivative is given by the Jacobian matrix

$$\frac{\partial \mathbf{f}_\alpha(\mathbf{x}_k, \mathbf{p}_\alpha)}{\partial \mathbf{x}_k} = \left[\frac{\partial f_{\alpha(i)}}{\partial x_{k(j)}}\right]_{n\times n}. \tag{2.13}$$

Definition 2.5 For $\Omega_\alpha \subseteq \mathscr{R}^n$ and $\Lambda \subseteq \mathscr{R}^m$, consider a vector function $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ with $\mathbf{f} : \Omega_\alpha \times \Lambda \to \mathscr{R}^n$, where $\mathbf{x}_k \in \Omega_\alpha$ and $\mathbf{p} \in \Lambda$ with $k \in \mathbb{Z}$. The vector function $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ satisfies the Lipschitz condition

$$||\mathbf{f}(\mathbf{y}_k, \mathbf{p}) - \mathbf{f}(\mathbf{x}_k, \mathbf{p})|| \le L||\mathbf{y}_k - \mathbf{x}_k|| \tag{2.14}$$

with $\mathbf{x}_k, \mathbf{y}_k \in \Omega_\alpha$ and L a constant. The constant L is called the Lipschitz constant.

2.2 Fixed Points and Stability

Definition 2.6 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4).

(i) A point $\mathbf{x}_k^* \in \Omega_\alpha$ is called a fixed point or a period-1 solution of a discrete nonlinear system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ under map P_k if for $\mathbf{x}_{k+1} = \mathbf{x}_k = \mathbf{x}_k^*$

$$\mathbf{x}_k^* = \mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) \tag{2.15}$$

The linearized system of the nonlinear discrete system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) at the fixed point $\mathbf{x}_k^*$ is given by

$$\mathbf{y}_{k+1} = DP(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k \tag{2.16}$$

where

$$\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^* \text{ and } \mathbf{y}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_{k+1}^*. \tag{2.17}$$

(ii) A set of points $\mathbf{x}_j^* \in \Omega_{\alpha_j}(\alpha_j \in \mathbb{Z})$ is called the fixed point set or period-1 point set of the total map $P_{(k;n)}$ with n-different submaps in nonlinear discrete system of Eq. (2.2) if

$$\begin{aligned} &\mathbf{x}_{k+j+1}^* = \mathbf{f}_{\alpha_{j'}}(\mathbf{x}_{k+j}^*, \mathbf{p}_{\alpha_{j'}}) \text{ for } j \in \mathbb{Z}_+ \text{ and } j' = \operatorname{mod}(j, n) + 1; \\ &\mathbf{x}_{k+\operatorname{mod}(j,n)}^* = \mathbf{x}_k^*. \end{aligned} \tag{2.18}$$

The linearized equation of the total map $P_{(k;n)}$ gives

$$\begin{aligned} &\mathbf{y}_{k+j+1} = DP_{\alpha_{j'}}(\mathbf{x}_{k+j}^*, \mathbf{p}_{\alpha_{j'}})\mathbf{y}_{k+j} = D\mathbf{f}_{\alpha_{j'}}(\mathbf{x}_{k+j}^*, \mathbf{p}_{\alpha_{j'}})\mathbf{y}_{k+j} \\ &\text{with } \mathbf{y}_{k+j+1} = \mathbf{x}_{k+j+1} - \mathbf{x}_{k+j+1}^* \text{ and } \mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_{k+j}^* \\ &\text{for } j \in \mathbb{Z}_+ \text{ and } j' = \operatorname{mod}(j, n) + 1. \end{aligned} \tag{2.19}$$

The resultant equation for the total map is

$$\mathbf{y}_{k+j+1} = DP_{(k,n)}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j} \text{ for } j \in \mathbb{Z}_+ \tag{2.20}$$

where

$$\begin{aligned} DP_{(k,n)}(\mathbf{x}_k^*, \mathbf{p}) &= \prod\nolimits_{j=n}^{1} DP_{\alpha_j}(\mathbf{x}_{k+j-1}^*, \mathbf{p}) \\ &= DP_{\alpha_n}(\mathbf{x}_{k+n-1}^*, \mathbf{p}_{\alpha_n}) \cdot \cdots \cdot DP_{\alpha_2}(\mathbf{x}_{k+1}^*, \mathbf{p}_{\alpha_2}) \cdot DP_{\alpha_1}(\mathbf{x}_k^*, \mathbf{p}_{\alpha_1}) \\ &= D\mathbf{f}_{\alpha_n}(\mathbf{x}_{k+n-1}^*, \mathbf{p}_{\alpha_n}) \cdot \cdots \cdot D\mathbf{f}_{\alpha_2}(\mathbf{x}_{k+1}^*, \mathbf{p}_{\alpha_2}) \cdot D\mathbf{f}_{\alpha_1}(\mathbf{x}_k^*, \mathbf{p}_{\alpha_1}). \end{aligned} \tag{2.21}$$

The fixed point $\mathbf{x}_k^*$ lies in the intersected set of two domains Ω_k and Ω_{k+1}, as shown in Fig. 2.2. In the vicinity of the fixed point $\mathbf{x}_k^*$, the incremental relations in the two domains Ω_k and Ω_{k+1} are different. In other words, setting $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$ and $\mathbf{y}_{k+1} = \mathbf{x}_{k+1} - \mathbf{x}_{k+1}^*$, the corresponding linearization is generated as in Eq. (2.16). Similarly, the fixed point of the total map with n-different submaps requires the intersection set of two domains Ω_k and Ω_{k+n}, which are a set of equations to obtain the fixed points from Eq. (2.18). The other values of fixed points lie in different domains, i.e., $\mathbf{x}_j^* \in \Omega_j$ $(j = k+1, k+2, \cdots, k+n-1)$, as shown in Fig. 2.3.

The corresponding linearized equations are given in Eq. (2.19). From Eq. (2.20), the local characteristics of the total map can be discussed as a single map. Thus, the dynamical characteristics for the fixed point of the single map will be discussed comprehensively, and the fixed points for the resultant map are applicable. The results can be extended to any period-m flows with $P^{(m)}$.

Definition 2.7 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The linearized system of the discrete nonlinear

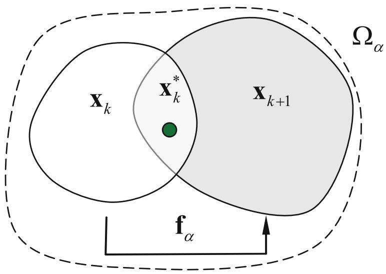

Fig. 2.2 A fixed point between domains Ω_k and Ω_{k+1} for a discrete dynamical system

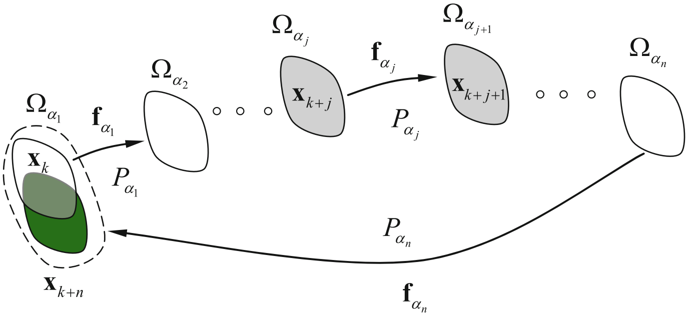

Fig. 2.3 Fixed points with n-maps for a discrete dynamical system

system in the neighborhood of $\mathbf{x}_k^*$ is $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_l = \mathbf{x}_l - \mathbf{x}_k^*$ and $l = k, k+1$) in Eq. (2.16). The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n_1 real eigenvalues $|\lambda_j| < 1$ $(j \in N_1)$, n_2 real eigenvalues $|\lambda_j| > 1$ $(j \in N_2)$, n_3 real eigenvalues $\lambda_j = 1$ $(j \in N_3)$, and n_4 real eigenvalues $\lambda_j = -1$ $(j \in N_4)$. $N = \{1, 2, \cdots, n\}$ and $N_i = \{l_1, l_2, \cdots, l_{n_i}\} \cup \varnothing$ $(i = 1, 2, 4)$ with $l_m \in N$ $(m = 1, 2, \cdots, n_i)$. $N_i \subseteq N \cup \varnothing$, $\cup_{i=1}^3 N_i = N$, $\cap_{i=1}^3 N_i = \varnothing$ and $\Sigma_{i=1}^3 n_i = n$. The corresponding eigenvectors for contraction, expansion, invariance and flip oscillation are $\{\mathbf{v}_{j_i}\}$ $(j_i \in N_i)$ $(i = 1, 2, 3, 4)$, respectively. The stable, unstable, invariant and flip subspaces of $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ in Eq. (2.16) are linear subspace spanned by $\{\mathbf{v}_{j_i}\}$ $(j_i \in N_i)$ $(i = 1, 2, 3, 4)$, respectively, i.e.,

$$
\begin{aligned}
\mathscr{E}^{\mathrm{s}} &= span\left\{\mathbf{v}_j \,\middle|\, \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ |\lambda_j| < 1, j \in N_1 \subseteq N \cup \varnothing \end{array}\right\}; \\
\mathscr{E}^{\mathrm{u}} &= span\left\{\mathbf{v}_j \,\middle|\, \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ |\lambda_j| > 1, j \in N_2 \subseteq N \cup \varnothing \end{array}\right\}; \\
\mathscr{E}^{\mathrm{i}} &= span\left\{\mathbf{v}_j \,\middle|\, \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j = 1, j \in N_3 \subseteq N \cup \varnothing \end{array}\right\}; \\
\mathscr{E}^{\mathrm{f}} &= span\left\{\mathbf{v}_j \,\middle|\, \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j = -1, j \in N_4 \subseteq N \cup \varnothing \end{array}\right\}.
\end{aligned}
\tag{2.22}
$$

where

$$\mathscr{E}^{\mathrm{s}} = \mathscr{E}^{\mathrm{s}}_{\mathrm{m}} \cup \mathscr{E}^{\mathrm{s}}_{\mathrm{o}} \text{ with}$$

$$\mathscr{E}^{\mathrm{s}}_{\mathrm{m}} = span\left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ 0 < \lambda_j < 1, j \in N_1^{\mathrm{m}} \subseteq N \cup \varnothing \end{array} \right. \right\}; \tag{2.23}$$

$$\mathscr{E}^{\mathrm{s}}_{\mathrm{o}} = span\left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ -1 < \lambda_j < 0, j \in N_1^{\mathrm{o}} \subseteq N \cup \varnothing \end{array} \right. \right\};$$

$$\mathscr{E}^{\mathrm{u}} = \mathscr{E}^{\mathrm{u}}_{\mathrm{m}} \cup \mathscr{E}^{\mathrm{u}}_{\mathrm{o}} \text{ with}$$

$$\mathscr{E}^{\mathrm{u}}_{\mathrm{m}} = span\left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ \lambda_j > 1, j \in N_2^{\mathrm{m}} \subseteq N \cup \varnothing \end{array} \right. \right\}; \tag{2.24}$$

$$\mathscr{E}^{\mathrm{s}}_{\mathrm{o}} = span\left\{ \mathbf{v}_j \left| \begin{array}{l} (D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - \lambda_j \mathbf{I})\mathbf{v}_j = \mathbf{0}, \\ -1 < \lambda_j, j \in N_2^{\mathrm{o}} \subseteq N \cup \varnothing \end{array} \right. \right\};$$

where subscripts "m" and "o" represent the monotonic and oscillatory evolutions.

Definition 2.8 Consider a $2n$-dimensional, discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The linearized system of the discrete nonlinear system in the neighborhood of $\mathbf{x}_k^*$ is $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_l = \mathbf{x}_l - \mathbf{x}_k^*$ and $l = k, k+1$) in Eq. (2.16). The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ has complex eigenvalues $\alpha_j \pm \mathbf{i}\beta_j$ with eigenvectors $\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j$ ($j \in \{1, 2, \cdots, n\}$) and the base of vector is

$$\mathbf{B} = \left\{ \mathbf{u}_1, \mathbf{v}_1, \cdots, \mathbf{u}_j, \mathbf{v}_j, \mathbf{u}_{j+1}, \cdots, \mathbf{u}_n, \mathbf{v}_n \right\}. \tag{2.25}$$

The stable, unstable and center subspaces of $\mathbf{y}_{k+1} = D\mathbf{f}_k(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ in Eq. (2.16) are linear subspaces spanned by $\{\mathbf{u}_{j_i}, \mathbf{v}_{j_i}\}$ ($j_i \in N_i$, $i = 1, 2, 3$), respectively. $N = \{1, 2, \cdots, n\}$ plus $N_i = \{l_1, l_2, \cdots, l_{n_i}\} \cup \varnothing \subseteq N \cup \varnothing$ with $l_m \in N$ ($m = 1, 2, \cdots, n_i$). $\cup_{i=1}^3 N_i = N$ with $\cap_{i=1}^3 N_i = \varnothing$ and $\Sigma_{i=1}^3 n_i = n$. The stable, unstable and center subspaces of $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ in Eq. (2.16) are defined by

$$\mathscr{E}^{\mathrm{s}} = span\left\{ (\mathbf{u}_j, \mathbf{v}_j) \left| \begin{array}{l} r_j = \sqrt{\alpha_j^2 + \beta_j^2} < 1, \\ \left(D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I}\right)(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_1 \subseteq \{1, 2, \cdots, n\} \cup \varnothing \end{array} \right. \right\};$$

$$\mathscr{E}^{\mathrm{u}} = span\left\{ (\mathbf{u}_j, \mathbf{v}_j) \left| \begin{array}{l} r_j = \sqrt{\alpha_j^2 + \beta_j^2} > 1, \\ \left(D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I}\right)(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_2 \subseteq \{1, 2, \cdots, n\} \cup \varnothing \end{array} \right. \right\};$$

$$\mathscr{E}^{c} = span\left\{ (\mathbf{u}_j, \mathbf{v}_j) \left| \begin{array}{l} r_j = \sqrt{\alpha_j^2 + \beta_j^2} = 1, \\ \left(D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) - (\alpha_j \pm \mathbf{i}\beta_j)\mathbf{I}\right)(\mathbf{u}_j \pm \mathbf{i}\mathbf{v}_j) = \mathbf{0}, \\ j \in N_3 \subseteq \{1, 2, \cdots, n\} \cup \varnothing \end{array} \right. \right\}. \tag{2.26}$$

Definition 2.9 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The linearized system of the discrete nonlinear system in the neighborhood of $\mathbf{x}_k^*$ is $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_l = \mathbf{x}_l - \mathbf{x}_k^*$ and $l = k, k+1$) in Eq. (2.16). The fixed point or period-1point is *hyperbolic* if no eigenvalues of $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ are on the unit circle (i.e., $|\lambda_j| \neq 1$ for $j = 1, 2, \cdots, n$).

Theorem 2.1 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *in Eq.* (2.4) *with a fixed point* $\mathbf{x}_k^*$*. The linearized system of the discrete nonlinear system in the neighborhood of* $\mathbf{x}_k^*$ *is* $\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k$ ($\mathbf{y}_l = \mathbf{x}_l - \mathbf{x}_k^*$ *and* $l = k, k+1$) *in Eq.* (2.16)*. The eigenspace of* $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ (*i.e.,* $\mathscr{E} \subseteq \mathscr{R}^n$) *in the linearized dynamical system is expressed by direct sum of three subspaces*

$$\mathscr{E} = \mathscr{E}^s \oplus \mathscr{E}^u \oplus \mathscr{E}^c \tag{2.27}$$

where $\mathscr{E}^s$, $\mathscr{E}^u$ and $\mathscr{E}^c$ *are the stable, unstable and center subspaces, respectively.*

Proof This proof is the same as the linear system in Appendix B. ■

Definition 2.10 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. Suppose there is a neighborhood of the equilibrium $\mathbf{x}_k^*$ as $U_k(\mathbf{x}_k^*) \subset \Omega_k$, and in the neighborhood,

$$\lim_{||\mathbf{y}_k|| \to 0} \frac{||\mathbf{f}(\mathbf{x}_k^* + \mathbf{y}_k, \mathbf{p}) - D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k||}{||\mathbf{y}_k||} = 0 \tag{2.28}$$

and

$$\mathbf{y}_{k+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_k. \tag{2.29}$$

(i) A C^r invariant manifold

$$\begin{aligned} \mathscr{S}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*) = \{\mathbf{x}_k \in U(\mathbf{x}_k^*) | \lim_{j \to +\infty} \mathbf{x}_{k+j} = \mathbf{x}_k^* \text{ and } \mathbf{x}_{k+j} \\ \in U(\mathbf{x}_k^*) \text{ with } j \in \mathbb{Z}_+\} \end{aligned} \tag{2.30}$$

is called the local stable manifold of $\mathbf{x}_k^*$, and the corresponding global stable manifold is defined as

$$\begin{aligned} \mathscr{S}(\mathbf{x}_k, \mathbf{x}_k^*) &= \cup_{j \in \mathbb{Z}_-} \mathbf{f}(\mathscr{U}_{loc}(\mathbf{x}_{k+j}, \mathbf{x}_{k+j}^*)) \\ &= \cup_{j \in \mathbb{Z}_-} \mathbf{f}^{(j)}(\mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)). \end{aligned} \tag{2.31}$$

(ii) A C^r invariant manifold $\mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$

$$\begin{aligned}\mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*) = \{\mathbf{x}_k \in U(\mathbf{x}_k^*) | \lim_{j\to-\infty} \mathbf{x}_{k+j} = \mathbf{x}_k^* \text{ and } \mathbf{x}_{k+j} \\ \in U(\mathbf{x}_k^*) \text{ with } j \in \mathbb{Z}_-\}\end{aligned} \tag{2.32}$$

is called the unstable manifold of $\mathbf{x}^*$, and the corresponding global stable manifold is defined as

$$\begin{aligned}\mathscr{U}(\mathbf{x}_k, \mathbf{x}_k^*) &= \cup_{j\in\mathbb{Z}_+} \mathbf{f}(\mathscr{U}_{loc}(\mathbf{x}_{k+j}, \mathbf{x}_{k+j}^*)) \\ &= \cup_{j\in\mathbb{Z}_+} \mathbf{f}^{(j)}(\mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)).\end{aligned} \tag{2.33}$$

(iii) A C^{r-1} invariant manifold $\mathscr{C}_{loc}(\mathbf{x}, \mathbf{x}^*)$ is called the center manifold of $\mathbf{x}^*$ if $\mathscr{C}_{loc}(\mathbf{x}, \mathbf{x}^*)$ possesses the same dimension of $\mathscr{E}^c$ for $\mathbf{x}^* \in \mathscr{S}(\mathbf{x}, \mathbf{x}^*)$, and the tangential space of $C_{loc}(\mathbf{x}, \mathbf{x}^*)$ is identical to $\mathscr{E}^c$.

As in continuous dynamical systems, the stable and unstable manifolds are unique, but the center manifold is not unique. If the nonlinear vector field $\mathbf{f}$ is C^∞-continuous, then a C^r center manifold can be found for any $r < \infty$.

Theorem 2.2 *Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a hyperbolic fixed point $\mathbf{x}_k^*$. The corresponding solution is $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the hyperbolic fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r ($r \geq 1$)-continuous in $U_k(\mathbf{x}_k^*)$. The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})y_{k+j}$ ($y_l = x_l - x_k^*$* and *$l = k+j, k+j+1$) *in $U_k(\mathbf{x}_k^*)$. If the homeomorphism between the local invariant subspace $E(\mathbf{x}_k^*) \subset U(\mathbf{x}_k^*)$ and the eigenspace $\mathscr{E}$ of the linearized system exists with the condition in Eq. (2.28), the local invariant subspace is decomposed by*

$$E(\mathbf{x}_k, \mathbf{x}_k^*) = \mathscr{S}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*) \oplus \mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*). \tag{2.34}$$

(a) *The local stable invariant manifold $\mathscr{S}_{loc}(\mathbf{x}, \mathbf{x}^*)$ possesses the following properties:*

(i) *for $\mathbf{x}_k^* \in \mathscr{S}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$, $\mathscr{S}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$ possesses the same dimension of $\mathscr{E}^s$ and the tangential space of $\mathscr{S}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$ is identical to $\mathscr{E}^s$;*
(ii) *for $\mathbf{x}_k \in \mathscr{S}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*$, $\mathbf{x}_{k+j} \in \mathscr{S}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$ and $\lim_{j\to\infty} \mathbf{x}_{k+j} = \mathbf{x}_k^*$ for all $j \in \mathbb{Z}_+$;*
(iii) *for $\mathbf{x}_k \notin \mathscr{S}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*$, $||\mathbf{x}_{k+j} - \mathbf{x}_k^*|| \geq \delta$ for $\delta > 0$ with $j, j_1 \in \mathbb{Z}_+$ and $j \geq j_1 \geq 0$.*

(b) *The local unstable invariant manifold $\mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$ possesses the following properties:*

(i) *for $\mathbf{x}_k^* \in \mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$, $\mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$ possesses the same dimension of $\mathscr{E}^u$ and the tangential space of $\mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$ is identical to $\mathscr{E}^u$;*
(ii) *for $\mathbf{x}_k \in \mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$, $\mathbf{x}_{k+j} \in \mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$ and $\lim_{j\to-\infty} \mathbf{x}_{k+j} = \mathbf{x}^*$ for all $j \in \mathbb{Z}_-$*

(iii) *for* $\mathbf{x}_k \notin \mathscr{U}_{loc}(\mathbf{x}, \mathbf{x}^*)$, $||\mathbf{x}_{k+j} - \mathbf{x}_k^*|| \geq \delta$ *for* $\delta > 0$ *with* $j_1, j \in \mathbb{Z}_-$ *and* $j \leq j_1 \leq 0$.

Proof See Hirtecki (1971). ∎

Theorem 2.3 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *in Eq.* (2.4) *with a fixed point* $\mathbf{x}_k^*$. *The corresponding solution is* $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ *with* $j \in \mathbb{Z}$. *Suppose there is a neighborhood of the fixed point* $\mathbf{x}_k^*$ *(i.e.,* $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$*), and* $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *is* C^r $(r \geq 1)$*−continuous in* $U_k(\mathbf{x}_k^*)$. *The linearized system is* $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. *If the homeomorphism between the local invariant subspace* $E(\mathbf{x}_k^*) \subset U(\mathbf{x}_k^*)$ *and the eigenspace* $\mathscr{E}$ *of the linearized system exists with the condition in Eq.* (2.28), *in addition to the local stable and unstable invariant manifolds, there is a* C^{r-1} *center manifold* $\mathscr{C}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$. *The center manifold possesses the same dimension of* $\mathscr{E}^c$ *for* $\mathbf{x}^* \in \mathscr{C}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*)$, *and the tangential space of* $\mathscr{C}_{loc}(\mathbf{x}, \mathbf{x}^*)$ *is identical to* $\mathscr{E}^c$. *Thus, the local invariant subspace is decomposed by*

$$E(\mathbf{x}_k, \mathbf{x}_k^*) = \mathscr{S}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*) \oplus \mathscr{U}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*) \oplus \mathscr{C}_{loc}(\mathbf{x}_k, \mathbf{x}_k^*). \tag{2.35}$$

Proof See Guckenhiemer and Holmes (1990). ∎

Definition 2.11 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) on domain $\Omega_\alpha \subseteq \mathscr{R}^n$. Suppose there is a metric space (Ω_α, ρ), then the map P under the vector function $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is called a contraction map if

$$\rho(\mathbf{x}_{k+1}^{(1)}, \mathbf{x}_{k+1}^{(2)}) = \rho(\mathbf{f}(\mathbf{x}_k^{(1)}, \mathbf{p}), \mathbf{f}(\mathbf{x}_k^{(2)}, \mathbf{p})) \leq \lambda\rho(\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)}) \tag{2.36}$$

for $\lambda \in (0, 1)$ and $\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)} \in \Omega_\alpha$ with $\rho(\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)}) = ||\mathbf{x}_k^{(1)} - \mathbf{x}_k^{(2)}||$.

Theorem 2.4 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *in Eq.* (2.4) on domain $\Omega_\alpha \subseteq \mathscr{R}^n$. *Suppose there is a metric space* (Ω_α, ρ), *if the map* P *under the vector function* $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *is a contraction map, then there is a unique fixed point* $\mathbf{x}_k^*$ *which is globally stable.*

Proof Consider

$$\begin{aligned}
\rho(\mathbf{x}_{k+j+1}^{(1)}, \mathbf{x}_{k+j+1}^{(2)}) &= \rho(\mathbf{f}(\mathbf{x}_{k+j}^{(1)}, \mathbf{p}), \mathbf{f}(\mathbf{x}_{k+j}^{(2)}, \mathbf{p})) \leq \lambda\rho(\mathbf{x}_{k+j}^{(1)}, \mathbf{x}_{k+j}^{(2)}) \\
&= \lambda\rho(\mathbf{f}(\mathbf{x}_{k+j-1}^{(1)}, \mathbf{p}), \mathbf{f}(\mathbf{x}_{k+j-1}^{(2)}, \mathbf{p})) \leq \lambda^2\rho(\mathbf{x}_{k+j-1}^{(1)}, \mathbf{x}_{k+j-1}^{(2)}) \\
&\vdots \\
&= \lambda^{j-1}\rho(\mathbf{f}(\mathbf{x}_{k+1}^{(1)}, \mathbf{p}), \mathbf{f}(\mathbf{x}_{k+1}^{(2)}, \mathbf{p})) \leq \lambda^j\rho(\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)})
\end{aligned}$$

As $j \to \infty$ and $0 < \lambda < 1$, thus, we have

$$\lim_{j\to\infty} \rho(\mathbf{x}_{k+j+1}^{(1)}, \mathbf{x}_{k+j+1}^{(2)}) = \lim_{j\to\infty} \lambda^j\rho(\mathbf{x}_k^{(1)}, \mathbf{x}_k^{(2)}) = 0$$

If $\mathbf{x}_{k+j+1}^{(2)} = \mathbf{x}_k^{(2)} = \mathbf{x}_k^*$, in domain $\Omega_\alpha \in \mathscr{R}^n$, we have

$$\lim_{j\to\infty} \rho(\mathbf{x}_{k+j+1}^{(1)}, \mathbf{x}_{k+j+1}^{(2)}) = \lim_{j\to\infty} ||\mathbf{x}_{k+j+1}^{(1)} - \mathbf{x}_k^*|| = 0$$

Consider two fixed points $\mathbf{x}_{k1}^*$ and $\mathbf{x}_{k2}^*$. The above equation gives

$$\begin{aligned} ||\mathbf{x}_{k1}^* - \mathbf{x}_{k2}^*|| &= \lim_{j\to\infty} ||\mathbf{x}_{k1}^* - \mathbf{x}_{k+j+1} + \mathbf{x}_{k+j+1} - \mathbf{x}_{k2}^*|| \\ &\le \lim_{j\to\infty} ||\mathbf{x}_{k1}^* - \mathbf{x}_{k+j+1}|| + \lim_{j\to\infty} ||\mathbf{x}_{k+j+1} - \mathbf{x}_{k2}^*|| = 0 \end{aligned}$$

Therefore, the fixed point is unique and globally stable. This theorem is proved. ■

Definition 2.12 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \ge 1)$-continuous in $U_k(\mathbf{x}_k^*)$. The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. Consider a real eigenvalue λ_i of matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ $(i \in N = \{1, 2, \cdots, n\})$ and there is a corresponding eigenvector $\mathbf{v}_i$. On the invariant eigenvector $\mathbf{v}_k^{(i)} = \mathbf{v}_i$, consider $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i = \lambda_i c_k^{(i)}\mathbf{v}_i$, thus, $c_{k+1}^{(i)} = \lambda_i c_k^{(i)}$.

(i) $\mathbf{x}_k^{(i)}$ on the direction $\mathbf{v}_i$ is stable if

$$\lim_{k\to\infty} |c_k^{(i)}| = \lim_{k\to\infty} |(\lambda_i)^k| \times |c_0^{(i)}| = 0 \text{ for } |\lambda_i| < 1. \tag{2.37}$$

(ii) $\mathbf{x}_k^{(i)}$ on the direction $\mathbf{v}_i$ is stable if

$$\lim_{k\to\infty} |c_k^{(i)}| = \lim_{k\to\infty} |(\lambda_i)^k| \times |c_0^{(i)}| = \infty \text{ for } |\lambda_i| > 1. \tag{2.38}$$

(iii) $\mathbf{x}_k^{(i)}$ on the direction $\mathbf{v}_i$ is invariant if

$$\lim_{k\to\infty} c_k^{(i)} = \lim_{k\to\infty} (\lambda_i)^k c_0^{(i)} = c_0^{(i)} \text{ for } \lambda_i = 1. \tag{2.39}$$

(iv) $\mathbf{x}_k^{(i)}$ on the direction $\mathbf{v}_i$ is flipped if

$$\left.\begin{aligned} \lim_{2k\to\infty} c_k^{(i)} &= \lim_{2k\to\infty} (\lambda_i)^{2k} \times c_0^{(i)} = c_0^{(i)} \\ \lim_{2k+1\to\infty} c_k^{(i)} &= \lim_{2k+1\to\infty} (\lambda_i)^{2k+1} \times c_0^{(i)} = -c_0^{(i)} \end{aligned}\right\} \text{for } \lambda_i = -1. \tag{2.40}$$

(v) $\mathbf{x}_k^{(i)}$ on the direction $\mathbf{v}_i$ is degenerate if

$$c_k^{(i)} = (\lambda_i)^k c_0^{(i)} = 0 \text{ for } \lambda_i = 0. \tag{2.41}$$

Definition 2.13 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$. Consider a pair of complex eigenvalues $\alpha_i \pm \mathbf{i}\beta_i$ of matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ $(i \in N = \{1, 2, \cdots, n\}, \mathbf{i} = \sqrt{-1})$ and there is a corresponding eigenvector $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_k^{(i)}, \mathbf{v}_k^{(i)}) = (\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{x}_k^{(i)} = \mathbf{x}_{k+}^{(i)} + \mathbf{x}_{k-}^{(i)}$ with

$$\mathbf{x}_k^{(i)} = c_k^{(i)}\mathbf{u}_i + d_k^{(i)}\mathbf{v}_i, \mathbf{x}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{u}_i + d_{k+1}^{(i)}\mathbf{v}_i. \tag{2.42}$$

Thus, $\mathbf{c}_k^{(i)} = (c_k^{(i)}, d_k^{(i)})^{\mathbf{T}}$ with

$$\mathbf{c}_{k+1}^{(i)} = \mathbf{E}_i\mathbf{c}_k^{(i)} = r_i\mathbf{R}_i\mathbf{c}_k^{(i)} \tag{2.43}$$

where

$$\begin{gathered}\mathbf{E}_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \text{ and } \mathbf{R}_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{bmatrix}, \\ r_i = \sqrt{\alpha_i^2 + \beta_i^2}, \cos\theta_i = \alpha_i/r_i \text{ and } \sin\theta_i = \beta_i/r_i;\end{gathered} \tag{2.44}$$

and

$$\mathbf{E}_i^k = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}^{\mathrm{k}} \text{ and } \mathbf{R}_i^k = \begin{bmatrix} \cos k\theta_i & \sin k\theta_i \\ -\sin k\theta_i & \cos k\theta_i \end{bmatrix}. \tag{2.45}$$

(i) $\mathbf{x}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable if

$$\lim_{k\to\infty} ||\mathbf{c}_k^{(i)}|| = \lim_{k\to\infty} r_i^k||\mathbf{R}_i^k|| \times ||\mathbf{c}_0^{(i)}|| = 0 \text{ for } r_i = |\lambda_i| < 1. \tag{2.46}$$

(ii) $\mathbf{x}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally unstable if

$$\lim_{k\to\infty} ||\mathbf{c}_k^{(i)}|| = \lim_{k\to\infty} r_i^k||\mathbf{R}_i^k|| \times ||\mathbf{c}_0^{(i)}|| = \infty \text{ for } r_i = |\lambda_i| > 1. \tag{2.47}$$

(iii) $\mathbf{x}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is on the invariant circles if,

$$||\mathbf{c}_k^{(i)}|| = r_i^k||\mathbf{R}_i^k|| \times ||\mathbf{c}_0^{(i)}|| = ||\mathbf{c}_0^{(i)}|| \text{ for } r_i = |\lambda_i| = 1. \tag{2.48}$$

(iv) $\mathbf{x}_k^{(i)}$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is degenerate in the direction of $\mathbf{u}_i$ if $\beta_i = 0$.

Definition 2.14 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n eigenvalues λ_k $(k = 1, 2, \cdots n)$.

(i) The fixed point $\mathbf{x}_k^*$ is called a hyperbolic point if $|\lambda_i| \neq 1$ $(i = 1, 2, \cdots, n)$.
(ii) The fixed point $\mathbf{x}_k^*$ is called a sink if $|\lambda_i| < 1$ $(i = 1, 2, \cdots, n)$.
(iii) The fixed point $\mathbf{x}_k^*$ is called a source if $|\lambda_i| > 1 (i \in \{1, 2, \cdots n\})$.
(iv) The fixed point $\mathbf{x}_k^*$ is called a center if $|\lambda_i| = 1$ $(i = 1, 2, \cdots n)$ with distinct eigenvalues.

Definition 2.15 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n eigenvalues λ_i $(i = 1, 2, \cdots n)$.

(i) The fixed point $\mathbf{x}_k^*$ is called a stable node if $|\lambda_i| < 1 (i = 1, 2, \cdots, n)$.
(ii) The fixed point $\mathbf{x}_k^*$ is called an unstable node if $|\lambda_i| > 1$ $(i = 1, 2, \cdots, n)$.
(iii) The fixed point $\mathbf{x}_k^*$ is called an $(l_1 : l_2)$-saddle if at least one $|\lambda_i| > 1 (i \in L_1 \subset \{1, 2, \cdots n\})$ and the other $|\lambda_j| < 1$ $(j \in L_2 \subset \{1, 2, \cdots n\})$ with $L_1 \cup L_2 = \{1, 2, \cdots, n\}$ and $L_1 \cap L_2 = \varnothing$.
(iv) The fixed point $\mathbf{x}_k^*$ is called an lth-order degenerate case if $\lambda_i = 0$ $(i \in L \subseteq \{1, 2, \cdots n\})$.

Definition 2.16 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n-pairs of complex eigenvalues λ_i $(i = 1, 2, \cdots, n)$.

(i) The fixed point $\mathbf{x}_k^*$ is called a spiral sink if $|\lambda_i| < 1$ $(i = 1, 2, \cdots, n)$ and $\operatorname{Im}\lambda_j \neq 0$ $(j \in \{1, 2, \cdots, n\})$.
(ii) fixed point $\mathbf{x}_k^*$ is called a spiral source if $|\lambda_i| > 1$ $(i = 1, 2, \cdots, n)$ with $\operatorname{Im}\lambda_j \neq 0$ $(j \in \{1, 2, \cdots, n\})$.
(iii) fixed point $\mathbf{x}_k^*$ is called a center if $|\lambda_i| = 1$ with distinct $\operatorname{Im}\lambda_i \neq 0$ $(i \in \{1, 2, \cdots, n\})$.

As in Appendix B, the refined classification of the linearized, discrete, nonlinear system at fixed points should be discussed. The generalized stability and bifurcation of flows in linearized, nonlinear dynamical systems in Eq. (2.4) will be discussed as follows.

Definition 2.17 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n eigenvalues λ_i $(i = 1, 2, \cdots n)$. Set $N = \{1, 2, \cdots, n\}$, $N_p = \{l_1, l_2, \cdots, l_{n_p}\} \cup \varnothing$ with $l_{q_p} \in N$ $(q_p = 1, 2, \cdots, n_p,\quad p = 1, 2, \cdots, 7)$ and

$\Sigma_{p=1}^{4} n_p + 2\Sigma_{p=5}^{7} n_p = n$. $\cup_{p=1}^{7} N_p = N$ and $\cap_{p=1}^{7} N_p = \varnothing$. $N_p = \varnothing$ if $n_{q_p} = 0$. $N_\alpha = N_\alpha^{\mathrm{m}} \cup N_\alpha^{\mathrm{o}}$ $(\alpha = 1, 2)$ and $N_\alpha^{\mathrm{m}} \cap N_\alpha^{\mathrm{o}} = \varnothing$ with $n_\alpha^{\mathrm{m}} + n_\alpha^{\mathrm{o}} = n_\alpha$, where superscripts "m" and "o" represent monotonic and oscillatory evolutions. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n_1-stable, n_2-unstable, n_3-invariant and n_4-flip real eigenvectors plus n_5-stable, n_6-unstable and n_7-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_i| = 1 (i \in N_3 \cup N_4 \cup N_7)$, an iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] \| n_5 : n_6 : n_7)$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$. With repeated complex eigenvalues of $|\lambda_i| = 1$ $(i \in N_3 \cup N_4 \cup N_7)$, an iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$, where $\kappa_p \in \{\varnothing, m_p\}$ $(p = 3, 4, 7)$. The meanings of notations in the aforementioned structures are defined as follows:

(i) $[n_1^{\mathrm{m}}, n_1^{\mathrm{o}}]$ represents that there are n_1-sinks with n_1^{m}-monotonic convergence and n_1^{o}-oscillatory convergence among n_1-directions of $\mathbf{v}_i$ $(i \in N_1)$ if $|\lambda_i| < 1$ $(k \in N_1$ and $1 \le n_1 \le n)$ with distinct or repeated eigenvalues.

(ii) $[n_2^{\mathrm{m}}, n_2^{\mathrm{o}}]$ represents that there are n_2-sources with n_2^{m}-monotonic divergence and n_2^{o}-oscillatory divergence among n_2-directions of $\mathbf{v}_i$ $(i \in N_2)$ if $|\lambda_i| > 1$ $(k \in N_2$ and $1 \le n_2 \le n)$ with distinct or repeated eigenvalues.

(iii) $n_3 = 1$ represents an invariant center on 1-direction of $\mathbf{v}_i$ $(i \in N_3)$ if $\lambda_k = 1$ $(i \in N_3$ and $n_3 = 1)$.

(iv) $n_4 = 1$ represents a flip center on 1-direction of $\mathbf{v}_i$ $(i \in N_4)$ if $\lambda_i = -1$ $(i \in N_4$ and $n_4 = 1)$.

(v) n_5 represents n_5-spiral sinks on n_5-pairs of $(\mathbf{u}_i, \mathbf{v}_i)$ $(i \in N_5)$ if $|\lambda_i| < 1$ and $\mathrm{Im}\,\lambda_i \ne 0$ $(i \in N_5$ and $1 \le n_5 \le n)$ with distinct or repeated eigenvalues.

(vi) n_6 represents n_6-spiral sources on n_6-directions of $(\mathbf{u}_i, \mathbf{v}_i)$ $(i \in N_6)$ if $|\lambda_i| > 1$ and $\mathrm{Im}\,\lambda_i \ne 0$ $(i \in N_6$ and $1 \le n_6 \le n)$ with distinct or repeated eigenvalues.

(vii) n_7 represents n_7-invariant centers on n_7-pairs of $(\mathbf{u}_i, \mathbf{v}_i)$ $(i \in N_7)$ if $|\lambda_i| = 1$ and $\mathrm{Im}\,\lambda_i \ne 0$ $(i \in N_7$ and $1 \le n_7 \le n)$ with distinct eigenvalues.

(viii) $\varnothing$ represents none if $n_j = 0$ $(j \in \{1, 2, \cdots, 7\})$.

(ix) $[n_3; \kappa_3]$ represents $(n_3 - \kappa_3)$ invariant centers on $(n_3 - \kappa_3)$ directions of $\mathbf{v}_{i_3}$ $(i_3 \in N_3)$ and κ_3-sources in κ_3-directions of $\mathbf{v}_{j_3}$ $(j_3 \in N_3$ and $j_3 \ne i_3)$ if $\lambda_i = 1$ $(i \in N_3$ and $n_3 \le n)$ with the $(\kappa_3 + 1)$th-order nilpotent matrix $\mathbf{N}_3^{\kappa_3+1} = \mathbf{0}$ $(0 < \kappa_3 \le n_3 - 1)$.

(x) $[n_3; \varnothing]$ represents n_3 invariant centers on n_3-directions of $\mathbf{v}_i (i \in N_3)$ if $\lambda_i = 1$ $(i \in N_3$ and $1 < n_3 \le n)$ with a nilpotent matrix $\mathbf{N}_3 = \mathbf{0}$.

(xi) $[n_4; \kappa_4]$ represents $(n_4 - \kappa_4)$ flip oscillatory centers on $(n_4 - \kappa_4)$ directions of $\mathbf{v}_{i_4}$ $(i_4 \in N_4)$ and κ_4-sources in κ_4-directions of $\mathbf{v}_{j_4}$ $(j_4 \in N_4$ and $j_4 \ne i_4)$ if $\lambda_i = -1$ $(i \in N_4$ and $n_4 \le n)$ with the $(\kappa_4 + 1)$th-order nilpotent matrix $\mathbf{N}_4^{\kappa_4+1} = \mathbf{0}$ $(0 < \kappa_4 \le n_4 - 1)$.

(xii) $[n_4; \varnothing]$ represents n_4 flip oscillatory centers on n_4-directions of $\mathbf{v}_i$ $(i \in N_3)$ if $\lambda_i = -1$ $(i \in N_4$ and $1 < n_4 \le n)$ with a nilpotent matrix $\mathbf{N}_4 = \mathbf{0}$.

(xiii) $[n_7, l; \kappa_7]$ represents $(n_7 - \kappa_7)$ invariant centers on $(n_7 - \kappa_7)$ pairs of $(\mathbf{u}_{i_7}, \mathbf{v}_{i_7})(i_7 \in N_7)$ and κ_7 sources on κ_7 pairs of $(\mathbf{u}_{j_7}, \mathbf{v}_{j_7})$ $(j_7 \in N_7$ and $j_7 \ne i_7)$ if $|\lambda_i| = 1$ and $\mathrm{Im}\,\lambda_i \ne 0$ $(i \in N_7$ and $n_7 \le n)$ for $(l + 1)$

pairs of repeated eigenvalues with the (κ_7+1)th-order nilpotent matrix $\mathbf{N}_7^{\kappa_7+1}=\mathbf{0}$ $(0<\kappa_7\le l)$.

(xiv) $[n_7, l; \varnothing]$ represents n_7-invariant centers on n_7-pairs of $(\mathbf{u}_i, \mathbf{v}_i)(i\in N_6)$ if $|\lambda_i|=1$ and $\mathrm{Im}\,\lambda_i\ne 0$ $(i\in N_7$ and $1\le n_7\le n)$ for $(l+1)$ pairs of repeated eigenvalues with a nilpotent matrix $\mathbf{N}_7=\mathbf{0}$.

Definition 2.18 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1}=\mathbf{f}(\mathbf{x}_k,\mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j}=\mathbf{f}(\mathbf{x}_{k+j-1},\mathbf{p})$ with $j\in\mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*)\subset\Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k,\mathbf{p})$ is C^r $(r\ge 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1}=D\mathbf{f}(\mathbf{x}_k^*,\mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j}=\mathbf{x}_{k+j}-\mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*,\mathbf{p})$ possesses n eigenvalues λ_i $(i=1,2,\cdots n)$. Set $N=\{1,2,\cdots,n\}$, $N_p=\{l_1,l_2,\cdots,l_{n_p}\}\cup\varnothing$ with $l_{q_p}\in N$ $(q_p=1,2,\cdots,n_p,\quad p=1,2,\cdots,7)$ and $\Sigma_{p=1}^4 n_p+2\,\Sigma_{p=5}^7\,n_p=n$. $\cup_{p=1}^7 N_p=N$ and $\cap_{p=1}^7 N_p=\varnothing$.$N_p=\varnothing$ if $n_{q_p}=0$. $N_\alpha=N_\alpha^{\mathrm{m}}\cup N_\alpha^{\mathrm{o}}$ $(\alpha=1,2)$ and $N_\alpha^{\mathrm{m}}\cap N_\alpha^{\mathrm{o}}=\varnothing$ with $n_\alpha^{\mathrm{m}}+n_\alpha^{\mathrm{o}}=n_\alpha$, where superscripts "m" and "o" represent monotonic and oscillatory evolutions. The matrix $D\mathbf{f}(\mathbf{x}_k^*,\mathbf{p})$ possesses n_1-stable, n_2-unstable, n_3-invariant, and n_4-flip real eigenvectors plus n_5-stable, n_6-unstable and n_7-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_k|=1(k\in N_3\cup N_4\cup N_7)$, an iterative response of $\mathbf{x}_{k+1}=\mathbf{f}(\mathbf{x}_k,\mathbf{p})$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:[n_3;\kappa_3]:[n_4;\kappa_4]\|\,n_5:n_6:n_7)$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$. With repeated complex eigenvalues of $|\lambda_i|=1$ $(i\in N_3\cup N_4\cup N_7)$, an iterative response of $\mathbf{x}_{k+1}=\mathbf{f}(\mathbf{x}_k,\mathbf{p})$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:[n_3;\kappa_3]:[n_4;\kappa_4]|n_5:n_6:[n_7,l;\kappa_7])$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$, where $\kappa_p\in\{\varnothing,m_p\}(p=3,4,7)$.

I. *Non-degenerate cases*

(i) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:\varnothing:\varnothing|n_5:n_6:\varnothing)$ hyperbolic point.

(ii) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[\varnothing,\varnothing]:\varnothing:\varnothing|n_5:\varnothing:\varnothing)$-sink.

(iii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:\varnothing:\varnothing|\varnothing:n_6:\varnothing)$-source.

(iv) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:\varnothing:\varnothing|\varnothing:\varnothing:n/2)$-circular center.

(v) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:\varnothing:\varnothing|\varnothing:\varnothing:[n/2,l;\varnothing])$-circular center.

(vi) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:\varnothing:\varnothing|\varnothing:\varnothing:[n/2,l;m])$-point.

(vii) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[\varnothing,\varnothing]:\varnothing:\varnothing|n_5:\varnothing:n_7)$-point.

(viii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:\varnothing:\varnothing|\varnothing:n_6:n_7)$-point.

(ix) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:\varnothing:\varnothing|n_5:n_6:n_7)$-point.

II. *Simple special cases*

(i) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[n;\varnothing]:\varnothing|\varnothing:\varnothing:\varnothing)$-invariant center (or static center).

(ii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[n;m]:\varnothing|\varnothing:\varnothing:\varnothing)$-point.

(iii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:\varnothing:[n;\varnothing]|\varnothing:\varnothing:\varnothing)$-flip center.

(iv) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:\varnothing:[n;m]|\varnothing:\varnothing:\varnothing)$-point.

(v) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[n_3;\kappa_3]:[n_4;\kappa_4]|\varnothing:\varnothing:\varnothing)$-point.
(vi) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[1;\varnothing]:[n_4;\kappa_4]|\varnothing:\varnothing:\varnothing)$-point.
(vii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[n_3;\kappa_3]:[1;\varnothing]|\varnothing:\varnothing:\varnothing)$-point.
(viii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[n_3;\kappa_3]:[\varnothing;\varnothing]|\varnothing:\varnothing:n_7)$-point.
(ix) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[1;\varnothing]:[\varnothing;\varnothing]|\varnothing:\varnothing:n_7)$-point.
(x) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[n_3;\kappa_3]:[\varnothing;\varnothing]|\varnothing:\varnothing:[n_7,l;\kappa_7])$-point.
(xi) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[\varnothing;\varnothing]:[n_4;\kappa_4]|\varnothing:\varnothing:n_7)$-point.
(xii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[\varnothing;\varnothing]:[n_4;\kappa_4]|\varnothing:\varnothing:[n_7,l;\kappa_7])$-point.
(xiii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[n_3;\kappa_3]:[n_4;\kappa_4]|\varnothing:\varnothing:n_7)$-point.
(xiv) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing,\varnothing]:[\varnothing,\varnothing]:[n_3;\kappa_3]:[n_4;\kappa_4]|\varnothing:\varnothing:[n_7,l;\kappa_7])$-point.

III. *Complex special cases*

(i) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:[1;\varnothing]:[\varnothing;\varnothing]|n_5:n_6:n_7)$-point.
(ii) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:[1;\varnothing]:[\varnothing;\varnothing]|n_5:n_6:[n_7,l;\kappa_7])$-point.
(iii) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:[\varnothing;\varnothing]:[1;\varnothing]|n_5:n_6:n_7)$-point.
(iv) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:[\varnothing;\varnothing]:[1;\varnothing]|n_5:n_6:[n_7,l;\kappa_7])$-point.
(v) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:[n_3;\kappa_3]:[n_4;\kappa_4]|n_5:n_6:n_7)$-point.
(vi) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:[n_3;\kappa_3]:[n_4;\kappa_4]|n_5:n_6:[n_7,l;\kappa_7])$-point.
(vii) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:[n_3;\kappa_3]:[n_4;\kappa_4]|n_5:n_6:n_7)$-point.
(viii) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\mathrm{m}},n_1^{\mathrm{o}}]:[n_2^{\mathrm{m}},n_2^{\mathrm{o}}]:[n_3;\kappa_3]:[n_4;\kappa_4]|n_5:n_6:[n_7,l;\kappa_7])$-point.

Definition 2.19 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1}=\mathbf{f}(\mathbf{x}_k,\mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j}=\mathbf{f}(\mathbf{x}_{k+j-1},\mathbf{p})$ with $j\in\mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*)\subset\Omega$), and $\mathbf{f}(\mathbf{x}_k,\mathbf{p})$ is C^r $(r\geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1}=D\mathbf{f}(\mathbf{x}_k^*,\mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j}=\mathbf{x}_{k+j}-\mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*,\mathbf{p})$ possesses n eigenvalues λ_i $(i=1,2,\cdots n)$. Set $N=\{1,2,\cdots,n\}$, $N_p=\{l_1,l_2,\cdots,l_{n_p}\}\cup\varnothing$ with $l_{q_p}\in N(q_p=1,2,\cdots,n_p,\ p=1,2,3,4)$ and $\Sigma_{p=1}^4\ n_p=n$. $\cup_{p=1}^4\ N_p=N$ and $\cap_{p=1}^4 N_p=\varnothing$. $N_p=\varnothing$ if $n_{q_p}=0$. $N_\alpha=N_\alpha^{\mathrm{m}}\cup$

$N_\alpha^{\rm o}$ $(\alpha = 1, 2)$ and $N_\alpha^{\rm m} \cap N_\alpha^{\rm o} = \varnothing$ with $n_\alpha^{\rm m} + n_\alpha^{\rm o} = n_\alpha$ where superscripts "m" and "o" represent monotonic and oscillatory evolutions. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n_1-stable, n_2-unstable, n_3-invariant and n_4-flip real eigenvectors. An iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : [n_3; \kappa_3] : [n_4; \kappa_4]|$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$. $\kappa_p \in \{\varnothing, m_p\}$ $(p = 3, 4)$.

I. *Non-degenerate cases*

(i) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|$ saddle.
(ii) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\rm m}, n_1^{\rm o}] : [\varnothing, \varnothing] : \varnothing : \varnothing|$-sink.
(iii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing, \varnothing] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|$-source.

II. *Simple special cases*

(i) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n; \varnothing] : \varnothing|$-invariant center (or static center).
(ii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n; m] : \varnothing|$-point.
(iii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : \varnothing : [n; \varnothing]|$-flip center.
(iv) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : \varnothing : [n; m]|$-point.
(v) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [n_4; \kappa_4]|$-point.
(vi) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [1; \varnothing] : [n_4; \kappa_4]|$-point.
(vii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [1; \varnothing]|$-point.
(viii) The fixed point $\mathbf{x}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [n_3; \kappa_3] : [\varnothing; \varnothing]|$-point.
(ix) T The fixed point $\mathbf{x}_k^*$ is an $([\varnothing, \varnothing] : [\varnothing, \varnothing] : [\varnothing; \varnothing] : [n_4; \kappa_4]|$-point.

III. *Complex special cases*

(i) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : [1; \varnothing] : [\varnothing; \varnothing]|$-point.
(ii) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : [\varnothing; \varnothing] : [1; \varnothing]|$-point.
(iii) The fixed point $\mathbf{x}_k^*$ is an $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : [n_3; \kappa_3] : [n_4; \kappa_4]|$-point.

Definition 2.20 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^{2n}$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega_\alpha$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses $2n$ eigenvalues λ_i $(i = 1, 2, \cdots, n)$. Set $N = \{1, 2, \cdots, n\}$, $N_p = \{l_1, l_2, \cdots, l_{n_p}\} \cup \varnothing$ with $l_{q_p} \in N$ $(q_p = 1, 2, \cdots, n_p,\ p = 5, 6, 7)$ and $\Sigma_{p=5}^{7}\, n_p = n$. $\cup_{p=5}^{7} N_p = N$ and $\cap_{p=5}^{7} N_p = \varnothing$. if $n_{q_p} = 0$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n_5-stable, n_6-unstable and n_7-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_k| = 1 (k \in N_7)$, an iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $|n_5 : n_6 : n_7)$ flow. With repeated complex eigenvalues of $|\lambda_k| = 1$ $(k \in N_7)$, an iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $|n_5 : n_6 : [n_7, l; \kappa_7])$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$, where $\kappa_p \in \{\varnothing, m_p\} (p = 7)$.

I. *Non-degenerate cases*

(i) The fixed point $\mathbf{x}_k^*$ is an $|n_5 : n_6 : \varnothing)$ spiral hyperbolic point.
(ii) The fixed point $\mathbf{x}_k^*$ is an $|n : \varnothing : \varnothing)$ spiral sink.

(iii) The fixed point $\mathbf{x}_k^*$ is an $|\varnothing : n : \varnothing)$ spiral source.
(iv) The fixed point $\mathbf{x}_k^*$ is an $|\varnothing : \varnothing : n)$-circular center.
(v) The fixed point $\mathbf{x}_k^*$ is an $|n_5 : \varnothing : n_7)$-point.
(vi) The fixed point $\mathbf{x}_k^*$ is an $|\varnothing : n_6 : n_7)$-point.
(vii) The fixed point $\mathbf{x}_k^*$ is an $|n_5 : n_6 : n_7)$-point.

II. *Special cases*

(i) The fixed point $\mathbf{x}_k^*$ is an $|\varnothing : \varnothing : [n, l; \varnothing])$-circular center.
(ii) The fixed point $\mathbf{x}_k^*$ is an $|\varnothing : \varnothing : [n, l; m])$-point.
(iii) The fixed point $\mathbf{x}_k^*$ is an $|n_5 : \varnothing : [n_7, l; \kappa_7])$-point.
(iv) The fixed point $\mathbf{x}_k^*$ is an $|\varnothing : n_6 : [n_7, l; \kappa_7])$-point.
(v) The fixed point $\mathbf{x}_k^*$ is an $|n_5 : n_6 : [n_7, l; \kappa_7])$-point.

2.3 Bifurcation and Stability Switching

To understand the qualitative changes of dynamical behaviors of discrete systems with parameters in the neighborhood of fixed points, the bifurcation theory for fixed points of nonlinear dynamical system in Eq. (2.4) will be investigated.

Definition 2.21 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n eigenvalues λ_i $(i = 1, 2, \cdots n)$. Set $N = \{1, 2, \cdots, n\}$, $N_p = \{l_1, l_2, \cdots, l_{n_p}\} \cup \varnothing$ with $l_{q_p} \in N$ $(q_p = 1, 2, \cdots, n_p$, $p = 1, 2, \cdots, 7)$ and $\Sigma_{p=1}^4\ n_p + 2\Sigma_{p=5}^7\ n_p = n$. $\cup_{p=5}^7 N_p = N$ and $\cap_{p=1}^7 N_p = \varnothing$. $N_p = \varnothing$ if $n_{q_p} = 0$. $N_\alpha = N_\alpha^{\mathrm{m}} \cup N_\alpha^{\mathrm{o}}$ $(\alpha = 1, 2)$ and $N_\alpha^{\mathrm{m}} \cap N_\alpha^{\mathrm{o}} = \varnothing$ with $n_\alpha^{\mathrm{m}} + n_\alpha^{\mathrm{o}} = n_\alpha$, where superscripts "m" and "o" represent monotonic and oscillatory evolutions. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n_1-stable, n_2-unstable, n_3-invariant and n_4-flip real eigenvectors plus n_5-stable, n_6-unstable and n_7-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_k| = 1 (k \in N_3 \cup N_4 \cup N_7)$, an iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4]|\ n_5 : n_6 : n_7)$ flow in the neighborhood of the fixed point x_k^*. With repeated complex eigenvalues of $|\lambda_k| = 1$ $(k \in N_3 \cup N_4 \cup N_7)$, an iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4]|n_5 : n_6 : [n_7, l; \kappa_7])$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$, where $\kappa_p \in \{\varnothing, m_p\} (p = 3, 4, 7)$.

I. *Simple switching and bifurcation*

(i) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : 1 : \varnothing|n_5 : n_6 : \varnothing)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + 1] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|n_5 : n_6 : \varnothing)$ spiral saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + 1, n_2^{\mathrm{o}}] : \varnothing : \varnothing|n_5 : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(ii) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : 1|n_5 : n_6 : \varnothing)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}, n_1^{\rm o}+1] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|\, n_5 : n_6 : \varnothing)$ spiral saddle and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}+1] : \varnothing : \varnothing|n_5 : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(iii) An $([n_1^{\rm m}, n_1^{\rm o}] : [\varnothing, \varnothing] : 1 : \varnothing|n_5 : \varnothing : \varnothing)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a stable saddle-node bifurcation of the $([n_1^{\rm m}+1, n_1^{\rm o}] : [\varnothing, \varnothing] : \varnothing : \varnothing|n_5 : \varnothing : \varnothing)$ spiral sink and $([n_1^{\rm m}, n_1^{\rm o}] : [1, \varnothing] : \varnothing : \varnothing|n_5 : \varnothing : \varnothing)$ spiral saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(iv) An $([n_1^{\rm m}, n_1^{\rm o}] : [\varnothing, \varnothing] : \varnothing : 1|n_5 : \varnothing : \varnothing)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a stable period-doubling bifurcation of the $([n_1^{\rm m}, n_1^{\rm o}+1] : [\varnothing, \varnothing] : \varnothing : \varnothing|n_5 : \varnothing : \varnothing)$ sink and $([n_1^{\rm m}, n_1^{\rm o}] : [\varnothing, 1] : \varnothing : \varnothing|n_5 : \varnothing : \varnothing)$ spiral saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(v) An $([\varnothing, \varnothing] : [n_2^{\rm m}, n_2^{\rm o}] : 1 : \varnothing|\varnothing : n_6 : \varnothing)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is an unstable saddle-node bifurcation of the $([\varnothing, \varnothing] : [n_2^{\rm m}+1, n_2^{\rm o}] : \varnothing : \varnothing|\varnothing : n_6 : \varnothing)$ spiral source and $([1, \varnothing] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|\varnothing : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(vi) An $([\varnothing, \varnothing] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : 1|\varnothing : n_6 : \varnothing)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is an unstable period-doubling bifurcation of the $([\varnothing, \varnothing] : [n_2^{\rm m}, n_2^{\rm o}+1] : \varnothing : \varnothing|\varnothing : n_6 : \varnothing)$ spiral source and $([\varnothing, 1] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|\varnothing : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(vii) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|n_5 : n_6 : 1)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|n_5+1 : n_6 : \varnothing)$ spiral saddle and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|n_5 : n_6+1 : \varnothing)$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(viii) An $([n_1^{\rm m}, n_1^{\rm o}] : [\varnothing, \varnothing] : \varnothing : \varnothing|n_5 : \varnothing : 1)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a Neimark bifurcation of the $([n_1^{\rm m}, n_1^{\rm o}] : [\varnothing, \varnothing] : \varnothing : \varnothing|n_5+1 : \varnothing : \varnothing)$ spiral sink and $([n_1^{\rm m}, n_1^{\rm o}] : [\varnothing, \varnothing] : \varnothing : \varnothing|n_5 : 1 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(ix) An $([\varnothing, \varnothing] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|\varnothing : n_6 : 1)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is an unstable Neimark bifurcation of the $([\varnothing, \varnothing] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|\varnothing : n_6+1 : \varnothing)$ spiral source and $([\varnothing, \varnothing] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|1 : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(x) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : 1 : \varnothing|n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}+1, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|n_5 : n_6 : n_7)$ state and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}+1, n_2^{\rm o}] : \varnothing : \varnothing|n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(xi) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : 1|n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}, n_1^{\rm o}+1] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|n_5 : n_6 : n_7)$ state and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}+1] : \varnothing : \varnothing|n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(xii) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : 1 : \varnothing|n_5 : n_6 : [n_7, l; \varnothing])$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}+1, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|n_5 : n_6 : [n_7, l; \kappa_7])$ state and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}+1, n_2^{\rm o}] : \varnothing : \varnothing|n_5 : n_6 : [n_7, l; \kappa_7])$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(xiii) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : 1|n_5 : n_6 : [n_7, l; \kappa_7])$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}, n_1^{\rm o}+1] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|n_5 : n_6 :$

$[n_7, l; \kappa_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}+1] : \varnothing : \varnothing | n_5 : n_6 : [n_7, l; \kappa_7]))$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(xiv) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7 + 1)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of its $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 + 1 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 + 1 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(xv) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : n_7 + 1)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | \; n_5 + 1 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 + 1 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

II. *Complex switching*

(i) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : \varnothing | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + n_3, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + n_3, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(ii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + n_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing | n_5 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + n_4] : \varnothing : \varnothing | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(iii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3 + k_3; \kappa_3] : \varnothing | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + k_3, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : \varnothing | n_5 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + k_3, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : \varnothing | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(iv) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4 + k_4; \kappa_4] | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + k_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4, \kappa_4] | n_5 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + k_4] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(v) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3 + k_3; \kappa_3] : [n_4 + k_4; \kappa_4] | n_5 : n_6 : n_7)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + k_3, n_1^{\mathrm{o}} + k_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : n_7)$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + k_3, n_2^{\mathrm{o}} + k_4] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(vi) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3 + k_3; \kappa_3] : \varnothing | n_5 : n_6 : [n_7, l; \kappa_7])$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + k_3, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : \varnothing | n_5 : n_6 : [n_7, l; \kappa_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + k_3, n_2^{\mathrm{o}}] : [n_3, \kappa_3] : \varnothing | n_5 : n_6 : [n_7, l; \kappa_7])$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(vii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4 + k_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + k_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + k_4] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(viii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4 + k_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + k_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + k_4] : \varnothing : [n_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(ix) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3 + k_3; \kappa_3] : [n_4 + k_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + k_3, n_1^{\mathrm{o}} + k_4] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + k_3, n_2^{\mathrm{o}} + k_4] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : [n_7, l; \kappa_7])$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(x) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 : [n_7 + k_7, l; \kappa_7])$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 + k_7 : n_6 : [n_7, l; \kappa_7])$ state and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : n_2^{\mathrm{o}} + k_4] : [n_3; \kappa_3] : [n_4; \kappa_4] | n_5 : n_6 + k_7 : [n_7, l; \kappa_7])$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

Definition 2.22 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n eigenvalues λ_i $(i = 1, 2, \cdots n)$. Set $N = \{1, 2, \cdots, n\}$, $N_p = \{l_1, l_2, \cdots, l_{n_p}\} \cup \varnothing$ with $l_{q_p} \in N$ $(q_p = 1, 2, \cdots, n_p,$ $p = 1, 2, 3, 4)$ and $\Sigma_{p=1}^4 n_p = n$. $\cup_{p=1}^4 N_p = N$ and $\cap_{p=1}^4 N_p = \varnothing . N_p = \varnothing$ if $n_{q_p} = 0$. $N_\alpha = N_\alpha^{\mathrm{m}} \cup N_\alpha^{\mathrm{o}}$ $(\alpha = 1, 2)$ and $N_\alpha^{\mathrm{m}} \cap N_\alpha^{\mathrm{o}} = \varnothing$ with $n_\alpha^{\mathrm{m}} + n_\alpha^{\mathrm{o}} = n_\alpha$ where superscripts "m" and "o" represent monotonic and oscillatory evolutions. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n_1-stable, n_2-unstable, n_3-invariant and n_4-flip real eigenvectors. An iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : [n_3; \kappa_3] : [n_4; \kappa_4]|$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$. $\kappa_p \in \{\varnothing, m_p\}$ $(p = 3, 4)$.

I. *Simple critical cases*

(i) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : 1 : \varnothing|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + 1, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + 1, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(ii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : 1|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + 1] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + 1] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(iii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : 1 : \varnothing|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a stable saddle-node bifurcation of the $([n_1^{\mathrm{m}} + 1, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : \varnothing|$ sink and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [1, \varnothing] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(iv) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, \varnothing] : \varnothing : 1|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a stable period-doubling bifurcation of the $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}} + 1] : [\varnothing, \varnothing] : \varnothing : \varnothing|$ sink and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [\varnothing, 1] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(v) An $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : 1 : \varnothing|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is an unstable saddle-node bifurcation of the $([\varnothing, \varnothing] : [n_2^{\mathrm{m}} + 1, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ source and $([1, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(vi) An $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : 1|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is an unstable period-doubling bifurcation of the $([\varnothing, \varnothing] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}} + 1] : \varnothing : \varnothing|$ source and $([\varnothing, 1] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(vii) An $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : 1 : \varnothing|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\mathrm{m}} + 1, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}}, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle and $([n_1^{\mathrm{m}}, n_1^{\mathrm{o}}] : [n_2^{\mathrm{m}} + 1, n_2^{\mathrm{o}}] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(viii) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : 1|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}, n_1^{\rm o}+1] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|$ saddle and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}+1] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

II. *Complex switching*

(i) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : [n_3; \kappa_3] : \varnothing|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}+n_3, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|$ saddle and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}+n_3, n_2^{\rm o}] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.
(ii) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : [n_4; \kappa_4]|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}, n_1^{\rm o}+n_4] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : \varnothing|$ saddle and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, \ n_2^{\rm o}+n_4] : \varnothing : \varnothing|$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.
(iii) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : [n_3+k_3; \kappa_3] : \varnothing|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}+k_3, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : [n_3; \kappa_3] : \varnothing|$ state and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}+k_3, n_2^{\rm o}] : [n_3; \kappa_3] : \varnothing|$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.
(iv) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : [n_4+k_4; \kappa_4]|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}, n_1^{\rm o}+k_4] : [n_2^{\rm m}, n_2^{\rm o}] : \varnothing : [n_4; \kappa_4]|$ state and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}+k_4] : \varnothing : [n_4; \kappa_4]|$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.
(v) An $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}, n_2^{\rm o}] : [n_3+k_3; \kappa_3] : [n_4+k_4; \kappa_4]|$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $([n_1^{\rm m}+k_3, n_1^{\rm o}+k_4] : [n_2^{\rm m}, n_2^{\rm o}] : [n_3; \kappa_3] : [n_4; \kappa_4]|$ state and $([n_1^{\rm m}, n_1^{\rm o}] : [n_2^{\rm m}+k_3, n_2^{\rm o}+k_4] : [n_3; \kappa_3] : [n_4; \kappa_4]|$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

Definition 2.23 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^{2n}$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses $2n$ eigenvalues λ_i $(i = 1, 2, \cdots, n)$. Set $N = \{1, 2, \cdots, n\}$, $N_p = \{l_1, l_2, \cdots, l_{n_p}\} \cup \varnothing$ with $l_{q_p} \in N$ $(q_p = 1, 2, \cdots, n_p$, $j = 5, 6, 7)$ and $\Sigma_{p=5}^7 n_p = n$. $\cup_{p=5}^7 N_p = N$ and $\cap_{p=5}^7 N_p = \varnothing$. $N_p = \varnothing$ if $n_{q_p} = 0$. The matrix $D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})$ possesses n_5-stable, n_6-unstable and n_7-center pairs of complex eigenvectors. Without repeated complex eigenvalues of $|\lambda_i| = 1 (i \in N_7)$, an iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $|n_5 : n_6 : n_7)$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$. With repeated complex eigenvalues of $|\lambda_i| = 1$ $(i \in N_7)$, an iterative response of $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is an $|n_5 : n_6 : [n_7, l; \kappa_7])$ flow in the neighborhood of the fixed point $\mathbf{x}_k^*$, $(\kappa_7 \in \{\varnothing, m_7\})$.

I. *Simple switching and bifurcation*

(i) An $|n_5 : n_6 : 1)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5+1 : n_6 : \varnothing)$ spiral saddle and $|n_5 : n_6+1 : \varnothing)$ saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.
(ii) An $|n_5 : \varnothing : 1)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a stable Neimark bifurcation of the $|n_5+1 : \varnothing : \varnothing)$ spiral sink and $|n_5 : 1 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

(iii) An $|\varnothing : n_6 : 1)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is an unstable Neimark bifurcation of the $|\varnothing : n_6 + 1 : \varnothing)$ spiral source and $|1 : n_6 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.
(iv) An $|n_5 : n_6 : n_7 + 1)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5 + 1 : n_6 : n_7)$ state and $|n_5 : n_6 + 1 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.
(v) An $|\varnothing : n_6 : n_7 + 1)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|1 : n_6 : n_7)$ state and $|n_5 : n_6 + 1 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.
(vi) An $|n_5 : \varnothing : n_7 + 1)$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5 + 1 : \varnothing : n_7)$ state and $|n_5 : 1 : n_7)$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

II. *Complex switching*

(i) An $|n_5 : n_6 : [n_7, l; \kappa_7])$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5 + n_7 : n_6 : \varnothing)$ spiral saddle and $|n_5 : n_6 + n_7 : \varnothing)$ spiral saddle for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.
(ii) An $|n_5 : n_6 : [n_7 + k_7, l; \kappa_7])$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5 + k_7 : n_6 : [n_7, l; \kappa_7])$ state and $|n_5 : n_6 + k_7 : [n_7, l; \kappa_7])$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.
(iii) An $|n_5 : n_6 : [n_7{+}k_5{-}k_6, l_2; \kappa_7])$ state of the fixed point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$ is a switching of the $|n_5 + k_5 : n_6 : [n_7, l_1; \kappa_7])$ state and $|n_5 : n_6 + k_6 : [n_7, l_3; \kappa_7])$ state for the fixed point $(\mathbf{x}_k^*, \mathbf{p})$.

2.3.1 Stability and Switching

To extend the idea of Definitions 2.11 and 2.12, a new function will be defined to determine the stability and the stability state switching.

Definition 2.24 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$ and there are n linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2, \cdots, n)$. For a perturbation of fixed point $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, let $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$,

$$s_k^{(i)} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*) \tag{2.49}$$

where $s_k^{(i)} = c_k^{(i)} ||\mathbf{v}_i||^2$. Define the following functions

$$G_i(\mathbf{x}_k, \mathbf{p}) = \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_k^*] \tag{2.50}$$

and

$$
\begin{aligned}
G^{(1)}_{s^{(i)}_k}(\mathbf{x}, \mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot D_{c^{(i)}_k} \mathbf{f}(\mathbf{x}_k(s^{(i)}_k), \mathbf{p}) \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k} \mathbf{f}(\mathbf{x}_k(s^{(i)}_k), \mathbf{p}) \partial_{c^{(i)}_k} \mathbf{x}_k \partial_{s_k} c^{(i)}_k \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}} \mathbf{f}(\mathbf{x}_k(s^{(i)}_k), \mathbf{p}) \mathbf{v}_i ||\mathbf{v}_i||^{-2}
\end{aligned} \tag{2.51}
$$

$$
\begin{aligned}
G^{(m)}_{s^{(i)}_k}(\mathbf{x}, \mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot D^{(m)}_{s^{(i)}_k} \mathbf{f}(\mathbf{x}_k(s^{(i)}_k), \mathbf{p}) \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot D_{s^{(i)}_k} (D^{(m-1)}_{s^{(i)}_k} \mathbf{f}(\mathbf{x}_k(s^{(i)}_k), \mathbf{p}))
\end{aligned} \tag{2.52}
$$

where $D_{s^{(i)}_k}(\cdot) = \partial(\cdot)/\partial s^{(i)}_k$ and $D^{(m)}_{s^{(i)}_k}(\cdot) = D_{s^{(i)}_k}(D^{(m-1)}_{s^{(i)}_k}(\cdot))$.

Definition 2.25 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}^*_k$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}^*_k$ (i.e., $U_k(\mathbf{x}^*_k) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}^*_k)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}^*_k, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}^*_k)$ in $U_k(\mathbf{x}^*_k)$ and there are n linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2, \cdots, n)$. For a perturbation of fixed point $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}^*_k$, let $\mathbf{y}^{(i)}_k = c^{(i)}_k \mathbf{v}_i$ and $\mathbf{y}^{(i)}_{k+1} = c^{(i)}_{k+1} \mathbf{v}_i$.

(i) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}^*_k$ on the direction $\mathbf{v}_i$ is stable if

$$
|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}^*_k)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}^*_k)| \tag{2.53}
$$

for $\mathbf{x}_k \in U(\mathbf{x}^*_k) \subset \Omega_\alpha$. The fixed point $\mathbf{x}^*_k$ is called the sink (or stable node) on the direction $\mathbf{v}_i$.

(ii) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}^*_k$ on the direction $\mathbf{v}_i$ is unstable if

$$
|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}^*_k)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}^*_k)| \tag{2.54}
$$

for $\mathbf{x}_k \in U(\mathbf{x}^*_k) \subset \Omega_\alpha$. The fixed point $\mathbf{x}^*_k$ is called the source (or unstable node) on the direction $\mathbf{v}_i$.

(iii) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}^*_k$ on the direction $\mathbf{v}_i$ is invariant if

$$
\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}^*_k) = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}^*_k) \tag{2.55}
$$

for $\mathbf{x}_k \in U(\mathbf{x}^*_k) \subset \Omega_\alpha$. The fixed point $\mathbf{x}^*_k$ is called to be degenerate on the direction $\mathbf{v}_i$.

(iv) $\mathbf{x}_{k+j}^{(i)}(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is symmetrically flipped if

$$\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*) = -\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*) \tag{2.56}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called to be degenerate on the direction $\mathbf{v}_i$.

The stability of fixed points for a specific eigenvector is presented in Fig. 2.4. The solid curve is $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_k, \mathbf{p})$. The circular symbol is the fixed point. The shaded regions are stable. The horizontal solid line is for a degenerate case. The vertical solid line is for a line with infinite slope. The monotonically stable node (sink) is presented in Fig. 2.4a. The dashed and dotted lines are for $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1}$ and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k$, respectively. The iterative responses approach the fixed point. However, the monotonically unstable (source) is presented in Fig. 2.4b. The iterative responses go away from the fixed point. Similarly, the oscillatory stable node (sink) is presented in Fig. 2.4c. The dashed and dotted lines are for $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k$ and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1}$, respectively. The oscillatory unstable node (source) is presented in Fig. 2.4d.

Theorem 2.5 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ *in Eq.* (2.4) with a fixed point $\mathbf{x}_k^*$. *The corresponding solution is given by* $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ *with* $j \in \mathbb{Z}$. *Suppose there is a neighborhood of the fixed point* $\mathbf{x}_k^*$ *(i.e.,* $U_k(\mathbf{x}_k^*) \subset \Omega$*), and* $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *is* C^r $(r \geq 1)$*-continuous in* $U_k(\mathbf{x}_k^*)$ *with Eq.* (2.28). *The linearized system is* $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ *in* $U_k(\mathbf{x}_k^*)$ *and there are n linearly independent vectors* $\mathbf{v}_i$ $(i = 1, 2, \cdots, n)$. *For a perturbation of fixed point* $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, *let* $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ *and* $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

(i) $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is stable if and only if*

$$G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i \in (-1, 1) \tag{2.57}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(ii) $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is unstable if and only if*

$$G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i \in (1, \infty) \text{ and } (-\infty, -1) \tag{2.58}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(iii) $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is invariant if and only if*

$$G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1 \text{ and } G_{s_k^{(i)}}^{(m_i)}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } m_i = 2, 3, \cdots \tag{2.59}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(iv) $\mathbf{x}_{k+j}^{(i)}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *on the direction* $\mathbf{v}_i$ *is symmetrically flipped if and only if*

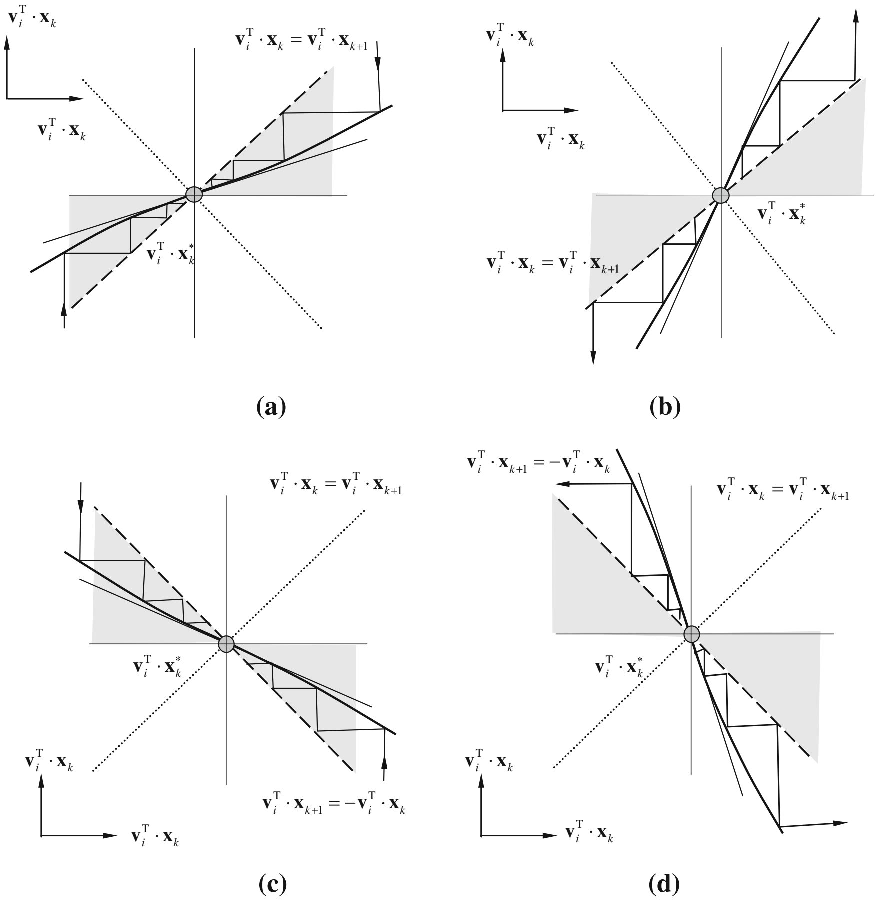

Fig. 2.4 Stability of fixed points: **a** monotonically stable node (sink), **b** monotonically unstable node (source), **c** oscillatory stable node (sink) and **d** oscillatory unstable node (sink). *Shaded areas* are stable zones

$$G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1 \text{ and } G^{(m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } m_i = 2, 3, \cdots \tag{2.60}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

Proof Because

$$\begin{aligned} s_{k+1}^{(i)} &= \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*) = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k^* + G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) s_k^{(i)} + o(s_k^{(i)}) - \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k^* \\ &= G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) s_k^{(i)} + o(s_k^{(i)}) \end{aligned}$$

due to any selection of $s_k^{(i)}$ and $s_{k+1}^{(i)}$ as an infinitesimal, we have

$$
\begin{aligned}
s_{k+1}^{(i)} &= G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) s_k^{(i)}, \\
G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}} \mathbf{f}(\mathbf{x}_k^*, \mathbf{p}) \mathbf{v}_i ||\mathbf{v}_i||^{-2} \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot \lambda_i \mathbf{v}_i ||\mathbf{v}_i||^{-2} = \lambda_i .
\end{aligned}
$$

(i) From the definition in Eq. (2.53), we have

$$
|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \Rightarrow |s_{k+1}^{(i)}| < |s_k^{(i)}|
$$

which gives

$$
|G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) s_k^{(i)}| < |s_k^{(i)}|
$$

Thus,

$$
|G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p})| < 1 \Rightarrow G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i \in (-1, 1).
$$

Therefore, $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is stable and vice versa.

(ii) From the definition in Eq. (2.54), we have

$$
|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \Rightarrow |s_{k+1}^{(i)}| > |s_k^{(i)}|
$$

which gives

$$
|G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) s_k^{(i)}| > |s_k^{(i)}|
$$

Thus,

$$
|G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p})| > 1 \Rightarrow G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i \in (-\infty, -1) \text{ and } (1, \infty).
$$

Therefore, $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is unstable and vice versa.

(iii) Because

$$
\begin{aligned}
s_{k+1}^{(i)} &= \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*) \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k^* + G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) s_k^{(i)} + \sum_{m_i=2}^{\infty} G_{s_k^{(i)}}^{(m_i)}(\mathbf{x}_k^*, \mathbf{p}) (s_k^{(i)})^{m_i} - \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k^* \\
&= G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) s_k^{(i)} + \sum_{m_i=2}^{\infty} G_{s_k^{(i)}}^{(m_i)}(\mathbf{x}_k^*, \mathbf{p}) (s_k^{(i)})^{m_i}
\end{aligned}
$$

From the definition in Eq. (2.55)

$$\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*) = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*) \Rightarrow s_{k+1}^{(i)} = s_k^{(i)}$$

Due to any selection of $s_k^{(i)}$ and $s_{k+1}^{(i)}$ as an infinitesimal, we have

$$G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1 \text{ and } G_{s_k^{(i)}}^{(m_i)}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } m_i = 2, 3, \cdots.$$

Therefore, $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is invariant and vice versa.

(iv) From the definition in Eq. (2.55)

$$\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*) = -\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*) \Rightarrow s_{k+1}^{(i)} = -s_k^{(i)}$$

Due to any selection of $s_k^{(i)}$ and $s_{k+1}^{(i)}$ as an infinitesimal, we have

$$G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1 \text{ and } G_{s_k^{(i)}}^{(m_i)}(\mathbf{x}_k^*, \mathbf{p}) = 0 \quad \text{for } m_i = 2, 3, \cdots.$$

Therefore, $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is flipped and vice versa. The theorem is proved. ■

Definition 2.26 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$ and there are n linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2, \cdots, n)$. For a perturbation of the fixed point $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, let $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

(i) $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is monotonically stable of the $(2m_i + 1)$th-order if

$$\begin{aligned}
&G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G_{s_k^{(i)}}^{(r_i)}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i,\\
&G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{x}_k^*, \mathbf{p}) \neq 0,\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|
\end{aligned} \tag{2.61}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called the monotonic sink (or stable node) of the $(2m_i + 1)$th-order in the direction $\mathbf{v}_i$.

(ii) $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i + 1)$th-order if

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i;\\
&G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0;\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|
\end{aligned} \tag{2.62}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called the monotonic source (or unstable node) of the $(2m_i + 1)$th-order in the direction $\mathbf{v}_i$.

(iii) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, lower saddle if

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1;\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0,\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \text{ for } s_k^{(i)} > 0\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \text{ for } s_k^{(i)} < 0
\end{aligned} \tag{2.63}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called the monotonic, lower saddle of the $(2m_i)$th-order in the direction $\mathbf{v}_i$.

(iv) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, upper saddle if

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1;\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0,\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \text{ for } s_k^{(i)} > 0\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \text{ for } s_k^{(i)} < 0
\end{aligned} \tag{2.64}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called the monotonic, upper saddle of the $(2m_i)$th-order in the direction $\mathbf{v}_i$.

(v) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is oscillatory stable of the $(2m_i + 1)$th-order if

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i;\\
&G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0;\\
&|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|
\end{aligned} \tag{2.65}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called the oscillatory sink (or stable node) of the $(2m_i + 1)$th-order in the direction $\mathbf{v}_i$.

(vi) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i + 1)$th-order if

$$\begin{aligned} &G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1;\\ &G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i;\\ &G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0;\\ &|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \end{aligned} \tag{2.66}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called the oscillatory source (or unstable node) of the $(2m_i + 1)$th-order in the direction $\mathbf{v}_i$.

(vii) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i)$th-order, lower saddle-node if

$$\begin{aligned} &G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1,\\ &G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1;\\ &G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0,\\ &|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \text{ for } s_k^{(i)} > 0,\\ &|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \text{ for } s_k^{(i)} < 0, \end{aligned} \tag{2.67}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called the oscillatory lower saddle of the $(2m_i)$th-order in the direction $\mathbf{v}_i$.

(viii) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i)$th-order, upper saddle-node if

$$\begin{aligned} &G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1,\\ &G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1;\\ &G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) \neq 0,\\ &|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \text{ for } s_k^{(i)} > 0,\\ &|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)| \text{ for } s_k^{(i)} < 0 \end{aligned} \tag{2.68}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$. The fixed point $\mathbf{x}_k^*$ is called the oscillatory, upper saddle of the $(2m_i)$th-order in the direction $\mathbf{v}_i$.

The monotonic stability of fixed points with higher order singularity for a specific eigenvector is presented in Fig. 2.5. The solid curve is $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_k, \mathbf{p})$.

The circular symbol is fixed pointed. The shaded regions are stable. The horizontal solid line is also for the degenerate case. The vertical solid line is for a line with infinite slope. The monotonically stable node (sink) of the $(2m_i + 1)$th order is sketched in Fig. 2.5a. The dashed and dotted lines are for $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1}$ and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k$, respectively. The nonlinear curve lies in the unstable zone, and the iterative responses approach the fixed point. However, the monotonically unstable (source) of the $(2m_i + 1)$th order is presented in Fig. 2.5b. The nonlinear curve lies in the unstable zone, and the iterative responses go away from the fixed point. The monotonically lower saddle of the $(2m_i)$th order is presented in Fig. 2.5c. The nonlinear curve is tangential to the line of $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1}$ with the $(2m_i)$th order, and one branch is in the stable zone while the other branch is in the unstable zone. Similarly, the monotonically upper saddle of the $(2m_i)$th order is presented in Fig. 2.5d.

Similar to Fig. 2.5, the oscillatory stability of fixed points with higher order singularity for a specific eigenvector is presented in Fig. 2.6. The oscillatory stable node (sink) of the $(2m_i + 1)$th order is sketched in Fig. 2.6a. The dashed and dotted lines are for $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k$ and $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1}$, respectively. The nonlinear curve lies in the unstable zone, and the iterative responses approach the fixed point. However, the oscillatory unstable (source) of the $(2m_i + 1)$th order is presented in Fig. 2.6b. The nonlinear curve lies in the unstable zone, and the iterative responses go away from the fixed point. The oscillatory lower saddle of the $(2m_i)$th order is presented in Fig. 2.6c. The nonlinear curve is tangential to and below the line of $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_{k+1} = -\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k$ with the $(2m_i)$th order, and one branch is in the stable zone while the branch is in the unstable zone. Finally, the oscillatory upper saddle of the $(2m_i)$th order is presented in Fig. 2.6d.

Theorem 2.6 *Consider a discrete, nonlinear dynamical system* $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ *in Eq.* (2.4) with a fixed point $\mathbf{x}_k^*$. *The corresponding solution is given by* $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ *with* $j \in \mathbb{Z}$. *Suppose there is a neighborhood of the fixed point* $\mathbf{x}_k^*$ *(i.e.,* $U_k(\mathbf{x}_k^*) \subset \Omega$*), and* $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ *is* C^r $(r \geq 1)$ *-continuous in* $U_k(\mathbf{x}_k^*)$ *with Eq.* (2.28). *The linearized system is* $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ *in* $U_k(\mathbf{x}_k^*)$ *and there are n linearly independent vectors* $\mathbf{v}_i$ $(i = 1, 2, \cdots, n)$. *For a perturbation of fixed point* $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, *let* $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ *and* $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

(i) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *in the direction* $\mathbf{v}_i$ *is monotonically stable of the* $(2m_i + 1)$th *-order if and only if*

$$
\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1, \\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i, \\
&G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0
\end{aligned}
\tag{2.69}
$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

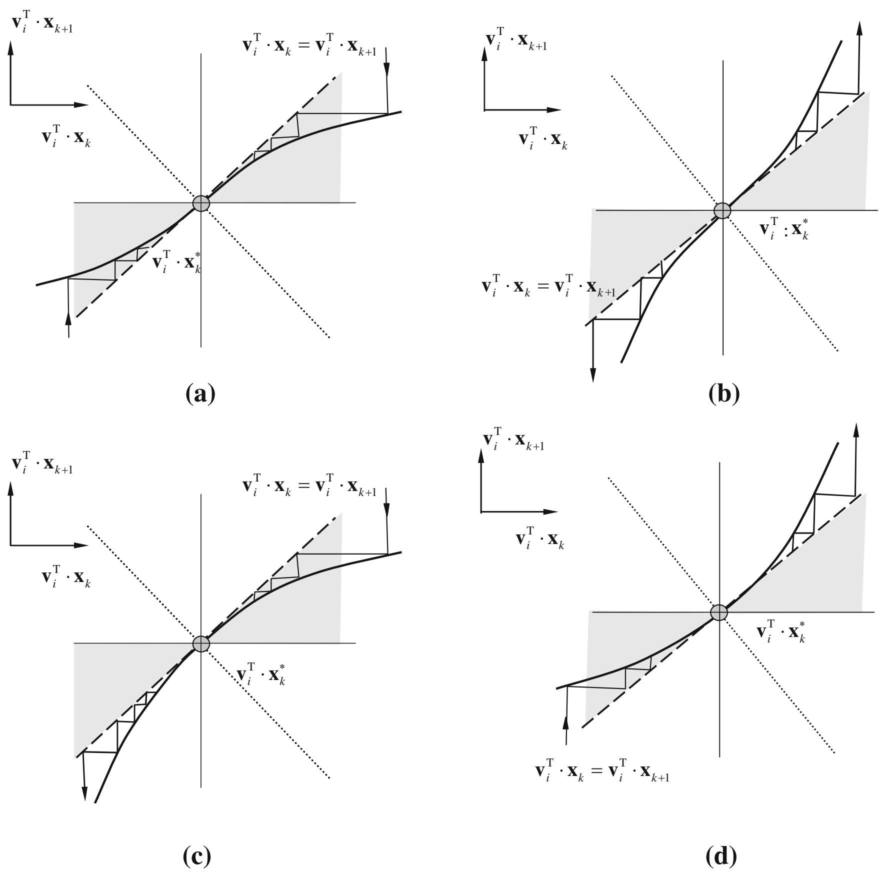

Fig. 2.5 Monotonic stability of fixed points with higher order singularity: **a** monotonically stable node (sink) of $(2m_i + 1)$th-order, **b** monotonically unstable node (source) of $(2m_i + 1)$th-order, **c** monotonically *lower* saddle of $(2m_i)$th-order and **d** monotonically upper saddle of $(2m_i)$th-order. *Shaded areas* are stable zones

(ii) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *in the direction* $\mathbf{v}_i$ *is monotonically unstable of the* $(2m_i + 1)$th *-order if and only if*

$$\begin{aligned} &G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1, \\ &G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i, \\ &G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0 \end{aligned} \tag{2.70}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(iii) $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *in the direction* $\mathbf{v}_i$ *is monotonically stable of the* $(2m_i)$th *-order, lower saddle if and only if*

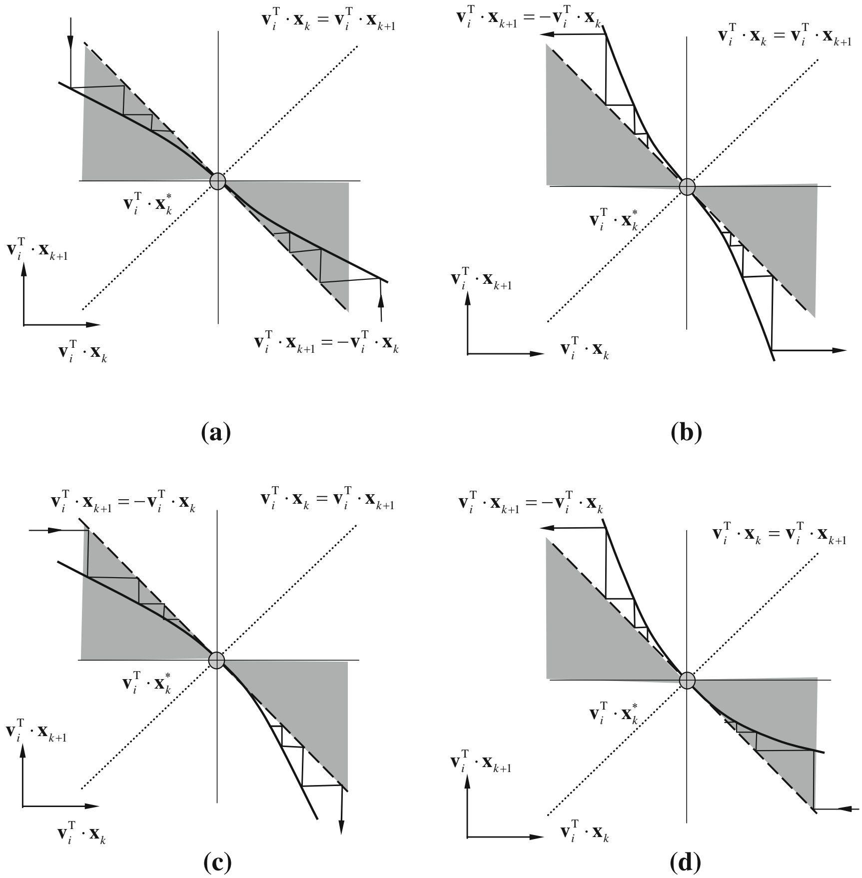

Fig. 2.6 Oscillatory stability of fixed points with higher order singularity: **a** oscillatory stable node (sink) of $(2m_i + 1)$th-order, **b** oscillatory unstable node (source) of $(2m_i + 1)$th-order, **c** oscillatory *lower* saddle of $(2m_i)$th-order and **d** oscillatory *upper* saddle of $(2m_i)$th-order. *Shaded areas* are stable zones

$$
\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1,\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0 \text{ stable for } s_k^{(i)} > 0;\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0 \text{ unstable for } s_k^{(i)} < 0
\end{aligned}
\tag{2.71}
$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(iv) $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *in the direction* $\mathbf{v}_i$ *is monotonically unstable of the* $(2m_i)$th *-order if and only if*

$$
\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = 1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1,\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0 \text{ unstable for } s_k^{(i)} > 0;\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0 \text{ stable for } s_k^{(i)} < 0
\end{aligned}
\tag{2.72}
$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(v) $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *in the direction* $\mathbf{v}_i$ *is oscillatory stable of the* $(2m_i + 1)$th *-order if and only if*

$$
\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i,\\
&G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0
\end{aligned}
\tag{2.73}
$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(vi) $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *in the direction* $\mathbf{v}_i$ *is oscillatory unstable of the* $(2m_i + 1)$th *-order if and only if*

$$
\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i,\\
&G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0
\end{aligned}
\tag{2.74}
$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(vii) $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *in the direction* $\mathbf{v}_i$ *is oscillatory unstable of the* $(2m_i)$th *-order, upper saddle if and only if*

$$
\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1,\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0 \text{ stable for } s_k^{(i)} > 0;\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0 \text{ unstable for } s_k^{(i)} < 0
\end{aligned}
\tag{2.75}
$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

(viii) $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ *at fixed point* $\mathbf{x}_k^*$ *in the direction* $\mathbf{v}_i$ *is oscillatory unstable of the* $(2m_i)$th *-order lower saddle if and only if*

$$\begin{aligned}
&G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = \lambda_i = -1,\\
&G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = 0 \text{ for } r_i = 2, 3, \cdots, 2m_i - 1,\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0 \text{ stable for } s_k^{(i)} < 0;\\
&G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0 \text{ unstable for } s_k^{(i)} > 0
\end{aligned} \tag{2.76}$$

for $\mathbf{x}_k \in U(\mathbf{x}_k^*) \subset \Omega_\alpha$.

Proof Because

$$\begin{aligned}
s_{k+1}^{(i)} =& \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)\\
=& \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k^* + G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) s_k^{(i)} + \sum_{r_i=2}^{2m_i+1} G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{r_i}\\
& - \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k^* + o((s_k^{(i)})^{2m_i+1})\\
=& G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) s_k^{(i)} + \sum_{r_i=2}^{2m_i} G^{(r_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{r_i}\\
& + G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i+1} + o((s_k^{(i)})^{2m_i+1})
\end{aligned}$$

and

$$s_k^{(i)} = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*).$$

(i) From the first two equations of Eq. (2.69), for the infinitesimal $s_k^{(i)}$, one obtains

$$s_{k+1}^{(i)} = [G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) + G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}] s_k^{(i)}.$$

Since

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|,$$

we have

$$
\begin{aligned}
|s_{k+1}^{(i)}| &= \left|[G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*,\mathbf{p}) + G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{x}_k^*,\mathbf{p})(s_k^{(i)})^{2m_i}]s_k^{(i)}\right| \\
&= \left|G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*,\mathbf{p}) + G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{x}_k^*,\mathbf{p})(s_k^{(i)})^{2m_i}\right| |s_k^{(i)}| \\
&< |s_k^{(i)}|
\end{aligned}
$$

For $G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*,\mathbf{p}) = 1$, we have

$$|1 + G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{x}_k^*,\mathbf{p})(s_k^{(i)})^{2m_i}| < 1.$$

Since the infinitesimal $s_k^{(i)}$ is arbitrarily selected, the foregoing equation gives

$$G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{x}_k^*,\mathbf{p}) < 0.$$

Therefore, $\mathbf{x}_{k+j}$ $(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ on the direction $\mathbf{v}_i$ is monotonically stable of the $(2m_i+1)$th-order, vice versa.

(ii) Similarly, since

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|,$$

we have

$$|1 + G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{x}_k^*,\mathbf{p})(s_k^{(i)})^{2m_i}| > 1.$$

For the arbitrarily infinitesimal $s_k^{(i)}$, the foregoing equation requires

$$G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{x}_k^*,\mathbf{p}) > 0.$$

Therefore, $\mathbf{x}_{k+j}(j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i+1)$th-order and vice versa.

(iii) The Taylor expansion of $s_{k+1}^{(i)}$ keeps up to the $(2m_i)$th term of $s_k^{(i)}$

$$
\begin{aligned}
s_{k+1}^{(i)} =& \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*) \\
=& \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k^* + G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*,\mathbf{p})s_k^{(i)} + \sum_{r_i=2}^{2m_i} G_{s_k^{(i)}}^{(r_i)}(\mathbf{x}_k^*,\mathbf{p})(s_k^{(i)})^{r_i} \\
& - \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{x}_k^* + o((s_k^{(i)})^{2m_i}) \\
=& G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*,\mathbf{p})s_k^{(i)} + \sum_{r_i=2}^{2m_i-1} G_{s_k^{(i)}}^{(r_i)}(\mathbf{x}_k^*,\mathbf{p})(s_k^{(i)})^{r_i} \\
& + G_{s_k^{(i)}}^{(2m_i)}(\mathbf{x}_k^*,\mathbf{p})(s_k^{(i)})^{2m_i} + o((s_k^{(i)})^{2m_i}).
\end{aligned}
$$

From the first two equations of Eq. (2.71), for the infinitesimal $s_k^{(i)}$, one obtains

$$s_{k+1}^{(i)} = s_k^{(i)} + G_{s_k^{(i)}}^{(2m_i)}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}.$$

Thus

$$\begin{aligned}|s_{k+1}^{(i)}| &= \left|[1 + G_{s_k^{(i)}}^{(2m_i)}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i-1}]s_k^{(i)}\right| \\ &= \left|1 + G_{s_k^{(i)}}^{(2m_i)}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i-1}\right| |s_k^{(i)}|.\end{aligned}$$

For $G_{s_k^{(i)}}^{(2m_i)}(\mathbf{x}_k^*, \mathbf{p}) < 0$, if $s_k^{(i)} > 0$, we have

$$|s_{k+1}^{(i)}| < |s_k^{(i)}| \Rightarrow |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|,$$

if $s_k^{(i)} < 0$, we have

$$|s_{k+1}^{(i)}| > |s_k^{(i)}| \Rightarrow |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|,$$

Thus, $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, lower saddle and vice versa.

(iv) Similar to (iii), for $G_{s_k^{(i)}}^{(2m_i)}(\mathbf{x}_k^*, \mathbf{p}) > 0$, if $s_k^{(i)} > 0$, we have

$$|s_{k+1}^{(i)}| > |s_k^{(i)}| \Rightarrow |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|,$$

if $s_k^{(i)} < 0$, we have

$$|s_{k+1}^{(i)}| < |s_k^{(i)}| \Rightarrow |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|,$$

Thus, $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, upper saddle and vice versa.

(v) Similar to case (i), consider

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|,$$

For $G_{s_k^{(i)}}^{(1)}(\mathbf{x}_k^*, \mathbf{p}) = -1$, we have

$$|-1 + G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}| < 1.$$

Since the infinitesimal $s_k^{(i)}$ is arbitrarily selected, the foregoing equation gives

$$G_{s_k^{(i)}}^{(2m_i+1)}(\mathbf{x}_k^*, \mathbf{p}) > 0.$$

Therefore, $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is oscillatory stable of the $(2m_i + 1)$th-order, vice versa.

(vi) Similar to case (ii), consider

$$|\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|,$$

For $G^{(1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) = -1$, we have

$$|-1 + G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}| > 1.$$

Since the infinitesimal $s_k^{(i)}$ is arbitrarily selected, the foregoing equation gives

$$G^{(2m_i+1)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0.$$

Therefore, $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i + 1)$th-order and vice versa.

(vii) Similar to (iii), from the first two equations of Eq. (2.69), for the infinitesimal $s_k^{(i)}$, one obtains

$$s_{k+1}^{(i)} = -s_k^{(i)} + G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i}$$

Thus

$$\begin{aligned} |s_{k+1}^{(i)}| &= \left|[-1 + G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i-1}]s_k^{(i)}\right| \\ &= \left|-1 + G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p})(s_k^{(i)})^{2m_i-1}\right| |s_k^{(i)}|. \end{aligned}$$

For $G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) > 0$, if $s_k^{(i)} > 0$, we have

$$|s_{k+1}^{(i)}| < |s_k^{(i)}| \Rightarrow |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|,$$

if $s_k^{(i)} < 0$, we have

$$|s_{k+1}^{(i)}| > |s_k^{(i)}| \Rightarrow |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|.$$

Thus, $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is oscillatory unstable of the $(2m_i)$th-order, upper saddle and vice versa.

(viii) Similar to (vii), for $G^{(2m_i)}_{s_k^{(i)}}(\mathbf{x}_k^*, \mathbf{p}) < 0$, if $s_k^{(i)} > 0$, we have

$$|s_{k+1}^{(i)}| > |s_k^{(i)}| \Rightarrow |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| > |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|,$$

if $s_k^{(i)} > 0$, we have

$$|s_{k+1}^{(i)}| < |s_k^{(i)}| \Rightarrow |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k^*)| < |\mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*)|.$$

Thus, $\mathbf{x}_{k+j} (j \in \mathbb{Z})$ at fixed point $\mathbf{x}_k^*$ in the direction $\mathbf{v}_i$ is monotonically unstable of the $(2m_i)$th-order, lower saddle and vice versa. This theorem is proved. ■

Definition 2.27 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. Consider a pair of complex eigenvalue $\alpha_i \pm \mathbf{i}\beta_i$ $(i \in N = \{1, 2, \cdots, n\},\ \mathbf{i} = \sqrt{-1})$ of matrix $D\mathbf{f}(\mathbf{x}^*, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{r}_k^{(i)} = \mathbf{y}_k^{(i)} = \mathbf{y}_{k+}^{(i)} + \mathbf{y}_{k-}^{(i)}$ with

$$\begin{aligned} \mathbf{r}_k^{(i)} &= c_k^{(i)}\mathbf{u}_i + d_k^{(i)}\mathbf{v}_i, \\ \mathbf{r}_{k+1}^{(i)} &= c_{k+1}^{(i)}\mathbf{u}_i + d_{k+1}^{(i)}\mathbf{v}_i. \end{aligned} \tag{2.77}$$

and

$$\begin{aligned} c_k^{(i)} &= \frac{1}{\Delta}[\Delta_2(\mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{y}_k) - \Delta_{12}(\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k)], \\ d_k^{(i)} &= \frac{1}{\Delta}[\Delta_1(\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k) - \Delta_{12}(\mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{y}_k)]; \\ \Delta_1 &= ||\mathbf{u}_i||^2,\ \Delta_2 = ||\mathbf{v}_i||^2,\ \Delta_{12} = \mathbf{u}_i^{\mathrm{T}} \cdot \mathbf{v}_i; \\ \Delta &= \Delta_1\Delta_2 - \Delta_{12}^2 \end{aligned} \tag{2.78}$$

Consider a polar coordinate of (r_k, θ_k) defined by

$$\begin{aligned} &c_k^{(i)} = r_k^{(i)}\cos\theta_k^{(i)},\ \text{and}\ d_k^{(i)} = r_k^{(i)}\sin\theta_k^{(i)}; \\ &r_k^{(i)} = \sqrt{(c_k^{(i)})^2 + (d_k^{(i)})^2},\ \text{and}\ \theta_k^{(i)} = \arctan d_k^{(i)}/c_k^{(i)}. \end{aligned} \tag{2.79}$$

Thus

$$\begin{aligned} c_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) - \Delta_{12} G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p})] \\ d_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) - \Delta_{12} G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p})] \end{aligned} \tag{2.80}$$

where

$$\begin{aligned} G_{c_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_k^*] = \sum\nolimits_{m_i=1}^{\infty} G_{c_k^{(i)}}^{(m_i)}(\theta_k^{(i)})(r_k^{(i)})^{m_i}, \\ G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_k^*] = \sum\nolimits_{m_i=1}^{\infty} G_{d_k^{(i)}}^{(m_i)}(\theta_k^{(i)})(r_k^{(i)})^{m_i}; \end{aligned} \tag{2.81}$$

$$\begin{aligned} G_{c_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) &= \mathbf{u}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(m_i)}\mathbf{f}(\mathbf{x}_k, \mathbf{p})[\mathbf{u}_i\cos\theta_k^{(i)} + \mathbf{v}_i\sin\theta_k^{(i)}]^{m_i}\Big|_{(\mathbf{x}_k^*, \mathbf{p})}, \\ G_{d_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(m)}\mathbf{f}(\mathbf{x}_k, \mathbf{p})[\mathbf{u}_i\cos\theta_k^{(i)} + \mathbf{v}_i\sin\theta_k^{(i)}]^{m_i}\Big|_{(\mathbf{x}_k^*, \mathbf{p})}. \end{aligned} \tag{2.82}$$

Thus

$$
\begin{aligned}
r_{k+1}^{(i)} &= \sqrt{(c_{k+1}^{(i)})^2 + (d_{k+1}^{(i)})^2} = \sqrt{\sum\nolimits_{m=2}^{\infty} (r_k^{(i)})^{m_i} G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)})} \\
&= \sqrt{G_{r_{k+1}^{(i)}}^{(2)}}\, r_k^{(i)} \sqrt{1 + (G_{r_{k+1}^{(i)}}^{(2)})^{-1} \sum\nolimits_{m=3}^{\infty} (r_k^{(i)})^{m_i-2} G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)})} \\
\theta_{k+1}^{(i)} &= \arctan(d_{k+1}^{(i)}/c_{k+1}^{(i)})
\end{aligned}
\tag{2.83}
$$

where

$$
G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)}) = [G_{c_{k+1}^{(i)}}^{(r_i)}(\theta_k^{(i)}) G_{c_{k+1}^{(i)}}^{(s_i)}(\theta_k^{(i)}) + G_{d_{k+1}^{(i)}}^{(r_i)}(\theta_k^{(i)}) G_{d_{k+1}^{(i)}}^{(s_i)}(\theta_k^{(i)})]\delta_{m_i}^{(r_i+s_i)}. \tag{2.84}
$$

and

$$
\begin{aligned}
G_{c_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)}) &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) - \Delta_{12} G_{d_k^{(i)}}^{(m_i)}(\theta_k^{(i)})], \\
G_{d_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)}) &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}^{(m_i)}(\theta_k^{(i)}) - \Delta_{12} G_{c_k^{(i)}}^{(m_i)}(\theta_k^{(i)})].
\end{aligned}
\tag{2.85}
$$

From the foregoing definition, consider the first order terms of G-function

$$
\begin{aligned}
G_{c_k^{(i)}}^{(1)}(\mathbf{x}_k, \mathbf{p}) &= G_{c_k^{(i)}1}^{(1)}(\mathbf{x}_k, \mathbf{p}) + G_{c_k^{(i)}2}^{(1)}(\mathbf{x}_k, \mathbf{p}), \\
G_{d_k^{(i)}}^{(1)}(\mathbf{x}_k, \mathbf{p}) &= G_{d_k^{(i)}1}^{(1)}(\mathbf{x}_k, \mathbf{p}) + G_{d_k^{(i)}2}^{(1)}(\mathbf{x}_k, \mathbf{p}),
\end{aligned}
\tag{2.86}
$$

where

$$
\begin{aligned}
G_{c_k^{(i)}1}^{(1)}(\mathbf{x}_k, \mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k, \mathbf{p})\partial_{c_k^{(i)}}\mathbf{x}_k = \mathbf{u}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k, \mathbf{p})\mathbf{u}_i \\
&= \mathbf{u}_i^{\mathrm{T}} \cdot (-\beta_i \mathbf{v}_i + \alpha_i \mathbf{u}_i) = \alpha_i \Delta_1 - \beta_i \Delta_{12}, \\
G_{c_k^{(i)}2}^{(1)}(\mathbf{x}_k, \mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k, \mathbf{p})\partial_{d_k^{(i)}}\mathbf{x}_k = \mathbf{u}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k, \mathbf{p})\mathbf{v}_i \\
&= \mathbf{u}_i^{\mathrm{T}} \cdot (\beta_i \mathbf{u}_i + \alpha_i \mathbf{v}_i) = \alpha_i \Delta_{12} + \beta_i \Delta_1;
\end{aligned}
\tag{2.87}
$$

and

$$
\begin{aligned}
G_{d_k^{(i)}1}^{(1)}(\mathbf{x}_k, \mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k, \mathbf{p})\partial_{c_k^{(i)}}\mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k, \mathbf{p})\mathbf{u}_i \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot (-\beta_i \mathbf{v}_i + \alpha_i \mathbf{u}_i) = -\beta_i \Delta_2 + \alpha_i \Delta_{12}, \\
G_{d_k^{(i)}2}^{(1)}(\mathbf{x}, \mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k, \mathbf{p})\partial_{d_k^{(i)}}\mathbf{x}_k = \mathbf{v}_i^{\mathrm{T}} \cdot D_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k, \mathbf{p})\mathbf{v}_i \\
&= \mathbf{v}_i^{\mathrm{T}} \cdot (\beta_i \mathbf{u}_i + \alpha_i \mathbf{v}_i) = \alpha_i \Delta_2 + \beta_i \Delta_{12}.
\end{aligned}
\tag{2.88}
$$

Substitution of Eqs. (2.85)–(2.87) into Eq. (2.82) gives

$$
\begin{aligned}
G^{(1)}_{c_k^{(i)}}(\theta_k^{(i)}) &= G^{(1)}_{c_k^{(i)}1}(\mathbf{x}_k,\mathbf{p})\cos\theta_k^{(i)} + G^{(1)}_{c_k^{(i)}2}(\mathbf{x}_k,\mathbf{p})\sin\theta_k^{(i)} \\
&= (\alpha_i\Delta_1 - \beta_i\Delta_{12})\cos\theta_k^{(i)} + (\alpha_i\Delta_{12} + \beta_i\Delta_1)\sin\theta_k^{(i)}, \\
G^{(1)}_{d_k^{(i)}}(\theta_k^{(i)}) &= G^{(1)}_{d_k^{(i)}1}(\mathbf{x}_k,\mathbf{p})\cos\theta_k^{(i)} + G^{(1)}_{d_k^{(i)}2}(\mathbf{x}_k,\mathbf{p})\sin\theta_k^{(i)} \\
&= (-\beta_i\Delta_2 + \alpha_i\Delta_{12})\cos\theta_k^{(i)} + (\alpha_i\Delta_2 + \beta_i\Delta_{12})\sin\theta_k^{(i)}.
\end{aligned}
\tag{2.89}
$$

From Eq. (2.84), we have

$$
\begin{aligned}
G^{(1)}_{c_{k+1}^{(i)}}(\theta_k^{(i)}) &= \frac{1}{\Delta}[\Delta_2 G^{(1)}_{c_k^{(i)}}(\theta_k^{(i)}) - \Delta_{12}G^{(1)}_{d_k^{(i)}}(\theta_k^{(i)})] \\
&= \alpha_i\cos\theta_k^{(i)} + \beta_i\sin\theta_k^{(i)} \\
G^{(1)}_{d_{k+1}^{(i)}}(\theta_k^{(i)}) &= \frac{1}{\Delta}[\Delta_1 G^{(1)}_{d_k^{(i)}}(\theta_k^{(i)}) - \Delta_{12}G^{(1)}_{c_k^{(i)}}(\theta_k^{(i)})] \\
&= \alpha_i\sin\theta_k^{(i)} - \beta_i\cos\theta_k^{(i)}.
\end{aligned}
\tag{2.90}
$$

Thus

$$
\begin{aligned}
G^{(2)}_{r_{k+1}^{(i)}}(\theta_k^{(i)}) &= [G^{(1)}_{c_{k+1}^{(i)}}(\theta_k^{(i)})G^{(1)}_{c_{k+1}^{(i)}}(\theta_k^{(i)}) + G^{(1)}_{d_{k+1}^{(i)}}(\theta_k^{(i)})G^{(1)}_{d_{k+1}^{(i)}}(\theta_k^{(i)})] \\
&= \alpha_i^2 + \beta_i^2.
\end{aligned}
\tag{2.91}
$$

Furthermore, Eq. (2.83) gives

$$
r_{k+1}^{(i)} = \rho_i r_k^{(i)} + o(r_k^{(i)}) \text{ and } \theta_{k+1}^{(i)} = \theta_k^{(i)} - \vartheta_i + o(r_k^{(i)}),
\tag{2.92}
$$

where

$$
\vartheta_i = \arctan(\beta_i/\alpha_i) \text{ and } \rho_i = \sqrt{\alpha_i^2 + \beta_i^2}.
\tag{2.93}
$$

As $r_k^{(i)} \ll 1$ and $r_k \to 0$, we have

$$
r_{k+1}^{(i)} = \rho_i r_k^{(i)} \text{ and } \theta_{k+1}^{(i)} = \vartheta_i - \theta_k^{(i)}.
\tag{2.94}
$$

With an initial condition of $r_k^{(i)} = r_k^0$ and $\theta_k^{(i)} = \theta_k^{(i)}$, the corresponding solution of Eq. (2.94) is

$$
r_{k+j}^{(i)} = (\rho_i)^j r_k^0 \text{ and } \theta_{k+j}^{(i)} = j\vartheta_i - \theta_k^{(i)}.
\tag{2.95}
$$

From Eq. (2.90), we have

$$
\begin{aligned}
c_{k+1}^{(i)} &= \alpha_i r_k^{(i)}\cos\theta_k^{(i)} + \beta_i r_k^{(i)}\sin\theta_k^{(i)} = \alpha_i c_k^{(i)} + \beta_i d_k^{(i)}, \\
d_{k+1}^{(i)} &= \alpha_i r_k^{(i)}\sin\theta_k^{(i)} - \beta_i r_k^{(i)}\cos\theta_k^{(i)} = -\beta_i c_k^{(i)} + \alpha_i d_k^{(i)}.
\end{aligned}
\tag{2.96}
$$

That is,

$$\begin{Bmatrix} c_{k+1}^{(i)} \\ d_{k+1}^{(i)} \end{Bmatrix} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \begin{Bmatrix} c_k^{(i)} \\ d_k^{(i)} \end{Bmatrix} = \rho_i \begin{bmatrix} \cos\vartheta_i & \sin\vartheta_i \\ -\sin\vartheta_i & \cos\vartheta_i \end{bmatrix} \begin{Bmatrix} c_k^{(i)} \\ d_k^{(i)} \end{Bmatrix}. \tag{2.97}$$

From the foregoing equation, we have

$$\begin{Bmatrix} c_{k+j}^{(i)} \\ d_{k+j}^{(i)} \end{Bmatrix} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}^j \begin{Bmatrix} c_k^{(i)} \\ d_k^{(i)} \end{Bmatrix} = (\rho_i)^j \begin{bmatrix} \cos j\vartheta_i & \sin j\vartheta_i \\ -\sin j\vartheta_i & \cos j\vartheta_i \end{bmatrix} \begin{Bmatrix} c_k^{(i)} \\ d_k^{(i)} \end{Bmatrix}. \tag{2.98}$$

Definition 2.28 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r ($r \geq 1$)-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ ($\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*$) in $U_k(\mathbf{x}_k^*)$. Consider a pair of complex eigenvalues $\alpha_i \pm \mathbf{i}\beta_i$ ($i \in N = \{1, 2, \cdots, n\}$, $\mathbf{i} = \sqrt{-1}$) of matrix $D\mathbf{f}(\mathbf{x}^*, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{r}_k^{(i)} = \mathbf{y}_k^{(i)} = \mathbf{y}_{k+}^{(i)} + \mathbf{y}_{k-}^{(i)}$ with Eqs. (2.77) and (2.79). For any arbitrarily small $\varepsilon > 0$, the stability of the fixed point $\mathbf{x}_k^*$ on the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$ can be determined.

(i) $\mathbf{x}^{(k)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable if

$$r_{k+1}^{(i)} - r_k^{(i)} < 0. \tag{2.99}$$

(ii) $\mathbf{x}^{(k)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally unstable if

$$r_{k+1}^{(i)} - r_k^{(i)} > 0. \tag{2.100}$$

(iii) $\mathbf{x}^{(k)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is stable with the m_ith-order singularity if for $\theta_k^{(i)} \in [0, 2\pi]$

$$\begin{aligned} &\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1, \\ &G_{r_{k+1}^{(i)}}^{(s_k^{(i)})}(\theta_k) = 0 \quad \text{for } s_k^{(i)} = 1, 2, \cdots, m_i - 1 \\ &r_{k+1}^{(i)} - r_k^{(i)} < 0. \end{aligned} \tag{2.101}$$

(iv) $\mathbf{x}^{(k)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally unstable with the m_ith-order singularity if for $\theta_k^{(i)} \in [0, 2\pi]$

$$\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1,$$
$$G^{(s_k^{(i)})}_{r_{k+1}^{(i)}}(\theta_k) = 0 \quad \text{for } s_k^{(i)} = 0, 1, 2, \cdots, m_i - 1 \tag{2.102}$$
$$r_{k+1}^{(i)} - r_k^{(i)} > 0.$$

(v) $\mathbf{x}^{(k)}$ at the fixed point $\mathbf{x}_k^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is circular if for $\theta_k^{(i)} \in [0, 2\pi]$

$$r_{k+1}^{(i)} - r_k^{(i)} = 0. \tag{2.103}$$

(vi) $\mathbf{x}^{(k)}$ at the fixed point $\mathbf{x}^*$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is degenerate in the direction of $\mathbf{u}_k$ if

$$\beta_i = 0 \text{ and } \theta_{k+1}^{(i)} - \theta_k^{(i)} = 0. \tag{2.104}$$

Theorem 2.7 *Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^{2n}$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. Consider a pair of complex eigenvalues $\alpha_i \pm \mathbf{i}\beta_i$ $(i \in N = \{1, 2, \cdots, n\},\ \mathbf{i} = \sqrt{-1})$ of matrix $D\mathbf{f}(\mathbf{x}^*, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{r}_k^{(i)} = \mathbf{y}_k^{(i)} = \mathbf{y}_{k+}^{(i)} + \mathbf{y}_{k-}^{(i)}$ with Eqs. (2.77) and (2.79). For any arbitrarily small $\varepsilon > 0$, the stability of the fixed point $\mathbf{x}_k^*$ on the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$ can be determined.*

(i) $\mathbf{x}^{(k)}$ *at the fixed point* $\mathbf{x}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is spirally stable if and only if*

$$\rho_i < 1. \tag{2.105}$$

(ii) $\mathbf{x}^{(k)}$ *at the fixed point* $\mathbf{x}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is spirally unstable if*

$$\rho_i > 1. \tag{2.106}$$

(iii) $\mathbf{x}^{(k)}$ *at the fixed point* $\mathbf{x}_k^*$ *on the plane of* $\mathbf{u}_i, \mathbf{v}_i$ *is stable with the* (m_k)th-*order singularity if and only if for* $\theta_k^{(i)} \in [0, 2\pi]$

$$\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1,$$
$$G^{(s_k^{(i)})}_{r_{k+1}^{(i)}}(\theta_k^{(i)}) = 0 \quad \text{for } s_k^{(i)} = 1, 2, \cdots, m_i - 1 \tag{2.107}$$
$$G^{(m_i)}_{r_{k+1}^{(i)}}(\theta_k^{(i)}) < 0.$$

(iv) $\mathbf{x}^{(k)}$ *at the fixed point* $\mathbf{x}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is spirally unstable with the* (m_i)th-*order singularity if and only if for* $\theta_k^{(i)} \in [0, 2\pi]$

$$\begin{aligned}
&\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1,\\
&G_{r_{k+1}^{(i)}}^{(s_k^{(i)})}(\theta_k^{(i)}) = 0 \quad \text{for } s_k^{(i)} = 0, 1, 2, \cdots, m_i - 1 \\
&G_{r_{k+1}^{(i)}}^{(m_i)}(\theta_k^{(i)}) > 0.
\end{aligned} \tag{2.108}$$

(v) $\mathbf{x}^{(k)}$ *at the fixed point* $\mathbf{x}_k^*$ *on the plane of* $(\mathbf{u}_i, \mathbf{v}_i)$ *is circular if and only if for* $\theta_k^{(i)} \in [0, 2\pi]$

$$\begin{aligned}
&\rho_i = \sqrt{\alpha_i^2 + \beta_i^2} = 1,\\
&G_{r_{k+1}^{(i)}}^{(s_k^{(i)})}(\theta_k^{(i)}) = 0 \quad \text{for } s_k^{(i)} = 0, 1, 2, \cdots.
\end{aligned} \tag{2.109}$$

Proof Since

$$\begin{aligned}
c_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) - \Delta_{12} G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p})]\\
d_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p}) - \Delta_{12} G_{d_k^{(i)}}(\mathbf{x}_k, \mathbf{p})]
\end{aligned}$$

For $\mathbf{x}_{k+1} = \mathbf{x}_k = \mathbf{x}_k^*,\ r_k = 0$. The first order approximation of $c_{k+1}^{(i)}$ and $d_{k+1}^{(i)}$ in the Taylor series expansion gives

$$\begin{aligned}
c_{k+1}^{(i)} &= G_{c_{k+1}^{(i)}}^{(1)}(\theta_k^{(i)}) r_k^{(i)} + o(r_k^{(i)}),\\
d_{k+1}^{(i)} &= G_{d_{k+1}^{(i)}}^{(1)}(\theta_k^{(i)}) r_k^{(i)} + o(r_k^{(i)})
\end{aligned}$$

where $r_k^{(i)} = \sqrt{(c_k^{(i)})^2 + (d_k^{(i)})^2}$ and $\theta_k^{(i)} = \arctan(d_k^{(i)}/c_k^{(i)})$

$$\begin{aligned}
G_{c_{k+1}^{(i)}}^{(1)}(\theta_k^{(i)}) &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}^{(1)}(\theta_k^{(i)}) - \Delta_{12} G_{d_k^{(i)}}^{(1)}(\theta_k^{(i)})]\\
G_{d_{k+1}^{(i)}}^{(1)}(\theta_k^{(i)}) &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}^{(1)}(\theta_k^{(i)}) - \Delta_{12} G_{c_k^{(i)}}^{(1)}(\theta_k^{(i)})]
\end{aligned}$$

and

$$\begin{aligned}
G_{c_k^{(i)}}^{(1)}(\theta_k^{(i)}) &= (\alpha_i \Delta_1 - \beta_i \Delta_{12}) \cos\theta_k^{(i)} + (\alpha_i \Delta_{12} + \beta_i \Delta_1) \sin\theta_k^{(i)},\\
G_{d_k^{(i)}}^{(1)}(\theta_k^{(i)}) &= (-\beta_i \Delta_2 + \alpha_i \Delta_{12}) \cos\theta_k^{(i)} + (\alpha_i \Delta_2 + \beta_i \Delta_{12}) \sin\theta_k^{(i)}.
\end{aligned}$$

Therefore,

$$G^{(1)}_{c^{(i)}_{k+1}}(\theta^{(i)}_k) = \alpha_i \cos\theta^{(i)}_k + \beta_i \sin\theta^{(i)}_k,$$
$$G^{(1)}_{d^{(i)}_{k+1}}(\theta^{(i)}_k) = \alpha_i \sin\theta^{(i)}_k - \beta_i \cos\theta^{(i)}_k.$$

Further

$$c^{(i)}_{k+1} = \alpha_i r^{(i)}_k \cos\theta^{(i)}_k + \beta_i r^{(i)}_k \sin\theta^{(i)}_k = \alpha_i c^{(i)}_k + \beta_i d^{(i)}_k,$$
$$d^{(i)}_{k+1} = \alpha_i r^{(i)}_k \sin\theta^{(i)}_k - \beta_i r^{(i)}_k \cos\theta^{(i)}_k = -\beta_i c^{(i)}_k + \alpha_i d^{(i)}_k.$$

That is,

$$\begin{Bmatrix} c^{(i)}_{k+1} \\ d^{(i)}_{k+1} \end{Bmatrix} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \begin{Bmatrix} c^{(i)}_k \\ d^{(i)}_k \end{Bmatrix} = \rho_i \begin{bmatrix} \cos\vartheta_i & \sin\vartheta_i \\ -\sin\vartheta_i & \cos\vartheta_i \end{bmatrix} \begin{Bmatrix} c^{(i)}_k \\ d^{(i)}_k \end{Bmatrix}.$$

From the foregoing equation, we have

$$r^{(i)}_{k+1} = \rho_i r^{(i)}_k + o(r^{(i)}_k) \text{ and } \theta^{(i)}_{k+1} = \theta^{(i)}_k - \vartheta_i + o(r^{(i)}_k).$$

where

$$\vartheta_i = \arctan(\beta_i/\alpha_i) \text{ and } \rho_i = \sqrt{\alpha_i^2 + \beta_i^2}.$$

As $r^{(i)}_k \ll 1$ and $r_k \to 0$, we have

$$r^{(i)}_{k+1} = \rho_i r^{(i)}_k \text{ and } \theta^{(i)}_{k+1} = \vartheta_i - \theta^{(i)}_k.$$

(i) For fixed point stability, if $\rho_i < 1$, then

$$r^{(i)}_{k+1} < r^{(i)}_k$$

which implies that $\mathbf{x}^{(i)}_k$ at the fixed point $\mathbf{x}^*_k$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable and vice versa.

(ii) If $\rho_i > 1$, then

$$r^{(i)}_{k+1} > r^{(i)}_k$$

which implies that $\mathbf{x}^{(i)}_k$ at the fixed point $\mathbf{x}^*_k$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable and vice versa.

(iii) If for $\theta^{(i)}_k \in [0, 2\pi]$ the following conditions exist

$$\rho_i = \sqrt{G^{(2)}_{r^{(i)}_{k+1}}} = \sqrt{\alpha_i^2 + \beta_i^2} = 1,$$

$$G^{(s^{(i)}_k)}_{r^{(i)}_{k+1}}(\theta^{(i)}_k) = 0 \quad \text{for } s^{(i)}_k = 1, 2, \cdots, m_i - 1$$

$$G^{(m_i)}_{r^{(i)}_{k+1}}(\theta^{(i)}_k) \neq 0, \quad \text{and } |G^{(s^{(i)}_k)}_{r^{(i)}_{k+1}}(\theta^{(i)}_k)| < \infty \text{ for } s^{(i)}_k = m_i + 1, m + 2 \cdots .$$

then the higher terms can be ignored, i.e.,

$$r^{(i)}_{k+1} = r^{(i)}_k \sqrt{1 + \sum\nolimits_{m_i=3}^{\infty} (r^{(i)}_k)^{m_i - 2} G^{(m_i)}_{r^{(i)}_{k+1}}(\theta^{(i)}_k)}.$$

If $G^{(m_i)}_{r^{(i)}_{k+1}}(\theta^{(i)}_k)$ is independent of θ_k (i.e., $G^{(m_i)}_{r^{(i)}_{k+1}}(\theta^{(i)}_k) = \text{const}$), it can be used to determine the equilibrium stability. If $G^{(m_i)}_{r^{(i)}_{k+1}}(\theta^{(i)}_k) < 0$, then

$$r^{(i)}_{k+1} < r^{(i)}_k$$

In other words this implies $\mathbf{x}^{(i)}_k$ at the fixed point $\mathbf{x}^*_k$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally stable and vice versa.

(iv) If $G^{(m_i)}_{r^{(i)}_{k+1}}(\theta^{(i)}_k) > 0$, then

$$r^{(i)}_{k+1} > r^{(i)}_k$$

That is, $\mathbf{x}^{(k)}$ at the fixed point $\mathbf{x}^*_k$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is spirally unstable with the $(m_k - 1)$th-order singularity and vice versa.

(v) If for $\theta^{(i)}_k \in [0, 2\pi]$ the following conditions exist

$$G^{(s^{(i)}_k)}_{r^{(i)}_{k+1}}(\theta^{(i)}_k) = 0 \text{ for } s^{(i)}_k = 1, 2, \cdots,$$

then

$$r^{(i)}_{k+1} = r^{(i)}_k$$

and vice versa. Therefore $\mathbf{x}^{(k)}$ at the fixed point $\mathbf{x}^*_k$ on the plane of $(\mathbf{u}_i, \mathbf{v}_i)$ is circular. This theorem is proved. ■

2.3.2 Bifurcations

Definition 2.29 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}^*_k$. The corresponding solution is given by

$\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$ and there are n linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2, \cdots, n)$. For a perturbation of the fixed point $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, let $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$.

$$s_i^{(i)} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_k = \mathbf{v}_i^{\mathrm{T}} \cdot (\mathbf{x}_k - \mathbf{x}_k^*) \tag{2.110}$$

where $s_k^{(i)} = c_k^{(i)}||\mathbf{v}_i||^2$.

$$s_{k+1}^{(i)} = \mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{y}_{k+1} = \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_k^*]. \tag{2.111}$$

In the vicinity of point $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$, $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ can be expended for $(0 < \theta < 1)$ as

$$\begin{aligned} \mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_{k(0)}^*] = {} & a_i(s_k^{(i)} - s_{k(0)}^{(i)*}) + \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) \\ & + \sum_{r=0}^{m>1} C_m^r \mathbf{a}_i^{(m-r,r)} (s_k^{(i)} - s_{k(0)}^{(i)*})^{m-r} (\mathbf{p} - \mathbf{p}_0)^r \\ & + [(s_k^{(i)} - s_{k(0)}^{(i)*})\partial_{s_k^{(i)}} + (\mathbf{p} - \mathbf{p}_0)\partial_{\mathbf{p}}]^{m+1} \\ & \times (\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_{k(0)}^* + \theta\Delta\mathbf{x}_k, \mathbf{p}_0 + \theta\Delta\mathbf{p})) \end{aligned} \tag{2.112}$$

where

$$\begin{aligned} a_i &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)}, \\ \mathbf{b}_i^{\mathrm{T}} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)}, \\ \mathbf{a}_i^{(r,s)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(r)} \partial_{\mathbf{p}}^{(s)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)}. \end{aligned} \tag{2.113}$$

If $a_i = 1$ and $\mathbf{p} = \mathbf{p}_0$, the stability of fixed point $\mathbf{x}_k^*$ on an eigenvector $\mathbf{v}_i$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold in the direction of $\mathbf{v}_i$ is determined by

$$\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + \sum_{r=0}^{m>1} C_m^r \mathbf{a}_i^{(m-r,r)} (s_k^{(i)*} - s_{k(0)}^{(i)*})^{m-r} (\mathbf{p} - \mathbf{p}_0)^r = 0. \tag{2.114}$$

In the neighborhood of $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$, when other components of fixed point $\mathbf{x}_k^*$ on the eigenvector of $\mathbf{v}_j$ for all $j \neq i$, $(i, j \in N)$ do not change their stability states, Eq. (2.114) possesses l-branch solutions of equilibrium $s_k^{(i)*}$ $(0 < l \leq m)$ with l_1-stable and l_2-unstable solutions $(l_1, l_2 \in \{0, 1, 2, \cdots, l\})$. Such l-branch solutions are called the bifurcation solutions of fixed point $\mathbf{x}_k^*$ on the eigenvector of $\mathbf{v}_i$ in the neighborhood of $(\mathbf{x}^*_{k(0)}, \mathbf{p}_0)$. Such a bifurcation at point $(\mathbf{x}^*_{k(0)}, \mathbf{p}_0)$ is called the hyperbolic bifurcation of mth-order on the eigenvector of $\mathbf{v}_i$. Consider two special cases herein.

(i) If

$$\mathbf{a}_i^{(1,1)} = \mathbf{0} \text{ and } \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + a_i^{(2,0)}(s_k^{(i)*} - s_{k0}^{(i)*})^2 = 0 \tag{2.115}$$

where

$$\begin{aligned} a_i^{(2,0)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial^{(2)}_{s_k^{(i)}} \partial^{(0)}_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}^*_{k(0)},\mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial^{(2)}_{s_k^{(i)}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}^*_{k(0)},\mathbf{p}_0)} \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial^{(2)}_{\mathbf{x}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})(\mathbf{v}_k\mathbf{v}_k)\Big|_{(\mathbf{x}^*_{k(0)},\mathbf{p}_0)} = G^{(2)}_{s_k^{(i)}}(\mathbf{x}^*_{k(0)}, \mathbf{p}_0) \neq 0, \\ \mathbf{b}_i^{\mathrm{T}} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}^*_{k(0)},\mathbf{p}_0)} \neq \mathbf{0}, \end{aligned} \tag{2.116}$$

$$a_i^{(2,0)} \times [\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0)] < 0, \tag{2.117}$$

such a bifurcation at point $(\mathbf{x}^*_0, \mathbf{p}_0)$ is called the *saddle-node* bifurcation on the eigenvector of $\mathbf{v}_i$.

(ii) If

$$\begin{aligned} &\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) = 0 \text{ and} \\ &\mathbf{a}_i^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0)(s_k^{(i)*} - s_{k(0)}^{(i)*}) + a_i^{(2,0)}(s_k^{(i)*} - s_{k(0)}^{(i)*})^2 = 0 \end{aligned} \tag{2.118}$$

where

$$\begin{aligned} a_i^{(2,0)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial^{(2)}_{s_k^{(i)}} \partial^{(0)}_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}^*_{k(0)},\mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial^{(2)}_{s_k^{(i)}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}^*_0,\mathbf{p}_0)} \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial^{(2)}_{\mathbf{x}_k} \mathbf{f}(\mathbf{x}_k, \mathbf{p})(\mathbf{v}_i\mathbf{v}_i)\Big|_{(\mathbf{x}^*_{k(0)},\mathbf{p}_0)} = G^{(2)}_{s_k^{(i)}}(\mathbf{x}^*_{k(0)}, \mathbf{p}_0) \neq 0, \\ \mathbf{a}_i^{(1,1)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial^{(1)}_{s_k^{(i)}} \partial^{(1)}_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}^*_{k(0)},\mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}} \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}^*_{k(0)},\mathbf{p}_0)} \\ &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k} \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\mathbf{v}_i\Big|_{(\mathbf{x}^*_{k(0)},\mathbf{p}_0)} \neq \mathbf{0}, \end{aligned} \tag{2.119}$$

$$a_i^{(2,0)} \times [\mathbf{a}_i^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0)] < 0, \tag{2.120}$$

such a bifurcation at point $(\mathbf{x}^*_{k(0)}, \mathbf{p}_0)$ is called the *transcritical* bifurcation on the eigenvector of $\mathbf{v}_i$.

Definition 2.30 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in \mathscr{R}^n$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is C^r $(r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}$ $(\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$ and there are n linearly independent vectors $\mathbf{v}_i$ $(i = 1, 2, \cdots, n)$. For a perturbation of fixed point $\mathbf{y}_k = \mathbf{x}_k - \mathbf{x}_k^*$, let $\mathbf{y}_k^{(i)} = c_k^{(i)}\mathbf{v}_i$ and $\mathbf{y}_{k+1}^{(i)} = c_{k+1}^{(i)}\mathbf{v}_i$. Equations (2.49)–(2.52) hold. In the vicinity of point $(\mathbf{x}_{k0}^*, \mathbf{p}_0)$, $\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_k, \mathbf{p})$ can be expended for $(0 < \theta < 1)$ as

$$\begin{aligned}
\mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_k, \mathbf{p}) - \mathbf{x}_{k+1(0)}^*] = {} & a_i(s_k^{(i)} - s_{k(0)}^{(i)*}) + \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) \\
& + \sum_{r=0}^{m>1} C_m^r \mathbf{a}_i^{(m-r,r)} (s_k^{(i)} - s_{k(0)}^{(i)*})^{m-r} (\mathbf{p} - \mathbf{p}_0)^r \\
& + [(s_k^{(i)} - s_{k(0)}^{(i)*})\partial_{s_k^{(i)}} + (\mathbf{p} - \mathbf{p}_0)\partial_{\mathbf{p}}]^{m+1} \\
& \times (\mathbf{v}_k^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_{k(0)}^* + \theta\Delta\mathbf{x}_k, \mathbf{p}_0 + \theta\Delta\mathbf{p}))
\end{aligned} \tag{2.121}$$

and

$$\begin{aligned}
\mathbf{v}_i^{\mathrm{T}} \cdot [\mathbf{f}(\mathbf{x}_{k+1}, \mathbf{p}) - \mathbf{x}_{k(0)}^*] = {} & a_i(s_{k+1}^{(i)} - s_{k+1(0)}^{(i)*}) + \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) \\
& + \sum_{r=0}^{m>1} C_m^r \mathbf{a}_i^{(m-r,r)} (s_{k+1}^{(i)} - s_{k+1(0)}^{(i)*})^{m-r} (\mathbf{p} - \mathbf{p}_0)^r \\
& + [(s_{k+1}^{(i)} - s_{k+1(0)}^{(i)*})\partial_{s_{k+1}^{(i)}} + (\mathbf{p} - \mathbf{p}_0)\partial_{\mathbf{p}}]^{m+1} \\
& \times (\mathbf{v}_i^{\mathrm{T}} \cdot \mathbf{f}(\mathbf{x}_{k+1(0)}^* + \theta\Delta\mathbf{x}_{k+1}, \mathbf{p}_0 + \theta\Delta\mathbf{p}))
\end{aligned} \tag{2.122}$$

If $a_i = -1$ and $\mathbf{p} = \mathbf{p}_0$, the stability of current equilibrium $\mathbf{x}_k^*$ on an eigenvector $\mathbf{v}_i$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold in the direction of $\mathbf{v}_i$ is determined by

$$\begin{aligned}
& \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + a_i(s_k^{(i)*} - s_{k(0)}^{(i)*}) + \sum_{r=0}^{m>1} C_m^r \mathbf{a}_i^{(m-r,r)} (s_k^{(i)*} - s_{k(0)}^{(i)*})^{m-r} (\mathbf{p} - \mathbf{p}_0)^r \\
& = (s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*}); \\
& \mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) + a_i(s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*}) + \sum_{r=0}^{m>1} C_m^r \mathbf{a}_i^{(m-r,r)} (s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*})^{m-r} (\mathbf{p} - \mathbf{p}_0)^r \\
& = (s_k^{(i)*} - s_{k(0)}^{(i)*}).
\end{aligned} \tag{2.123}$$

In the neighborhood of $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$, when other components of fixed point $\mathbf{x}_{k(0)}^*$ on the eigenvector of $\mathbf{v}_j$ for all $j \neq i$, $(j, i \in N)$ do not change their stability states, Eq. (2.123) possesses l-branch solutions of equilibrium $s_k^{(i)*}$ $(0 < l \leq m)$ with l_1-stable and l_2-unstable solutions $(l_1, l_2 \in \{0, 1, 2, \cdots, l\})$. Such l-branch solutions are called the bifurcation solutions of fixed point $\mathbf{x}_k^*$ on the eigenvector of $\mathbf{v}_i$ in the neighborhood of $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$. Such a bifurcation at point $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$ is called the *hyperbolic bifurcation* of mth-order with doubling iterations on the eigenvector of $\mathbf{v}_i$. Consider a special case. If

$$\begin{aligned}
&\mathbf{b}_i^{\mathrm{T}} \cdot (\mathbf{p} - \mathbf{p}_0) = 0, a_i = -1, a_i^{(2,0)} = 0, \mathbf{a}_i^{(2,1)} = 0, \mathbf{a}_i^{(1,2)} = 0,\\
&[\mathbf{a}^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0) + a_i](s_k^{(i)*} - s_{k(0)}^{(i)*}) + a_i^{(3,0)}(s_k^* - s_{k(0)}^*)^3\\
&= (s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*}),\\
&[\mathbf{a}^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0) + a_i](s_{k+1}^{(i)*} - s_{k+1(0)}^{(i)*})] + a_i^{(3,0)}(s_{k+1}^* - s_{k+1(0)}^*)^3\\
&= (s_k^{(i)*} - s_{k(0)}^{(i)*})
\end{aligned} \tag{2.124}$$

where

$$\begin{aligned}
a_i^{(3,0)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(3)} \partial_{\mathbf{p}}^{(0)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(3)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)}\\
&= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k}^{(3)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})(\mathbf{v}_i\mathbf{v}_i\mathbf{v}_i)\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = G_{s_k^{(i)}}^{(3)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0) \neq 0,\\
\mathbf{a}_i^{(1,1)} &= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}}^{(1)} \partial_{\mathbf{p}}^{(1)} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} = \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{s_k^{(i)}} \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)}\\
&= \mathbf{v}_i^{\mathrm{T}} \cdot \partial_{\mathbf{x}_k} \partial_{\mathbf{p}} \mathbf{f}(\mathbf{x}_k, \mathbf{p})\mathbf{v}_i\Big|_{(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)} \neq \mathbf{0},
\end{aligned} \tag{2.125}$$

$$a_i^{(3,0)} \times [\mathbf{a}_i^{(1,1)} \cdot (\mathbf{p} - \mathbf{p}_0)] < 0, \tag{2.126}$$

such a bifurcation at point $(\mathbf{x}_{k(0)}^*, \mathbf{p}_0)$ is called the *pitchfork* bifurcation (or period-doubling bifurcation) on the eigenvector of $\mathbf{v}_i$.

The three types of special cases can be discussed through 1-D systems and intuitive illustrations are presented in Fig. 2.7 for a better understanding of bifurcation for nonlinear discrete maps. Similarly, other cases on the eigenvector of $\mathbf{v}_i$ can be discussed from Eqs. (2.114) and (2.123). In Fig. 2.7, the bifurcation point is also represented by a solid circular symbol. The stable and unstable fixed point branches are given by solid and dashed curves, respectively. The vector fields are represented by lines with arrows. If no fixed points exist, such a region is shaded.

Consider a saddle-node bifurcation in 1-D system

$$x_{k+1} = f(x_k, p) \equiv x_k + p - x_k^2. \tag{2.127}$$

For $x_{k+1} = x_k$, the fixed points of the foregoing equation are $x_k^* = \pm\sqrt{p}$ $(p > 0)$ and no fixed points exist for $p < 0$. From Eq. (2.127), the linearized equation in the vicinity of the fixed points with $y_k = x_k - x_k^*$ is

$$y_{k+1} = Df(x_k^*, p)y_k = (1 - 2x_k^*)y_k. \tag{2.128}$$

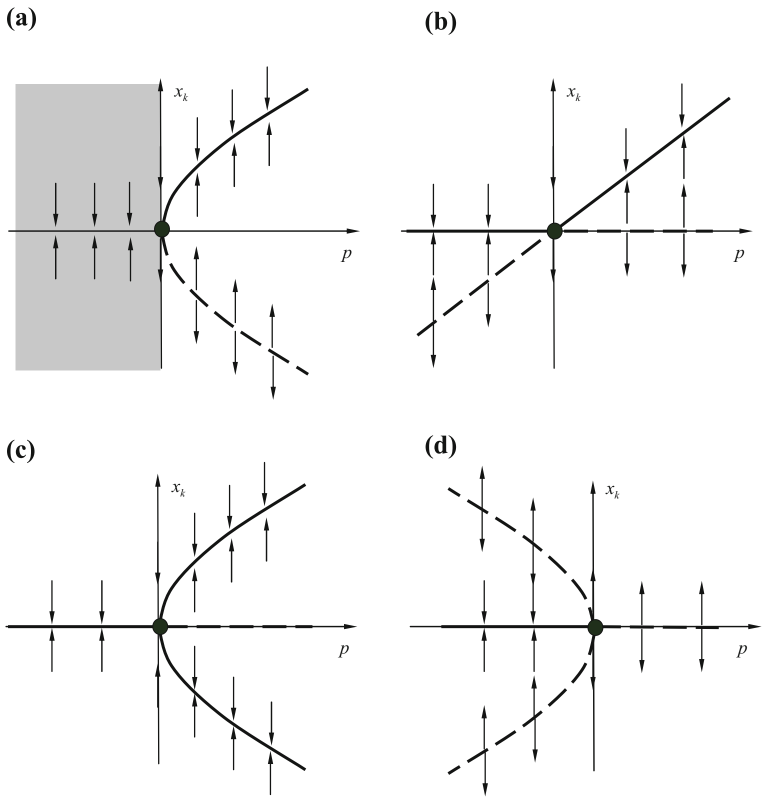

Fig. 2.7 Bifurcation diagrams: **a** saddle-node bifurcation of the first kind, **b** transcritical bifurcation, **c** pitchfork bifurcation for stable-symmetry (or saddle-node bifurcation of the second kind) and **d** pitchfork bifurcation for unstable-symmetry (or unstable saddle-node bifurcation of the second kind)

For the branch of $x_k^* = +\sqrt{p}$ $(p > 0)$, the fixed point is stable due to $|y_{k+1}| < |y_k|$. However, for the branch of $x_k^* = -\sqrt{p}$ $(p > 0)$, such a fixed point is unstable due to $|y_{k+1}| > |y_k|$, For $p = p_0 = 0$, we have $x_k^* = x_{k(0)}^* = 0$ and $Df(x_{k(0)}^*, p_0) = 1$. Since $D^2 f(x_{k(0)}^*, p_0) = -2 < 0$, is needed. Thus

$$y_{k+1} = y_k + D^2 f(x_{k(0)}^*, p_0) y_k^2 = (1 - 2y_k) y_k. \tag{2.129}$$

At $(x_{k(0)}^*, p_0) = (0, 0)$, $|y_{k+1}| < |y_k|$ for $y_k > 0$ and $|y_{k+1}| > |y_k|$ for $y_k < 0$. The fixed point $(x_{k(0)}^*, p_0) = (0, 0)$ is bifurcation point, which is a decreasing saddle of the second order. For $p < 0$, from Eq. (2.127), $p - (x_k^*)^2 < 0$. Thus, no fixed point exists. The fixed point x_k^* varying with parameter p is sketched in Fig. 2.7a. On the left side of x_k-axes, no fixed point exists. So only the vector field of the map in Eq. (2.127) is presented.

Consider a transcritical bifurcation through a 1-D discrete system as

$$x_{k+1} = f(x_k, p) \equiv x_k + px_k - x_k^2. \tag{2.130}$$

The fixed points of the map in the foregoing equation are $x_k^* = 0, p$. From Eq. (2.130), the linearized equation in the vicinity of the fixed points with $y_k = x_k - x_k^*$ is

$$y_{k+1} = Df(x_k^*, p)y_k = (1 + p - 2x_k^*)y_k. \tag{2.131}$$

For the branch of $x_k^* = 0$ $(p > 0)$, the fixed point is unstable due to $|y_{k+1}| > |y_k|$. For the branch of $x_k^* = p$ $(p > 0)$, such a fixed point is stable because of $|y_{k+1}| < |y_k|$. However, for the branch of $x_k^* = 0$ $(p < 0)$, the fixed point is stable due to $|y_{k+1}| > |y_k|$. For the branch of $x_k^* = p$ $(p < 0)$, such a fixed point is unstable owing to $|y_{k+1}| < |y_k|$. For $p = p_0 = 0$, $x_k^* = x_{k(0)}^* = 0$ and $Df(x_{k(0)}^*, p_0) = 1$ are obtained. $D^2 f(x_{k(0)}^*, p_0) = -2$ is needed. Thus the variational equation at the fixed point is given by $y_{k+1} = y_k - 2y_k^2 = (1 - 2y_k)y_k$. From this equation, at $(x_{k(0)}^*, p_0) = (0, 0)$, $|y_{k+1}| < |y_k|$ for $y_k > 0$ and $|y_{k+1}| > |y_k|$ for $y_k < 0$. The fixed point $(x_{k(0)}^*, p_0) = (0, 0)$ is a bifurcation point, which is a decreasing saddle of the second order. The fixed point varying with parameter p is sketched in Fig. 2.7b.

Consider the pitchfork bifurcation with stable-symmetry (or saddle-node bifurcation of the second kind, or period-doubling bifurcation) with a 1-D system as

$$x_{k+1} = (-1 - p)x_k + x_k^3. \tag{2.132}$$

For $x_{k+1} = x_k = x_k^*$, the corresponding fixed point are $x_k^* = 0, \pm\sqrt{p}$ $(p > 0)$ and $x_k^* = 0$ $(p \le 0)$. From Eq. (2.132), the linearized equation in the vicinity of the fixed point with $y_k = x_k - x_k^*$ is

$$y_k = Df(x_k^*, p)y_k = [-1 - p + 3(x_k^*)^2]y_k. \tag{2.133}$$

For the branch of $x_k^* = 0$ $(p > 0)$, the fixed point is unstable due to $|y_{k+1}| > |y_k|$. For the branches of $x_k^* = \pm\sqrt{p}$ $(p > 0)$, such two fixed points are stable because of $|y_{k+1}| < |y_k|$. However, for the branch of $x_k^* = 0$ $(p < 0)$, the fixed point is stable due to $|y_{k+1}| < |y_k|$. For $p = p_0 = 0$, $x_k^* = x_{k(0)}^* = 0$ and $Df(x_{k(0)}^*, p_0) = -1$ are obtained. However, $D^2 f(x_{k(0)}^*, p_0) = 6x_{k(0)}^* = 0$ is also obtained. Further, $\mathrm{D}^3 f(x_{k(0)}^*, p_0) = 6 > 0$ is computed. Thus the variational equation at the fixed point is

$$y_{k+1} = -y_k + D^3 f(x_{k(0)}^*, p_0)y_k^3 = (-1 + 6y_k^2)y_k. \tag{2.134}$$

At $(x_{k(0)}^*, p_0) = (0, 0)$, $|y_{k+1}| < |y_k|$ exists always. The fixed point $(x_{k0}^*, p_0) = (0, 0)$ is as bifurcation point, which is an oscillatory sink of the third order due to $D^3 f > 0$. The fixed point varying with parameter p is sketched in Fig. 2.7c.

Consider the pitchfork bifurcation for unstable-symmetry (or unstable saddle-node bifurcation of the second kind, or unstable period-doubling bifurcation) with a ID system as

$$x_{k+1} = (-1-p)x_k - x_k^3 \tag{2.135}$$

For $x_{k+1} = x_k = x_k^*$, the fixed points are $x_k^* = 0, \pm\sqrt{-p}\ (p<0)$ and $x_k^* = 0\ (p \geq 0)$. From Eq. (2.135), the linearized equation in vicinity of fixed points with $y_k = x_k - x_k^*$ is

$$y_{k+1} = Df(x_k^*, p)y_k = [-1-p-3(x_k^*)^2]y_k. \tag{2.136}$$

For the branch of $x_k^* = 0\ (p<0)$, the fixed point is stable due to $|y_{k+1}| < |y_k|$. For the branches of $x_k^* = \pm\sqrt{-p}\ (p<0)$, such two fixed points are unstable due to $|y_{k+1}| > |y_k|$. However, for the branch of $x_k^* = 0\ (p>0)$, the fixed point is unstable due to $|y_{k+1}| > |y_k|$. For $p = p_0 = 0,\ x_k^* = x_{k(0)}^* = 0$ and $Df(x_{k(0)}^*, p_0) = -1$ are obtained. $D^2 f(x_{k(0)}^*, p_0) = -6x_{k(0)}^* = 0$ are obtained. Furthermore, $D^3 f(x_{k(0)}^*, p_0) = -6 < 0$. Thus the variational equation at the fixed point is

$$y_{k+1} = -y_k + D^3 f(x_{k(0)}^*, p_0)y_k^3 = (-1-6y_k^2)y_k. \tag{2.137}$$

At $(x_{k(0)}^*, p_0) = (0,0)$, $|y_{k+1}| > |y_k|$ exists always. The fixed point $(x_0^*, p_0) = (0,0)$ is a bifurcation point, which is an oscillatory source of the third order. The fixed point varying with parameter p is sketched in Fig. 2.7d.

From the proceeding analysis, the bifurcation points possess the higher-order singularity of the flow in discrete dynamical system. For the saddle-node bifurcation of the first kind, the $(2m)$th order singularity of the flow at the bifurcation point exists as a saddle of the $(2m)$th order. For the transcritical bifurcation, the $(2m)$th order singularity of the flow at the bifurcation point exists as a saddle of of the $(2m)$th order. However, for the stable pitchfork bifurcation (or saddle-node bifurcation of the second kind, or period-doubling bifurcation), the $(2m+1)$th order singularity of the flow at the bifurcation point exists as an oscillatory sink of the $(2m+1)$th order. For the unstable pitchfork bifurcation (or the unstable saddle- node bifurcation of the second kind, or unstable period-doubling bifurcation), the $(2m+1)$th order singularity of the flow at the bifurcation point exists as an oscillatory source of the $(2m+1)$th order.

Definition 2.31 Consider a discrete, nonlinear dynamical system $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}) \in R^{2n}$ in Eq. (2.4) with a fixed point $\mathbf{x}_k^*$. The corresponding solution is given by $\mathbf{x}_{k+j} = \mathbf{f}(\mathbf{x}_{k+j-1}, \mathbf{p})$ with $j \in \mathbb{Z}$. Suppose there is a neighborhood of the fixed point $\mathbf{x}_k^*$ (i.e., $U_k(\mathbf{x}_k^*) \subset \Omega$), and $\mathbf{f}(\mathbf{x}_k, \mathbf{p})$ is $C^r\ (r \geq 1)$-continuous in $U_k(\mathbf{x}_k^*)$ with Eq. (2.28). The linearized system is $\mathbf{y}_{k+j+1} = D\mathbf{f}(\mathbf{x}_k^*, \mathbf{p})\mathbf{y}_{k+j}\ (\mathbf{y}_{k+j} = \mathbf{x}_{k+j} - \mathbf{x}_k^*)$ in $U_k(\mathbf{x}_k^*)$. Consider a pair of complex eigenvalues $\alpha_i \pm \mathbf{i}\beta_i\ (i \in N = \{1, 2, \cdots, n\},\ \mathbf{i} = \sqrt{-1})$ of matrix $D\mathbf{f}(\mathbf{x}^*, \mathbf{p})$ with a pair of eigenvectors $\mathbf{u}_i \pm \mathbf{i}\mathbf{v}_i$. On the invariant plane of $(\mathbf{u}_i, \mathbf{v}_i)$, consider $\mathbf{r}_k^{(i)} = \mathbf{y}_k^{(i)} = \mathbf{y}_{k+}^{(i)} + \mathbf{y}_{k-}^{(i)}$ with

$$\begin{aligned}\mathbf{r}_k^{(i)} &= c_k^{(i)}\mathbf{u}_i + d_k^{(i)}\mathbf{v}_i,\\ \mathbf{r}_{k+1}^{(i)} &= c_{k+1}^{(i)}\mathbf{u}_i + d_{k+1}^{(i)}\mathbf{v}_i.\end{aligned} \tag{2.138}$$

and

$$\begin{aligned}c_k^{(i)} &= \frac{1}{\Delta}[\Delta_2(\mathbf{u}_i^{\mathrm{T}}\cdot\mathbf{y}_k) - \Delta_{12}(\mathbf{v}_i^{\mathrm{T}}\cdot\mathbf{y}_k)],\\ d_k^{(i)} &= \frac{1}{\Delta}[\Delta_1(\mathbf{v}_i^{\mathrm{T}}\cdot\mathbf{y}_k) - \Delta_{12}(\mathbf{u}_i^{\mathrm{T}}\cdot\mathbf{y}_k)];\\ \Delta_1 &= ||\mathbf{u}_i||^2,\ \Delta_2 = ||\mathbf{v}_i||^2,\ \Delta_{12} = \mathbf{u}_i^{\mathrm{T}}\cdot\mathbf{v}_i;\\ \Delta &= \Delta_1\Delta_2 - \Delta_{12}^2\end{aligned} \tag{2.139}$$

Consider a polar coordinate of (r_k, θ_k) defined by

$$\begin{aligned}c_k^{(i)} &= r_k^{(i)}\cos\theta_k^{(i)},\ \text{and}\ d_k^{(i)} = r_k^{(i)}\sin\theta_k^{(i)};\\ r_k^{(i)} &= \sqrt{(c_k^{(i)})^2 + (d_k^{(i)})^2},\ \text{and}\ \theta_k^{(i)} = \arctan d_k^{(i)}/c_k^{(i)}.\end{aligned} \tag{2.140}$$

Thus

$$\begin{aligned}c_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_2 G_{c_k^{(i)}}(\mathbf{x}_k,\mathbf{p}) - \Delta_{12}G_{d_k^{(i)}}(\mathbf{x}_k,\mathbf{p})]\\ d_{k+1}^{(i)} &= \frac{1}{\Delta}[\Delta_1 G_{d_k^{(i)}}(\mathbf{x}_k,\mathbf{p}) - \Delta_{12}G_{d_k^{(i)}}(\mathbf{x}_k,\mathbf{p})]\end{aligned} \tag{2.141}$$

where

$$\begin{aligned}G_{c_k^{(i)}}(\mathbf{x}_k,\mathbf{p}) &= \mathbf{u}_i^{\mathrm{T}}\cdot[\mathbf{f}(\mathbf{x}_k,\mathbf{p}) - \mathbf{x}_{k(0)}^*]\\ &= \mathbf{a}_i^{\mathrm{T}}\cdot(\mathbf{p}-\mathbf{p}_0) + a_{i11}(c_k^{(i)} - c_{k(0)}^{(i)*}) + a_{i12}(d_k^{(i)} - d_{k(0)}^{(i)*})\\ &\quad + \sum\nolimits_{r_i=0}^{m_i>1} C_{m_i}^{r_i}\mathbf{G}_{c_k^{(i)}}^{(m_i-r_i,r_i)}(\mathbf{x}_k^*,\mathbf{p}_0)(\mathbf{p}-\mathbf{p}_0)^{r_i}(r_k^{(i)})^{m_i-r_i}\\ &\quad + [(c_k^{(i)} - c_{k(0)}^{(i)*})\partial_{c_k^{(i)}} + (d_k^{(i)} - d_{k(0)}^{(i)*})\partial_{d_k^{(i)}} + (\mathbf{p}-\mathbf{p}_0)\partial_{\mathbf{p}}]^{m_i+1}\\ &\quad \times(\mathbf{u}_i^{\mathrm{T}}\cdot\mathbf{f}(\mathbf{x}_{k0}^* + \theta\Delta\mathbf{x}_k, \mathbf{p}_0 + \theta\Delta\mathbf{p})),\\ G_{d_k^{(i)}}(\mathbf{x}_k,\mathbf{p}) &= \mathbf{v}_i^{\mathrm{T}}\cdot[\mathbf{f}(\mathbf{x}_k,\mathbf{p}) - \mathbf{x}_{k(0)}^*]\\ &= \mathbf{b}_i^{\mathrm{T}}\cdot(\mathbf{p}-\mathbf{p}_0) + a_{i21}(c_k^{(i)} - c_{k(0)}^{(i)*}) + a_{i22}(d_k^{(i)} - d_{k(0)}^{(i)*})\\ &\quad + \sum\nolimits_{r_i=0}^{m_i>1} C_{m_i}^{r_i}\mathbf{G}_{d_k^{(i)}}^{(m_i-r_i,r_i)}(\mathbf{x}_k^*,\mathbf{p}_0)(\mathbf{p}-\mathbf{p}_0)^{r_i} r_k^{m_i-r_i}\\ &\quad + [(c_k^{(i)} - c_{k(0)}^{(i)*})\partial_{c_k^{(i)}} + (d_k^{(i)} - d_{k(0)}^{(i)*})\partial_{d_k^{(i)}} + (\mathbf{p}-\mathbf{p}_0)\partial_{\mathbf{p}}]^{m+1}\\ &\quad \times(\mathbf{v}_i^{\mathrm{T}}\cdot\mathbf{f}(\mathbf{x}_{k(0)}^* + \theta\Delta\mathbf{x}, \mathbf{p}_0 + \theta\Delta\mathbf{p}));\end{aligned} \tag{2.142}$$

and

$$
\begin{aligned}
&\mathbf{G}^{(s,r)}_{c_k^{(i)}}(\mathbf{x}^*_{k(0)},\mathbf{p}_0)\\
&=\mathbf{u}_i^{\mathrm{T}}\cdot[\partial_{\mathbf{x}_k}()\mathbf{u}_i\cos\theta_k^{(i)}+\partial_{\mathbf{x}_k}()\mathbf{v}_i\sin\theta_k^{(i)}]^s\partial_{\mathbf{p}}^{(r)}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\Big|_{(\mathbf{x}^*_{k(0)},\mathbf{p}_0)},\\
&\mathbf{G}^{(s,r)}_{d_k^{(i)}}(\mathbf{x}^*_{k(0)},\mathbf{p}_0)\\
&=\mathbf{v}_i^{\mathrm{T}}\cdot[\partial_{\mathbf{x}_k}()\mathbf{u}_i\cos\theta_k^{(i)}+\partial_{\mathbf{x}_k}()\mathbf{v}_i\sin\theta_k^{(i)}]^s\partial_{\mathbf{p}}^{(r)}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\Big|_{(\mathbf{x}^*_{k(0)},\mathbf{p}_0)};
\end{aligned}
\tag{2.143}
$$

$$
\begin{aligned}
\mathbf{a}_i^{\mathrm{T}}&=\mathbf{u}_i^{\mathrm{T}}\cdot\partial_{\mathbf{p}}\mathbf{f}(\mathbf{x}_k,\mathbf{p}),\ \mathbf{b}_i^{\mathrm{T}}=\mathbf{v}_i^{\mathrm{T}}\cdot\partial_{\mathbf{p}}\mathbf{f}(\mathbf{x}_k,\mathbf{p});\\
a_{i11}&=\mathbf{u}_i^{\mathrm{T}}\cdot\partial_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{u}_i,\ a_{i12}=\mathbf{u}_i^{\mathrm{T}}\cdot\partial_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{u}_i;\\
a_{i21}&=\mathbf{v}_i^{\mathrm{T}}\cdot\partial_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{u}_i,\ a_{i22}=\mathbf{v}_i^{\mathrm{T}}\cdot\partial_{\mathbf{x}_k}\mathbf{f}(\mathbf{x}_k,\mathbf{p})\mathbf{v}_i.
\end{aligned}
\tag{2.144}
$$

Suppose

$$
\mathbf{a}_i=\mathbf{0}\text{ and }\mathbf{b}_i=\mathbf{0}
\tag{2.145}
$$

then

$$
\begin{aligned}
r_{k+1}^{(i)}&=\sqrt{(c_{k+1}^{(i)})^2+(d_{k+1}^{(i)})^2}=\sqrt{\sum\nolimits_{m=2}^{\infty}(r_k^{(i)})^m G^{(m)}_{r_{k+1}^{(i)}}}\\
&=\sqrt{G^{(2,0)}_{r_{k+1}^{(i)}}}\,r_k^{(i)}\sqrt{1+\lambda^{(i)}+\sum\nolimits_{m=3}^{\infty}\lambda_m^{(i)}(r_k^{(i)})^{m-2}}\\
\theta_{k+1}^{(i)}&=\arctan(d_{k+1}^{(i)}/c_{k+1}^{(i)})
\end{aligned}
\tag{2.146}
$$

where

$$
\begin{aligned}
G^{(2)}_{r_{k+1}^{(i)}}&=G^{(2,0)}_{r_{k+1}^{(i)}}+G^{(1,1)}_{r_{k+1}^{(i)}}\text{ and }\lambda^{(i)}=G^{(1,1)}_{r_{k+1}^{(i)}}/G^{(2,0)}_{r_{k+1}^{(i)}}\text{ with}\\
G^{(2,0)}_{r_{k+1}^{(i)}}&=[G^{(1,0)}_{c_{k+1}^{(i)}}(\theta_k^{(i)},\mathbf{p}_0)]^2+[G^{(1,0)}_{d_{k+1}^{(i)}}(\theta_k^{(i)},\mathbf{p}_0)]^2,\\
G^{(1,1)}_{r_{k+1}^{(i)}}&=[\mathbf{G}^{(1,1)}_{c_{k+1}^{(i)}}(\theta_k^{(i)},\mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)]^2+[\mathbf{G}^{(1,1)}_{d_{k+1}^{(i)}}(\theta_k^{(i)},\mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)]^2;
\end{aligned}
\tag{2.147}
$$

and

$$
\begin{aligned}
\lambda_m^{(i)}&=G^{(m)}_{r_{k+1}^{(i)}}/G^{(2,0)}_{r_{k+1}^{(i)}}\text{ with}\\
G^{(m)}_{r_{k+1}^{(i)}}&=\sum\nolimits_{m_i=0}^{\infty}\sum\nolimits_{m_j=0}^{\infty}[\mathbf{G}^{(m_i-r_i,r_i)}_{c_{k+1}^{(i)}}(\theta_k^{(i)},\mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_i-r_i}]\\
&\quad\times\mathbf{G}^{(m_j-s_j,s_j)}_{c_{k+1}^{(i)}}(\theta_k^{(i)},\mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_j-s_j}
\end{aligned}
$$

$$\begin{aligned}&+ \mathbf{G}^{(m_i-r_i,r_i)}_{d^{(i)}_{k+1}}(\theta^{(i)}_k,\mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_i-r_i}]\\&\times \mathbf{G}^{(m_j-s_j,s_j)}_{d^{(i)}_{k+1}}(\theta^{(i)}_k,\mathbf{p}_0)\cdot(\mathbf{p}-\mathbf{p}_0)^{m_j-s_j}]\delta_m^{(r_i+s_j)}.\end{aligned} \tag{2.148}$$

$$\begin{aligned}\mathbf{G}^{(m-r,r)}_{c^{(i)}_{k+1}}(\theta_k,\mathbf{p}_0)&=\frac{1}{\Delta}[\Delta_2\mathbf{G}^{(m-r,r)}_{c^{(i)}_k}(\mathbf{x}^*_{k(0)},\mathbf{p}_0)-\Delta_{12}\mathbf{G}^{(m-r,r)}_{d^{(i)}_k}(\mathbf{x}^*_{k(0)},\mathbf{p}_0)],\\\mathbf{G}^{(m-r,r)}_{d^{(i)}_{k+1}}(\theta_k,\mathbf{p}_0)&=\frac{1}{\Delta}[\Delta_{12}\mathbf{G}^{(m-r,r)}_{c^{(i)}_k}(\mathbf{x}^*_{k(0)},\mathbf{p}_0)-\Delta_1\mathbf{G}^{(m-r,r)}_{d^{(i)}_k}(\mathbf{x}^*_{k(0)},\mathbf{p}_0)].\end{aligned} \tag{2.149}$$

If $G^{(2,0)}_{r^{(i)}_{k+1}}=1$ and $\mathbf{p}=\mathbf{p}_0$, the stability of current fixed point $\mathbf{x}^*_k$ on an eigenvector plane of $(\mathbf{u}_i,\mathbf{v}_i)$ changes from stable to unstable state (or from unstable to stable state). The bifurcation manifold in the direction of $\mathbf{v}_k$ is determined by

$$\lambda^{(i)}+\sum_{m=3}^{\infty}\lambda^{(i)}_m(r^{(i)}_k)^{m-2}=0. \tag{2.150}$$

Such a bifurcation at the fixed point $(\mathbf{x}^*_{k(0)},\mathbf{p}_0)$ is called the generalized Neimark bifurcation on the eigenvector plane of $(\mathbf{u}_i,\mathbf{v}_i)$.

For a special case, if

$$\lambda^{(i)}+\lambda^{(i)}_4(r^{(i)}_k)^2=0,\ \text{for}\ \lambda^{(i)}\times\lambda^{(i)}_4<0\ \text{and}\ \lambda^{(i)}_3=0 \tag{2.151}$$

such a bifurcation at the fixed point $(\mathbf{x}^*_{k(0)},\mathbf{p}_0)$ is called the Neimark bifurcation on the eigenvector plane of $(\mathbf{u}_i,\mathbf{v}_i)$.

For the repeating eigenvalues of $DP(\mathbf{x}^*_k,\mathbf{p})$, the bifurcation of fixed point $\mathbf{x}^*_k$ can be similarly discussed in the foregoing two Theorems 2.5 and 2.6. Herein, such a procedure will not be repeated.

Consider a dynamical system

$$\begin{aligned}x_{k+1}&=\alpha[1+\lambda+a(x_k^2+y_k^2)]x_k+\beta[1+\lambda+a(x_k^2+y_k^2)]y_k,\\y_{k+1}&=-\beta[1+\lambda+a(x_k^2+y_k^2)]x_k+\alpha[1+\lambda+a(x_k^2+y_k^2)]y_k.\end{aligned} \tag{2.152}$$

Setting

$$r_k^2=x_k^2+y_k^2\ \text{with}\ x_k=r_k\cos\theta_k\ \text{and}\ y_k=r_k\sin\theta_k, \tag{2.153}$$

we have

$$\begin{aligned}r_{k+1}&=\sqrt{x_{k+1}^2+y_{k+1}^2}=\rho r_k(1+\lambda+ar_k^2),\\\theta_{k+1}&=\arctan\frac{-\beta\cos\theta_k+\alpha\sin\theta_k}{\beta\sin\theta_k+\alpha\cos\theta_k}=\theta_k-\vartheta\\\rho&=\sqrt{\alpha^2+\beta^2}\ \text{and}\ \vartheta=\arctan\frac{\beta}{\alpha}\end{aligned} \tag{2.154}$$

If $\rho = 1$, the fixed point is

$$\begin{aligned} r^*_{k(1)} &= 0 \text{ for } \lambda \in (-\infty, +\infty) \\ r^*_{k(2)} &= (-\lambda/a)^{1/2} \text{ for } \lambda \times a < 0. \end{aligned} \tag{2.155}$$

If $\lambda \neq 0$, we have $Df_r(r^*_k, \lambda) = 1 + \lambda + 3a(r^*_k)^2$, the variational equation is

$$s_{k+1} = Df_r(r^*_k, \lambda)s_k = [1 + \lambda + 3a(r^*_k)^2]s_k \text{ with } s_k = r_k - r^*_k. \tag{2.156}$$

For $r^*_{k(1)} = 0$, $Df_r = 1 + \lambda$. This fixed point is stable for $\lambda < 0$ owing to $s_{k+1} < s_k$ or unstable for $\lambda > 0$ owing to $s_{k+1} > s_k$. The fixed point is a critical point for $\lambda = 0$. The fixed point of $r^*_{k(2)} = (-\lambda/a)^{1/2}$ requires $a\lambda < 0$. For $a > 0$, such a fixed point exists for $\lambda < 0$. For $a < 0$, the fixed point existence condition is $\lambda > 0$. From $Df_r = 1 - 2\lambda$, the fixed point is stable for $\lambda > 0$ owing to $s_{k+1} < s_k$ and unstable for $\lambda < 0$ owing to $s_{k+1} > s_k$. For $\lambda = 0$, we have $r^*_{k(0)} = 0$ and

$$Df_r(r^*_k, \lambda) = 1 \text{ and } D_\lambda Df_r(r^*, \alpha) = 1 \neq 0. \tag{2.157}$$

For $r^*_{k(0)} = 0$ and $\lambda = 0$, $Df_r(r^*_k, \lambda) = 1$ and $D^2 f_r(r^*_k, \lambda) = 6ar^*_k = 0$ exists. So we have $D^3 f_r(r^*_k, \lambda) = 6a$. The variational equation is given by $s_{k+1} = (1 + 6as^2_k)s_k$. For $a < 0$, $s_{k+1} < s_k$, the fixed point $(r^*_{k(0)}, \lambda) = (0, 0)$ is sprially stable of the third order. The bifurcation of the fixed point $(r^*_{k(0)}, \lambda) = (0, 0)$ is the Neimark bifurcation. The Neimark bifurcation with stable focus ($a < 0$) is called a supercritical case. For $a > 0$, $s_{k+1} > s_k$, the fixed point $(r^*_{k(0)}, \lambda) = (0, 0)$ is spirally unstable of the third order. The bifurcation of the fixed point $(r^*_{k(0)}, \lambda) = (0, 0)$ is the Neimark bifurcation. The Neimark bifurcation with unstable focus ($a > 0$) is called a subcritical case. The supercritical and subcritical Neimark bifurcation is shown in Fig. 2.8a and b. The solid lines and curves represent stable fixed point. The dashed lines and curves represent unstable fixed point. The phase shift is determined by $\theta_{k+1} = \theta_k - \vartheta$ and $r^*_{k(2)} \neq 0$, one get a unstable or unstable periodic solution on the circle.

From the foregoing analysis of the Neimark bifurcation, the Neimark bifurcation points possess the higher-order singularity of the flow in discrete dynamical system in the radius direction. For the stable Neimark bifurcation, the mth order singularity of the flow at the bifurcation point exists as a sink of the mth order in the radius direction. For the unstable Neimark bifurcation, the mth order singularity of the flow at the bifurcation point exists as a source of the mth order in the radius direction.

2.4 Lower Dimensional Discrete Systems

For a better understanding, the stability and bifurcation of 1-D and 2-D maps will be discussed.

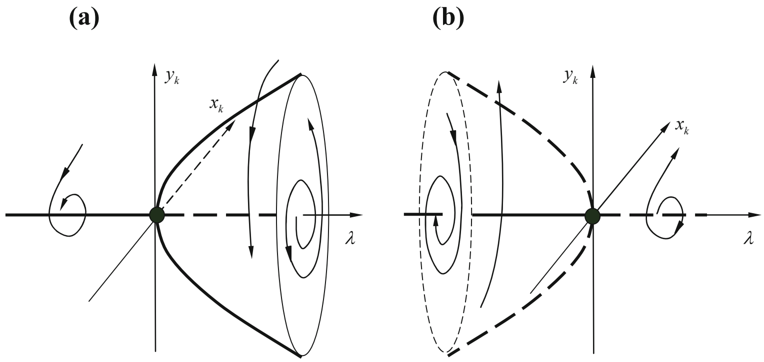

Fig. 2.8 Neimark bifurcations: **a** supercritical ($a < 0$) and **b** subcritical ($a > 0$)

2.4.1 One-Dimensional Maps

Consider a 1-D map,

$$P : x_k \to x_{k+1} \text{ with } x_{k+1} = f(x_k, \mathbf{p}) \tag{2.158}$$

where $\mathbf{p}$ is a parameter vector. To determine the period-1 solution (fixed point) of Eq. (2.158), substitution of $x_{k+1} = x_k$ into Eq. (2.158) yields the periodic solution $x_k = x_k^*$. The stability and bifurcation of the period-1solution is presented.

(i) Pitchfork bifurcation (period-doubling bifurcation)

$$\frac{dx_{k+1}}{dx_k} = \left.\frac{df(x_k, \mathbf{p})}{dx_k}\right|_{x_k = x_k^*} = -1. \tag{2.159}$$

(ii) Tangent (saddle-node) bifurcation

$$\frac{dx_{k+1}}{dx_k} = \left.\frac{df(x_k, \mathbf{p})}{dx_k}\right|_{x_k = x_k^*} = 1. \tag{2.160}$$

With two such conditions and fixed points $x_k = x_k^*$, the critical parameter vector $\mathbf{p}_0$ on the corresponding parameter manifolds can be determined. The two kinds of bifurcations for 1-D iterative maps are depicted in Fig. 2.9. Note that the most common pitchfork bifurcation involves an infinite cascade of period-doubling bifurcations with universal scalings. An exact renormalization theory for period-doubling bifurcation was developed in terms of a functional equation by Feigenbaum (1978), and Collet and Eckmann (1980). Helleman (1980a, b) employed an *algebraic* renormalization procedure to determine the rescaling constants. It is assumed that $f(x_k, \mathbf{p})$ has a quadratic maximum at $x_k = x_k^0$. If chaotic solution ensues at $\mathbf{p}_\infty$ via the period-doubling bifurcation, the function $x_{k+1} = f(x_k, \mathbf{p}_\infty)$ is rescaled by a scale factor α

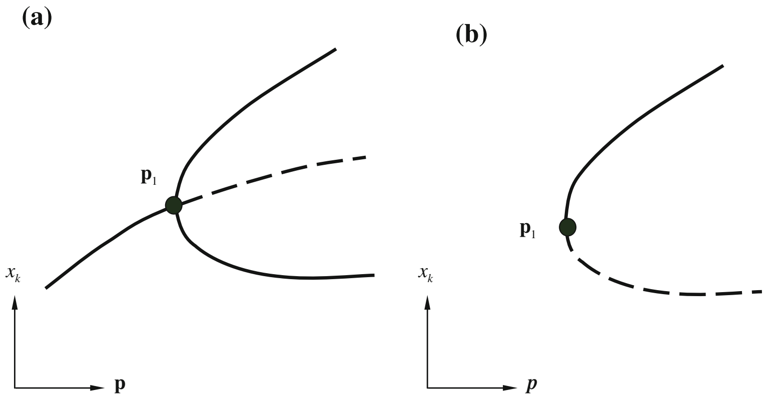

Fig. 2.9 Bifurcation types: **a** period-doubling and **b** saddle-node

and a self-similar structure exists near $x_k = x_k^0$. Under the transition to chaos, the period doubling bifurcation will be discussed where two renormalization procedures will be presented in next section, namely, the renormalization group approach via the functional equation method as outlined by Feigenbaum (1978) (see also, Schuster 1988; Lichtenberg and Lieberman 1992), and the algebraic renormalization technique as described by Helleman (1980a, b). In next section, the quasiperiodicity route to chaos and the intermittency route to chaos will be discussed.

2.4.2 *Two-Dimensional Maps*

Consider a 2-D map

$$P : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ with } \mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}), \tag{2.161}$$

where $\mathbf{x}_k = (x_k, y_k)^{\mathrm{T}}$ and $\mathbf{f} = (f_1, f_2)^{\mathrm{T}}$ with a parameter vector $\mathbf{p}$. The period-n fixed point for Eq. (2.161) is $(\mathbf{x}_k^*, \mathbf{p})$, i.e., $P^{(n)}\mathbf{x}_k^* = \mathbf{x}_{k+n}^*$, where $P^{(n)} = P \circ P^{(n-1)}$ and $P^{(0)} = 1$, and its stability and bifurcation conditions are given as follows:

(i) period-doubling (flip or pitchfork) bifurcation

$$\mathrm{tr}(DP^{(n)}) + \det(DP^{(n)}) + 1 = 0 \tag{2.162}$$

(ii) saddle-node bifurcation

$$\det(DP^{(n)}) + 1 = \mathrm{tr}(DP^{(n)}) \tag{2.163}$$

(iii) Neimark bifurcation

$$\det(DP^{(n)}) = 1, \tag{2.164}$$

where

$$DP^{(n)}(\mathbf{x}_k^*) = \prod\nolimits_{j=n-1}^{0} DP(\mathbf{x}_{k+j}^*) = \underbrace{\left[\frac{\partial \mathbf{x}_{k+n}}{\partial \mathbf{x}_{k+n-1}}\right]_{\mathbf{x}_{k+n-1}^*} \cdot \cdots \cdot \left[\frac{\partial \mathbf{x}_{k+1}}{\partial \mathbf{x}_k}\right]_{\mathbf{x}_k^*}}_{n} \tag{2.165}$$

For $n = 1$, we have

$$DP(\mathbf{x}_k^*) = \left[\frac{\partial \mathbf{x}_{k+1}}{\partial \mathbf{x}_k}\right]_{\mathbf{x}_k^*} = \begin{bmatrix} \partial_{x_k} f_1 & \partial_{y_k} f_1 \\ \partial_{x_k} f_2 & \partial_{y_k} f_2 \end{bmatrix}_{\mathbf{x}_k^*} \tag{2.166}$$

$$\begin{aligned} \mathrm{tr}(DP) &= \partial_{x_k} f_1 + \partial_{y_k} f_2 \\ \det(DP) &= \partial_{x_k} f_1 \cdot \partial_{y_k} f_2 - \partial_{y_k} f_1 \cdot \partial_{x_k} f_2 \end{aligned} \tag{2.167}$$

and

$$\begin{aligned} \partial_{x_k} f_1 &= \partial f_1(\mathbf{x}_k, p)/\partial x_k|_{\mathbf{x}_k=\mathbf{x}_k^*}, \\ \partial_{y_k} f_1 &= \partial f_1(\mathbf{x}_k, p)/\partial y_k|_{\mathbf{x}_k=\mathbf{x}_k^*}, \\ \partial_{x_k} f_2 &= \partial f_2(\mathbf{x}_k, p)/\partial x_k|_{\mathbf{x}_k=\mathbf{x}_k^*}, \\ \partial_{y_k} f_2 &= \partial f_2(\mathbf{x}_k, p)/\partial y_k|_{\mathbf{x}_k=\mathbf{x}_k^*}. \end{aligned} \tag{2.168}$$

The bifurcation and stability conditions for the solution of period-n for Eq. (2.161) are summarized in Fig. 2.10. The stability and bifurcation for 2-D discrete system are summarized in Fig. 2.10 with $\det(DP^{(n)}) = \det(DP^{(n)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0))$ and $\mathrm{tr}(DP^{(n)}) = \mathrm{tr}(DP^{(n)}(\mathbf{x}_{k(0)}^*, \mathbf{p}_0))$. The thick dashed lines are bifurcation lines. The stability of fixed point is given by the eigenvalues in complex plane. The stability of fixed point for higher dimensional systems can be identified by using a naming of stability for linear dynamical systems in Appendix B. The saddle-node bifurcation possesses stable saddle-node bifurcation (critical) and unstable saddle-node bifurcation (degenerate).

2.4.3 Finite-Dimensional Maps

Consider an m-D map

$$P : \mathbf{x}_k \to \mathbf{x}_{k+1} \text{ with } \mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k, \mathbf{p}), \tag{2.169}$$

where $\mathbf{x}_k = (x_{1k}, x_{2k}, \cdots x_{mk})^{\mathrm{T}}$ and $\mathbf{f} = (f_1, f_2, \cdots, f_m)^{\mathrm{T}}$ with a parameter $\mathbf{p}$. The period-n fixed point for Eq. (2.169) is $(\mathbf{x}_k^*, \mathbf{p})$, and its stability and bifurcation

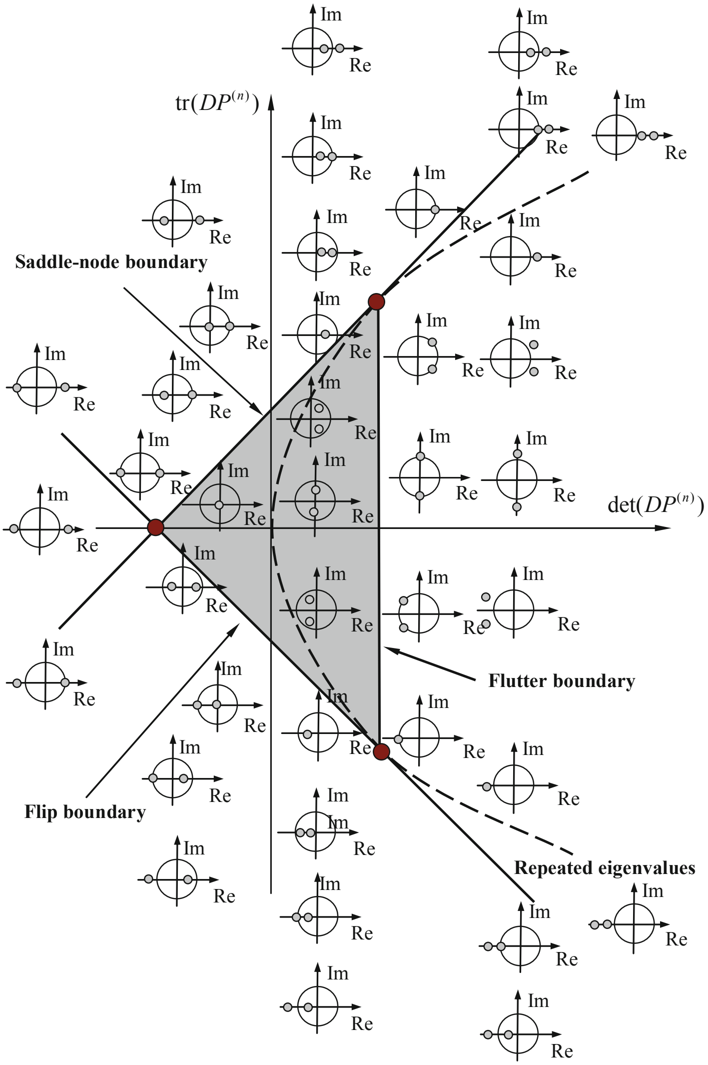

Fig. 2.10 Stability and bifurcation diagrams through the complex plane of eigenvalues for 2D-discrete dynamical systems

conditions are given as follows. Similarly, $P^{(n)}\mathbf{x}_k^* = \mathbf{x}_{k+n}^*$, where $P^{(n)} = P \circ P^{(n-1)}$ and $P^{(0)} = 1$.

(i) period-doubling (flip or pitchfork) bifurcation

$$|DP^{(n)} + \mathbf{I}_{m \times m}| = 0, \tag{2.170}$$

(ii) saddle-node bifurcation

$$|DP^{(n)} - \mathbf{I}_{m\times m}| = 0, \tag{2.171}$$

(iii) Neimark bifurcation

$$\begin{vmatrix} DP^{(n)} - \alpha\mathbf{I}_{m\times m} & -\beta\mathbf{I}_{m\times m} \\ \beta\mathbf{I}_{m\times m} & DP^{(n)} - \alpha\mathbf{I}_{m\times m} \end{vmatrix} = 0 \text{ with } \alpha^2 + \beta^2 = 1 \tag{2.172}$$

where

$$DP^{(n)}(\mathbf{x}_k^*) = \prod\nolimits_{j=n-1}^{0} DP(\mathbf{x}_{k+j}^*) = \underbrace{\left[\frac{\partial\mathbf{x}_{k+n}}{\partial\mathbf{x}_{k+n-1}}\right]_{\mathbf{x}_{k+n-1}^*} \cdot \cdots \cdot \left[\frac{\partial\mathbf{x}_{k+1}}{\partial\mathbf{x}_k}\right]_{\mathbf{x}_k^*}}_{n} \tag{2.173}$$

with

$$DP(\mathbf{x}_{k+j}^*) = \left[\frac{\partial\mathbf{x}_{k+j+1}}{\partial\mathbf{x}_{k+j}}\right]_{\mathbf{x}_{k+j}^*} = \begin{bmatrix} \partial_{x_{1(k+j)}} f_1 & \partial_{x_{2(k+j)}} f_1 & \cdots & \partial_{x_{m(k+j)}} f_1 \\ \partial_{x_{1(k+j)}} f_2 & \partial_{x_{2(k+j)}} f_2 & \cdots & \partial_{x_{m(k+j)}} f_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{x_{1(k+j)}} f_m & \partial_{x_{2(k+j)}} f_m & \cdots & \partial_{x_{m(k+j)}} f_m \end{bmatrix}_{\mathbf{x}_{k+j}^*} \tag{2.174}$$

for $j = 0, 1, \cdots, n-1$

and

$$\partial_{x_{\alpha(k+j)}} f_\beta = \frac{\partial f_\beta(\mathbf{x}_{k+j}, \mathbf{p})}{\partial x_{\alpha(k+j)}} \text{ for } \alpha, \beta = 1, 2, \cdots, m. \tag{2.175}$$

2.5 Routes to Chaos

The routes to chaos will be discussed. The 1-D discrete system will be discussed first, then we will discuss the 2-D discrete systems.

2.5.1 One-Dimensional Maps

(A) Period doubling route to chaos

(i) *Functional renormalization theory*. Consider a universal function as

$$g^*(x) = \lim_{n\to\infty} \alpha^n f^{(2^n)}(x/\alpha^n, \mathbf{p}_\infty) \tag{2.176}$$

where g^* must satisfy the rescaling equation of the geometry, that is,

$$g^* = \alpha g^*(g^*(x/\alpha)) = Tg^* \tag{2.177}$$

in which T is a period-doubling operator. From Eq. (2.177), the universality of the scale factor α is obtained. The linearization of $f(x, \mathbf{p}_n)$ at $\mathbf{p}_n = \mathbf{p}_\infty$ yields the universal constant δ.

$$f(x, \mathbf{p}_n) = f(x, \mathbf{p}_\infty) + \left.\frac{\partial f(x, \mathbf{p}_n)}{\partial \mathbf{p}_n}\right|_{\mathbf{p}_n = \mathbf{p}_\infty} (\mathbf{p}_n - \mathbf{p}_\infty) + o(\mathbf{p}_n - \mathbf{p}_\infty). \tag{2.178}$$

Applying the period-doubling operator n times to Eq. (2.178) yields,

$$\lim_{n\to\infty} T^n f(x, \mathbf{p}_n) = g^*(x) + L_{g^*}^n \left(\frac{\partial f(x, \mathbf{p}_n)}{\partial \mathbf{p}_n}\right)\Big|_{\mathbf{p}_n = \mathbf{p}_\infty} (\mathbf{p}_n - \mathbf{p}_\infty). \tag{2.179}$$

Substitution of the unstable eigenvalue of L_{g^*} into Eq. (2.179) gives

$$\lim_{n\to\infty} T^n f(x, \mathbf{p}_n) = g^*(x) + \delta^n \left(\frac{\partial f(x, \mathbf{p}_n)}{\partial \mathbf{p}_n}\right)\Big|_{\mathbf{p}_n = \mathbf{p}_\infty} (\mathbf{p}_n - \mathbf{p}_\infty). \tag{2.180}$$

Transformation of the point of origin to $x = x_0$ and normalization of Eq. (2.158) by setting $g^*(0) = 1$, the condition is

$$f^{(2^n)}(0, \mathbf{p}_n) = 0. \tag{2.181}$$

From Eqs. (2.180) and (2.181), the universal constant is proportional to

$$||\mathbf{p}_n - \mathbf{p}_\infty|| \sim \delta^{-n}. \tag{2.182}$$

(ii) *Algebraic renormalization theory.* Taking into account the period-2 solutions of Eq. (2.158), we can solve for $x_{1\pm}, x_{2\pm}$ at $x_k = x_{k+2}$:

$$f(x_k, \mathbf{p}) = x_{k+1} \text{ and } f(x_{k+1}, \mathbf{p}) = x_{k+2}. \tag{2.183}$$

Using a Taylor series expansion, we can apply a perturbation to Eq. (2.183) at $x_k = x_{k(2)\pm} + \Delta x_k$, $x_{k+1} = x_{k(1)\pm} + \Delta x_{k+1}$ and $x_{k+2} = x_{k(2)\pm} + \Delta x_{k+2}$, that is,

$$\Delta x_{k+1} = f_1(\Delta x_k, \mathbf{p}), \tag{2.184}$$

$$\Delta x_{k+2} = f_2(\Delta x_{k+1}, \mathbf{p}). \tag{2.185}$$

Substitution of Eq. (2.184) into Eq. (2.185) yields

$$\Delta x_{k+2} = f_2(f_1(\Delta x_k, \mathbf{p}), \mathbf{p}) = f(\Delta x_k, \bar{\mathbf{p}}). \tag{2.186}$$

Rescaling Eq. (2.186) with

$$x'_k = \alpha \Delta x_k \tag{2.187}$$

gives the corresponding renormalized equation, i.e.,

$$x'_{k+2} = f(x'_k, \bar{\mathbf{p}}_{2^{k+1}}), \tag{2.188}$$

where

$$\bar{p}_{2^{k+1}} = \mathbf{g}(\bar{\mathbf{p}}_{2^k}). \tag{2.189}$$

Equation (2.189) presents a relationship of the bifurcation values between two period-doubling bifurcations. The rescaling factor α is determined by comparing Eq (2.188) with Eq. (2.161). If chaos appears via the period-doubling cascade, i. e., $\bar{\mathbf{p}}_{2^{k+1}} = \bar{\mathbf{p}}_{2^k} = \mathbf{p}_\infty$, the universal parameter manifolds are determined.

(B) Quasiperiodicity route to chaos

Consider a mapping defined on the unit interval $0 \le x \le 1$, that is,

$$x_{k+1} = x_k + \Omega + f(x_k, \mathbf{p}) = F(x_k, \Omega, \mathbf{p}), \tag{2.190}$$

where $f(x_k, \mathbf{p})$ is a periodic modulo, i. e., $f(x_k + 1, \mathbf{p}) = f(x_k, \mathbf{p})$; and Ω is a prescribed parameter defined in the interval $0 \le \Omega \le 1$. In Eq. (2.190), parameters $(\Omega, \mathbf{p})$ can be adjusted to generate a transition from quasiperiodicity to chaos. We can increase the parameter vector amplitude $||\mathbf{p}||$ first under a rational winding number $w = p/q$ fixed to a selected value, and we will have to increase Ω as well. The winding number w is an important quantity for describing the dynamics, which is defined by

$$w(\Omega, \mathbf{p}) = \lim_{k\to\infty} \frac{x_k - x_0}{k}. \tag{2.191}$$

Define a quantity $\Omega_{p,q}(\mathbf{p})$ which belongs to a q-cycle of the map $f(x_k, \mathbf{p})$ and shifted by p. This quantity generates a rational winding number $w = p/q$ and for a fixed value of $\mathbf{p}$, it can be determined from

$$F^{(q)}(0, \Omega_{p,q}, \mathbf{p}) = p, \tag{2.192}$$

where $F^{(q)} = F(F^{(q-1)})$. Choosing the winding number equal to the golden mean $w^* = (\sqrt{5} - 1)/2$, the universal constants for chaos can be computed.

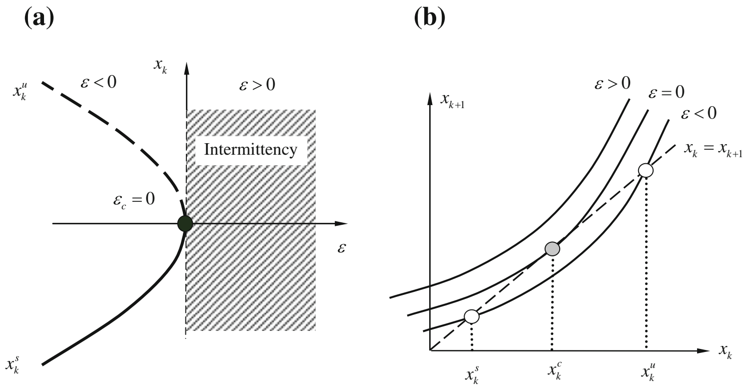

Fig. 2.11 **a** bifurcation and **b** iterative map for Eq. (2.193)

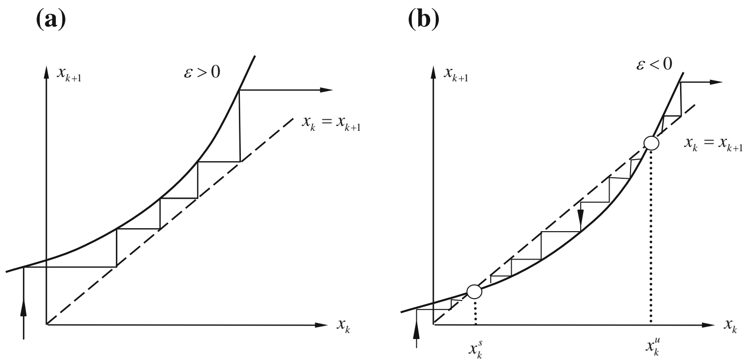

Fig. 2.12 **a** Intermittency and **b** stable and unstable fixed points for Eq. (2.193)

(C) Intermittency route to chaos
There are three types of intermittencies, Types I, II and III. In this section, we will present only Type I and III intermittencies. The Type II intermittency will be discussed in a later section under 2-D maps.

(i) *Type I intermittency*. Consider an iterative map with a small perturbation defined by

$$x_{k+1} = f(x_k, \varepsilon) = \varepsilon + x_k + \eta x_k^2, \qquad (2.193)$$

where ε is a control parameter and η is a prescribed parameter. This mapping results in the Type I intermittency caused by the tangent bifurcation which occurs when a real eigenvalue of Eq. (2.193) crosses the unit circle at +1. In other words,

$$x_k^* = \pm(-\varepsilon/\eta)^{1/2} \text{ and } Df(x_k^*) = df/dx_k|_{x_k=x_k^*} = 1 + 2\eta x_k^*. \qquad (2.194)$$

For $\eta > 0$, if $\varepsilon > 0$, no fixed point exists. If $\varepsilon = 0$, $x_k^* = x_k^c = 0$ with $Df(x_k^*) = 1$. Since $D^2 f(x_k^*) = 2\eta \neq 0$, the saddle-node bifurcation occurs. For $\eta > 0$, if $\varepsilon > 0$, $x_k^* = (-\varepsilon/\eta)^{1/2} \equiv x_k^u$ with $Df(x_k^*) > 1$ and $x_k^* = -(-\varepsilon/\eta)^{1/2} \equiv x_k^s$ with $Df(x_k^*) < 1$. The tangent bifurcation and iterative map for the Type I intermittency is shown in Fig. 2.11. The intermittency and the stable and unstable fixed points are presented in Fig. 2.12. This case includes the Poincare map for the Lorenz model and the iterative map for the window of period-3 solution in the chaotic band. The renormalization procedure of Eq. (2.193) has been presented in Hu and Rudnick (1982). Also, interested readers can refer to Guckenheimer and Holmes (1990), and Schuster (1988) for details.

(ii) *Type III intermittency*. Consider the following an iterative map

$$x_{k+1} = f(x_k, \varepsilon) = -(1+\varepsilon)x_k - \eta x_k^3, \tag{2.195}$$

which produces the Type III intermittency caused by the inverse pitchfork bifurcation. The fixed points are

$$\begin{aligned} &x_k^* = 0 \text{ and } x_k^* = \pm(-\varepsilon/\eta)^{1/2} \text{ with} \\ &Df(x_k^*) = df/dx_k|_{x_k = x_k^*} = -(1+\varepsilon) - 3\eta x_k^2. \end{aligned} \tag{2.196}$$

For $\eta > 0$, if $\varepsilon > 0$, only one unstable fixed point exists. $x_k^* = x_k^u = 0$ because of $Df(x_k^*) < -1$. If $\varepsilon < 0$, there are three fixed points. $x_k^* = x_k^s = 0$ and $Df(x_k^*) > -1$; $x_k^* \equiv x_k^s = \pm(-\varepsilon/\eta)^{1/2}$ are stable with $Df(x_k^*) < -1$. If $\varepsilon = 0$, $x_k^* = x_k^c = 0$ with $Df(x_k^*) = -1$. $D^2 f(x_k^*) = -6\eta x_k^* = 0$, however, $D^3 f(x_k^*) = -6\eta < 0$. This is an inverse pitchfork bifurcation (unstable period-doubling bifurcation). The bifurcation diagram and the iterative map for the Type III intermittency is presented in Fig. 2.13. In addition, 4he intermittency and the stable and unstable fixed points are presented in Fig. 2.14.

2.5.2 Two-Dimensional Systems

For 2-D invertible maps, the transition from regular motion to chaos takes place via a series of cascades of period-doubling bifurcations. The renormalization procedure of the period-doubling route to chaos for a 2-D map appears in next section through an example. The quasi-periodic transition to chaos and the intermittence to chaos are briefly presented through an example. The quasiperiodic transition to chaos and the intermittence to chaos are presented briefly.

(A) Quasiperiodic transition to chaos
This route to chaos is presented via the standard map as

$$x_{k+1} = x_k + K \sin\theta_k \text{ and } \theta_{k+1} = \theta_k + x_{k+1}. \tag{2.197}$$

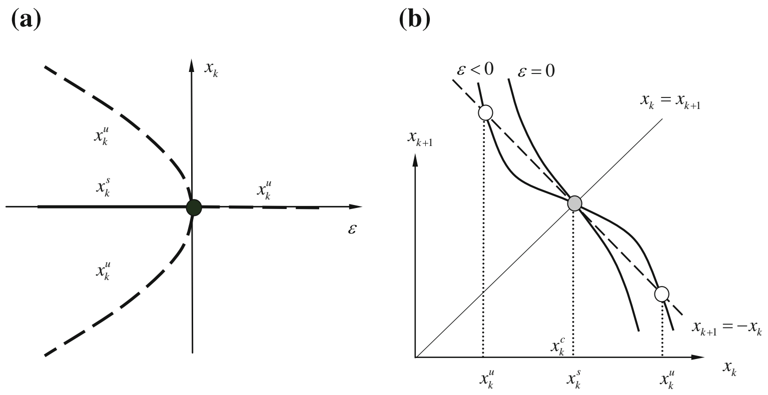

Fig. 2.13 **a** Inverse pitchfork bifurcation and **b** iterative map for Eq. (2.195)

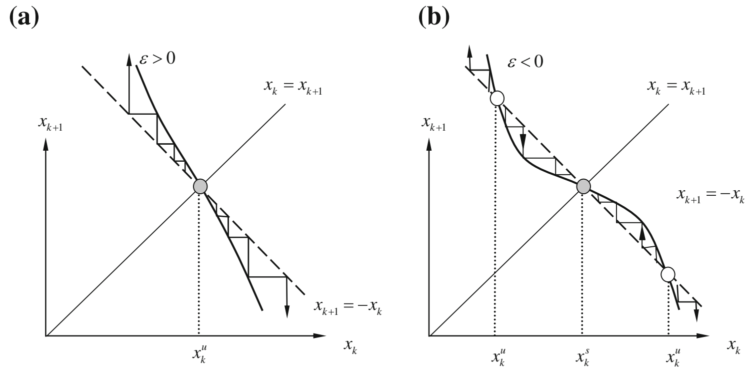

Fig. 2.14 **a** Intermittency and **b** stable and unstable fixed points for Eq. (2.195)

The critical condition of Eq. (2.197) for transition from local to global stochasticity is $K_{\mathrm{cr}} \approx 0.9716 \cdots$. For a dissipative standard map, consider

$$x_{k+1} = (1 - \delta)x_k + K \sin\theta_k \text{ and } \theta_{k+1} = \theta_k + x_{k+1}, \tag{2.198}$$

where δ is the dissipative coefficient. Some results are given in Lichtenberg and Lieberman (1992).

(B) Type II intermittency to chaos
Consider the following mapping which represents Type II intermittency to chaos,

$$r_{k+1} = (1 + \varepsilon)r_k + \eta r_k^3 \text{ and } \theta_{k+1} = \theta_k + \Omega, \tag{2.199}$$

$$x_k = r_k \cos\theta_k \text{ and } y_k = r_k \sin\theta_k. \tag{2.200}$$

When a pair of complex eigenvalues of Eq. (2.199) passes over the unit circle, the subcritical Neimark bifurcation occurs. Hence, Type II intermittency

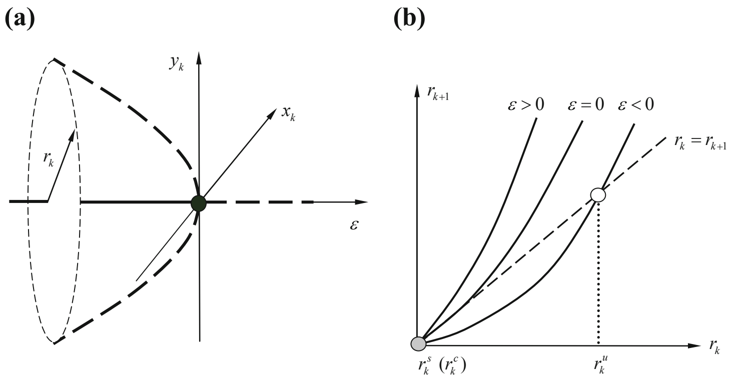

Fig. 2.15 **a** Subcritical Neimark bifurcation and **b** iterative map for Eq. (2.199)

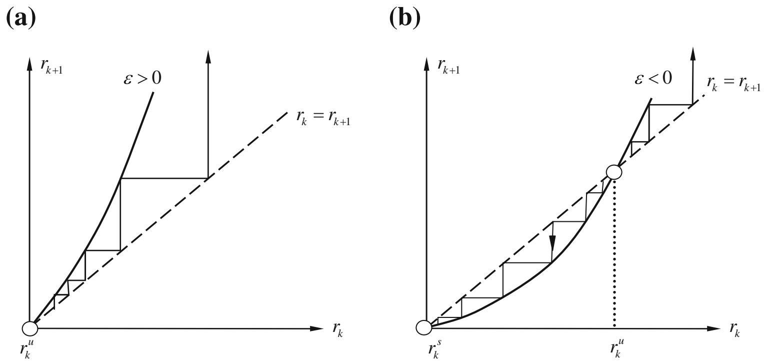

Fig. 2.16 **a** Intermittency and **b** stable and unstable fixed points for Eq. (2.199)

results from the subcritical Neimark bifurcation as shown in Fig. 2.15, and the corresponding intermittency and stable and unstable fixed points are presented in Fig. 2.16.

2.6 Universality for Discrete Duffing Systems

Consider a Duffing oscillator

$$\ddot{x} + \delta\dot{x} + \alpha_1 x + \alpha_2 x^3 = Q_0 \cos \Omega t, \tag{2.201}$$

where system parameters are $\delta, \alpha_1, \alpha_2, Q_0$ and Ω. Discretizing it with respect to time yields a discrete map to investigate qualitatively its universal behavior. Here, by means of the Naive discretization of the time derivative, Eq. (2.201) is discretized at $x_k = x(t_k)$ and $t_k = 2k\pi/\Omega$ using time difference $\Delta t = 2\pi/\Omega$ for external excitation. Therefore

$$x_{k+1} - 2x_k + x_{k-1} + (1-b)(x_k - x_{k-1}) + cx_k + dx_k^3 = (1-b)\varpi, \quad (2.202)$$

where all parameters are defined as

$$b = 1 - \frac{2\pi}{\Omega}\delta,\ c = (\frac{2\pi}{\Omega})^2\alpha_1,\ d = (\frac{2\pi}{\Omega})^2\alpha_2,\ \varpi = \frac{Q_0}{\alpha_1}\frac{2\pi}{\Omega}. \quad (2.203)$$

From Eq. (2.202), a Duffing map is constructed as

$$P : x_{k+1} = x_k + \varpi + y_{k+1} \text{ and } y_{k+1} = by_k - cx_k - dx_k^3. \quad (2.204)$$

To qualitatively investigate the Feigenbaum cascade of Eq. (2.201), the renormalization of Eq. (2.204) will be presented via period-doubling bifurcation cascade. Consider a transformation as

$$x_k = X_k + \delta \text{ and } y_k = Y_k - \varpi. \quad (2.205)$$

Substitution of the foregoing equation into Eq. (2.204) yields

$$X_{k+1} = X_k + Y_{k+1} \text{ and } Y_{k+1} = BY_k + CX_k + DX_k^2 - EX_k^3, \quad (2.206)$$

where

$$-d\bar{\delta}^3 + c\bar{\delta} + (-1+b)\varpi = 0, \quad (2.207)$$

$$B = b,\ C = -(c + 3d\bar{\delta}^2),\ D = -3d\bar{\delta},\ \text{and } E = d. \quad (2.208)$$

From Eq. (2.207), the parameter $\bar{\delta}$ is determined. Using Eq. (2.206), its period-1 solution is determined by

$$\begin{aligned} &Y^*_{k(1)} = 0 \text{ and } X^*_{k(1)} = 0; \\ &Y^*_{k(2,3)} = 0 \text{ and } X^*_{k(2,3)} = \frac{D \pm \sqrt{D^2 + 4CE}}{2E}. \end{aligned} \quad (2.209)$$

Deformation of Eq. (2.206) is

$$X_{k+1} + BX_{k-1} = (1 + B + C)X_k + DX_k^2 - EX_k^3. \quad (2.210)$$

The second iteration of Eq. (2.206) gives

$$X_{k+2} + BX_k = (1 + B + C)X_{k+1} + DX_{k+1}^2 - EX_{k+1}^3. \quad (2.211)$$

The period-2 of Eq. (2.206) requires $X_{k+2} = X_k$ and $X_{k+1} = X_{k-1}$. Thus simplification of Eqs. (2.210) and (2.211) gives

$$a_6X_k^6 + a_5X_k^5 + a_4X_k^4 + a_3X_k^3 + a_2X_k^2 + a_1X_k + a_0 = 0, \quad (2.212)$$

where

$$
\begin{aligned}
&a_0 = (1+B)^2[C+2(1+B)],\ a_1 = D(1+B)[C+2(1+B)],\\
&a_2 = D^2(1+B) + EC^2 - 3(1+B)E[C+2(1+B)],\\
&a_3 = 2DE[C-(1+B)],\ a_4 = E^2[3(1+B)+2C] - D^2E,\ a_6 = -E^3.
\end{aligned} \tag{2.213}
$$

For all the given parameters, solving Eq. (2.212) numerically obtains $X_k = X_k^*$, meanwhile, using one of Eqs. (2.210) or (2.211) and $X_{k+2} = X_k$ and $X_{k+1} = X_{k-1}$, the second solution $X_{k+1} = X_{k+1}^*$ can be determined. With the periodicity of period-2, solving of Eqs.(2.210) and (2.211) directly gives the period-2 solutions via Newton-Raphson method. In the neighborhoods of solutions X_{k+j}^*, consider a perturbation as

$$
X_{k+j} = X_{k+j}^* + \Delta X_{k+j} \text{ for } j = -1, 0, 1, 2. \tag{2.214}
$$

Substitution of them into Eq. (2.210) yields a group of iterative equations as:

$$
\begin{aligned}
\Delta X_k + B\Delta X_{k-2} &= e_{11}\Delta X_{k-1} + e_{12}\Delta X_{k-1}^2 - E\Delta X_{k-1}^3,\\
\Delta X_{k+1} + B\Delta X_{k-1} &= e_{21}\Delta X_k + e_{22}\Delta X_k^2 - E\Delta X_k^3,\\
\Delta X_{k+2} + B\Delta X_k &= e_{11}\Delta X_{k+1} + e_{12}\Delta X_{k+1}^2 - E\Delta X_{k+1}^3
\end{aligned} \tag{2.215}
$$

where e_{11}, e_{12} and e_{21}, e_{22} are

$$
\begin{aligned}
&e_{11} = 1 + B + C + 2DX_n^* - 3E(X_k^*)^2,\ e_{12} = D - 3EX_k^*\\
&e_{21} = 1 + B + C + 2DX_{n+1}^* - 3E(X_{k+1}^*)^2,\ e_{22} = D - 3EX_{k+1}^*.
\end{aligned} \tag{2.216}
$$

Multiplication of the first equation by B and *the second equation* by e_{11} of Eq. (2.215), and adding both equations into the third equation yields

$$
\begin{aligned}
\Delta X_{k+2} + B^2\Delta X_{k-2} =& (e_{11}e_{21} - 2B)\Delta X_k + e_{11}e_{22}\Delta X_k^2\\
&- e_{11}\Delta X_k^3 + e_{12}(\Delta X_{k+1}^2 + B\Delta X_{k-1}^2)\\
&+ e_{13}(\Delta X_{k+1}^3 + B\Delta X_{k-1}^3).
\end{aligned} \tag{2.217}
$$

For a small vicinity of the bifurcation of period-2, ΔX_{k+1} and ΔX_{k-1} are quite close. Therefore a similar linear scale ratio is introduced as

$$
r = \Delta X_{k+1}/\Delta X_{k-1}. \tag{2.218}
$$

The nonlinear terms can be ignored because ΔX_{k+j} $(j = -1, 0, 1, 2)$ is the infinitesimal quantity. The second equation of Eq. (2.215) gives an approximate relationship as

$$
\Delta X_{k-1} \approx \frac{e_{21}}{r + B}\Delta X_k. \tag{2.219}
$$

From Eqs. (2.212) and (2.219), equation (2.217) becomes

$$\begin{aligned}\Delta X_{k+2}+B^2\Delta X_{k-2}=&(e_{11}e_{21}-2B)\Delta X_k\\&+[e_{11}e_{22}+e_{12}e_{21}^2(r^2+B^2)(r+B)^{-2}]\Delta X_k^2\\&-[e_{11}E-e_{13}e_{21}^3(r^3+B^3)(r+B)^{-3}]\Delta X_k^3.\end{aligned}\tag{2.220}$$

From the renormalization theory, the rescaling length of the variables should be adopted.

$$X_k'=\varepsilon\Delta X_k \text{ and } X_{k\pm 1}'=\varepsilon\Delta X_{k\pm 2},\tag{2.221}$$

where ε is a scaling constant. The foregoing equation makes Eq. (2.220) have an algebraically similar structure to Eq. (2.210), i.e.,

$$X_{k+1}'+B'X_{k-1}'=C_1'X_k'+D'(X_k')^2-E'(X_k')^3,\tag{2.222}$$

where

$$\begin{aligned}&B'=B^2,\ C_1'=e_{11}e_{21}-2B,\ C_1=1+B+C,\\&D'=\varepsilon[e_{11}e_{22}+e_{12}e_{21}^2(r^2+B^2)(r+B)^{-2}],\\&E'=\varepsilon^2[e_{11}E-e_{13}e_{21}^3(r^3+B^3)(r+B)^{-3}].\end{aligned}\tag{2.223}$$

If Eq. (2.222) has a self-similarity with Eq. (2.210), the similar scaling ratio r in Eq. (2.218) should be one, i.e., $r=1$ in Eq. (2.218). The similar parameters have the same property as the scaling of variable Δx_k, and this property indicates that the cascade of bifurcations will be accumulated. Therefore the chaos is generated via period-doubling bifurcation at $B=B'=B_\infty$, $C_1=C_1'=C_{1\infty}$, $D=D'=D_\infty$ and $E=E'=E_\infty$. Thus Eq. (2.223) becomes

$$\begin{aligned}B&=0 \text{ or } 1,\\C_1+2B&=[C_1+2DX_k^*-3E(X_k^*)^2][C_1+2DX_{k+1}^*-3E(X_{k+1}^*)^2],\\D&=\varepsilon[e_{11}e_{22}+e_{12}e_{21}^2(r^2+B^2)(r+B)^{-2}],\\E&=\varepsilon^2[e_{11}E-e_{13}e_{21}^3(r^3+B^3)(r+B)^{-3}].\end{aligned}\tag{2.224}$$

where $C_1=1+B+C$. To solve five parameters from four equations, one parameter should be given. However, from Eqs. (2.206) and (2.208), the parameter D is determined if parameters B, C and E have already been computed. Employing Eqs. (2.212), (2.213), (2.223) and (2.224), the universal parameter values for Eq. (2.206) are determined. To verify these values, the corresponding universal values are determined numerically via iteration of Eq. (2.206). Taking the parameters $D=1.0$ and $E=1.0$ into account, the universalized parameter C_1, computed via *R*enormalization *G*roup (*RG*) and *N*umerical *S*imulation (*NS*), versus the damping parameter B are shown in Fig. 2.17a with solid and circular-symbol curves, respectively. *RG* values are close

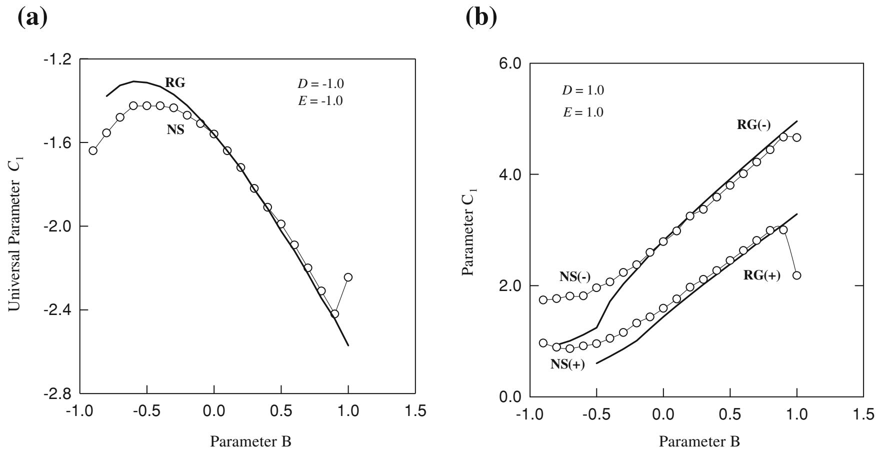

Fig. 2.17 Universal parameters via renormalization: **a** Duffing map with a soft spring and **b** Duffing map with a double-potential well

to the *NS* values. For the parameters $D = -1.0$ and $E = 1.0$, the universalized parameter C_1 values $RG(+)$, $NS(+)$, $RG(-)$ and $NS(-)$ are the $RG(-)$, $NS(-)$, $RG(+)$ and $NS(+)$ of the situation of $D = 1.0$ and $E = 1.0$, respectively. As for the situation of $D = -1.0$ and $E = -1.0$, the universalized parameter C_1 versus the damping parameter B is also plotted in Fig. 2.17b. This renormalization group can provide a good prediction for $D = 1.0$ and $E = 1.0$ as the parameter B corresponding to the damping is in the range of $B = (-0.5, 0.9)$. However, a good prediction for $D = -1.0$ and $E = -1.0$ is given for $B = (-0.8, 0.9)$. $B = 1$ implies the conservative system, and $B > 1$ implies the negative damping system. From such universal parameters for chaos generated by the period-doubling, the system parameters are $\delta, \alpha_1, \alpha_2, Q_0$ and Ω can be determined.

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