

Preface

Inverse limits have played a crucial role in the development of the theory of continua in the past 50 years or so. Particularly useful is their inherent ability to produce complicated spaces from simple ones. Nowhere is this feature more evident than in a couple of papers, one by G. W. Henderson and one by Howard Cook. Henderson shows that the pseudo-arc (an hereditarily indecomposable chainable continuum) is the inverse limit on the interval $[0, 1]$ with a single bonding map. Cook employs inverse limits to produce a continuum having only one nonconstant self-map, the identity.

In dynamics the inverse limit construction allows the study of a dynamical system consisting of a topological space and a map of that space into itself to be turned into the study of a (likely more complicated) space and one of its self-homeomorphisms.

We believe one of the best ways to learn about inverse limits is by means of a thorough study of inverse limits on $[0, 1]$. For this reason, we begin this monograph with a detailed look at inverse limits on $[0, 1]$ in our first chapter. There we develop the basic properties of inverse limits employing only theorems from topology normally found in a senior-level undergraduate or a beginning-level graduate course in topology. However, for the convenience of the reader, most of the background is developed in an appendix.

The origins of the study of inverse limits date back to the 1920s and 1930s. In the 1950s and 1960s the field exploded with a torrent of results. In 1954 C. E. Capel showed that monotone maps on arcs (respectively, simple closed curves) produce arcs (respectively, simple closed curves). An extremely important paper by R. D. Anderson and Gustav Choquet appeared in 1959 showing just how useful inverse limits can be in describing complicated examples. They constructed an example of a planar tree-like continuum no two of whose nondegenerate subcontinua are homeomorphic. Employing the techniques of Anderson and Choquet, J. J. Andrews produced a chainable continuum with the same property. In 1967 Mahavier showed that the Andrews example is not homeomorphic to an inverse limit on $[0, 1]$ using a single

bonding map. The example of Cook mentioned earlier also appeared in 1967. In the early 1970s Ingram used inverse limits to describe an example of a nonchainable tree-like continuum such that all of its nondegenerate proper subcontinua are arcs (and consequently the continuum does not contain a triod). Late in the decade of the 1970s Bellamy used inverse limits in producing his example of a tree-like continuum without the fixed point property.

R. F. Williams introduced many dynamicists to the value of inverse limits in his 1967 paper on nonwandering sets. In 1990 Marcy Barge and Joe Martin promoted interest in inverse limits in dynamics using inverse limits to construct global attractors in the plane. Earlier they had demonstrated connections between dynamics and continuum theory showing, for example, that an inverse limit on $[0, 1]$ with a single bonding map contains an indecomposable continuum if the bonding map has a periodic point whose period is not a power of 2. Studies of inverse limits on intervals with a single bonding map chosen from the logistic family (or the tent family) have led to interesting results. Major unsolved problems remain in determining the nature of inverse limits with bonding maps from these simple families of interval maps, although quite recently Barge, Bruin, and Štimac have solved one of these by showing that two tent maps on $[0, 1]$ that have different maximum values in the interval $[1/2, 1]$ produce nonhomeomorphic inverse limits.

A fundamental example in dynamics is the Smale horseshoe. The continuum most naturally associated with the horseshoe is a chainable continuum first described by Janiszewski who was reacting to the first example of an indecomposable continuum that Brouwer had constructed a few years earlier. Subsequently, Knaster gave us a beautiful geometric description of Janiszewski's example that we depict in Section 3 of Chapter 1. This example remains at the heart of the connection between dynamical systems and continuum theory. Recent work of Judy Kennedy, David Stockman, and Jim Yorke as well as work of Alfredo Medio and Brian Raines has connected inverse limits to the theories of backward dynamics in economics.

Some recent research in inverse limits has been, in part, devoted to studying inverse limits on compact Hausdorff spaces with upper semi-continuous bonding functions. Inverse limits of systems over directed sets other than the set of positive integers have proved useful at times, as well. Chapter 2 includes a development of the theory of inverse limits in this very general setting where the factor spaces are compact Hausdorff spaces, the bonding functions are upper semi-continuous, and the underlying directed set may be any directed set.

Anyone interested in dynamical systems or continuum theory or in exploring relationships between the two disciplines could benefit from a careful study of the first chapter. The reader who is already familiar with inverse limits may choose to skim through or skip the first chapter and move directly to the second. The third chapter is devoted to a few topics in continuum

theory not addressed in the first two chapters, and Chapter 4 is devoted to a fundamental approximation theorem of Morton Brown that has seen much use in inverse limits over the years.

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