Chapter 2
Materials with Constitutive Equations That Are Local in Time

2.1 Introduction

We now consider the constitutive equations relating to fluids and solids for which memory effects are negligible. This contrasts with subsequent chapters, which are devoted almost entirely to materials with memory. In fact, however, one example included in the discussion, namely viscous fluids, can be visualized as possessing very short-term memory, expressed by the presence of time derivatives of certain field quantities. This is consistent with the general correlation that will arise throughout the present work between memory effects and energy dissipation.

In what follows we will consider three types of constitutive equations: (i) constraints on the possible deformations the body may undergo; (ii) assumptions concerning the form of the stress tensor; (iii) constitutive equations relating the stress to the deformation. Each of these three types of constitutive equations is appropriate for a certain class of materials, and its validity is verified by experiment.

As an example of a constitutive equation of type (i) we give the constraint that only rigid motions are possible, a constraint that underlies rigid-body mechanics. Another example in this class is the assumption of incompressibility, in which only isochoric deformations are permissible. Such an assumption is realistic for liquids such as water under normal flow conditions. An example of a constitutive equation of type (ii) is the widely used assumption that the stress is a pressure, an assumption appropriate for most fluids when viscous effects are negligible. Finally, an example of a constitutive equation of type (iii) is Hooke's law relating the deformation of a body to the state of stress, which is appropriate for linear elastic materials.

2.2 Fluids. Ideal Fluids

In this section, the constitutive equations for various classes of fluids are discussed. To this end we note that the contact forces in such materials are most naturally

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considered in the Eulerian description of deformation (Definition 1.2.2), that is, they are regarded as depending on the variables $(\mathbf{x}, t)$. From a physical point of view, this means that the contact forces are determined by the kinematic properties of the fluid at the present time.

Definition 2.2.1. By a *dynamical process* we mean a pair $(\mathbf{x}, \mathbf{T})$ where $\mathbf{x}$ is a deformation in the sense of $(1.2.3)_1$ and $\mathbf{T}$ a symmetric tensor field on the trajectory of $\mathbf{x}$. The quantity $\mathbf{T}(\mathbf{x}, t)$ is a smooth function of $\mathbf{x}$ on $\varphi_t(\mathcal{B})$.

Definition 2.2.2. A dynamical process $(\mathbf{x}, \mathbf{T})$ is *isochoric* if $(\mathbf{x}, t)$ is an isochoric or volume-preserving deformation at each t. A material body is *incompressible* if each $(\mathbf{x}, \mathbf{T})$ is isochoric, so that for every subbody A of the body, we have

$$\mathrm{vol}\,(\varphi_t(A)) = \mathrm{vol}\,(\varphi_0(A)), \quad \text{for all } t. \tag{2.2.1}$$

The above relation states that every deformation preserves volume, and moreover, the volume of any subbody throughout the deformation must be the same as its volume in the reference configuration. In view of relations (1.2.31), (1.2.32), and $(1.3.2)_1$, the condition of incompressibility (2.2.1) takes one of the following forms:

$$J = \det \mathbf{F} = 1 \quad \text{or} \quad \mathrm{div}_{\mathbf{x}}\, \mathbf{v} = 0 \quad \text{or} \quad \rho = \rho_0.$$

Definition 2.2.3. A dynamical process $(\mathbf{x}, \mathbf{T})$ is *Eulerian* if the Cauchy stress is a pressure, given by

$$\mathbf{T} = -p\mathbf{1},$$

where p is a scalar field on the trajectory of $\mathbf{x}$.

Definition 2.2.4. An *ideal fluid* is a material body that can support only isochoric Eulerian dynamical processes and whose density ρ_0 is constant.

The tension on an arbitrary elementary surface $d\sigma$, with unit normal vector $\mathbf{n}$, is given by $\mathbf{t}(\mathbf{n}) = -p\mathbf{n}$, so that it is parallel to the normal (see Figure 2.2.1). We observe that an ideal fluid is an incompressible material body for which the Cauchy stress is a pressure in every flow. Furthermore, the pressure is not determined uniquely by the deformation; there exists an infinite number of pressure fields corresponding to the same deformation. That such a property is physically reasonable can be inferred from the following example. Consider a ball composed of an ideal fluid under a time–independent uniform pressure p and assume, for the moment, that all body forces are absent. Then the ball should remain in the same configuration for all time. Moreover, since the ball is incompressible, an increase or decrease in pressure should not result in a deformation. Thus, the same deformation corresponds to all uniform pressure fields.

Summarizing, the equations of motion for an ideal fluid with density ρ_0 (see (1.3.25) and $(1.3.2)_2$) are

$$\rho_0 \dot{\mathbf{v}} = -\nabla_{\mathbf{x}} p + \rho_0 \mathbf{b}, \qquad \mathrm{div}_{\mathbf{x}}\, \mathbf{v} = 0. \tag{2.2.2}$$

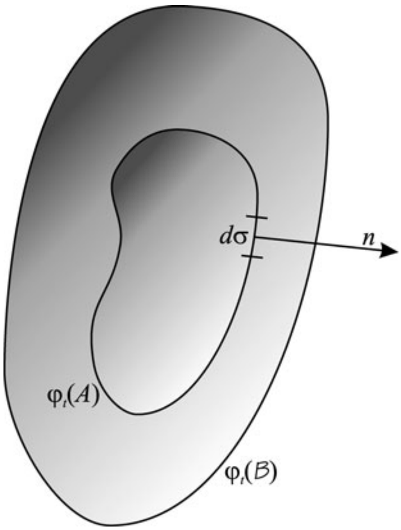

Fig. 2.2.1 The normal unit vector to an elementary surface element.

We note that for a conservative body force with the potential energy $\mathcal{V}$, relation $(2.2.2)_1$ becomes

$$\dot{\mathbf{v}} = -\nabla_{\mathbf{x}}\left(\frac{p}{\rho_0} + \mathcal{V}\right),$$

so that the acceleration is the gradient of a potential .

On the basis of Bernoulli's theorem (Theorem 1.3.13), we can formulate the following result.

Theorem 2.2.5. (Bernoulli's theorem for ideal fluids) *Let* $\{\mathbf{v}, \rho_0, -p\mathbf{1}\}$ *be a flow of an ideal fluid under a conservative body force with potential energy* $\mathcal{V}$. *Then:*

(i) If the flow is potential ($\mathbf{v} = \nabla_{\mathbf{x}}\phi$*), then*

$$\nabla_{\mathbf{x}}\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{v}^2 + \frac{p}{\rho_0} + \mathcal{V}\right) = \mathbf{0}. \tag{2.2.3}$$

(ii) If the flow is steady, then

$$\frac{d}{dt}\left(\frac{1}{2}\mathbf{v}^2 + \frac{p}{\rho_0} + \mathcal{V}\right) = 0. \tag{2.2.4}$$

(iii) If the flow is steady and irrotational, then

$$\frac{1}{2}\mathbf{v}^2 + \frac{p}{\rho_0} + \mathcal{V} = constant. \tag{2.2.5}$$

Proof. Since $\rho = \rho_0 = constant$, the relation (1.3.40) implies (2.2.3). Furthermore, if we set

$$\eta = \frac{1}{2}\mathbf{v}^2 + \frac{p}{\rho_0} + \mathcal{V},$$

then for a steady flow, $\frac{\partial \eta}{\partial t} = 0$, and moreover, by (1.3.41), we have $\mathbf{v} \cdot \nabla_{\mathbf{x}} \eta = 0$, so that

$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \nabla_x \eta = 0,$$

which is (2.2.4). Finally, for a steady and irrotational flow, (1.3.42) yields $\nabla_{\mathbf{x}} \eta = 0$, which, with $\frac{\partial \eta}{\partial t} = 0$, implies that η is a constant in space and time, or (2.2.5). □

Remark 2.2.6. With the aid of Bernoulli's theorem, we see that for a steady and irrotational flow under a conservative body force, equations $(2.2.2)_1$ and $(2.2.2)_2$ reduce to

$$\mathrm{div}_{\mathbf{x}}\, \mathbf{v} = 0, \quad \mathrm{curl}_{\mathbf{x}} \mathbf{v} = \mathbf{0}, \quad \frac{1}{2}\mathbf{v}^2 + \frac{p}{\rho_0} + \mathcal{V} = constant. \tag{2.2.6}$$

In a steady motion the velocity is tangent to the boundary, so that (2.2.6) should be supplemented by the boundary condition

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial \varphi_0(\mathcal{B}).$$

Remark 2.2.7. For an unsteady flow, we have to solve the system of differential equations described by (2.2.2). Using (1.2.6), we can write this system in the form of *Euler's equations*

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla_{\mathbf{x}} \mathbf{v})\, \mathbf{v} = -\nabla_{\mathbf{x}} \left(\frac{p}{\rho_0} \right) + \mathbf{b},$$
$$\mathrm{div}_{\mathbf{x}}\, \mathbf{v} = 0,$$

which is a nonlinear differential system because of the presence of the term $(\nabla_{\mathbf{x}} \mathbf{v})\, \mathbf{v}$.

2.2.1 Elastic Fluids

In what follows we consider a material body in which compressibility effects are not ignored and for which the pressure is completely specified by the deformation.

Definition 2.2.8. An elastic fluid is a material body for which the constitutive class is defined by the smooth response function $\hat{p} : \mathbb{R}^+ \to \mathbb{R}$ giving the pressure when the mass density is known:

$$p = \hat{p}(\rho)\,. \tag{2.2.7}$$

For such a fluid, the constitutive class is the set of all Eulerian dynamical processes $(\mathbf{x}, -p\mathbf{1})$ (Definition 2.2.1) that obey the constitutive equation (2.2.7).

The basic equations for the flow $\{\mathbf{v}, \rho, -p\mathbf{1}\}$ of an elastic fluid are the equations of motion

$$\rho\dot{\mathbf{v}} = -\nabla_{\mathbf{x}} p + \rho \mathbf{b}, \tag{2.2.8}$$

conservation of mass

$$\frac{\partial \rho}{\partial t} + \mathrm{div}_{\mathbf{x}}\,(\rho \mathbf{v}) = 0, \tag{2.2.9}$$

and the constitutive equation (2.2.7).

We assume that $\hat{p}$ has a strictly positive derivative and define the functions $\kappa > 0$ and ε on $\mathbb{R}^+$ by

$$\kappa^2(\rho) = \frac{d\hat{p}(\rho)}{d\rho}, \qquad \varepsilon(\rho) = \int_{\rho_*}^{\rho} \frac{\kappa^2(\xi)}{\xi} d\xi,$$

where ρ_* is an arbitrarily chosen value of the mass density. The function $\kappa(\rho)$ is called the *speed of sound*. By the chain rule, we obtain

$$\nabla_{\mathbf{x}} \varepsilon(\rho) = \frac{\kappa^2(\rho)}{\rho} \nabla_{\mathbf{x}} \rho = \frac{1}{\rho}\frac{d\hat{p}(\rho)}{d\rho} \nabla_{\mathbf{x}} \rho = \frac{1}{\rho} \nabla_{\mathbf{x}} p,$$

so that for a conservative body force with potential energy $\mathcal{V}$, the equation of motion (2.2.8) takes the form

$$\dot{\mathbf{v}} = -\nabla_{\mathbf{x}} \left[\varepsilon(\rho) + \mathcal{V}\right],$$

and hence the acceleration is the gradient of a potential. Thus, by Bernoulli's theorem we get the following result.

Theorem 2.2.9. (Bernoulli's theorem for elastic fluids) *Let* $\{\mathbf{v}, \rho, -p\mathbf{1}\}$ *be a flow for an elastic fluid under a conservative body force with potential energy* $\mathcal{V}$*. Then:*

(i) If the flow is potential, then

$$\nabla_{\mathbf{x}} \left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{v}^2 + \varepsilon(\rho) + \mathcal{V}\right] = \mathbf{0}.$$

(ii) If the flow is steady, then

$$\frac{d}{dt}\left[\frac{1}{2}\mathbf{v}^2 + \varepsilon(\rho) + \mathcal{V}\right] = 0.$$

(iii) If the flow is steady and irrotational, then

$$\frac{1}{2}\mathbf{v}^2 + \varepsilon(\rho) + \mathcal{V} = constant.$$

Remark 2.2.10. If we set

$$\alpha(\rho) = \frac{\kappa^2(\rho)}{\rho},$$

then the basic equations (2.2.7)–(2.2.9) lead to

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\nabla_{\mathbf{x}} \mathbf{v})\, \mathbf{v} + \alpha(\rho)\, \nabla_{\mathbf{x}} \rho = \mathbf{b}, \\ \frac{\partial \rho}{\partial t} + \mathrm{div}_{\mathbf{x}}\, (\rho \mathbf{v}) = 0. \end{aligned} \tag{2.2.10}$$

The relations (2.2.10) furnish a nonlinear differential system for ρ and $\mathbf{v}$. Concerning such a system we can establish the following result.

Proposition 2.2.11. *In a steady flow of an elastic fluid under vanishing body forces we have*

$$\frac{d}{dt}(\rho v) = \rho\left(1 - m^2\right)\frac{dv}{dt}, \tag{2.2.11}$$

where $v = |\mathbf{v}|$ *is assumed different from zero and*

$$m = \frac{v}{\kappa(\rho)} \tag{2.2.12}$$

is the Mach number.

Proof. Since we have a steady flow, it follows that $\frac{\partial \rho}{\partial t} = 0$ and hence, from (1.2.6),

$$\dot{\rho} = \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho.$$

Furthermore, with $\mathbf{b} = \mathbf{0}$, equation $(2.2.10)_1$ implies that

$$\mathbf{v} \cdot \dot{\mathbf{v}} = -\frac{\kappa^2(\rho)}{\rho} \mathbf{v} \cdot \nabla_{\mathbf{x}} \rho = -\frac{\kappa^2(\rho)}{\rho} \dot{\rho},$$

and therefore, with the aid of (2.2.12), we have

$$\mathbf{v} \cdot \frac{d}{dt}(\rho \mathbf{v}) = \mathbf{v} \cdot (\rho \dot{\mathbf{v}} + \dot{\rho} \mathbf{v}) = \rho\, (\mathbf{v} \cdot \dot{\mathbf{v}}) \left(1 - m^2\right).$$

This last relation, when combined with the observation that

$$\mathbf{v} \cdot \dot{\mathbf{v}} = \frac{d}{dt}\left(\frac{1}{2} v^2\right) = v \dot{v}$$

and hence

$$\mathbf{v} \cdot \frac{d}{dt}(\rho \mathbf{v}) = v(\dot{\rho} + \rho \dot{v}) = v \frac{d}{dt}(\rho v),$$

proves (2.2.11).

It follows that for $m < 1$, the mass flow $\rho(\mathbf{x}) v(\mathbf{x})$ increases, while for $m > 1$, the mass flow decreases. □

Proposition 2.2.11 motivates the following definition.

Definition 2.2.12. A flow is subsonic, sonic, or supersonic at $(\mathbf{x}, t)$ according to whether $m(\mathbf{x}, t)$ is less than, equal to, or greater than 1, respectively.

For a steady and irrotational flow under a conservative body force with potential energy $\mathcal{V}$, we conclude, from Bernoulli's theorem, that the basic equations characterizing such a flow are

$$\mathrm{div}_{\mathbf{x}}\,(\rho\mathbf{v}) = 0, \quad \mathrm{curl}_{\mathbf{x}}\mathbf{v} = \mathbf{0}, \quad \frac{1}{2}\mathbf{v}^2 + \varepsilon\,(\rho) + \mathcal{V} = constant.$$

2.2.2 Newtonian Fluids. The Navier–Stokes Equations

The ideal and elastic fluids previously discussed never exhibit shearing stress, and therefore they are incapable of describing frictional force. Friction in fluids generally manifests itself through shearing forces that retard the relative motion of fluid particles. A measure of the relative motion of fluid particles is furnished by the velocity gradient.

Definition 2.2.13. A Newtonian fluid is an incompressible material with constitutive equation

$$\mathbf{T}(\mathbf{x}, t) = -p(\mathbf{x}, t)\mathbf{1} + 2\mu\mathbf{D}(\mathbf{x}, t), \tag{2.2.13}$$

where $\mathbf{D}$ is the stretching, and the scalar constant μ is known as the *viscosity* of the fluid. The term $\mathbf{T}_0 = 2\mu\mathbf{D} = \mathbf{T} + p\mathbf{1}$ is referred to as the *extra stress*. Since $\mathrm{tr}\mathbf{D} = \mathrm{div}_{\mathbf{x}}\,\mathbf{v} = 0$, we have that $\mathbf{T}_0 = \mathbf{T} - \frac{1}{3}\,(tr\mathbf{T})\,\mathbf{1}$.

Remark 2.2.14. Since $\mathbf{D}$ vanishes for a material at rest, by (1.2.23), it follows from (2.2.13) that a Newtonian fluid at rest behaves like an ideal fluid.

The basic equations for a Newtonian fluid are

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\nabla_{\mathbf{x}}\mathbf{v})\,\mathbf{v}\right] = \mathrm{div}_{\mathbf{x}}\,\mathbf{T} + \rho\mathbf{b}, \quad \mathbf{T} = -p\mathbf{1} + 2\mu\mathbf{D}, \quad \mathrm{div}_{\mathbf{x}}\,\mathbf{v} = 0. \tag{2.2.14}$$

We have $\rho = \rho_0$, because of $(1.3.2)_2$ and $(2.2.14)_3$. Also

$$2\,\mathrm{div}_{\mathbf{x}}\,\mathbf{D} = \mathrm{div}_{\mathbf{x}}\left[\nabla_{\mathbf{x}}\mathbf{v} + (\nabla_{\mathbf{x}}\mathbf{v})^T\right] = \Delta\mathbf{v} + \nabla_{\mathbf{x}}\,\mathrm{div}_{\mathbf{x}}\,\mathbf{v} = \Delta\mathbf{v},$$

and equations (2.2.14) reduce to

$$\frac{\partial\mathbf{v}}{\partial t} + (\nabla_{\mathbf{x}}\mathbf{v})\,\mathbf{v} = \nu\Delta\mathbf{v} - \nabla_{\mathbf{x}}p_0 + \mathbf{b}, \qquad \mathrm{div}_{\mathbf{x}}\,\mathbf{v} = 0, \tag{2.2.15}$$

where

$$\nu = \frac{\mu}{\rho_0}, \quad p_0 = \frac{p}{\rho_0}.$$

The scalar constant ν is known as the *kinematic viscosity*, and equations (2.2.15) are the *Navier–Stokes equations*. These constitute a nonlinear system of partial differential equations for the velocity $\mathbf{v}$ and the pressure p.

Suppose that the flow takes place in a region R. To the Navier–Stokes equations we add the restriction that the fluid adhere, without slipping, to the boundary ∂R. For a stationary boundary this means that $\mathbf{v} = \mathbf{0}$ on ∂R. If the boundary moves, then at each point on the boundary the fluid velocity must coincide with the velocity of the boundary.

Theorem 2.2.15. (Balance of energy for a viscous fluid) *For any flow of a Newtonian fluid we have*

$$\begin{aligned}\frac{d}{dt}\int_{\varphi_t(A)} \frac{1}{2}\rho_0 \mathbf{v}^2 dv_t + 2\mu \int_{\varphi_t(A)} |\mathbf{D}|^2 \, dv_t \\ = \int_{\varphi_t(A)} \rho_0 \mathbf{b}\cdot\mathbf{v} dv_t + \int_{\partial\varphi_t(A)} \mathbf{T}\mathbf{n}\cdot\mathbf{v} da_t,\end{aligned} \tag{2.2.16}$$

for every part A of the fluid.

Proof. The constitutive equation (2.2.13), together with the fact that $\mathbf{D}$ is traceless, gives

$$\mathbf{T}\cdot\mathbf{D} = -p\mathrm{tr}\mathbf{D} + 2\mu\,|\mathbf{D}|^2 = 2\mu\,|\mathbf{D}|^2\,,$$

and hence the theorem of power expended, expressed by relation (1.3.34), yields (2.2.16). □

Remark 2.2.16. The term

$$2\mu \int_{\varphi_t(A)} |\mathbf{D}|^2 \, dv_t$$

represents the rate at which the fluid in A dissipates energy. The energy equation (2.2.16) asserts that the total power expended on A must equal the rate of change of kinetic energy plus the rate of energy dissipation.

Corollary 2.2.17. *For a flow of a Newtonian fluid in a finite region $\mathcal{B}$ under the hypotheses of zero body force $\mathbf{b} = \mathbf{0}$, $\mathbf{v} = \mathbf{0}$ on $\partial\varphi_t(\mathcal{B})$ at all times and $\mu > 0$, we have*

$$\frac{d}{dt}\int_{\varphi_t(\mathcal{B})} \frac{1}{2}\rho_0 \mathbf{v}^2 dv_t \leq 0,$$

so that the kinetic energy decreases with time.

It is useful to write the Navier–Stokes equations in dimensionless form. For convenience we will assume that $\mathbf{b} = \mathbf{0}$. Consider l a typical length (such as the diameter when a cylindrical body is considered) and v a typical velocity. Let us further identify points $\mathbf{x}$ with their position vectors from a given origin O and introduce the dimensionless position vector $\bar{\mathbf{x}} = \frac{\mathbf{x}}{l}$, the dimensionless time $\bar{t} = \frac{tv}{l}$, the dimensionless velocity $\bar{\mathbf{v}}(\bar{\mathbf{x}}, \bar{t}) = \frac{1}{v}\mathbf{v}(\mathbf{x}, t)$, and the dimensionless pressure $\bar{p}_0(\bar{\mathbf{x}}, \bar{t}) = \frac{1}{v^2} p_0(\mathbf{x}, t)$. Thus, we have

$$\nabla_{\bar{\mathbf{x}}}\bar{\mathbf{v}} = \frac{l}{v}\nabla_{\mathbf{x}}\mathbf{v}, \quad \frac{\partial\bar{\mathbf{v}}}{\partial\bar{t}} = \frac{l}{v^2}\frac{\partial\mathbf{v}}{\partial t}, \quad \nabla_{\bar{\mathbf{x}}}\bar{p}_0 = \frac{l}{v^2}\nabla_{\bar{\mathbf{x}}} p_0.$$

Hence, the Navier–Stokes equations (2.2.15) become

$$\begin{gathered}\frac{\partial \bar{\mathbf{v}}}{\partial \bar{t}} + (\nabla_{\bar{\mathbf{x}}} \bar{\mathbf{v}})\, \bar{\mathbf{v}} = \frac{1}{Re} \bar{\Delta} \bar{\mathbf{v}} - \nabla_{\bar{\mathbf{x}}} \bar{p}_0, \\ \operatorname{div}_{\bar{\mathbf{x}}} \bar{\mathbf{v}} = 0,\end{gathered} \tag{2.2.17}$$

where $\bar{\Delta}$ is the Laplacian in dimensionless coordinates and

$$\mathrm{Re} = \frac{lv}{\nu}$$

is a dimensionless quantity known as the *Reynolds number* of the flow. The Navier–Stokes equations in the dimensionless form (2.2.17) show that a solution of the Navier–Stokes equations with a given Reynolds number can be used to generate solutions that have different length and velocity scales, but the same Reynolds number. This fact allows one to model a given flow situation in the laboratory by adjusting the length and velocity scales and the viscosity to give an experimentally tractable problem with the same Reynolds number of the effective flow.

Remark 2.2.18. Returning to the general Navier–Stokes equations, we note that if the flow is steady and if we neglect the nonlinear term $(\nabla_{\mathbf{x}} \mathbf{v})\, \mathbf{v}$, then relations (2.2.15) reduce to

$$\nu \Delta \mathbf{v} = \nabla_{\mathbf{x}} p_0 - \mathbf{b}, \qquad \operatorname{div}_{\mathbf{x}} \mathbf{v} = 0.$$

Solutions of this equation are called *Stokes flows* and are presumed to describe slow or creeping flows of Newtonian fluids.

For the compressible flow of a fluid, the constitutive equation (2.2.13) is replaced by

$$\mathbf{T}(\mathbf{x}, t) = -p(\mathbf{x}, t)\mathbf{1} + \lambda\, (\operatorname{div}_{\mathbf{x}} \mathbf{v})\, \mathbf{1} + 2\mu \mathbf{D}(x, t), \tag{2.2.18}$$

where λ and μ are the coefficients of viscosity. The basic equations (2.2.14) for such a flow become

$$\begin{aligned}&\rho \frac{\partial \mathbf{v}}{\partial t} + \rho\, (\nabla_{\mathbf{x}} \mathbf{v})\, \mathbf{v} = \mu \Delta \mathbf{v} + (\lambda + \mu)\, \nabla_{\mathbf{x}}\, (\operatorname{div}_{\mathbf{x}} \mathbf{v}) - \nabla_{\mathbf{x}} p + \rho \mathbf{b}, \\ &\frac{\partial \rho}{\partial t} + \operatorname{div}_{\mathbf{x}}\, (\rho \mathbf{v}) = 0.\end{aligned}$$

Now let us study the steady flow of a Newtonian fluid in a pipeline of cylindrical form having as cross-section a circle of radius R. Body forces are neglected. A reference frame is chosen such that the x_3-axis is parallel to the generators of the cylinder. We seek a solution of the following boundary value problem:

$$\nu \Delta \mathbf{v} - \nabla_{\mathbf{x}} p_0 = (\nabla_{\mathbf{x}} \mathbf{v}) \mathbf{v}, \qquad \operatorname{div}_{\mathbf{x}} \mathbf{v} = 0, \tag{2.2.19}$$

in the cylinder $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^3 : x_1^2 + x_2^2 < R^2,\ x_3 \in \mathbb{R}\}$ with

$$\mathbf{v} = \mathbf{0} \quad \text{on} \quad \partial \mathcal{C},$$

which is the frequently adopted no-slip assumption, noted before Theorem 2.2.15, that no relative motion can take place between the viscous fluid and the solid cylinder.

Let us try a solution of the above boundary value problem in the form

$$\mathbf{v} = v(x_1, x_2)\mathbf{i}_3,$$

for which the relation $\mathrm{div}_{\mathbf{x}}\, \mathbf{v} = 0$ is identically satisfied and the right-hand side of $(2.2.19)_1$ vanishes. This last equation then gives

$$\frac{\partial p_0}{\partial x_\alpha} = 0 \quad (\alpha = 1, 2),$$
$$\nu \Delta_0 v = \frac{\partial p_0}{\partial x_3},$$

where $\Delta_0 v = \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2}$ is the Laplacian in two dimensions. Thus, from the first two equations, we conclude that $p_0 = p_0(x_3)$, while the third equation implies that $\frac{dp_0}{dx_3}$ is independent of x_3. Therefore, we deduce that

$$\frac{dp_0}{dx_3} = -m, \quad \nu \Delta_0 v = -m,$$

where m is an unknown constant. Hence,

$$p_0 = p_1 - m x_3,$$

where p_1 is a constant of integration. Moreover, the function v is the solution of the following boundary value problem:

$$\Delta_0 v = -\frac{m}{\nu} \;\; \text{in } \Sigma, \quad v = 0 \quad \text{on } \partial\Sigma,$$

where Σ denotes the cross-section $x_1^2 + x_2^2 < R^2$. We try a solution of this boundary value problem in the form

$$v = v(r), \quad r = \sqrt{x_1^2 + x_2^2},$$

so that

$$\frac{\partial v}{\partial x_\alpha} = \frac{x_\alpha}{r}\frac{dv}{dr} = \frac{x_\alpha}{r} v', \qquad \frac{\partial^2 v}{\partial x_\alpha \partial x_\beta} = \left(\frac{1}{r}\delta_{\alpha\beta} - \frac{x_\alpha x_\beta}{r^3}\right) v' + \frac{x_\alpha x_\beta}{r^2} v''$$

and hence

$$\Delta_0 v = \frac{1}{r} v' + v''.$$

Thus, the function $v(r)$ satisfies the following differential equation:

$$(rv')' = -\frac{m}{\nu} r, \quad \text{with } v(R) = 0.$$

The general solution of this differential equation is

$$v(r) = -\frac{m}{4\nu} r^2 + C_1 \ln r + C_2,$$

where C_1 and C_2 are arbitrary constants. Because v is finite for $r = 0$, it follows that $C_1 = 0$, while the condition $v(R) = 0$ gives $C_2 = \frac{m}{4\nu} R^2$. Thus, the solution is

$$\mathbf{v} = \frac{m}{4\nu} \left(R^2 - r^2\right) \mathbf{i}_3, \qquad p_0 = p_1 - m x_3.$$

The above motion is known as *Poiseuille flow*.

2.2.3 Uniqueness of Solutions

For the classical viscous flow problem, we assume that the following data are given: a bounded regular region R, a kinematic viscosity $\nu > 0$, a body force field $\mathbf{b}$ on $R \times [0, \infty)$, an initial velocity distribution $\mathbf{v}_0$ on R, and a boundary velocity distribution $\hat{\mathbf{v}}$ on $\partial R \times [0, \infty)$. The problem is to find a class C^2 velocity field $\mathbf{v}$ and a smooth pressure field p on $R \times [0, \infty)$ that satisfy the Navier–Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\nabla_{\mathbf{x}} \mathbf{v})\, \mathbf{v} = \nu \Delta \mathbf{v} - \nabla_{\mathbf{x}} p + \mathbf{b}, \qquad \operatorname{div}_{\mathbf{x}} \mathbf{v} = 0, \tag{2.2.20}$$

with the initial condition

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \quad \text{for every } \ \mathbf{x} \in R, \tag{2.2.21}$$

and the boundary condition

$$\mathbf{v} = \hat{\mathbf{v}} \quad \text{on} \quad \partial R \times [0, \infty). \tag{2.2.22}$$

A pair $(\mathbf{v}, p)$ with these properties will be called a *solution* of the above initial–boundary value problem. We now prove the following result.

Theorem 2.2.19. (Uniqueness of solution) *Let* $(\mathbf{v}_1, p_1)$ *and* $(\mathbf{v}_2, p_2)$ *be solutions of the viscous flow problem corresponding to the same data. Then we have*

$$\mathbf{v}_1 = \mathbf{v}_2, \quad p_1 = p_2 + \alpha, \tag{2.2.23}$$

where α *does not depend on* $\mathbf{x}$.

Proof. By setting

$$\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_2, \quad \alpha = p_1 - p_2,$$

we obtain, from relations (2.2.20)–(2.2.22),

$$\mathbf{u}(\mathbf{x},0)=\mathbf{0}, \quad \mathbf{u}=\mathbf{0} \quad \text{on} \quad \partial R\times[0,\infty), \quad \operatorname{div}_{\mathbf{x}}\mathbf{u}=0. \tag{2.2.24}$$

Moreover, by subtracting $(2.2.20)_1$ with $\mathbf{v}=\mathbf{v}_2$, $p=p_2$ from $(2.2.20)_1$ with $\mathbf{v}=\mathbf{v}_1$, $p=p_1$, we obtain

$$\frac{\partial\mathbf{u}}{\partial t}+(\nabla_{\mathbf{x}}\mathbf{v}_1)\,\mathbf{v}_1-(\nabla_{\mathbf{x}}\mathbf{v}_2)\,\mathbf{v}_2=\nu\Delta\mathbf{u}-\nabla_{\mathbf{x}}\alpha.$$

Since

$$(\nabla_{\mathbf{x}}\mathbf{v}_1)\,\mathbf{v}_1=(\nabla_{\mathbf{x}}\mathbf{u})\,\mathbf{v}_1+(\nabla_{\mathbf{x}}\mathbf{v}_2)\,\mathbf{v}_1,$$

we have

$$\frac{\partial\mathbf{u}}{\partial t}+(\nabla_{\mathbf{x}}\mathbf{u})\,\mathbf{v}_1+(\nabla_{\mathbf{x}}\mathbf{v}_2)\,\mathbf{u}=\nu\Delta\mathbf{u}-\nabla_{\mathbf{x}}\alpha. \tag{2.2.25}$$

Note the following identities:

$$\begin{aligned}
\mathbf{u}\cdot\Delta\mathbf{u}&=\operatorname{div}_{\mathbf{x}}\left[(\nabla_{\mathbf{x}}\mathbf{u})^T\,\mathbf{u}\right]-|\nabla_{\mathbf{x}}\mathbf{u}|^2,\\
\mathbf{u}\cdot(\nabla_{\mathbf{x}}\mathbf{u})\,\mathbf{v}_1&=\mathbf{v}_1\cdot(\nabla_{\mathbf{x}}\mathbf{u})^T\,\mathbf{u}=\mathbf{v}_1\cdot\nabla_{\mathbf{x}}\left(\frac{1}{2}\mathbf{u}^2\right),\\
\mathbf{u}\cdot(\nabla_{\mathbf{x}}\mathbf{v}_2)\,\mathbf{u}&=\mathbf{u}\cdot\mathbf{D}_2\mathbf{u},
\end{aligned} \tag{2.2.26}$$

where (see (1.2.23))

$$\mathbf{D}_2=\frac{1}{2}\left[\nabla_{\mathbf{x}}\mathbf{v}_2+(\nabla_{\mathbf{x}}\mathbf{v}_2)^{\top}\right]. \tag{2.2.27}$$

In view of relation (2.2.26), we conclude from (2.2.25) that

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{u}^2\right)+\mathbf{v}_1\cdot\nabla_{\mathbf{x}}\left(\frac{1}{2}\mathbf{u}^2\right)+\mathbf{u}\cdot\mathbf{D}_2\mathbf{u}=\nu\operatorname{div}_{\mathbf{x}}\left[(\nabla_{\mathbf{x}}\mathbf{u})^T\,\mathbf{u}\right]-\nu\,|\nabla_{\mathbf{x}}\mathbf{u}|^2-\mathbf{u}\cdot\nabla_{\mathbf{x}}\alpha.$$

If we integrate this relation over R, then from the divergence theorem and the boundary conditions $(2.2.24)_{1,2}$, together with $(2.2.20)_2$ (for $\mathbf{v}_1$), we conclude that

$$\frac{1}{2}\frac{d}{dt}\,||\mathbf{u}||^2+\int_R\mathbf{u}\cdot\mathbf{D}_2\mathbf{u}dv=-\nu\int_R|\nabla_{\mathbf{x}}\mathbf{u}|^2\,dv\leq 0, \tag{2.2.28}$$

where

$$||\mathbf{u}||^2\,(t)=\int_R\mathbf{u}^2(\mathbf{x},t)dv.$$

Since $\operatorname{div}_{\mathbf{x}}\mathbf{v}_2=0$ also, it follows that $\mathrm{tr}\mathbf{D}_2=0$, and thus the lowest eigenvalue of the symmetric tensor $\mathbf{D}_2(\mathbf{x},t)$ defined by (2.2.27) will be nonpositive. Let $-\gamma(\mathbf{x},t)$ denote this eigenvalue (with $\gamma\geq 0$), so that

$$\mathbf{u}\cdot\mathbf{D}_2\mathbf{u}\geq-\gamma\mathbf{u}^2.$$

Let us choose $\tau>0$ and put

$$\lambda=2\sup_{(\mathbf{x},t)\in R\times[0,\tau]}\gamma(\mathbf{x},t).$$

This quantity is finite by virtue of the C^2 property of $\mathbf{v}$. Then

$$\mathbf{u} \cdot \mathbf{D}_2\mathbf{u} \geq -\frac{\lambda}{2}\mathbf{u}^2,$$

so that the relation (2.2.28) gives

$$\frac{d}{dt}\,||\mathbf{u}||^2 - \lambda\,||\mathbf{u}||^2 \leq 0 \quad \text{on} \quad [0, \tau],$$

or

$$\frac{d}{dt}\left(||\mathbf{u}||^2\, e^{-\lambda t}\right) \leq 0 \quad \text{on} \quad [0, \tau]$$

and hence

$$||\mathbf{u}||^2\,(\tau) \leq ||\mathbf{u}||^2\,(0)e^{\lambda\tau}.$$

Since $||\mathbf{u}||^2\,(0) = 0$, the above relation implies that $||\mathbf{u}||^2\,(\tau) = 0$ and hence

$$\mathbf{u}(\mathbf{x}, \tau) = \mathbf{0} \quad \text{for every} \;\; \mathbf{x} \in R.$$

Since τ was arbitrarily chosen, it follows that $\mathbf{u} \equiv \mathbf{0}$, and hence relation $(2.2.23)_1$ holds. Finally, (2.2.25) implies that $\nabla_{\mathbf{x}}\alpha = \mathbf{0}$, and the proof is complete. □

2.3 Elastic Solids

The force on an elastic spring depends only on the change in length of the spring, and it is independent of the past history of the length as well as the rate at which the length is changing with time. We have seen previously that the deformation gradient $\mathbf{F}$ measures local changes in distance. Thus, it seems natural to define an elastic body as one for which the constitutive equation prescribes the Cauchy stress $\mathbf{T}(\mathbf{x}, t)$ at $\mathbf{x} = \tilde{\chi}(\mathbf{X}, t)$ when the deformation gradient $\mathbf{F}$ is known, that is,

$$\mathbf{T}(\mathbf{x}, t) = \hat{\mathbf{T}}(\mathbf{F}(\mathbf{X}, t), \mathbf{X}). \tag{2.3.1}$$

We now proceed to make more precise the dependence of the Cauchy stress on the deformation gradient.

2.3.1 Finite Elasticity

We assume that the response of the material body is independent of the observer, so that by $(1.4.13)_3$ (see (1.4.4) and (1.4.5)), we have

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T = \hat{\mathbf{T}}(\mathbf{Q}\mathbf{F}), \tag{2.3.2}$$

for every tensor $\mathbf{F}$ with $\det \mathbf{F} > 0$ and every orthogonal tensor $\mathbf{Q}$ with $\det \mathbf{Q} = 1$.

The polar decomposition of $\mathbf{F}$ is given by (1.2.14). Let us choose $\mathbf{Q} = \mathbf{R}^\top$ such that (2.3.2) becomes

$$\mathbf{R}^\top \hat{\mathbf{T}}(\mathbf{F})\mathbf{R} = \hat{\mathbf{T}}(\mathbf{U}),$$

or

$$\hat{\mathbf{T}}(\mathbf{F}) = \mathbf{R}\hat{\mathbf{T}}(\mathbf{U})\mathbf{R}^\top = \mathbf{F}\mathbf{U}^{-1}\hat{\mathbf{T}}(\mathbf{U})\mathbf{U}^{-1}\mathbf{F}^\top = \mathbf{F}\tilde{\mathbf{T}}(\mathbf{C})\mathbf{F}^\top,$$

by virtue of $(1.2.15)_1$, on putting $\mathbf{U} = \mathbf{C}^{1/2}$. This final form motivates the following definition.

Definition 2.3.1. An *elastic solid* is a material body characterized by a constitutive equation of the form

$$\mathbf{T}(\mathbf{x}, t) = \hat{\mathbf{T}}(\mathbf{F}(\mathbf{X}, t), \mathbf{X}), \tag{2.3.3}$$

where the response function $\hat{\mathbf{T}}$ is completely determined by a tensor function $\tilde{\mathbf{T}}(\mathbf{C}, \mathbf{X})$ according to the formula

$$\hat{\mathbf{T}}(\mathbf{F}, \mathbf{X}) = \mathbf{F}\tilde{\mathbf{T}}(\mathbf{C}, \mathbf{X})\mathbf{F}^T,$$

with $\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$, the right Cauchy–Green strain tensor corresponding to $\mathbf{F}$.

The above definition emphasizes the importance of the strain tensors $\mathbf{U}$ and $\mathbf{C}$ for describing the deformation of an elastic solid.

Remark 2.3.2. The complete system of field equations for an elastic solid consists of the constitutive equation

$$\mathbf{T} = \mathbf{F}\tilde{\mathbf{T}}(\mathbf{C}, \mathbf{X})\mathbf{F}^T, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \tag{2.3.4}$$

the equation of motion

$$\rho\dot{\mathbf{v}} = \operatorname{div}_{\mathbf{x}} \mathbf{T} + \rho\mathbf{b}, \tag{2.3.5}$$

and the balance of mass

$$\rho \det \mathbf{F} = \rho_0, \tag{2.3.6}$$

where ρ_0 is the density in the reference configuration.

Definition 2.3.3. An elastic material is *homogeneous* provided both $\rho_0(\mathbf{X})$ and $\hat{\mathbf{T}}(\mathbf{F}, \mathbf{X})$ are independent of the material point. In other cases the elastic material is *inhomogeneous*.

Definition 2.3.4. A *symmetry transformation* at $\mathbf{X}$ is an orthogonal tensor $\mathbf{Q}$ with $\det \mathbf{Q} = 1$ such that

$$\hat{\mathbf{T}}(\mathbf{F}, \mathbf{X}) = \hat{\mathbf{T}}(\mathbf{F}\mathbf{Q}, \mathbf{X}).$$

An elastic material is *isotropic* if every rotation is a symmetry transformation. Otherwise, the elastic material is *anisotropic*.

Remark 2.3.5. Assume that the material at $\mathbf{X}$ is isotropic. Then the constitutive equation can be written in the form (for example [127])

$$\mathbf{T} = \beta_0(\mathcal{I}_\mathbf{B})\mathbf{1} + \beta_1(\mathcal{I}_\mathbf{B})\mathbf{B} + \beta_2(\mathcal{I}_\mathbf{B})\mathbf{B}^{-1}, \tag{2.3.7}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy–Green strain tensor and β_0, β_1, and β_2 are scalar functions of the principal invariants of $\mathbf{B}$ (see (1.2.22)), denoted by $\mathcal{I}_\mathbf{B}$.

We further note that

$$\det \mathbf{C} = \det\left(\mathbf{F}^T\mathbf{F}\right) = (\det \mathbf{F})^2 .$$

The second Piola–Kirchhoff stress tensor $\widehat{\mathbf{S}}$, defined by (1.3.54), is given by

$$\widehat{\mathbf{S}} = \det \mathbf{F}\tilde{\mathbf{T}}(\mathbf{C}, \mathbf{X}) = \sqrt{\det \mathbf{C}}\tilde{\mathbf{T}}(\mathbf{C}, \mathbf{X}). \tag{2.3.8}$$

As shown in general by Proposition 1.4.1, this quantity is an objective scalar. In terms of the first Piola–Kirchhoff stress tensor defined by (1.3.51), the constitutive equation (2.3.3) becomes

$$\mathbf{S} = \mathbf{F}\widehat{\mathbf{S}}(\mathbf{C}, \mathbf{X}), \tag{2.3.9}$$

on using $(2.3.4)_1$ and (2.3.8).

In view of relations (1.3.52)–(1.3.53) with (2.3.9), we can rewrite the basic equations (2.3.4)–(2.3.6) in the form

$$\mathbf{S} = \mathbf{F}\widehat{\mathbf{S}}(\mathbf{C}, \mathbf{X}), \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad \mathbf{F} = \nabla_\mathbf{X}\mathbf{x}, \quad \rho_0\ddot{\mathbf{x}} = \mathrm{Div}_\mathbf{X}\mathbf{S} + \rho_0\mathbf{b}. \tag{2.3.10}$$

Note that relation (1.3.53) follows automatically from the fact that $\widehat{\mathbf{S}}$ is a symmetric tensor. Moreover, since the density enters into (2.3.10) only through its reference value ρ_0, which is assumed known a priori, the balance of mass (2.3.6) need not be included in the list of field equations.

In what follows we assume that the material body is identified with the regular region $\mathcal{B}$ occupied by the body in the reference configuration. All the fields in (2.3.10) are defined on $\mathcal{B} \times \mathbb{R}$, and the operator $\mathrm{Div}_\mathbf{X}$ is with respect to the material point $\mathbf{X}$ in $\mathcal{B}$. In contrast, some of the fields in (2.3.4)–(2.3.6) are defined on the trajectory of $\mathbf{x}$, and moreover, the operator $\mathrm{div}_\mathbf{x}$ is with respect to the position $\mathbf{x}$ in the current configuration. For this reason the formulation (2.3.10) is more convenient than that furnished by (2.3.4)–(2.3.6) in problems for which the trajectory is not known in advance.

The initial–boundary value problems of finite elasticity are obtained by including with (2.3.10) suitable initial and boundary conditions. As initial conditions one usually specifies the initial position and velocity

$$\mathbf{x}(\mathbf{X}, 0) = \mathbf{x}_0(\mathbf{X}), \quad \dot{\mathbf{x}}(\mathbf{X}, 0) = \mathbf{v}_0(\mathbf{X}), \tag{2.3.11}$$

where $\mathbf{x}_0(\mathbf{X})$ and $\mathbf{v}_0(\mathbf{X})$ are prescribed functions on $\mathcal{B}$. As boundary conditions one usually specifies

$$\mathbf{x}(\mathbf{X}, t) = \hat{\mathbf{x}}(\mathbf{X}, t) \ \text{ on } \Sigma_1 \times [0, \infty), \ \ \mathbf{S}(\mathbf{X}, t)\mathbf{n} = \hat{\mathbf{s}}(\mathbf{X}, t) \ \text{ on } \Sigma_2 \times [0, \infty), \tag{2.3.12}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{s}}$ are prescribed vector fields on $\Sigma_1 \times [0, \infty)$ and $\Sigma_2 \times [0, \infty)$, respectively, and Σ_1 and Σ_2 are regular subsets of $\partial\mathcal{B}$ such that $\overline{\Sigma_1 \cup \Sigma_2} = \partial\mathcal{B}$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$.

In the static theory, all fields are independent of time, and the underlying boundary value problem consists in finding a deformation $\mathbf{x} = \tilde{\chi}(\mathbf{X})$ that satisfies the field equations

$$\mathbf{S} = \mathbf{F}\widehat{\mathbf{S}}(\mathbf{C}, \mathbf{X}), \quad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \quad \mathbf{F} = \nabla_{\mathbf{X}}\mathbf{x}, \quad \mathrm{Div}_{\mathbf{X}}\mathbf{S} + \rho_0\mathbf{b} = \mathbf{0} \tag{2.3.13}$$

and the boundary conditions

$$\mathbf{x}(\mathbf{X}) = \hat{\mathbf{x}}(\mathbf{X}) \quad \text{on} \quad \Sigma_1, \quad \mathbf{S}(\mathbf{X})\mathbf{n} = \hat{\mathbf{s}}(\mathbf{X}) \quad \text{on} \quad \Sigma_2, \tag{2.3.14}$$

where again $\hat{\mathbf{x}}$ and $\hat{\mathbf{s}}$ are prescribed functions on Σ_1 and Σ_2, respectively.

When the traction is prescribed over the entire boundary, that is, when we have the boundary condition

$$\mathbf{S}\mathbf{n} = \hat{\mathbf{s}} \quad \text{on} \quad \partial\mathcal{B}, \tag{2.3.15}$$

then an integration on $\mathcal{B}$ of (2.3.13) implies that

$$\int_{\mathcal{B}} \rho_0 \mathbf{b} dV + \int_{\partial\mathcal{B}} \hat{\mathbf{s}} dA = \mathbf{0}. \tag{2.3.16}$$

This relation is a particular case of (1.3.47), and it involves only the prescribed data. It furnishes a necessary condition for the existence of a solution. On the other hand, (1.3.48) yields

$$\int_{\mathcal{B}} \rho_0 (\mathbf{x} - \mathbf{x}_0) \times \mathbf{b} dV + \int_{\partial\mathcal{B}} (\mathbf{x} - \mathbf{x}_0) \times \hat{\mathbf{s}} dA = \mathbf{0}, \tag{2.3.17}$$

which, because of the presence of $\mathbf{x}$, is not a restriction on the data, but rather a compatibility condition automatically satisfied by any solution of the boundary value problem.

2.3.2 Hyperelastic Bodies

We now characterize a class of elastic materials for which the Piola–Kirchhoff stress is given by the derivative of a scalar function. To this end we consider a dynamical process $(\mathbf{x}, \mathbf{T})$ of the body $\mathcal{B}$ (Definition 2.2.1) corresponding to the body force $\mathbf{b}$. Then, given any subbody A of $\mathcal{B}$, the work on A during a time interval $[t_0, t_1]$ is given by

$$\begin{aligned} &\int_{t_0}^{t_1} \left\{ \int_{\varphi_t(A)} \rho \mathbf{b} \cdot \mathbf{v} dv_t + \int_{\partial\varphi_t(A)} \mathbf{T}\mathbf{n} \cdot \mathbf{v} da_t \right\} dt \\ &\quad = \int_{t_0}^{t_1} \left\{ \int_{\varphi_0(A)} \rho_0 \mathbf{b} \cdot \mathbf{v} dv_0 + \int_{\partial\varphi_0(A)} \mathbf{S}\mathbf{N} \cdot \mathbf{v} da_0 \right\} dt. \end{aligned} \tag{2.3.18}$$

Definition 2.3.6. The dynamical process $(\mathbf{x}, \mathbf{T})$ is closed during the interval $[t_0, t_1]$ if

$$\mathbf{x}(\mathbf{X}, t_0) = \mathbf{x}(\mathbf{X}, t_1), \quad \dot{\mathbf{x}}(\mathbf{X}, t_0) = \dot{\mathbf{x}}(\mathbf{X}, t_1), \tag{2.3.19}$$

for all $\mathbf{X} \in \varphi_0(\mathcal{B})$.

For such a process, it follows from (1.2.8) and (2.3.19) that

$$\mathbf{F}(\mathbf{X}, t_0) = \mathbf{F}(\mathbf{X}, t_1), \qquad \dot{\mathbf{F}}(\mathbf{X}, t_0) = \dot{\mathbf{F}}(\mathbf{X}, t_1). \tag{2.3.20}$$

If we integrate (1.3.56) between t_0 and t_1 (recalling (1.3.50)) and use (2.3.19), then we deduce that for closed processes, (2.3.18) reduces to

$$\int_{t_0}^{t_1} \left\{ \int_{\varphi_0(A)} \rho_0 \mathbf{b} \cdot \mathbf{v} dv_0 + \int_{\partial\varphi_0(A)} \mathbf{SN} \cdot \mathbf{v} da_0 \right\} dt = \int_{t_0}^{t_1} \int_{\varphi_0(A)} \mathbf{S} \cdot \dot{\mathbf{F}} dv_0 dt.$$

Definition 2.3.7. We say that the *work is nonnegative in closed processes* if given any subbody A and any time interval $[t_0, t_1]$, we have

$$\int_{t_0}^{t_1} \int_{\varphi_0(A)} \mathbf{S} \cdot \dot{\mathbf{F}} dv_0 dt \geq 0$$

for any process that is closed during $[t_0, t_1]$.

We can conclude, from the above definition, the following result.

Proposition 2.3.8. *The work is nonnegative in closed processes if and only if, given any* $\mathbf{X} \in \varphi_0(\mathcal{B})$ *and any time interval* $[t_0, t_1]$*, we have*

$$\int_{t_0}^{t_1} \mathbf{S}(\mathbf{X}, t) \cdot \dot{\mathbf{F}}(\mathbf{X}, t) dt \geq 0, \tag{2.3.21}$$

for any process that is closed during $[t_0, t_1]$.

Definition 2.3.9. An elastic material is *hyperelastic* if there exists a scalar function $\hat{e} : \mathrm{Lin}^+ \times \varphi_0(\mathcal{B}) \to \mathbb{R}$ such that the first Piola–Kirchhoff stress tensor $\mathbf{S}$ is the derivative of $\hat{e}$ with respect to $\mathbf{F}$:

$$\mathbf{S}(\mathbf{X}, t) = \hat{\mathbf{S}}(\mathbf{F}, \mathbf{X}) = D_{\mathbf{F}} \hat{e}(\mathbf{F}, \mathbf{X}) \quad \text{or} \quad S_{ij} = \frac{\partial \hat{e}}{\partial F_{ij}}, \tag{2.3.22}$$

where the derivative is with respect to $\mathbf{F}$, holding $\mathbf{X}$ fixed. The scalar function $\hat{e}(\mathbf{F}, \mathbf{X})$ is known as the *strain-energy density*. It is assumed to be an objective scalar.

Remark 2.3.10. Observe that material objectivity as expressed by $(1.4.13)_1$ gives that

$$\hat{e}(\mathbf{F}, \mathbf{X}) = \hat{e}(\mathbf{Q}(t)\mathbf{F}, \mathbf{X}), \tag{2.3.23}$$

for all $\mathbf{F} \in \mathrm{Lin}^+$ and orthogonal $\mathbf{Q}$. It follows from Proposition 1.4.2 that

$$\hat{e}(\mathbf{F}, \mathbf{X}) = \tilde{e}(\mathbf{C}, \mathbf{X}), \tag{2.3.24}$$

where $\tilde{e}$ is also an objective scalar function and (2.3.22) yields (2.3.9). In fact, the second Piola–Kirchhoff stress tensor is given by

$$\widehat{\mathbf{S}}(\mathbf{C}, \mathbf{X}) = \frac{1}{2}\frac{\partial}{\partial \mathbf{C}}\tilde{e}(\mathbf{C}, \mathbf{X}). \tag{2.3.25}$$

We now give a characterization of the class of hyperelastic materials in terms of the work in closed processes.

Theorem 2.3.11. *An elastic material is hyperelastic if and only if the work is zero in closed processes.*

Proof. Suppose first that the body is hyperelastic so that there exists the strain–energy density $\hat{e}$ such that (2.3.22) holds true. Then, for a closed process during the time interval $[t_0, t_1]$, we have

$$\frac{d}{dt}\hat{e}(\mathbf{F}, \mathbf{X}) = D_{\mathbf{F}}\hat{e}(\mathbf{F}, \mathbf{X}) \cdot \dot{\mathbf{F}}(\mathbf{X}, t) = \hat{\mathbf{S}}(\mathbf{F}, \mathbf{X}) \cdot \dot{\mathbf{F}}(\mathbf{X}, t), \tag{2.3.26}$$

and since $\mathbf{F}(\mathbf{X}, t_0) = \mathbf{F}(\mathbf{X}, t_1)$ by virtue of $(2.3.20)_1$, we obtain

$$\begin{aligned}\int_{t_0}^{t_1} \hat{\mathbf{S}}(\mathbf{F}(\mathbf{X}, t), \mathbf{X}) \cdot \dot{\mathbf{F}}(\mathbf{X}, t)dt &= \int_{t_0}^{t_1} \frac{d}{dt}\hat{e}\,(\mathbf{F}(\mathbf{X}, t), \mathbf{X})\,dt \\ &= \hat{e}\,(\mathbf{F}(\mathbf{X}, t_1), \mathbf{X}) - \hat{e}\,(\mathbf{F}(\mathbf{X}, t_0), \mathbf{X}) = 0.\end{aligned}$$

Thus, the work is zero in closed processes.

We assume now that the work is zero in closed processes, so that

$$\int_{t_0}^{t_1} \hat{\mathbf{S}}(\mathbf{F}) \cdot \dot{\mathbf{F}}dt = 0. \tag{2.3.27}$$

Relation (2.3.27) shows that the integral of $\hat{\mathbf{S}}$ over any piecewise smooth, closed curve in Lin^+ vanishes. Since Lin^+ is an open and connected subset of the vector space Lin, a standard theorem in vector analysis tells us that $\hat{\mathbf{S}}$ is the derivative of a smooth scalar function $\hat{e}$ on Lin^+, which is the strain-energy density. Clearly, $\hat{\mathbf{S}}$ determines $\hat{e}$ only up to an arbitrary function of $\mathbf{X}$ alone. □

Theorem 2.3.12. *If the work is nonnegative in closed processes, then it is zero in such processes.*

Proof. We define

$$\mathbf{F}^*(t) = \mathbf{F}(t_0 + t_1 - t). \tag{2.3.28}$$

The quantity $\mathbf{F}^*$ represents the reversal in time of $\mathbf{F}$. In view of $(2.3.20)_1$ and (2.3.28), we have

$$\mathbf{F}^*(t_0) = \mathbf{F}(t_1) = \mathbf{F}(t_0) = \mathbf{F}^*(t_1),$$

and moreover,

$$\dot{\mathbf{F}}^*(t) = \frac{d}{dt}\mathbf{F}(t_0 + t_1 - t) = -\dot{\mathbf{F}}(t_0 + t_1 - t),$$

giving

$$\dot{\mathbf{F}}^*(t_0) = -\dot{\mathbf{F}}(t_1) = -\dot{\mathbf{F}}(t_0) = \dot{\mathbf{F}}^*(t_1).$$

Therefore, $\mathbf{F}^*(t)$ is a closed process, and hence the work is nonnegative on this process, so that

$$\begin{aligned} 0 \leq \int_{t_0}^{t_1} \hat{\mathbf{S}}(\mathbf{F}^*) \cdot \dot{\mathbf{F}}^* dt &= -\int_{t_0}^{t_1} \hat{\mathbf{S}}(\mathbf{F}(t_0 + t_1 - t)) \cdot \dot{\mathbf{F}}(t_0 + t_1 - t) dt \\ &= -\int_{t_0}^{t_1} \hat{\mathbf{S}}(\mathbf{F}(\tau)) \cdot \dot{\mathbf{F}}(\tau)\, d\tau. \end{aligned}$$

Thus, we can conclude that for every $\mathbf{F}$ satisfying (2.3.20), we have

$$\int_{t_0}^{t_1} \hat{\mathbf{S}}(\mathbf{F}) \cdot \dot{\mathbf{F}} dt = 0. \qquad (2.3.29)$$

□

It can be shown that the condition $(2.3.20)_2$ can be avoided without affecting the validity of the result (2.3.29) [127].

Combining Theorems 2.3.11 and 2.3.12, we see that the property that the work is nonnegative in closed processes implies that the material is hyperelastic.

Remark 2.3.13. Theorem 2.3.11 proves that the work is zero in closed processes for hyperelastic materials. Moreover, if we set $e(\mathbf{X}, t) = \hat{e}(\mathbf{F}(\mathbf{X}, t), \mathbf{X})$, then we have

$$\dot{e} = \mathbf{S} \cdot \dot{\mathbf{F}},$$

and so the theorem of power expended (1.3.56) leads to the following important corollary.

Theorem 2.3.14. (Balance of energy for hyperelastic materials) *Each dynamical process for a hyperelastic body satisfies the energy equation*

$$\frac{d}{dt}\int_{\varphi_0(A)} \left(e + \frac{1}{2}\rho_0 \mathbf{v}^2\right) dv_0 = \int_{\varphi_0(A)} \rho_0 \mathbf{b} \cdot \mathbf{v} dv_0 + \int_{\partial\varphi_0(A)} \mathbf{S}\mathbf{n} \cdot \mathbf{v} da_0, \qquad (2.3.30)$$

for each subbody A.

Remark 2.3.15. The term

$$\int_{\varphi_0(A)} e dv_0$$

represents the *strain energy* of the subbody A. The energy equation (2.3.30) asserts that the rate at which the total energy of A is changing must equal the power expended on A.

A direct consequence of the above balance of energy is the following result concerning the conservation of energy .

Proposition 2.3.16. (Conservation of energy) *For a dynamical process in a hyperelastic finite body with body force* $\mathbf{b} = \mathbf{0}$ *and subject to the condition* $\mathbf{Sn} \cdot \mathbf{v} = \mathbf{0}$ *on the boundary of* $\mathcal{B}$*, the total energy is constant, that is,*

$$\int_{\varphi_0(\mathcal{B})} \left(e + \frac{1}{2}\rho_0 \mathbf{v}^2 \right) dv_0 = constant.$$

2.4 Linear Elasticity

Let us consider an elastic material described by the general constitutive equation

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F}). \tag{2.4.1}$$

We now consider the linearized theory appropriate to situations in which the displacement vector is small, in the sense described in Section 1.2.2. The crucial point is the linearization of the general constitutive equation (2.4.1) near $\mathbf{F} = \mathbf{1}$, and of great importance in this context is the elasticity tensor.

Definition 2.4.1. The *elasticity tensor* $\mathbb{C}$ for the material point $\mathbf{X}$ is the derivative of the first Piola–Kirchhoff stress with respect to $\mathbf{F}$ at $\mathbf{F} = \mathbf{1}$:

$$\mathbb{C} = D_{\mathbf{F}}\hat{\mathbf{S}}(\mathbf{1}), \quad \text{or} \quad C_{ijkl} = \frac{\partial S_{ij}}{\partial F_{kl}}(\mathbf{1}). \tag{2.4.2}$$

The derivation of the linearized form of the general constitutive equation (2.4.1) requires the following two fundamental assumptions:

- the displacement vector $\mathbf{u}$ is small;
- the residual stress vanishes, i.e.,

$$\hat{\mathbf{S}}(\mathbf{1}) = \mathbf{0}. \tag{2.4.3}$$

In order to derive this linearized form, we note that

$$\mathbf{x} = \mathbf{X} + \mathbf{u}, \qquad \mathbf{F} = \mathbf{1} + \mathbf{H}, \qquad \mathbf{H} = \nabla_{\mathbf{X}}\mathbf{u}, \tag{2.4.4}$$

and consider $\hat{\mathbf{S}}(\mathbf{F})$ as a function of $\mathbf{H}$. In view of the hypothesis that the displacement vector $\mathbf{u}$ is small, we have the infinitesimal theory of deformation and recall that the measures of deformation reduce to the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ defined by (1.2.33) with $\nabla\mathbf{u} = \nabla_{\mathbf{X}}\mathbf{u} = \nabla_{\mathbf{x}}\mathbf{u}$. Moreover, the powers of $\mathbf{H} = \nabla_{\mathbf{X}}\mathbf{u} = \nabla_{\mathbf{x}}\mathbf{u} = \nabla\mathbf{u}$ greater than or equal to two are all negligible. In view of relations (1.3.51) and (2.4.3), it follows that

$$\hat{\mathbf{T}}(\mathbf{1}) = \mathbf{0}$$

and

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{F}) = \hat{\mathbf{T}}(\mathbf{F}) + \mathbf{o}(\mathbf{H}) = \widehat{\mathbf{S}}(\mathbf{C}) + \mathbf{o}(\mathbf{H}). \tag{2.4.5}$$

Thus, the various stress tensors coincide at the limit of the linearized theory, and we can use **S**, **T** interchangeably. Moreover, on the basis of relations (2.4.1)–(2.4.4), we deduce the following form of the constitutive relation:

$$\hat{\mathbf{S}}(\mathbf{F}) = \mathbb{C}\boldsymbol{\varepsilon} + \mathbf{o}(\mathbf{H}), \tag{2.4.6}$$

as $\mathbf{H} \to \mathbf{0}$, where $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor.

The importance of the elasticity tensor now becomes apparent from the linearized constitutive relation (2.4.6). Let us outline some of its properties. In view of relations (1.3.51) and (2.4.2), we deduce that

$$\mathbb{C} = D_{\mathbf{F}}\hat{\mathbf{T}}(\mathbf{1}), \quad \text{or} \quad C_{ijkl} = \frac{\partial T_{ij}}{\partial F_{kl}}(\mathbf{1}),$$

so that

$$C_{ijkl} = C_{jikl}. \tag{2.4.7}$$

If the material at **X** is hyperelastic, then

$$\mathbb{C} = D_{\mathbf{F}}\hat{\mathbf{S}}(\mathbf{1}) = D^2_{\mathbf{F}}\hat{e}(\mathbf{1}),$$

and therefore we have the supplementary symmetry

$$C_{ijkl} = C_{klij}. \tag{2.4.8}$$

In view of the symmetry of ε in (2.4.6), we need only consider C_{ijkl} with the further property

$$C_{ijkl} = C_{ijlk}. \tag{2.4.9}$$

Thus, $\mathbb{C} \in \mathrm{Lin(Sym)}$.

We have the following definitions (see Section A.2).

Definition 2.4.2. We say that $\mathbb{C}$ is *symmetric* if

$$\mathbf{H} \cdot \mathbb{C}\mathbf{G} = \mathbf{G} \cdot \mathbb{C}\mathbf{H}$$

for all tensors **H** and **G**. This is the case if (2.4.8) holds. Also, $\mathbb{C}$ is *positive definite* if

$$\boldsymbol{\varepsilon} \cdot \mathbb{C}\boldsymbol{\varepsilon} > 0$$

for all symmetric tensors $\boldsymbol{\varepsilon} \neq \mathbf{0}$. We call $\mathbb{C}$ *strongly elliptic* if

$$\mathbf{A} \cdot \mathbb{C}\mathbf{A} > 0$$

whenever **A** has the form $\mathbf{A} = \mathbf{a} \otimes \mathbf{c}$, with $\mathbf{a} \neq \mathbf{0}$, $\mathbf{c} \neq \mathbf{0}$.

An important consequence of the linearized constitutive equation (2.4.6) and relation (2.3.7) is the following result.

Theorem 2.4.3. *Assume that the elastic material at* $\mathbf{X}$ *is isotropic. Then there exist scalars* μ *and* λ *such that*

$$\mathbb{C}\boldsymbol{\varepsilon} = 2\mu\boldsymbol{\varepsilon} + \lambda\,(tr\,\boldsymbol{\varepsilon})\,\mathbf{1},$$

for every symmetric tensor $\boldsymbol{\varepsilon}$*. The scalars* $\mu = \mu(\mathbf{X})$ *and* $\lambda = \lambda(\mathbf{X})$ *are the* Lamé moduli *at* $\mathbf{X}$.

Summarizing, we conclude that the basic equations of the linear theory of elasticity consist of the stress–strain relation

$$\mathbf{S} = \mathbb{C}\boldsymbol{\varepsilon}, \tag{2.4.10}$$

the strain–displacement relation (see (1.2.33))

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \tag{2.4.11}$$

and the equation of motion

$$\rho_0\ddot{\mathbf{u}} = \mathrm{Div}\mathbf{S} + \rho_0\mathbf{b}, \tag{2.4.12}$$

where ∇ and Div are with respect to $\mathbf{X}$ or $\mathbf{x}$. Note that these equations are expressed in terms of the displacement $\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$, rather than the motion $\mathbf{x}$. Given $\mathbb{C}$, ρ_0, and $\mathbf{b}$, the system described by relations (2.4.10)–(2.4.12) is a linear system of partial differential equations for the fields $\mathbf{u}$, $\boldsymbol{\varepsilon}$, and $\mathbf{S}$.

When the body is isotropic, the constitutive equation (2.4.10) is replaced by

$$\mathbf{S} = 2\mu\boldsymbol{\varepsilon} + \lambda\,(\mathrm{tr}\,\boldsymbol{\varepsilon})\,\mathbf{1}. \tag{2.4.13}$$

Moreover, when the body is homogeneous, then ρ_0, μ, and λ are constants.

Sometimes it is convenient to have the stress–strain law (2.4.13) inverted to give $\boldsymbol{\varepsilon}$ as a function of $\mathbf{S}$. This inversion is easily accomplished upon noting that (2.4.13) gives

$$\mathrm{tr}\,\mathbf{S} = (3\lambda + 2\mu)\,\mathrm{tr}\,\boldsymbol{\varepsilon}, \tag{2.4.14}$$

and hence we have

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\left[\mathbf{S} - \frac{\lambda}{3\lambda + 2\mu}\,(\mathrm{tr}\,\mathbf{S})\,\mathbf{1}\right],$$

or

$$\boldsymbol{\varepsilon} = \frac{1}{E}\,[(1+\nu)\,\mathbf{S} - \nu\,(\mathrm{tr}\,\mathbf{S})\,\mathbf{1}]\,, \tag{2.4.15}$$

where

$$E = \frac{\mu\,(3\lambda + 2\mu)}{\lambda + \mu}, \qquad \nu = \frac{\lambda}{2\,(\lambda + \mu)}.$$

The modulus E is known as *Young's modulus*, while ν is *Poisson's ratio*.

Let us assume that the body is homogeneous and isotropic. Then, since

$$\mathrm{Div}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) = \Delta\mathbf{u} + \nabla\,(\mathrm{Div}\,\mathbf{u})\,, \quad tr\,\boldsymbol{\varepsilon} = \mathrm{Div}\,\mathbf{u},$$

equations (2.4.11), (2.4.12), and (2.4.13) are easily combined to give the displacement equations of motion

$$\rho_0 \ddot{\mathbf{u}} = \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla (\mathrm{Div}\, \mathbf{u}) + \rho_0 \mathbf{b}. \tag{2.4.16}$$

In the case of static theory we have $\ddot{\mathbf{u}} = \mathbf{0}$, and the displacement equations of equilibrium are

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla (\mathrm{Div}\, \mathbf{u}) + \rho_0 \mathbf{b} = \mathbf{0},$$

which hold approximately for slow deformations.

We now discuss some particular solutions of the equilibrium equations, in the absence of body force, for a homogeneous and isotropic body.

- *Pure shear*

Let us consider the following state of displacement:

$$u_1 = \gamma X_2, \quad u_2 = u_3 = 0.$$

The matrices for the corresponding $\boldsymbol{\varepsilon}$ and $\mathbf{S}$ are

$$(\boldsymbol{\varepsilon}) = \frac{1}{2}\begin{pmatrix} 0 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (\mathbf{S}) = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tau = \mu\gamma.$$

Thus, μ determines the response of the body in shear and for this reason is called the *shear modulus*.

- *Uniform compression or expansion*

The state of displacement

$$u_1 = \epsilon X_1, \quad u_2 = \epsilon X_2, \quad u_3 = \epsilon X_3$$

corresponds to

$$\boldsymbol{\varepsilon} = \epsilon \mathbf{1}, \quad \mathbf{S} = \pi \mathbf{1}, \quad \pi = 3\kappa\epsilon,$$

where

$$\kappa = \frac{2}{3}\mu + \lambda \tag{2.4.17}$$

is the *modulus of compression*.

- *Pure tension*

We consider the following state of stress:

$$(\mathbf{S}) = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which obviously satisfies the equilibrium equations. We want to find the corresponding state of displacement. Because we know the state of stress, it is convenient to use (2.4.15), which gives

$$(\boldsymbol{\varepsilon}) = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & l \end{pmatrix} \quad \text{with} \quad \epsilon = \frac{\sigma}{E}, \quad l = -\nu\epsilon.$$

Furthermore, we note that this state of strain satisfies Saint-Venant's conditions of compatibility (1.2.35), and the corresponding state of displacement is given by

$$u_1 = \epsilon X_1, \quad u_2 = l X_2, \quad u_3 = l X_3.$$

Remark 2.4.4. Since an elastic solid should increase its length when pulled, decrease its volume when acted on by a pure pressure, and respond to a positive shearing strain by a positive shearing stress, one would expect that

$$E > 0, \quad \kappa > 0, \quad \mu > 0.$$

Also, a pure tensile stress should produce a contraction in the direction perpendicular to it and hence $\nu > 0$.

2.4.1 Linear Elastostatics

The basic system of field equations for the static behavior of an elastic material consists of the strain–displacement relation

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \tag{2.4.18}$$

the stress–strain relations

$$\mathbf{S} = \mathbb{C}\boldsymbol{\varepsilon}, \tag{2.4.19}$$

and the equations of equilibrium

$$\mathrm{Div}\mathbf{S} + \rho\mathbf{b} = \mathbf{0}. \tag{2.4.20}$$

The body is assumed to occupy a volume $B \in \mathbb{R}^3$ corresponding to the reference volume $\mathcal{B}$ but also closely approximating the current deformed shape.

Definition 2.4.5. We call a list $[\mathbf{u}, \boldsymbol{\varepsilon}, \mathbf{S}]$ of fields that are smooth on B and satisfy (2.4.18)–(2.4.20) an *elastic state corresponding to* $\mathbf{b}$.

Let Σ_1 and Σ_2 denote complementary regular subsets of the boundary of B, so that $\overline{\Sigma}_1 \cup \Sigma_2 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \emptyset$. The *mixed problem of elastostatics* can be formulated as follows:

- *given:* B, Σ_1, Σ_2, an elasticity tensor $\mathbb{C}$ on B, a body force field $\mathbf{b}$ on B, surface displacements $\hat{\mathbf{u}}$ on Σ_1, surface tractions $\hat{\mathbf{s}}$ on Σ_2;
- *find:* an elastic state $[\mathbf{u}, \boldsymbol{\varepsilon}, \mathbf{S}]$ that corresponds to $\mathbf{b}$ and satisfies the boundary conditions

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \Sigma_1, \qquad \mathbf{S}\mathbf{n} = \hat{\mathbf{s}} \quad \text{on } \Sigma_2. \tag{2.4.21}$$

An elastic state with these properties will be called a *solution of the mixed problem of elastostatics*.

Three fundamental theorems of elastostatics are now proved.

Theorem 2.4.6. (Theorem of work and energy) *Let* $[\mathbf{u}, \boldsymbol{\varepsilon}, \mathbf{S}]$ *be an elastic state corresponding to the body force* $\mathbf{b}$. *Then*

$$2\mathfrak{U}(\boldsymbol{\varepsilon}) = \int_B \rho\mathbf{b} \cdot \mathbf{u} dv + \int_{\partial B} \mathbf{S}\mathbf{n} \cdot \mathbf{u} da, \tag{2.4.22}$$

where $\mathfrak{U}(\boldsymbol{\varepsilon})$ *is the strain energy of the body defined by*

$$\mathfrak{U}(\boldsymbol{\varepsilon}) = \frac{1}{2}\int_B \boldsymbol{\varepsilon} \cdot \mathbb{C}\boldsymbol{\varepsilon} dv.$$

Proof. By the symmetry of $\mathbf{S}$ and the divergence theorem, we obtain

$$\int_{\partial B} \mathbf{S}\mathbf{n} \cdot \mathbf{u} da = \int_{\partial B} \mathbf{S}\mathbf{u} \cdot \mathbf{n} da = \int_B \mathrm{Div}\,(\mathbf{S}\mathbf{u})\, dv = \int_B (\mathbf{u} \cdot \mathrm{Div}\mathbf{S} + \mathbf{S} \cdot \nabla\mathbf{u})\, dv,$$

and further, from (2.4.18) and (2.4.19),

$$\mathbf{S} \cdot \nabla\mathbf{u} = \mathbf{S}^T \cdot \nabla\mathbf{u}^T = \mathbf{S} \cdot \left\{\frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)\right\} = \mathbf{S} \cdot \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \cdot \mathbb{C}\boldsymbol{\varepsilon}.$$

These relations, when combined with the equations of equilibrium (2.4.20), give (2.4.22). □

Theorem 2.4.7. (Uniqueness theorem) *Assume that the elasticity tensor is positive definite. Let* $[\mathbf{u}_1, \boldsymbol{\varepsilon}_1, \mathbf{S}_1]$ *and* $[\mathbf{u}_2, \boldsymbol{\varepsilon}_2, \mathbf{S}_2]$ *be solutions of the same mixed problem of linear elastostatics. Then*

$$\mathbf{u}_1 = \mathbf{u}_2 + \mathbf{u}^*, \quad \boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_2, \quad \mathbf{S}_1 = \mathbf{S}_2,$$

where $\mathbf{u}^*$ *is an infinitesimal rigid displacement of* B.

Proof. Let

$$\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2, \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2, \quad \mathbf{S} = \mathbf{S}_1 - \mathbf{S}_2.$$

Then $[\mathbf{u}, \boldsymbol{\varepsilon}, \mathbf{S}]$ is an elastic state that corresponds to a null body force and satisfies the following boundary conditions:

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Sigma_1, \quad \mathbf{S}\mathbf{n} = \mathbf{0} \quad \text{on } \Sigma_2,$$

which give

$$\mathbf{S}\mathbf{n} \cdot \mathbf{u} = 0 \quad \text{on} \quad \partial B.$$

In view of relation (2.4.22), we conclude that

$$\int_B \boldsymbol{\varepsilon} \cdot \mathbb{C}\boldsymbol{\varepsilon} dv = 0.$$

Since $\mathbb{C}$ is positive definite, this last relation can hold only if $\boldsymbol{\varepsilon} = \mathbf{0}$; this in turn implies that $\mathbf{S} = \mathbf{0}$ and $\mathbf{u} = \mathbf{u}^*$, where $\mathbf{u}^*$ is an infinitesimal rigid displacement of B. Thus, the proof is complete. □

Theorem 2.4.8. (Betti's reciprocal theorem) *Assume that the elasticity tensor is symmetric. Let* $[\mathbf{u}_1, \boldsymbol{\varepsilon}_1, \mathbf{S}_1]$ *and* $[\mathbf{u}_2, \boldsymbol{\varepsilon}_2, \mathbf{S}_2]$ *be elastic states in* B *corresponding to body force fields* $\mathbf{b}_1$ *and* $\mathbf{b}_2$*, respectively. Then*

$$\int_B \rho\mathbf{b}_1 \cdot \mathbf{u}_2 dv + \int_{\partial B} \mathbf{S}_1\mathbf{n} \cdot \mathbf{u}_2 da = \int_B \rho\mathbf{b}_2 \cdot \mathbf{u}_1 dv + \int_{\partial B} \mathbf{S}_2\mathbf{n} \cdot \mathbf{u}_1 da. \tag{2.4.23}$$

Proof. In view of the equations of equilibrium (2.4.20) and the divergence theorem, we have

$$\int_B \rho\mathbf{b}_1 \cdot \mathbf{u}_2 dv + \int_{\partial B} \mathbf{S}_1\mathbf{n} \cdot \mathbf{u}_2 da = \int_B \mathbf{S}_1 \cdot \nabla\mathbf{u}_2 dv. \tag{2.4.24}$$

Since the elasticity tensor is symmetric, we conclude from the strain–displacement relation that

$$\mathbf{S}_1 \cdot \nabla\mathbf{u}_2 = \mathbf{S}_1 \cdot \left[\frac{1}{2}\left(\nabla\mathbf{u}_2 + \nabla\mathbf{u}_2^T\right)\right] = \mathbf{S}_1 \cdot \boldsymbol{\varepsilon}_2,$$

so that relation (2.4.24) becomes

$$\int_B \rho\mathbf{b}_1 \cdot \mathbf{u}_2 dv + \int_{\partial B} \mathbf{S}_1\mathbf{n} \cdot \mathbf{u}_2 da = \int_B \mathbf{S}_1 \cdot \boldsymbol{\varepsilon}_2 dv. \tag{2.4.25}$$

On the other hand, the symmetry of the elasticity tensor and the stress–strain relation give

$$\mathbf{S}_1 \cdot \boldsymbol{\varepsilon}_2 = \mathbb{C}\boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}_2 = \boldsymbol{\varepsilon}_1 \cdot \mathbb{C}\boldsymbol{\varepsilon}_2 = \mathbf{S}_2 \cdot \boldsymbol{\varepsilon}_1. \tag{2.4.26}$$

Then, (2.4.25) and (2.4.26) give relation (2.4.23), and the proof is complete. □

When $\Sigma_1 = \partial B$ (that is, $\Sigma_2 = \emptyset$), the boundary condition (2.4.21) takes the form

$$\mathbf{u} = \hat{\mathbf{u}} \quad \text{on } \partial B.$$

We refer to this as the *displacement problem*.

When $\Sigma_2 = \partial B$ (that is, $\Sigma_1 = \emptyset$), the boundary condition (2.4.20) takes the form

$$\mathbf{S}\mathbf{n} = \hat{\mathbf{s}} \quad \text{on } \partial B,$$

and we have the *traction problem*.

Proposition 2.4.9. *A necessary condition that the traction problem have a solution is that*

$$\int_B \rho\mathbf{b} dv + \int_{\partial B} \hat{\mathbf{s}} da = \mathbf{0}, \qquad \int_B \rho\mathbf{r} \times \mathbf{b} dv + \int_{\partial B} \mathbf{r} \times \hat{\mathbf{s}} da = \mathbf{0}, \tag{2.4.27}$$

where $\mathbf{r}$ *is the position vector, with respect to the origin* O*, of the point of evaluation of* $\mathbf{b}$ *and* $\hat{\mathbf{s}}$*.*

Proof. The relation (2.4.27) is a direct consequence of the equilibrium equations (2.4.20) and the divergence theorem. □

2.4.2 Saint-Venant's Problem

We consider a homogeneous and isotropic cylindrical bar with generators parallel to the x_3-axis. Let the end faces $\mathcal{S}_1$ and $\mathcal{S}_2$ be located at $x_3 = 0$ and $x_3 = l$, respectively, with the origin at the centroid of $\mathcal{S}_1$ and with the x_1- and x_2-axes coincident with the principal axes of inertia, so that

$$\int_{\mathcal{S}_1} x_1 da = \int_{\mathcal{S}_1} x_2 da = 0, \quad \int_{\mathcal{S}_1} x_1 x_2 da = 0. \tag{2.4.28}$$

We assume that the bar is loaded only on the end faces, so that the lateral surface $\mathcal{L}$ is traction-free. Moreover, we assume that the body forces are zero.

Saint-Venant's problem consists in the determination of an equilibrium displacement field $\mathbf{u}$ on B (that is, a displacement field satisfying the basic equations of elastostatics with null body force), subjected to the requirements

$$\mathbf{Sn} = \mathbf{0} \quad \text{on } \mathcal{L},$$

or equivalently, since $n_3 = 0$ on $\mathcal{L}$,

$$S_{\alpha\beta} n_\beta = 0, \quad \alpha = 1, 2, \qquad \text{on } \mathcal{L}, \tag{2.4.29}$$

and

$$\mathbf{Sn} = \hat{\mathbf{s}}^{(\alpha)} \quad \text{on } \mathcal{S}_\alpha, \quad \alpha = 1, 2. \tag{2.4.30}$$

Necessary conditions for the existence of a solution to this problem are

$$\int_{\mathcal{S}_1} \hat{\mathbf{s}}^{(1)} da + \int_{\mathcal{S}_2} \hat{\mathbf{s}}^{(2)} da = \mathbf{0}, \quad \int_{\mathcal{S}_1} \mathbf{r} \times \hat{\mathbf{s}}^{(1)} da + \int_{\mathcal{S}_2} \mathbf{r} \times \hat{\mathbf{s}}^{(2)} da = \mathbf{0}. \tag{2.4.31}$$

These are an immediate consequence of (2.4.27).

Under suitable smoothness hypotheses on the given data, a solution of Saint-Venant's problem exists and it is uniquely determined.

In the *relaxed formulation of Saint-Venant's problem*, the local conditions (2.4.30) are replaced by the following global conditions:

$$\int_{\mathcal{S}_1} \mathbf{Sn} da = \mathbf{R}, \quad \int_{\mathcal{S}_1} \mathbf{r} \times \mathbf{Sn} da = \mathbf{M},$$

or equivalently,

$$\int_{\mathcal{S}_1} S_{3i} da = -R_i, \quad \int_{\mathcal{S}_1} e_{ijk} x_j S_{3k} da = -M_i, \tag{2.4.32}$$

where **R** and **M** represent the resultant force and the resultant moment about O of the tractions acting on $\mathcal{S}_1$. We do not specify the loading on $\mathcal{S}_2$, since balance of forces and moments require that (2.4.31) be satisfied.

Definition 2.4.10. By a solution of Saint-Venant's relaxed problem we mean any equilibrium displacement field that satisfies the conditions (2.4.29) and (2.4.32).

Remark 2.4.11. It is obvious that the relaxed statement of the problem fails to characterize the solution uniquely. However, we recall the so-called Saint-Venant's principle (e.g., [137]), which states in effect that any solution obeying (2.4.32) will not differ significantly from any other solution with the same property, except in the vicinity of $\mathcal{S}_1$. Thus, if this principle is accepted, it suffices to outline an appropriate representative solution of this class. The classification of the relaxed problem rests on various assumptions concerning the resultants **R** and **M**. We will exemplify this with some particular cases.

- **Extension problem**

The total force on the end $\mathcal{S}_1$ is equipollent to a force directed along the negative x_3-axis, of magnitude R_3 and null moment about O, that is,

$$\int_{\mathcal{S}_1} S_{3\alpha} da = 0, \quad \int_{\mathcal{S}_1} S_{33} da = -R_3, \quad \int_{\mathcal{S}_1} e_{ijk} x_j S_{3k} da = 0. \tag{2.4.33}$$

The *problem of extension* consists in the determination of an equilibrium displacement field **u** satisfying the boundary conditions (2.4.29) and (2.4.33). We seek a representative solution of this problem by assuming the following state of stress:

$$S_{11} = S_{22} = S_{12} = S_{23} = S_{31} = 0, \quad S_{33} = a_1, \tag{2.4.34}$$

with a_1 a constant. This state of stress satisfies the equilibrium equations and the lateral boundary condition (2.4.29). From relations (2.4.33) and (2.4.34), we obtain

$$a_1 = -\frac{R_3}{A},$$

where A is the area of $\mathcal{S}_1$. To find a solution of the extension problem we need to find the displacement field corresponding to the state of stress (2.4.34). We use the stress–strain relation in the form expressed by (2.4.15) to obtain the state of strain

$$\varepsilon_{11} = \varepsilon_{22} = -\frac{\nu}{E} a_1, \quad \varepsilon_{33} = \frac{1}{E} a_1, \quad \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = 0.$$

This state of strain satisfies the compatibility conditions (1.2.35), and the corresponding displacement field is given (up to an arbitrary infinitesimal rigid displacement) by

$$u_1 = \frac{\nu R_3}{EA} x_1, \quad u_2 = \frac{\nu R_3}{EA} x_2, \quad u_3 = -\frac{R_3}{EA} x_3.$$

- **Bending of a beam**

The total force on the end $\mathcal{S}_1$ is equipollent to a moment of magnitude M_2 about the negative x_2-axis, so that

$$\int_{\mathcal{S}_1} S_{3i} da = 0, \quad \int_{\mathcal{S}_1} x_2 S_{33} da = \int_{\mathcal{S}_1} (x_1 S_{32} - x_2 S_{31})\, da = 0, \tag{2.4.35}$$

and

$$\int_{\mathcal{S}_1} x_1 S_{33} da = -M_2. \tag{2.4.36}$$

The *problem of bending* involves determining an equilibrium displacement field **u** satisfying the boundary conditions (2.4.29), (2.4.35), and (2.4.36). We seek a representative solution of this problem by assuming the following state of stress:

$$S_{11} = S_{22} = S_{12} = S_{23} = S_{31} = 0, \quad S_{33} = a_2 x_1,$$

where a_2 is a constant. This field satisfies the lateral boundary conditions (2.4.29) and the conditions (2.4.35), by virtue of (2.4.28). Moreover, we have

$$\mathrm{Div}\mathbf{S} = \mathbf{0}.$$

Furthermore, the condition (2.4.36) implies that

$$a_2 = -\frac{M_2}{I},$$

where

$$I = \int_{\mathcal{S}_1} x_1^2 da$$

represents the moment of inertia of $\mathcal{S}_1$ about the x_2-axis. To obtain a solution of the bending problem we need only construct the displacement field corresponding to this state of stress. Using the stress–strain relation in the form (2.4.15), we see that

$$\begin{aligned} \varepsilon_{23} &= \varepsilon_{31} = \varepsilon_{12} = 0, \\ \varepsilon_{11} &= \varepsilon_{22} = \frac{\nu M_2 x_1}{EI}, \quad \varepsilon_{33} = -\frac{M_2 x_1}{EI}. \end{aligned}$$

The compatibility relations (1.2.35) are satisfied. Moreover, the corresponding displacement field is given by

$$u_1 = \frac{M_2}{2EI}\left[x_3^2 + \nu\left(x_1^2 - x_2^2\right)\right], \quad u_2 = \frac{M_2 \nu}{EI} x_1 x_2, \quad u_3 = -\frac{M_2}{EI} x_1 x_3.$$

- **Torsion of a cylinder**

The total force on the end $\mathcal{S}_1$ is equipollent to

$$\int_{\mathcal{S}_1} S_{3i} da = 0, \quad \int_{\mathcal{S}_1} x_\alpha S_{33} da = 0, \quad \int_{\mathcal{S}_1} (x_1 S_{32} - x_2 S_{31})\, da = -M_3. \quad (2.4.37)$$

The *problem of torsion* involves determining an equilibrium displacement field **u** satisfying the boundary conditions (2.4.29) and (2.4.37). We seek such a solution by assuming the following state of displacement:

$$u_1 = -\tau x_2 x_3, \quad u_2 = \tau x_1 x_3, \quad u_3 = \tau\psi(x_1, x_2), \quad (2.4.38)$$

where τ is a constant and ψ is termed the *warping function*. The state of stress corresponding to the displacements given by (2.4.38) is

$$\begin{aligned} &S_{\alpha\beta} = S_{33} = 0, \\ &S_{31} = \tau\mu\left(\psi_{,1} - x_2\right), \quad S_{23} = \tau\mu\left(\psi_{,2} + x_1\right). \end{aligned}$$

The equilibrium equations are satisfied if and only if

$$\psi_{,11} + \psi_{,22} = 0 \quad \text{in} \;\; \mathcal{S}_1, \quad (2.4.39)$$

while the lateral boundary conditions are equivalent to

$$\frac{\partial\psi}{\partial n} = x_2 n_1 - x_1 n_2 \quad \text{on} \quad \partial\mathcal{S}_1, \quad (2.4.40)$$

where $\frac{\partial\psi}{\partial n}$ is the normal derivative of ψ on $\partial\mathcal{S}_1$. The condition (2.4.37) implies

$$\tau D = -M_3, \quad (2.4.41)$$

where

$$D = \mu \int_{\mathcal{S}_1} \left(x_1^2 + x_2^2 + x_1\psi_{,2} - x_2\psi_{,1}\right) da$$

is the *torsional rigidity of the cross-section* $\mathcal{S}_1$. Thus, the torsion solution is given by (2.4.38) with τ determined by (2.4.41) and where ψ is the solution of the Neumann problem defined by (2.4.39) and (2.4.40).

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