2

Preliminaries

1 Poincaré and Sobolev Inequalities

Let E be a bounded domain in $\mathbb{R}^N$ with boundary ∂E. If $f \in L^q(E)$ for some $1 \le q \le \infty$, denote by $\|f\|_{q,E}$ the $L^q(E)$-norm of f over E. We also write $\|f\|_q$ whenever the specification of the domain E is unambiguous from the context. The function $f \in L^q_{\mathrm{loc}}(E)$ if $\|f\|_{q,K} < \infty$, for all compact subsets $K \subset E$. For $f \in C^1(E)$ denote by $Df = (f_{x_1}, \dots, f_{x_N})$ its gradient and set

$$\|f\|_{1,p;E} = \|f\|_{p,E} + \|Df\|_{p,E}.$$

The spaces $W^{1,p}(E)$ and $W^{1,p}_o(E)$ for $p \ge 1$ are defined as

$$\begin{array}{ll} W^{1,p}(E) & \text{the completion of } C^\infty(E) \text{ under } \|\cdot\|_{1,p;E} \\ W^{1,p}_o(E) & \text{the completion of } C^\infty_o(E) \text{ under } \|\cdot\|_{1,p;E}. \end{array}$$

Equivalently $W^{1,p}(E)$ is the Banach space of functions $f \in L^p(E)$ whose generalized derivatives f_{x_i} belong to $L^p(E)$ for all $i = 1, \dots, N$.

A function $f \in W^{1,p}_{\mathrm{loc}}(E)$ if $\|f\|_{1,p;K} < \infty$ for every compact subset $K \subset E$.

Let $W^{1,\infty}(E)$ denote the Banach space of functions $f \in L^\infty(E)$ whose distributional derivatives $f_{x_i} \in L^\infty(E)$, for $i = 1, \dots, N$.

The space $W^{1,\infty}_{\mathrm{loc}}(E)$ is defined analogously.

Theorem 1.1 (Gagliardo–Nirenberg [71, 124]) *Let $v \in W^{1,p}_o(E)$ for some $p \ge 1$. For every $s \ge 1$ there exists a constant C depending only on N, p, q, and s, and independent of E, such that*

$$\|v\|_{q,E} \le C \|Dv\|^{\alpha}_{p,E} \|v\|^{1-\alpha}_{s,E} \tag{1.1}$$

where $\alpha \in [0,1]$ and $p, q \ge 1$, are linked by

$$\alpha = \left(\frac{1}{s} - \frac{1}{q}\right)\left(\frac{1}{N} - \frac{1}{p} + \frac{1}{s}\right)^{-1}$$

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and their admissible range is

$$\begin{aligned}
&\text{if } N=1, && \alpha\in[0,\tfrac{p}{p+s(p-1)}], && q\in[s,\infty];\\
&\text{if } 1\le p<N, && \alpha\in[0,1], && \begin{cases} q\in\big[s,\frac{Np}{N-p}\big] & \text{if } s\le\frac{Np}{N-p},\\[1ex] q\in\big[\frac{Np}{N-p},s\big] & \text{if } s\ge\frac{Np}{N-p};\end{cases}\\
&\text{if } 1<N\le p, && \alpha\in\big[0,\tfrac{Np}{Np+s(p-N)}\big), && q\in[s,\infty).
\end{aligned}$$

Corollary 1.1 *Let $v\in W_o^{1,p}(E)$, and assume $p\in[1,N)$. There exists a constant $\gamma=\gamma(N,p)$ such that*

$$\|v\|_{q,E}\le\gamma\|Dv\|_{p,E},\qquad \textit{where}\qquad q=\frac{Np}{N-p}. \tag{1.1$'$}$$

The boundary ∂E is *piecewise smooth* if it is the union of finitely many portions of $(N-1)$-dimensional hypersurfaces of class $C^{1,\lambda}$, for some $\lambda\in(0,1)$.

If ∂E is piecewise smooth, functions v in $W^{1,p}(E)$ are defined up to ∂E via their traces denoted by $v\,\big|_{\partial E}$.

Theorem 1.2 *Let ∂E be piecewise smooth. There exists a constant C depending only on N,p and the structure of ∂E such that*

$$\|v\|_{q,\partial E}\le C\|v\|_{W^{1,p}(E)},$$

where

$$\begin{aligned}
&q\in\big[1,\tfrac{(N-1)p}{N-p}\big], && \text{if } 1<p<N\\
&q\in[1,\infty), && \text{if } p=N.
\end{aligned}$$

If ∂E is piecewise smooth, the space $W_o^{1,p}(E)$ can be defined equivalently as the set of functions $v\in W^{1,p}(E)$ whose trace on ∂E is zero.

Remark 1.1 The embedding inequalities of Theorem 1.1 and Corollary 1.1 continue to hold for functions v in $W^{1,p}(E)$, not necessarily vanishing on ∂E in the sense of the traces, provided ∂E is piecewise smooth and

$$\int_E v(x)dx=0.$$

In such a case the constant C depends on s,p,q,N and the structure of ∂E. However, it does not depend on the *size* of E, and in particular it does not change by dilations of E.

2 Cuts and Truncations of Functions in $W^{1,p}(E)$ and Their Embeddings

Let k be any real number and for a function $v \in W^{1,p}(E)$ consider the truncations of v given by

$$(v-k)_+ = \max\{(v-k); 0\}$$
$$(v-k)_- = \max\{-(v-k); 0\}.$$

Lemma 2.1 (Stampacchia [144]) *Let $v \in W^{1,p}(E)$. Then $(v-k)_\pm \in W^{1,p}(E)$ for all $k \in \mathbb{R}$. If in addition the trace of v on ∂E is essentially bounded and*

$$\|v\|_{\infty,\partial E} \le M \qquad \textit{for some } \; M > 0,$$

then $(v-k)_\pm \in W_o^{1,p}(E)$ for all $k \ge M$.

Corollary 2.1 *Let $v_i \in W^{1,p}(E)$ for $i = 1, \dots, n \in \mathbb{N}$. Then*

$$w = \min\{v_1, \dots, v_n\} \in W^{1,p}(E).$$

For a function v defined in E and real numbers $k < \ell$, set

$$[v > \ell] = \{x \in E \big| v(x) > \ell\}$$
$$[v < k] = \{x \in E \big| v(x) < k\}$$
$$[k < v < \ell] = \{x \in E \big| k < v(x) < \ell\}.$$

For $\rho > 0$ and $y \in \mathbb{R}^N$, denote by $B_\rho(y)$ the ball of radius ρ centered at y, and by $K_\rho(y)$ the cube of edge ρ, centered at y and with faces parallel to the coordinate planes. If y is the origin, let $B_\rho(0) = B_\rho$, and $K_\rho(0) = K_\rho$.

For a Lebesgue measurable set $A \subset \mathbb{R}^N$ denote by $|A|$ its measure.

Lemma 2.2 (DeGiorgi [36]) *Let $v \in W^{1,1}(K_\rho(y))$, and let $k < \ell$ be real numbers. There exists a constant γ depending only on N, p and independent of k, ℓ, v, y, ρ, such that*

$$(\ell - k)|[v > \ell]| \le \gamma \frac{\rho^{N+1}}{|[v < k]|} \int_{[k<v<\ell]} |Dv| dx. \tag{2.1}$$

Remark 2.1 The conclusion of the lemma continues to hold for functions $v \in W^{1,1}(E)$ provided E is *convex*. It can be used for balls $B_\rho(y)$.

The embedding $(1.1)'$ of Corollary 1.1 gives a majorization of the $L^q(E)$-norm of u solely in terms of the $L^p(E)$-norm of its gradient. This is possible because u vanishes on ∂E in the sense of the traces.

A Poincaré-type inequality bounds some integral norm of a function $u \in W^{1,p}(E)$ in terms *only* of some integral norm of its gradient, provided some information is available on the set where u vanishes.

Proposition 2.1 *Let $E \subset \mathbb{R}^N$ be bounded and convex and let $\varphi \in C(\bar{E})$ satisfy*

$$0 \le \varphi \le 1, \qquad \text{and the sets } [\varphi > k] \text{ are convex for all } k \in \mathbb{R}_+.$$

Let $v \in W^{1,p}(E)$ and assume that the set

$$\mathcal{E} = [v = 0] \cap [\varphi = 1]$$

has positive measure. There exists a constant C depending only on N and p and independent of v and φ, such that

$$\Big(\int_E \varphi |v|^p dx\Big)^{\frac{1}{p}} \le C \frac{(\operatorname{diam} E)^N}{|\mathcal{E}|^{\frac{N-1}{N}}} \Big(\int_E \varphi |Dv|^p dx\Big)^{\frac{1}{p}}. \qquad (2.2)$$

Remark 2.2 Inequality (2.1) follows from this by applying (2.2) with $\varphi = 1$ and $p = 1$ to the function

$$w = \begin{cases} \min\{v\,,\, \ell\} - k & \text{if } v > k \\ 0 & \text{if } v \le k. \end{cases}$$

By Lemma 2.1 such a function is in $W^{1,1}(E)$.

3 A Measure-Theoretical Lemma ([48])

If $u \in C(E)$ and $u(y) = 1$ for some $y \in E$, for every $\sigma \in (0,1)$ there exists a ball $B_\rho(y) \subset E$, such that $u \ge 1-\sigma$ in $B_\rho(y)$, with ρ being determined by σ and the modulus of continuity of u. A similar statement valid for measurable functions follows from the Severini–Egorov theorem ([142], [64]), where, however, one cannot, in general, quantify the size and shape of the neighborhood of y where, roughly speaking, u is near 1. The following measure-theoretical lemma can be regarded as a quantitative version of the Severini–Egorov theorem, for functions $u \in W^{1,1}_{\mathrm{loc}}(E)$.

Lemma 3.1 *Let $u \in W^{1,1}(K_\rho)$ satisfy*

$$\|u\|_{W^{1,1}(K_\rho)} \le \gamma \rho^{N-1} \qquad \text{and} \qquad |[u > 1]| \ge \alpha |K_\rho|$$

for some $\gamma > 0$ and $\alpha \in (0,1)$. Then, for every $\delta \in (0,1)$ and $0 < \lambda < 1$ there exist $y \in K_\rho$ and $\varepsilon = \varepsilon(\alpha, \delta, \gamma, \lambda, N) \in (0,1)$, such that

$$|[u > \lambda] \cap K_{\varepsilon\rho}(y)| > (1-\delta)|K_{\varepsilon\rho}(y)|.$$

Roughly speaking the lemma asserts that if the set where u is bounded away from zero occupies a sizable portion of K_ρ, then there exist at least one point y and a neighborhood $K_{\varepsilon\rho}(y)$ where u remains large in a large portion of $K_{\varepsilon\rho}(y)$. Thus the set where u is positive clusters about at least one point $y \in K_\rho$.

Proof It suffices to establish the lemma for u smooth and $\rho = 1$. For $n \in \mathbb{N}$ partition K_1 into n^N cubes, with pairwise disjoint interior and each of edge $1/n$. Divide these cubes into two finite subcollections $\mathbf{Q}^+$ and $\mathbf{Q}^-$ by

$$\begin{aligned} Q_j \in \mathbf{Q}^+ \quad &\Longleftrightarrow \quad |[u > 1] \cap Q_j| > \tfrac{1}{2}\alpha |Q_j| \\ Q_i \in \mathbf{Q}^- \quad &\Longleftrightarrow \quad |[u > 1] \cap Q_i| \le \tfrac{1}{2}\alpha |Q_i| \end{aligned}$$

and denote by $\#(\mathbf{Q}^+)$ the number of cubes in $\mathbf{Q}^+$. By the assumption

$$\sum_{Q_j \in \mathbf{Q}^+} |[u > 1] \cap Q_j| + \sum_{Q_i \in \mathbf{Q}^-} |[u > 1] \cap Q_i| > \alpha |K_1| = \alpha n^N |Q|$$

where $|Q|$ is the common measure of the Q_ℓ. From the definition of $\mathbf{Q}^\pm$

$$\begin{aligned} \alpha n^N &< \sum_{Q_j \in \mathbf{Q}^+} \frac{|[u > 1] \cap Q_j|}{|Q_j|} + \sum_{Q_i \in \mathbf{Q}^-} \frac{|[u > 1] \cap Q_i|}{|Q_i|} \\ &< \#(\mathbf{Q}^+) + \frac{\alpha}{2}(n^N - \#(\mathbf{Q}^+)). \end{aligned}$$

Therefore

$$\#(\mathbf{Q}^+) > \frac{\alpha}{2 - \alpha} n^N. \tag{3.1}$$

Fix $\delta, \lambda \in (0, 1)$. The integer n can be chosen depending on $\alpha, \delta, \lambda, \gamma$, and N, such that

$$|[u > \lambda] \cap Q_j| \ge (1 - \delta)|Q_j| \quad \text{for some } \; Q_j \in \mathbf{Q}^+. \tag{3.2}$$

This would establish the lemma for $\varepsilon = 1/n$. Let $Q \in \mathbf{Q}^+$ satisfy

$$|[u > \lambda] \cap Q| < (1 - \delta)|Q|. \tag{3.3}$$

We will show that for such a cube, there exists a constant $c = c(\delta, \lambda, N)$ such that

$$\|u\|_{W^{1,1}(Q)} \ge \alpha c(\delta, \lambda, N) \frac{1}{n^{N-1}}. \tag{3.4}$$

From the assumptions

$$|[u \le \lambda] \cap Q| \ge \delta |Q| \qquad \text{and} \qquad \left| \left[u > \frac{1 + \lambda}{2} \right] \cap Q \right| > \frac{\alpha}{2} |Q|.$$

For fixed $x \in [u \le \lambda] \cap Q$ and $y \in [u > (1 + \lambda)/2] \cap Q$,

$$\frac{1 - \lambda}{2} \le u(y) - u(x) = \int_0^{|y - x|} Du(x + t\mathbf{n}) \cdot \mathbf{n} \, dt$$

where

$$\mathbf{n} = \frac{y - x}{|x - y|}, \qquad \text{for} \qquad x \ne y.$$

Let $R(x,\omega)$ be the polar representation of ∂Q with pole at x, for the solid angle ω. Integrate the previous relation in dy over $[u > (1+\lambda)/2]\cap Q$. Minorize the resulting left-hand side, by using the lower bound on the measure of such a set, and majorize the resulting integral on the right-hand side by extending the integration over Q. Expressing such integration in polar coordinates with pole at $x \in [u \le \lambda] \cap Q$ gives

$$\begin{aligned}
\frac{\alpha(1-\lambda)}{4}|Q| &\le \int_{|\mathbf{n}|=1}\int_0^{R(x,\mathbf{n})} r^{N-1}\int_0^{|y-x|}|Du(x+t\mathbf{n})|dt\,dr\,d\mathbf{n}\\
&\le N^{N/2}|Q|\int_{|\mathbf{n}|=1}\int_0^{R(x,\mathbf{n})}|Du(x+t\mathbf{n})|dt\,d\mathbf{n}\\
&= N^{N/2}|Q|\int_Q\frac{|Du(z)|}{|z-x|^{N-1}}dz.
\end{aligned}$$

Integrate now in dx over $[u \le \lambda]\cap Q$. Minorize the resulting left-hand side by using the lower bound on the measure of such a set, and majorize the resulting right-hand side, by extending the integration to Q. This gives

$$\begin{aligned}
\frac{\alpha\delta(1-\lambda)}{4N^{N/2}}|Q| &\le \|u\|_{W^{1,1}(Q)}\sup_{z\in Q}\int_Q\frac{1}{|z-x|^{N-1}}dx\\
&\le C(N)|Q|^{1/N}\|u\|_{W^{1,1}(Q)}
\end{aligned}$$

for a constant $C(N)$ depending only on N, thereby establishing (3.4).

If (3.2) does not hold for any cube $Q_j \in \mathbf{Q}^+$, then (3.3) and hence (3.4) is verified for all such Q_j. Adding over such cubes and taking into account (3.1),

$$\frac{\alpha^2}{2-\alpha}c(\delta,\lambda,N)n \le \|u\|_{W^{1,1}(K_1)} \le \gamma. \qquad \blacksquare$$

Remark 3.1 Following the various steps of the proof, the dependence of the reducing parameter ε on the measure-theoretical parameter α, and on the constant γ appearing in the assumptions of the lemma, can be traced to be of the form

$$\varepsilon = B^{-1}\frac{\alpha^2}{\gamma} \tag{3.5}$$

for a constant $B > 1$ depending on δ, N, and λ and independent of α.

4 Parabolic Spaces and Embeddings

For $0 < T < \infty$ let E_T denote the cylindrical domain $E \times (0,T]$. The space $L^{r,q}(E_T)$ for $q, r \ge 1$ is the collection of functions f defined and measurable in E_T such that

$$\|f\|_{q,r;E_T} = \Big(\int_0^T\Big(\int_E|f|^q dx\Big)^{\frac{r}{q}}d\tau\Big)^{\frac{1}{r}} < \infty.$$

Also $f \in L^{q,r}_{\rm loc}(E_T)$, if for every compact subset $K \subset E$ and every subinterval $[t_1, t_2] \subset (0, T]$

$$\int_{t_1}^{t_2} \Big(\int_K |f|^q dx \Big)^{\frac{r}{q}} d\tau < \infty.$$

Whenever $q = r$ we set $L^{q,q}(E_T) = L^q(E_T)$. These definitions are extended in the obvious way when either q or r is infinity.

We introduce spaces of functions, depending on $(x, t) \in E_T$, that exhibit different behavior in the space and time variables. These are spaces where typically solutions to parabolic equations in divergence form are found.

Let $m, p \geq 1$ and consider the Banach spaces

$$V^{m,p}(E_T) = L^\infty\big(0, T; L^m(E)\big) \cap L^p\big(0, T; W^{1,p}(E)\big)$$
$$V_o^{m,p}(E_T) = L^\infty\big(0, T; L^m(E)\big) \cap L^p\big(0, T; W_o^{1,p}(E)\big)$$

both equipped with the norm

$$\|v\|_{V^{m,p}(E_T)} = \operatorname*{ess\,sup}_{0<t<T} \|v(\cdot, t)\|_{m,E} + \|Dv\|_{p,E_T}.$$

When $m = p$, set $V_o^{p,p}(E_T) = V_o^p(E_T)$ and $V^{p,p}(E_T) = V^p(E_T)$. Both spaces are embedded in $L^q(E_T)$ for some $q > p$. In a precise way we have

Proposition 4.1 *There exists a constant γ depending only on N, p, m such that for every $v \in V_o^{m,p}(E_T)$*

$$\begin{aligned}\iint_{E_T} |v(x,t)|^q dx\, dt \leq \gamma^q \Big(\iint_{E_T} |Dv(x,t)|^p dx\, dt \Big) \\ \times \Big(\operatorname*{ess\,sup}_{0<t<T} \int_E |v(x,t)|^m dx \Big)^{\frac{p}{N}}\end{aligned} \tag{4.1}$$

where

$$q = p\frac{N+m}{N}.$$

Moreover

$$\|v\|_{q,E_T} \leq \gamma \|v\|_{V^{m,p}(E_T)}. \tag{4.2}$$

Remark 4.1 The multiplicative inequality (4.1) and the embedding (4.2) continue to hold for functions $v \in V^{m,p}(E_T)$ such that

$$\int_E v(x,t) dx = 0 \qquad \text{for a.e. } t \in (0, T)$$

provided ∂E is piecewise smooth. In such a case the constant γ depends also on the structure of ∂E, but not on its size.

The next corollary follows from Proposition 4.1 by taking $m = p$ and by applying Hölder's inequality.

Corollary 4.1 *Let $p > 1$. There exists a constant γ depending only on N and p, such that for every $v \in V_o^p(E_T)$,*

$$\|v\|_{p,E_T}^p \le \gamma \big|[|v| > 0]\big|^{\frac{p}{N+p}} \|v\|_{V^p(E_T)}^p.$$

When $m = p$, Proposition 4.1 takes the form

Proposition 4.2 *There exists a constant γ depending only on N and p such that for every $v \in V_o^p(E_T)$,*

$$\|v\|_{q,r;E_T} \le \gamma \|v\|_{V^p(E_T)},$$

where the numbers $q, r \ge 1$ are linked by

$$\frac{1}{r} + \frac{N}{pq} = \frac{N}{p^2},$$

and their admissible range is

$$\begin{array}{lll} \textit{if } N = 1, & q \in (p, \infty], & r \in [p^2, \infty); \\ \textit{if } 1 \le p < N, & q \in \big[p, \frac{Np}{N-p}\big], & r \in [p, \infty]; \\ \textit{if } 1 < N \le p, & q \in [p, \infty), & r \in \big(\frac{p^2}{N}, \infty\big]. \end{array}$$

We conclude this section by stating a parabolic version of Lemma 2.1 and Corollary 2.1 concerning the truncated functions $(v - k)_\pm$.

Lemma 4.1 *Let $v \in V^{m,p}(E_T)$. Then $(v - k)_\pm \in V^{m,p}(E_T)$ for all $k \in \mathbb{R}$. Assume in addition that ∂E is piecewise smooth and that the trace of $v(\cdot, t)$ on ∂E is essentially bounded and*

$$\operatorname*{ess\,sup}_{0<t<T} \|v(\cdot, t)\|_{\infty,\partial E} \le M \qquad \textit{for some } M > 0.$$

Then $(v - k)_\pm \in V_o^{m,p}(E_T)$ for all $k \ge M$.

5 Some Technical Facts

5.1 A Lemma on Fast Geometric Convergence

Lemma 5.1 *Let $\{Y_n\}$ for $n = 0, 1, \ldots,$ be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{n+1} \le C b^n Y_n^{1+\alpha},$$

where $C, b > 1$ and $\alpha > 0$ are given numbers. If

$$Y_o \le C^{-1/\alpha} b^{-1/\alpha^2},$$

then $\{Y_n\} \to 0$ as $n \to \infty$.

5.2 An Interpolation Lemma

Lemma 5.2 *Let $\{Y_n\}$ for $n = 0, 1, \ldots,$ be a sequence of equi-bounded positive numbers satisfying the recursive inequalities*

$$Y_n \leq C b^n Y_{n+1}^{1-\alpha},$$

where $C, b > 1$ and $\alpha \in (0,1)$ are given constants. Then

$$Y_o \leq \left(\frac{2C}{b^{1-\frac{1}{\alpha}}} \right)^{\frac{1}{\alpha}}.$$

Remark 5.1 The lemma turns the *qualitative* information of equi-boundedness of the sequence $\{Y_n\}$ into a *quantitative* a priori estimate for Y_o.

5.3 Steklov Averages

Let $v \in L^1(E_T)$ and let $0 < h < T$. The Steklov averages $v_h(\cdot, t)$ and $v_{\bar{h}}(\cdot, t)$ are defined by

$$v_h = \begin{cases} \dfrac{1}{h} \displaystyle\int_t^{t+h} v(\cdot, \tau) d\tau & \text{for } \ t \in (0, T-h], \\ 0, & \text{for } \ t > T - h. \end{cases}$$

$$v_{\bar{h}} = \begin{cases} \dfrac{1}{h} \displaystyle\int_{t-h}^{t} v(\cdot, \tau) d\tau & \text{for } \ t \in (h, T], \\ 0, & \text{for } \ t < h. \end{cases}$$

Lemma 5.3 *Let $v \in L^{q,r}(E_T)$. Then, as $h \to 0$, $v_h \to v$ in $L^{q,r}(E_{T-\varepsilon})$ for every $\varepsilon \in (0,T)$. If $v \in C(0,T; L^q(E))$, then $v_h(\cdot,t) \to v(\cdot,t)$ in $L^q(E)$ for every $t \in (0, T-\varepsilon)$ for all $\varepsilon \in (0,T)$.*

A similar statement holds for $v_{\bar{h}}$. The proof of the lemma is straightforward from the theory of L^p spaces.

6 Remarks and Bibliographical Notes

The proofs of the multiplicative embedding of Theorem 1.1 and the embeddings of Theorem 1.2, are in a number of monographs ([102, 114, 5, 42]).

The best constants in (1.1) are traced by Talenti [146].

Theorem 1.2 is due to Sobolev and Nikol'ski [143]. The dependence of the constant C of the structure of ∂E is traced in [42].

Poincaré first stated and proved the inequality that was later named after him in [128]; he then gave a second and more refined proof in [129].

A thorough treatment of Sobolev inequalities and their connection with Harnack-type estimates is in [136].

In the context of partial differential equations, the truncations $(v-k)_\pm$ were introduced by Bernstein ([17]) and effectively used by Stampacchia ([144]) and DeGiorgi ([36]).

The proof of Lemma 2.1, on the truncations $(v-k)_\pm$, is in [144]. A simpler proof is reported in [76].

Inequality (2.1) is due to DeGiorgi [36] and it is referred to as a "discrete" isoperimetric inequality. A continuous version is in [67].

The proof of Proposition 2.1 is in [101] and it follows essentially DeGiorgi's proof of Lemma 2.2.

A version of the measure-theoretical Lemma 3.1 was first established in [63] for $u \in W^{1,p}(K_\rho)$ and $p > 1$. Such a limitation on p was essential to the proof. The proof presented here, taken from [48], removes such a restriction and is simpler.

The parabolic spaces $V^{m,p}(E_T)$ and $V_o^{m,p}(E_T)$ are generalizations of the spaces $V^2(E_T)$ introduced in [101]. The generalizations are introduced to track down the notion of degenerate and singular parabolic equations. The embeddings of § 4 of these spaces are established in [41].

The proof of Lemma 5.1 on fast geometric convergence is in [36] and reported in [102, 101]. A simpler proof is in [41].

The interpolation Lemma 5.2 is taken from [26, 27].

In a series of papers published at the beginning of the 20th century, Steklov studied completeness problems, making a large use of integral averaging of functions. This later prompted the use of the term Steklov averages.

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