

Chapter 2

Inverse Limits in a General Setting

Abstract In this chapter we investigate inverse limits in a very general setting: over directed sets with factor spaces that are compact Hausdorff spaces using upper semi-continuous closed set-valued bonding functions. Basic existence and connectedness theorems are proved and examples are provided that illustrate limitations to the generality of the theorems. One section is devoted to examples in the case where the factor spaces are all the interval $[0, 1]$. Basic theorems on mappings of inverse limits are included as well. As the chapter progresses additional hypotheses are added to the factor spaces (up to compact metric) and the bonding functions (continuous single-valued or unions of such). The chapter concludes with considerations of a few miscellaneous topics including dimension and a proof that a 2-cell is not an inverse limit with a single upper semi-continuous function on $[0, 1]$.

2.1 Introduction

Inverse limits are normally defined for a pair of sequences X_1, X_2, X_3, \dots and f_1, f_2, f_3, \dots such that, for each i , X_i is a topological space and f_i is a mapping (i.e., continuous function) from X_{i+1} into X_i . Such a pair of sequences is often denoted $\{X_i, f_i\}$ and is called an *inverse limit sequence* and the mappings f_i are called *bonding maps* and the spaces X_i are referred to as *factor spaces*. More generally an inverse limit system is defined to be a triple that consists of a directed set D , a collection of topological spaces $\{X_\alpha\}_{\alpha \in D}$, and a collection of mappings $\{f_{\alpha\beta} : X_\beta \rightarrow X_\alpha \mid \alpha, \beta \in D \text{ and } \alpha \preceq \beta\}$. Often this triple is shortened to $\{X_\alpha, f_{\alpha\beta}, D\}$. This more general form has proved useful at times. See Howard Cook's article [1] for a particularly nice construction that employs this type of inverse limit system. Throughout, we use the term inverse limit sequence when the underlying directed set is the

set of positive integers and the term inverse limit system when the underlying directed set may be a more general directed set.

Inverse limit systems in which the bonding functions are mappings have been studied for quite some time, particularly those for which the underlying directed set is the set of positive integers. Recently, inverse limit sequences in which the bonding functions are upper semi-continuous set-valued functions were introduced in [8] as inverse limits of closed subsets of $[0, 1] \times [0, 1]$. These were generalized in [7] to inverse limit sequences where the spaces were compact Hausdorff spaces and the bonding functions were upper semi-continuous set-valued functions. A natural next step is to examine inverse limits in a setting that encompasses all of these scenarios. In Section 2.5 we prove the basic existence theorems in this very general setting over a directed set where the factor spaces are compact Hausdorff spaces and the bonding functions are upper semi-continuous. In Sections 2.6 and 2.9 we address the connectedness of the inverse limit. In Section 2.7 we include some examples that demonstrate some of the variety of spaces that result as an inverse limit with upper semi-continuous bonding functions. In Section 2.8 we examine some basic theorems concerning mappings between inverse limits of inverse limit systems. In Section 2.10 of this chapter we contrast a couple of major differences between inverse limits of ordinary inverse limit sequences and these more general inverse limit systems through examples showing that certain basic theorems do not hold in this setting. Finally, in Section 2.11 we include some theorems requiring metric factor spaces.

We recommend that the reader with little or no experience with inverse limits first read Chapter 1 to get a better feel for inverse limits before reading the present chapter.

2.2 Definitions and a basic theorem

A *relation* on a set D is a subset of $D \times D$ such that each member of D is a first term of some pair in the relation. If \preceq is a relation on a set D and (x, y) is in \preceq then we write $x \preceq y$. A *directed set* is a pair (D, \preceq) where \preceq is a relation on D such that (a) if $\alpha \in D$ then $\alpha \preceq \alpha$; (b) if α, β , and γ are in D and $\alpha \preceq \beta$ and $\beta \preceq \gamma$ then $\alpha \preceq \gamma$; and (c) if α and β are in D then there is a member γ of D such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. If (D, \preceq) is a directed set, for short, we usually say simply that D is a directed set. If D is a directed set and α and β are elements of D such that $\alpha \preceq \beta$ we say that α precedes β in D . A directed set is called *totally ordered* provided if α and β belong to D then $\alpha \preceq \beta$ or $\beta \preceq \alpha$ and in the case where $\alpha \preceq \beta$ and $\beta \preceq \alpha$ we have $\alpha = \beta$.

If Y is a topological space, then 2^Y denotes the collection of nonempty closed subsets of Y whereas we denote by $C(Y)$ the elements of 2^Y that are connected. Let each of X and Y be a topological space and let f be a function from X into 2^Y . The function $f : X \rightarrow 2^Y$ is *upper semi-continuous at the point* $x \in X$ if and only if for each open set V in Y containing $f(x)$ there is an open set U in X containing x such that if u is in U then $f(u) \subseteq V$; f is called *upper semi-continuous* provided it is upper semi-continuous at each point of X . If $A \subseteq X$, $f(A)$ denotes $\{y \in Y \mid y \in f(x) \text{ for some } x \in A\}$. The *graph* of f is denoted by $G(f)$ and is the set of all points $(x, y) \in X \times Y$ such that y is in $f(x)$. If $f : X \rightarrow 2^Y$ and $g : Y \rightarrow 2^Z$, we denote by $g \circ f : X \rightarrow 2^Z$ the closed set-valued function given by $(g \circ f)(x) = \{z \in Z \mid \text{there is an element } y \in Y \text{ such that } y \in f(x) \text{ and } z \in g(y)\}$. In the case where f is a singleton-valued upper semi-continuous function, we do not distinguish between f and the corresponding mapping associated with f . For example, if $f : X \rightarrow 2^X$ is given by $f(x) = \{x\}$ for each $x \in X$, we still refer to f as the identity on X .

Suppose D is a directed set and, for each α in D , X_α is a topological space. Suppose further, for each α and β in D with $\alpha \preceq \beta$, $f_{\alpha\beta} : X_\beta \rightarrow X_\alpha$ is an upper semi-continuous function from X_β into 2^{X_α} such that $f_{\alpha\alpha}$ is the identity on X_α and if $\alpha \preceq \beta \preceq \gamma$ then $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$. The triple $\{X_\alpha, f_{\alpha\beta}, D\}$ is called an *inverse limit system*. The spaces X_α are called *factor spaces* and the functions $f_{\alpha\beta}$ are called *bonding functions*. The inverse limit of the system $\{X_\alpha, f_{\alpha\beta}, D\}$ is a subspace of $\mathbf{\Pi} = \prod_{\alpha \in D} X_\alpha$ with the product topology. We denote elements of $\mathbf{\Pi}$ using boldface type. If $\mathbf{x} \in \mathbf{\Pi}$, x_α denotes the α -coordinate of \mathbf{x} (i.e., $\mathbf{x}(\alpha) = x_\alpha$). The points of the *inverse limit* are the elements \mathbf{x} of $\mathbf{\Pi}$ such that if $\alpha \preceq \beta$ in D then $x_\alpha \in f_{\alpha\beta}(x_\beta)$. We denote the inverse limit by $\varprojlim \{X_\alpha, f_{\alpha\beta}, D\}$. Consistent with our use of boldface type to denote sequences of bonding maps in Chapter 1, we also use boldface type to denote collections of bonding functions $f_{\alpha\beta}$. Also consistent with the notation in Chapter 1, if \mathbf{f} is the collection of all of the functions $f_{\alpha\beta}$ in an inverse limit system $\{X_\alpha, f_{\alpha\beta}, D\}$ we normally denote the inverse limit of this inverse limit system by $\varprojlim \mathbf{f}$. If, for each $\alpha \preceq \beta$ in D and each point t of X_β , $f_{\alpha\beta}(t)$ is degenerate then this definition reduces to the usual one for systems over directed sets. If, in addition, the directed set is the set of positive integers, this definition reduces to the usual one for inverse limit sequences.

The most commonly used directed set in inverse limits is the set of positive integers. Often, in the case where D is the set of positive integers, instead of specifying all of the bonding functions f_{ij} in the system, only the terms of a sequence of functions are specified. This was the practice we used in Chapter 1 and is the way we present most of the examples in this chapter. The commonly used convention for expanding a sequence f_1, f_2, f_3, \dots of functions into the functions of an inverse sequence is to define f_{ij} to be the composition $f_i \circ f_{i+1} \circ \dots \circ f_{j-1}$ for $i < j$ and to let f_{ii} be the identity for each

i. In the case where the inverse limit system over the set of positive integers is specified by a sequence of mappings $f_i : X_{i+1} \rightarrow X_i$, as mentioned in the introduction to this chapter, we may denote the inverse limit sequence by the pair, $\{X_i, f_i\}$, and its inverse limit by $\varprojlim \{X_i, f_i\}$, or simply $\varprojlim \mathbf{f}$. In the specific instance where $f : X \rightarrow X$ is a function, $X_i = X$ and $f_i = f$ for each positive integer i , we may denote the system by $\{X, f\}$ and its inverse limit by $\varprojlim \mathbf{f}$. Such systems determined by a single space and a single bonding function are often called inverse limits with a single bonding function.

The reader who is only interested in the proofs of these theorems for inverse sequences may assume throughout that D is the set of positive integers. Moreover, for the basic results on inverse limit sequences using single-valued continuous functions, the reader may also assume the bonding functions are mappings.

If $\{X_\alpha \mid \alpha \in D\}$ is a collection of topological spaces, we denote the projection of $\prod_{\alpha \in D} X_\alpha$ onto the factor space X_α by p_α (i.e., $p_\alpha(\mathbf{x}) = x_\alpha$). We are usually more interested in the inverse limit space than the product space, therefore we denote by π_α the restriction of p_α to the inverse limit space. The projection p_α is an open mapping on the product space, but π_α is not normally open.

A useful feature of inverse limits lies in the interaction between the bonding functions and the projection mappings. The proof of the following analogue of Theorem 2 from Chapter 1 is an immediate consequence of the definitions and is left to the reader.

Theorem 104 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system and the inverse limit, M , of the system is nonempty. If $\mathbf{x} \in M$ and $\alpha \preceq \beta$, $\pi_\alpha(\mathbf{x}) \in f_{\alpha\beta}(\pi_\beta(\mathbf{x}))$. If $f_{\alpha\beta}$ is a mapping, $\pi_\alpha(\mathbf{x}) = f_{\alpha\beta} \circ \pi_\beta(\mathbf{x})$.*

2.3 Graphs of upper semi-continuous functions

That there is a close connection between closed subsets of product spaces and upper semi-continuous set-valued functions can be seen from the following theorem.

Theorem 105 *Suppose each of X and Y is a compact Hausdorff space and M is a subset of $X \times Y$ such that if x is in X then there is a point y in Y such that (x, y) is in M . Then M is closed if and only if there is an upper semi-continuous function $f : X \rightarrow 2^Y$ such that $M = G(f)$.*

Proof. We first show that if $f : X \rightarrow 2^Y$ is an upper semi-continuous function then $G(f)$ is closed. Let $p = (p_1, p_2)$ be a point of $X \times Y$ that is not in $G(f)$. Then, $p_2 \notin f(p_1)$, so, because compact Hausdorff spaces are regular, there are mutually exclusive open sets V and W in Y such that $p_2 \in V$ and $f(p_1) \subseteq W$. Because f is upper semi-continuous, there is an open subset U of X containing p_1 such that if $t \in U$ then $f(t) \subseteq W$. Thus, $U \times V$ is an open subset of $X \times Y$ containing p that misses $G(f)$. It follows that $G(f)$ is closed.

Assume that M is closed and, for each x in X , define $f(x)$ to be $\{y \in Y \mid (x, y) \in M\}$. Because M is closed, $f(x)$ is closed for each x in X . To see that f is upper semi-continuous, suppose x is in X and V is an open set in Y containing $f(x)$. If f is not upper semi-continuous at x , then for each open set U containing x there exist points z of U and (z, y) of M such that y is not in V . For each open set U containing x , denote by M_U the set of all points (p, q) of M such that p is in \overline{U} and q is not in V . Observe that if U and U' are open sets containing x and $U \subseteq U'$ then $M_U \subseteq M_{U'}$. From this it follows that the collection \mathcal{M} of all the closed sets M_U has the finite intersection property. $X \times Y$ is compact, thus there is a point (a, b) common to all the sets in \mathcal{M} . Each element of \mathcal{M} is a subset of M , therefore (a, b) belongs to M so $b \in f(a)$. Because x is the only point common to all the sets \overline{U} , $a = x$. Furthermore, b is not in V . This contradicts the fact that b belongs to $f(x)$. \square

2.4 Consistent systems

In general, an inverse limit system with upper semi-continuous bonding functions over a directed set may fail to produce a nonempty inverse limit even if the factor spaces are compact. Consider the following example. In this example and hereafter we denote the *identity* from X into X by Id_X . If no confusion should arise with respect to the domain, we may shorten this to Id .

Example 106 Let $D = \{1, 2, 3, \dots\} \cup \{a, b\}$ where, if i and j are positive integers, then $i \preceq j$ if and only if $i \leq j$, $a \preceq j$ if and only if $j \geq 2$, $b \preceq j$ if and only if $j \geq 3$, $1 \preceq b$, and $a \preceq b$ (1 and a do not compare nor do 2 and b). Let $X_\alpha = \{0, 1\}$ for each $\alpha \in D$. If $3 \preceq i \preceq j$, let $f_{ij} = Id$. Let $f_{12} = f_{a2} = f_{ab} = Id$ as well, and let $f_{1b} = 1 - Id$. Let $f_{23}(t) = \{0, 1\}$ for $t \in \{0, 1\}$ and $f_{b3} = f_{23}$. We expand this into a system by composition. Then, $\varprojlim \mathbf{f} = \emptyset$.

Proof. Suppose $\mathbf{x} \in \varprojlim \mathbf{f}$. If $x_1 = 0$, then $x_2 = 0$ and $x_b = 1$. Because $x_2 = 0$, $x_a = 0$; but $x_a = 1$ because $x_b = 1$. This is a contradiction. Because $x_1 \neq 0$,

$x_1 = 1$. Then, $x_2 = 1$ and $x_b = 0$. But, $x_2 = 1$ yields $x_a = 1$ whereas $x_b = 0$ yields $x_a = 0$, again a contradiction. So, $\varprojlim \mathbf{f} = \emptyset$. \square

We call an inverse limit system $\{X_\alpha, f_{\alpha\beta}, D\}$ *consistent* provided for each $\eta \in D$ and each $t \in X_\eta$ there is a point \mathbf{x} of \mathbf{II} such that $x_\eta = t$ and if $\alpha \preceq \beta \preceq \eta$ then $x_\alpha \in f_{\alpha\beta}(x_\beta)$. We now show that two important classes of inverse limit systems are consistent: those in which all the bonding functions are mappings and those for which the directed set is the set of positive integers.

Theorem 107 *Each inverse limit system $\{X_\alpha, f_{\alpha\beta}, D\}$ where each function $f_{\alpha\beta}$ in the system is a mapping is consistent.*

Proof. Suppose $\eta \in D$ and $t \in X_\eta$. Let \mathbf{z} be a point of \mathbf{II} . For each γ such that $\gamma \preceq \eta$, let $x_\gamma = f_{\gamma\eta}(t)$ and let $x_\gamma = z_\gamma$ otherwise. Suppose $\alpha \preceq \beta \preceq \eta$. Inasmuch as $f_{\alpha\eta}(t) = f_{\alpha\beta}(f_{\beta\eta}(t))$ we have $x_\alpha = f_{\alpha\beta}(x_\beta)$ and we see that the system is consistent. \square

Theorem 108 *Each inverse limit system $\{X_i, f_{ij}, D\}$ where D is the set of positive integers is consistent.*

Proof. Suppose n is a positive integer and $t \in X_n$. Let $\mathbf{z} \in \mathbf{II}$. Let $x_k = z_k$ if $k > n$. Let $x_n = t$ and x_1 be a point belonging to $f_{1n}(t)$. Inductively, suppose $1 \leq j < n$ is an integer such that x_1, x_2, \dots, x_j have been chosen so that $x_i \in f_{in}(t)$ for $1 \leq i \leq j$ and if $l \leq m \leq j$ then $x_l \in f_{lm}(x_m)$. Because $f_{jn} = f_{jj+1} \circ f_{j+1n}$ there exists an element x_{j+1} of X_{j+1} such that $x_{j+1} \in f_{j+1n}(t)$ and $x_j \in f_{jj+1}(x_{j+1})$. It follows that the system $\{X_i, f_{ij}\}$ is consistent. \square

2.5 Compact inverse limits

In this section we assume that $\{X_\alpha, f_{\alpha\beta}, D\}$ is a consistent inverse limit system over a directed set D with upper semi-continuous bonding functions and that X_α is a compact Hausdorff space for each $\alpha \in D$. Our goal in this section is to prove that under these conditions the inverse limit is nonempty and compact (Theorem 111).

In the absence of assumptions of some sort, the inverse limit may be empty even if D is the set of positive integers, the factor spaces metric, and there is only one bonding map. Consider the following example.

Example 109 Let D be the set of positive integers and, for each i , $X_i = (0, 1)$, and $f_i = f$ where $f : (0, 1) \rightarrow (0, 1)$ is given by $f(x) = x/2$ for each $x \in (0, 1)$. One can see that $\varprojlim \mathbf{f} = \emptyset$ for if n is a positive integer no point of $\varprojlim \mathbf{f}$ can have a point with first coordinate greater than $1/2^{n-1}$ because in that case its n th coordinate would have to be greater than 1.

Recall that we use \mathbf{I} to denote $\prod_{\alpha \in D} X_\alpha$. It is convenient to introduce the following notation: if η is an element of D , G_η denotes the set of all points \mathbf{x} of \mathbf{I} such that if α and β are elements of D and $\alpha \preceq \beta \preceq \eta$ then $x_\alpha \in f_{\alpha\beta}(x_\beta)$. The reader should note that this notation differs slightly from the notation employed in Chapter 1 for $D = \mathbb{N}$.

Theorem 110 Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is a consistent inverse limit system such that X_α is a compact Hausdorff space for each α in D . Then, for each $\eta \in D$, G_η is a nonempty compact set.

Proof. The system is consistent, therefore G_η is nonempty for each $\eta \in D$. Because \mathbf{I} is compact it suffices to show that G_η is closed. Let \mathbf{x} be a point of \mathbf{I} that is not in G_η . There exist α and β in D with $\alpha \preceq \beta \preceq \eta$ such that x_α is not in $f_{\alpha\beta}(x_\beta)$. By Theorem 105, the graph of $f_{\alpha\beta}$ is closed, so there is an open set $U_\beta \times U_\alpha \subseteq X_\beta \times X_\alpha$ such that $(U_\beta \times U_\alpha) \cap G(f_{\alpha\beta}) = \emptyset$. Let $O = U_\beta \times U_\alpha \times (\prod_{\gamma \in D - \{\alpha, \beta\}} X_\gamma)$. Then O contains \mathbf{x} and if $\mathbf{y} \in O$ then $y_\alpha \notin f_{\alpha\beta}(y_\beta)$; that is, $\mathbf{y} \notin G_\eta$. So G_η is closed and therefore compact. \square

As an immediate consequence of Theorem 110 we have the result we sought in the following theorem.

Theorem 111 Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is a consistent inverse limit system such that X_α is a compact Hausdorff space for each α in D . Then, $K = \varprojlim \mathbf{f}$ is nonempty and compact.

Proof. The collection of all the sets G_η for $\eta \in D$ is a collection of nonempty compact sets in the compact Hausdorff space \mathbf{I} . Note that if α and β are in D there is a member γ of D such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. Because $G_\gamma \subseteq G_\alpha$ and $G_\gamma \subseteq G_\beta$, it follows that the collection $\{G_\eta \mid \eta \in D\}$ has the finite intersection property. From this we see that $\bigcap_{\eta \in D} G_\eta$ is nonempty and compact. Clearly $K = \bigcap_{\eta \in D} G_\eta$. \square

If $f : X \rightarrow 2^Y$ is a set-valued function and A is a subset of X , recall that $f(A) = \{y \in Y \mid \text{there exists } x \in A \text{ such that } y \in f(x)\}$. If $f : X \rightarrow 2^Y$, we call f *surjective* if $f(X) = Y$. This is consistent with the usual definition of surjective for mappings. In the case where D is the set of positive integers and there is a positive integer n such that, for each $i \geq n$, f_i is surjective then for each positive integer $j \geq n$ and for each point t of X_n there is a point

\mathbf{x} in the inverse limit with $t \in \pi_n(\mathbf{x})$. Thus, in the case where D is the set of positive integers, one does not need Theorem 111 to see that the inverse limit is nonempty. Indeed, we have the following theorem the proof of which is left to the reader.

Theorem 112 *Suppose X_1, X_2, X_3, \dots is a sequence of compact Hausdorff spaces and $f_k : X_{k+1} \rightarrow 2^{X_k}$ is an upper semi-continuous function for each positive integer k . If there is a positive integer n such that f_k is surjective for each positive integer $k \geq n$, i and j are positive integers not less than n with $i < j$, s is a point of X_j , and $t \in f_{ij}(s)$, then there is a point \mathbf{x} of $\varprojlim \mathbf{f}$ such that $x_i = t$ and $x_j = s$.*

In the absence of compactness of the factor spaces or assumptions about the directed set such as $D = \{1, 2, 3, \dots\}$, even with surjective bonding functions, the inverse limit of a consistent system may be empty. An example of an inverse limit system with surjective bonding maps that has an empty inverse limit is given by Henkin in [3].

We conclude this section with a proof that the inverse limit of a consistent inverse limit system on compact Hausdorff spaces is the inverse limit of an inverse limit system with surjective bonding functions.

Theorem 113 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is a consistent inverse limit system such that X_α is a compact Hausdorff space for each α in D . Then, $\varprojlim \mathbf{f}$ is the inverse limit of an inverse limit system $\{Y_\alpha, g_{\alpha\beta}, D\}$ where, for each α in D , $Y_\alpha = \bigcap_{\alpha \preceq \beta} f_{\alpha\beta}(X_\beta)$, and if $\alpha \preceq \beta$ in D , $g_{\alpha\beta} = f_{\alpha\beta}|_{Y_\beta}$ is surjective.*

Proof. Choose $\alpha \in D$. Observe that if $\alpha \preceq \beta \preceq \gamma$, then $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ so $f_{\alpha\gamma}(X_\gamma) \subseteq f_{\alpha\beta}(f_{\beta\gamma}(X_\gamma)) \subseteq f_{\alpha\beta}(X_\beta)$. From this and the fact that D is a directed set it follows that the collection $\mathcal{C} = \{f_{\alpha\beta}(X_\beta) \mid \alpha \preceq \beta\}$ has the finite intersection property. Because \mathcal{C} is a collection of nonempty compact sets, $Y_\alpha = \bigcap_{\alpha \preceq \beta} f_{\alpha\beta}(X_\beta)$ is a nonempty compact set.

We now show that if $\alpha \preceq \beta$ then $f_{\alpha\beta}(Y_\beta) \subseteq Y_\alpha$. Let x be an element of Y_β and suppose γ is an element of D such that $\alpha \preceq \gamma$. There is an element η of D such that $\beta \preceq \eta$ and $\gamma \preceq \eta$. Because $x \in f_{\beta\eta}(X_\eta)$, $f_{\alpha\beta}(x) \subseteq f_{\alpha\beta}(f_{\beta\eta}(X_\eta)) = f_{\alpha\eta}(X_\eta) \subseteq f_{\alpha\gamma}(X_\gamma)$. Thus, $f_{\alpha\beta}(x) \subseteq Y_\alpha$.

Next we show that if $\alpha \preceq \beta$ and y is in Y_α then there is an element x of Y_β such that $y \in f_{\alpha\beta}(x)$. Suppose $\alpha \preceq \beta$ and let y be an element of Y_α . If γ is an element of D such that $\beta \preceq \gamma$, because y belongs to $f_{\alpha\gamma}(X_\gamma)$ and $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$, there is a point t of X_β such that $t \in f_{\beta\gamma}(X_\gamma)$ and $y \in f_{\alpha\beta}(t)$. For each $\gamma \in D$ such that $\beta \preceq \gamma$, let $N_\gamma = \{t \in f_{\beta\gamma}(X_\gamma) \mid y \in f_{\alpha\beta}(t)\}$ and let $M_\gamma = \overline{N_\gamma}$. We now show that if $\gamma_1 \preceq \gamma_2$ then $N_{\gamma_2} \subseteq N_{\gamma_1}$. To see this note that if $t \in N_{\gamma_2}$ then $t \in f_{\beta\gamma_1}(X_{\gamma_1})$ which follows from the fact that $t \in f_{\beta\gamma_2}(X_{\gamma_2})$ and the definition of $f_{\beta\gamma_1} \circ f_{\gamma_1\gamma_2}$. So the collection of compact sets M_γ has

the finite intersection property. Thus, $\bigcap_{\beta \preceq \gamma} M_\gamma$ is a nonempty subset of X_β . Let x be in $\bigcap_{\beta \preceq \gamma} M_\gamma$. To see that x is in \overline{Y}_β , let U be an open set containing x and suppose $\beta \preceq \gamma$. Because x is a point of $M_\gamma = \overline{N}_\gamma$, there is a point t of N_γ in U . Because t is in N_γ , $t \in f_{\beta\gamma}(X_\gamma)$. It follows that $x \in \overline{f_{\beta\gamma}(X_\gamma)}$. But, $f_{\beta\gamma}(X_\gamma)$ is closed so $x \in f_{\beta\gamma}(X_\gamma)$. Because $x \in f_{\beta\gamma}(X_\gamma)$ for each γ such that $\beta \preceq \gamma$, $x \in Y_\beta$. Finally, $y \in f_{\alpha\beta}(x)$ for suppose $y \notin f_{\alpha\beta}(x)$. There exist mutually exclusive open sets O_y and O containing y and $f_{\alpha\beta}(x)$, respectively. There is an open set U containing x such that if $z \in U$ then $f_{\alpha\beta}(z) \subseteq O$. If $\gamma \in D$ and $\beta \preceq \gamma$, U contains a point t of N_γ so $f_{\alpha\beta}(t) \subseteq O$, a contradiction to the fact that $y \in f_{\alpha\beta}(t)$.

For α and β in D with $\alpha \preceq \beta$, let $g_{\alpha\beta} = f_{\alpha\beta}|Y_\beta$. It is clear that $\varprojlim g \subseteq \varprojlim f$. If $\mathbf{x} \in \varprojlim f$ and $\alpha \in D$, $x_\alpha \in f_{\alpha\beta}(x_\beta)$ for each β such that $\alpha \preceq \beta$. Thus, x_α belongs to Y_α for each α , and so $\mathbf{x} \in \varprojlim g$. \square

2.6 Connected inverse limits

We next turn our attention to conditions under which inverse limits are connected. One might suspect that imposing a natural condition such as all of the bonding functions have connected graphs would be sufficient to guarantee that the inverse limit is connected. That this condition is not sufficient even if D is the set of positive integers, each factor space is the interval $[0, 1]$, and the sequence of bonding functions is constant may be seen from the following example. The reader will recall that Theorem 105 allows us to specify an upper semi-continuous closed set-valued function by identifying its graph as a closed subset of a product of two spaces that projects onto the first factor space. In the examples of this chapter, I denotes the interval $[0, 1]$ and \mathcal{Q} denotes the Hilbert cube I^∞ .

Example 114 (A function with a connected graph that yields an inverse limit that is not connected) *Let $G(f)$ be the union of the four straight line intervals, $I \times \{0\}$, $\{1\} \times I$, the interval from $(0, 0)$ to $(1/4, 1/4)$, and the interval from $(3/4, 1/4)$ to $(1, 1)$ in $I \times I$ (see Figure 2.1). Then, $G(f)$ is connected but $\varprojlim f$ is not connected.*

Proof. Let N be the set of all points \mathbf{p} of $K = \varprojlim f$ such that $p_1 = p_2 = 1/4$ and $p_3 = 3/4$. Note that N is closed. Let \mathbf{x} be a point of N . Let $R = R_1 \times R_2 \times R_3 \times \mathcal{Q}$ be the region in \mathcal{Q} where $R_1 = R_2 = (1/8, 3/8)$ and $R_3 = (5/8, 7/8)$, and note that R contains \mathbf{x} . Assume that the point \mathbf{y} is in $R \cap K$. Then y_1 and y_2 are in $(1/8, 3/8)$. It follows that $y_2 \leq 1/4$. But if $y_2 < 1/4$, inasmuch as $y_3 > 5/8$ we have $y_3 = 1$ and \mathbf{y} is not in R . We

conclude that $\mathbf{y} \in N$, so N and $K - N$ are closed and mutually exclusive, thus K is not connected. \square

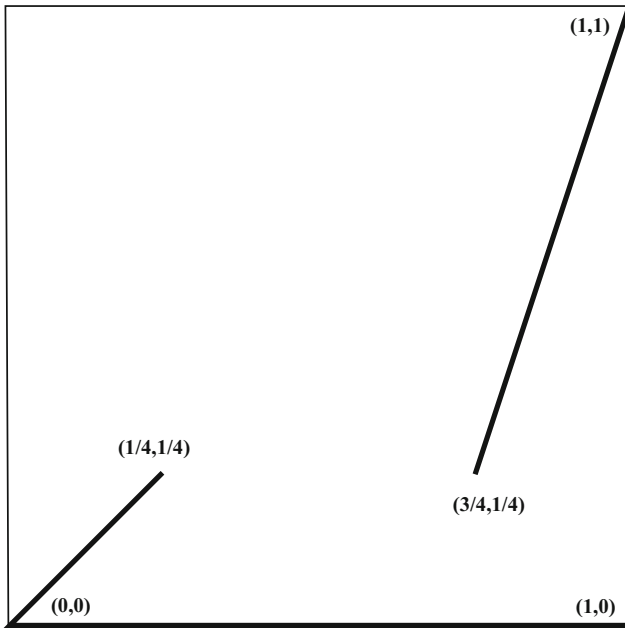


Fig. 2.1 The function from Example 114

Another natural condition one could impose with the expectation of obtaining a connected inverse limit would be that the bonding functions have connected values. However, even for inverse systems over directed sets in which every factor space is the interval $[0, 1]$ this may not be the case as may be seen from the following example.

Example 115 Let D be the directed set given in Example 106; that is, $D = \{1, 2, 3, \dots\} \cup \{a, b\}$ where, if i and j are positive integers, then $i \preceq j$ if and only if $i \leq j$, $a \preceq j$ if and only if $j \geq 2$, $b \preceq j$ if and only if $j \geq 3$, $1 \preceq b$, and $a \preceq b$ (1 and a do not compare nor do 2 and b). Let $X_\alpha = [0, 1]$ for each $\alpha \in D$. Let $f_{a2} = f_{12} = f_{ab} = \text{Id}$. Let f_{1b} be the full tent map; that is, $f_{1b}(t) = 2t$ for $0 \leq t \leq 1/2$ and $f_{1b}(t) = 2 - 2t$ for $1/2 < t \leq 1$. Let f_{23} be given by $f_{23}(t) = [0, 1]$ for each $t \in [0, 1]$ and $f_{b3} = f_{23}$. Finally, let $f_{i \ i+1} = \text{Id}$ for $i \geq 3$. The inverse limit of the consistent system $\{X_\alpha, f_{\alpha\beta}, D\}$ is the union of two mutually exclusive arcs.

Proof. We first note that if $\mathbf{x} \in \varprojlim \mathbf{f}$ and $x_1 = p$ then $x_2 = p$. Thus, $x_a = p$ and, as a consequence, $x_b = p$. It follows that $f_{1b}(p) = p$. There are two fixed points for f_{1b} so if $\mathbf{x} \in \varprojlim \mathbf{f}$, then $x_1 = x_2 = x_a = x_b = 0$ or $x_1 = x_2 = x_a = x_b = 2/3$. In either case, x_3 may be any element of $[0, 1]$, but $x_i = x_3$ for all $i \geq 3$. It may now be seen that the two arcs whose union is $\varprojlim \mathbf{f}$ are $A = \{\mathbf{x} \in \varprojlim \mathbf{f} \mid x_1 = x_2 = x_a = x_b = 0, x_3 = t, x_i = x_3 \text{ for } i \geq 3 \text{ where } t \in [0, 1]\}$ and $B = \{\mathbf{x} \in \varprojlim \mathbf{f} \mid x_1 = x_2 = x_a = x_b = 2/3, x_3 = t, x_i = x_3 \text{ for } i \geq 3 \text{ where } t \in [0, 1]\}$. \square

Recall our notation from Section 2.5 that if $\eta \in D$ then $G_\eta = \{\mathbf{x} \in \mathbf{II} \mid \text{if } \alpha \preceq \beta \preceq \eta \text{ then } x_\alpha \in f_{\alpha\beta}(x_\beta)\}$. By a *Hausdorff continuum* we mean a compact connected subset of a Hausdorff space. This distinction allows us to continue to use the term *continuum* to mean a compact connected subset of a metric space. Our next theorem provides a sufficient condition that an inverse limit system in which the factor spaces are compact Hausdorff spaces produces a connected inverse limit.

Theorem 116 *Suppose that $\{X_\alpha, f_{\alpha\beta}, D\}$ is a consistent inverse limit system such that X_α is a compact Hausdorff space for each α in D . If, for each η in D , G_η is connected, then $\varprojlim \mathbf{f}$ is a Hausdorff continuum.*

Proof. By Theorem 110, for each $\eta \in D$, G_η is a nonempty compact set. This theorem is thus an immediate consequence of the observation that the collection $\{G_\eta \mid \eta \in D\}$ of compact and connected subsets of the compact Hausdorff space \mathbf{II} has the finite intersection property. \square

We now turn to two important cases in which Theorem 116 applies. These are the case where all the bonding functions in the system are mappings and the case where the directed set is totally ordered.

2.6.1 Systems in which all of the bonding functions are mappings

Theorem 117 *Suppose that $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system such that X_α is a Hausdorff continuum for each $\alpha \in D$ and each $f_{\alpha\beta}$ is a mapping. Then $\varprojlim \mathbf{f}$ is a Hausdorff continuum.*

Proof. Suppose $\eta \in D$ and let $E = \{\eta\} \cup \{\gamma \in D \mid \gamma \not\preceq \eta\}$. Let $\varphi : \prod_{\alpha \in E} X_\alpha \rightarrow G_\eta$ be given by $\varphi(\mathbf{x}) = \mathbf{y}$ where $y_\alpha = x_\alpha$ for each $\alpha \in E$ and $y_\alpha = f_{\alpha\eta}(x_\eta)$ otherwise. Suppose \mathbf{x} and \mathbf{z} are in $\prod_{\alpha \in E} X_\alpha$. If $\mathbf{x} \neq \mathbf{z}$ there is an element $\gamma \in E$ such that $x_\gamma \neq z_\gamma$. Then, $\varphi(\mathbf{x}) \neq \varphi(\mathbf{z})$, so φ is 1-1. We now show that φ is surjective and continuous. If $\mathbf{y} \in G_\eta$, let \mathbf{x} be the member of $\prod_{\alpha \in E} X_\alpha$ such that $x_\alpha = y_\alpha$ for each $\alpha \in E$. Then, $\varphi(\mathbf{x}) = \mathbf{y}$,

so φ is surjective. Finally, φ is continuous because, for each $\alpha \in D$, $\pi_\alpha \circ \varphi$ is continuous. From the compactness of \mathbf{I} it follows that φ is a surjective homeomorphism, so G_η is connected being homeomorphic to the connected set $\prod_{\alpha \in E} X_\alpha$. The theorem now follows from Theorem 116. \square

2.6.2 Systems in which the directed set is totally ordered

Next we consider systems in which the underlying directed set is totally ordered although some of the theorems of this subsection do not depend on this assumption.

Theorem 118 *Suppose that each of X and Y is a compact Hausdorff space, X is connected, f is an upper semi-continuous function from X into 2^Y and, for each x in X , $f(x)$ is a Hausdorff continuum. Then $G(f)$ is a Hausdorff continuum.*

Proof. Recall that $G(f) = \{(x, y) \in X \times Y \mid y \in f(x)\}$. Note that $G(f)$ is closed by Theorem 105. Assume that $G(f)$ is not connected. There are then two nonempty mutually exclusive closed sets H and K whose union is $G(f)$. If x is in X , then $\{x\} \times f(x)$ is a connected subset of $G(f)$ and thus a subset of one of H and K . Let H_1 be the set of all points x of X such that $\{x\} \times f(x)$ lies in H and let K_1 be the points x of X such that $\{x\} \times f(x)$ lies in K . Because H_1 and K_1 are nonempty compact sets whose union is the connected set X , they have a common point z . But this is impossible because $\{z\} \times f(z)$ would then be a connected subset of both H and K . \square

If M is a subset of the product $X \times Y$ of compact Hausdorff spaces, then the inverse of M is the subset of $Y \times X$ consisting of all points (y, x) such that (x, y) is in M . We denote this inverse by M^{-1} . If f is an upper semi-continuous function, by the *inverse* of f , denoted f^{-1} , we mean the function from $f(X) \subseteq Y$ into 2^X such that if $y \in f(X)$, $f^{-1}(y) = \{x \in X \mid y \in f(x)\}$. One of the consequences of Theorem 105 is that if $f : X \rightarrow Y$ is an upper semi-continuous function, then $f^{-1} : f(X) \rightarrow 2^X$ is upper semi-continuous. We often use this observation in this section.

Lemma 119 *Suppose X and Y are compact Hausdorff spaces, $f : X \rightarrow 2^Y$ is an upper semi-continuous function, and $M = G(f^{-1})$. Then, $M^{-1} = G(f)$.*

Proof. Note that $(x, y) \in M^{-1}$ if and only if $(y, x) \in M = G(f^{-1})$ if and only if $x \in f^{-1}(y)$ if and only if $y \in f(x)$ if and only if $(x, y) \in G(f)$. \square

Using Lemma 119, the following theorem is an immediate consequence of Theorem 105.

Theorem 120 *If X and Y are compact Hausdorff spaces and $f : X \rightarrow 2^Y$ is an upper semi-continuous function then $f^{-1} : f(X) \rightarrow 2^X$ is an upper semi-continuous function.*

Proof. Let $M = G(f^{-1})$. By Lemma 119, $M^{-1} = G(f)$ which is closed by Theorem 105. Inasmuch as M^{-1} is closed, M is closed so f^{-1} is upper semi-continuous. \square

Theorem 120 yields a corollary to Theorem 118 for certain upper semi-continuous functions whose inverses are Hausdorff continuum-valued.

Theorem 121 *Suppose that X and Y are compact Hausdorff spaces and $f : X \rightarrow 2^Y$ is an upper semi-continuous function such that $f(X)$ is connected and $f^{-1} : f(X) \rightarrow 2^X$ is Hausdorff continuum-valued. Then $G(f)$ is connected.*

Proof. Let $M = G(f^{-1})$. By Theorem 120, f^{-1} is upper semi-continuous. By hypothesis, f^{-1} is Hausdorff continuum-valued, so by Theorem 118, M is connected. Because M is connected, M^{-1} is connected. By Lemma 119, $M^{-1} = G(f)$, so $G(f)$ is connected. \square

For the next few theorems it is convenient to generalize the definition of the graph $G(f)$ of an upper semi-continuous function. Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system with upper semi-continuous bonding functions and X_α is a compact Hausdorff space for each α in D . Suppose $\{\beta_1, \beta_2, \dots, \beta_n\}$ with $n \geq 2$ is a finite subset of D . Let $G(\beta_1, \beta_2, \dots, \beta_n) = \{x \in X_{\beta_1} \times X_{\beta_2} \times \dots \times X_{\beta_n} \mid x_{\beta_i} \in f_{\beta_i \beta_j}(x_{\beta_j}) \text{ whenever } \beta_i \preceq \beta_j\}$. Note that if $\beta_1 \preceq \beta_2$ then $G(\beta_1, \beta_2)^{-1}$ is the graph of $f_{\beta_1 \beta_2}$. This slight twist in the notation is convenient for our intended use of these sets in the proofs of theorems leading to Theorem 125.

Theorem 122 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is a consistent inverse limit system such that, for each $\alpha \in D$, X_α is a compact Hausdorff space, $n \geq 2$, and $B = \{\beta_1, \beta_2, \dots, \beta_n\}$ is a finite subset of D . Then $G(\beta_1, \beta_2, \dots, \beta_n)$ is nonempty and compact.*

Proof. B is a finite subset of a directed set D , thus there is an element η of D such that $\beta_i \preceq \eta$ for $i = 1, 2, 3, \dots, n$. The system is consistent, therefore if $t \in X_\eta$, there is an element \mathbf{x} of \mathbf{II} such that $x_\eta = t$ and if $\alpha \preceq \beta \preceq \eta$ then $x_\alpha \in f_{\alpha\beta}(x_\beta)$. Then, $(x_{\beta_1}, x_{\beta_2}, \dots, x_{\beta_n}) \in G(\beta_1, \beta_2, \dots, \beta_n)$ so this set is nonempty.

$X_{\beta_1} \times X_{\beta_2} \times \dots \times X_{\beta_n}$ is compact, therefore it is sufficient to show that $G(\beta_1, \beta_2, \dots, \beta_n)$ is closed. If $(x_{\beta_1}, x_{\beta_2}, \dots, x_{\beta_n}) \notin G(\beta_1, \beta_2, \dots, \beta_n)$, there exist integers i and j such that $\beta_i \preceq \beta_j$ and $x_{\beta_i} \notin f_{\beta_i \beta_j}(x_{\beta_j})$. Because the

graph of $f_{\beta_i \beta_j}$ is closed, there are open sets U_{β_j} and U_{β_i} containing x_{β_j} and x_{β_i} , respectively, such that $U_{\beta_j} \times U_{\beta_i}$ does not intersect the graph of $f_{\beta_i \beta_j}$. Then, $O = O_1 \times O_2 \times \cdots \times O_n$, where $O_i = U_{\beta_i}$, $O_j = U_{\beta_j}$, and $O_k = X_{\beta_k}$ if $k \in \{1, 2, \dots, n\} - \{i, j\}$, is an open set containing $(x_{\beta_1}, x_{\beta_2}, \dots, x_{\beta_n})$ and no point of $G(\beta_1, \beta_2, \dots, \beta_n)$. \square

Theorem 123 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is a consistent inverse limit system such that, for each $\alpha \in D$, X_α is a Hausdorff continuum. Further suppose $n \geq 2$ and $B = \{\beta_1, \beta_2, \dots, \beta_n\}$ is a finite subset of D such that $\beta_i \preceq \beta_j$ if and only if $i \leq j$ and if $\beta_i \preceq \beta_j$ then $f_{\beta_i \beta_j}$ is Hausdorff continuum-valued or $f_{\beta_i \beta_j}(X_{\beta_j})$ is connected with $f_{\beta_i \beta_j}^{-1} : f_{\beta_i \beta_j}(X_{\beta_j}) \rightarrow 2^{X_{\beta_j}}$ Hausdorff continuum-valued. Then $G(\beta_1, \beta_2, \dots, \beta_n)$ is a Hausdorff continuum.*

Proof. By Theorem 122, $G(\beta_1, \beta_2, \dots, \beta_n)$ is nonempty and compact so we only need to show that $G(\beta_1, \beta_2, \dots, \beta_n)$ is connected. We proceed by induction on the number of elements in B . If there are only two elements of B , β_1 and β_2 with $\beta_1 \preceq \beta_2$, $G(\beta_1, \beta_2) = G(f_{\beta_1 \beta_2}^{-1})$. The graph of $f_{\beta_1 \beta_2}$ is connected if and only if the graph of its inverse is connected. So it follows that $G(\beta_1, \beta_2)$ is connected by Theorem 118 if $f_{\beta_1 \beta_2}^{-1}$ is Hausdorff continuum-valued and by Theorem 121 if $f_{\beta_1 \beta_2}$ is Hausdorff continuum-valued. Suppose the conclusion holds for any subset of D with n elements and let $\beta_1, \beta_2, \dots, \beta_{n+1}$ be $n+1$ elements of D such that $\beta_i \preceq \beta_j$ if and only if $i \leq j$. The proof reduces to two cases.

Case (1): $f_{\beta_1 \beta_2}$ is Hausdorff continuum-valued. By the inductive hypothesis, $G(\beta_2, \dots, \beta_{n+1})$ is connected. Suppose H and K are closed sets whose union is $G(\beta_1, \dots, \beta_{n+1})$ and let $h : G(\beta_1, \dots, \beta_{n+1}) \rightarrow G(\beta_2, \dots, \beta_{n+1})$ be the mapping defined by $h((x_{\beta_1}, x_{\beta_2}, \dots, x_{\beta_{n+1}})) = (x_{\beta_2}, \dots, x_{\beta_{n+1}})$. We shall show that $h(H \cup K) = G(\beta_2, \dots, \beta_{n+1})$. Suppose $(x_{\beta_2}, x_{\beta_3}, \dots, x_{\beta_{n+1}})$ is in $G(\beta_2, \dots, \beta_{n+1})$ and let x_{β_1} be an element of $f_{\beta_1 \beta_2}(x_{\beta_2})$. If $2 \leq i \leq n+1$, $x_{\beta_2} \in f_{\beta_2 \beta_i}(x_{\beta_i})$, so $x_{\beta_1} \in f_{\beta_1 \beta_2}(f_{\beta_2 \beta_i}(x_{\beta_i})) = f_{\beta_1 \beta_i}(x_{\beta_i})$. Thus $(x_{\beta_1}, x_{\beta_2}, \dots, x_{\beta_{n+1}}) \in G(\beta_1, \beta_2, \dots, \beta_{n+1})$ and $h((x_{\beta_1}, \dots, x_{\beta_{n+1}})) = (x_{\beta_2}, \dots, x_{\beta_{n+1}})$. Because $h(H)$ and $h(K)$ are closed and $h(H) \cup h(K)$ is connected, there is a point p belonging to $h(H)$ and $h(K)$. Then, $C = \{x \in G(\beta_1, \dots, \beta_{n+1}) \mid x_{\beta_1} \in f_{\beta_1 \beta_2}(p_{\beta_2}), x_{\beta_i} = p_{\beta_i} \text{ for } 2 \leq i \leq n+1\}$ is connected because $f_{\beta_1 \beta_2}(p_{\beta_2})$ is connected. There exist elements t of H and s of K such that $h(t) = h(s) = p$. Because $t_{\beta_1} \in f_{\beta_1 \beta_2}(p_{\beta_2})$ and $t_{\beta_k} = p_{\beta_k}$ for $2 \leq k \leq n+1$, t is in C . Similarly, s is in C . Because C is a connected subset of $H \cup K$ intersecting both H and K , there is a point common to H and K . Consequently, H and K are not mutually separated.

Case (2): $f_{\beta_1 \beta_2}^{-1}$ is Hausdorff continuum-valued. By the inductive hypothesis $G(\beta_1, \beta_3, \dots, \beta_{n+1})$ is connected. Suppose H and K are closed sets whose union is $G(\beta_1, \dots, \beta_{n+1})$ and let $h : G(\beta_1, \dots, \beta_{n+1}) \rightarrow G(\beta_1, \beta_3, \dots, \beta_{n+1})$ be the mapping defined by $h((x_{\beta_1}, x_{\beta_2}, \dots, x_{\beta_{n+1}})) = (x_{\beta_1}, x_{\beta_3}, \dots, x_{\beta_{n+1}})$. Because $x_{\beta_1} \in f_{\beta_1 \beta_3}(x_{\beta_3})$ there is an element x_{β_2} of X_{β_2} such that

$x_{\beta_1} \in f_{\beta_1 \beta_2}(x_{\beta_2})$ and $x_{\beta_2} \in f_{\beta_2 \beta_3}(x_{\beta_3})$. If $3 \leq i \leq n+1$, because $x_{\beta_3} \in f_{\beta_3 \beta_i}(x_{\beta_i})$, we see that $x_{\beta_2} \in f_{\beta_2 \beta_i}(x_{\beta_i})$ so $(x_{\beta_1}, x_{\beta_2}, \dots, x_{\beta_{n+1}}) \in G(\beta_1, \beta_2, \dots, \beta_{n+1})$ and $h((x_{\beta_1}, x_{\beta_2}, \dots, x_{\beta_{n+1}})) = (x_{\beta_1}, x_{\beta_3}, \dots, x_{\beta_{n+1}})$. It follows that $h(H \cup K) = G(\beta_1, \beta_3, \dots, \beta_{n+1})$. Thus, there is a point p belonging to $h(H)$ and $h(K)$. Then, $C = \{x \in G(\beta_1, \dots, \beta_{n+1}) \mid x_{\beta_2} \in f_{\beta_1 \beta_2}^{-1}(p_{\beta_1}), x_{\beta_i} = p_{\beta_i} \text{ for } 1 \leq i \leq n+1, i \neq 2\}$ is connected inasmuch as $f_{\beta_1 \beta_2}^{-1}(p_{\beta_1})$ is connected. There exist elements t of H and s of K such that $h(t) = h(s) = p$. Note that $t_{\beta_1} \in f_{\beta_1 \beta_2}(t_{\beta_2})$, so $t_{\beta_2} \in f_{\beta_1 \beta_2}^{-1}(p_{\beta_1})$. Because $t_{\beta_k} = p_{\beta_k}$ for $k = 1, 3, \dots, n+1$, t is in C . Similarly, s is in C . Because C is a connected subset of $H \cup K$ intersecting both H and K , there is a point common to H and K . Consequently, H and K are not mutually separated. \square

Theorem 124 Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is a consistent inverse limit system such that, for each $\alpha \in D$, X_α is a Hausdorff continuum and D is totally ordered. If, for each α and β in D such that $\alpha \preceq \beta$, $f_{\alpha\beta}$ is Hausdorff continuum-valued or $f_{\alpha\beta}(X_\beta)$ is connected with $f_{\alpha\beta}^{-1} : f_{\alpha\beta}(X_\beta) \rightarrow 2^{X_\beta}$ Hausdorff continuum-valued, then G_η is a Hausdorff continuum for each $\eta \in D$.

Proof. Suppose $\eta \in D$. If $\{\beta_1, \beta_2, \dots, \beta_n\}$ is a finite subset of D , because D is totally ordered, we may assume that $\beta_i \preceq \beta_j$ if and only if $i \leq j$. Let $G'(\beta_1, \beta_2, \dots, \beta_n) = \{x \in \prod \mid x_{\beta_i} \in f_{\beta_i \beta_j}(x_{\beta_j}) \text{ whenever } i \leq j\}$. Let $D' = D - \{\beta_1, \beta_2, \dots, \beta_n\}$. By Theorem 123, $G(\beta_1, \dots, \beta_n)$ is connected. Therefore $G'(\beta_1, \dots, \beta_n)$, which is homeomorphic to $G(\beta_1, \dots, \beta_n) \times \prod_{\alpha' \in D'} X_{\alpha'}$, is connected. Let $\mathcal{G} = \{G'(\beta_1, \beta_2, \dots, \beta_n) \mid \{\beta_1, \dots, \beta_n\} \text{ is a finite subset of } D \text{ such that } \beta_i \preceq \eta \text{ for } 1 \leq i \leq n\}$ and note that \mathcal{G} is a collection of Hausdorff continua by Theorem 123. If $G'(\beta_1, \dots, \beta_n)$ and $G'(\gamma_1, \dots, \gamma_m)$ are in \mathcal{G} , then $G'(\zeta_1, \dots, \zeta_k) \subseteq G'(\beta_1, \dots, \beta_n) \cap G'(\gamma_1, \dots, \gamma_m)$ where $\{\zeta_1, \dots, \zeta_k\} = \{\beta_1, \dots, \beta_n\} \cup \{\gamma_1, \dots, \gamma_m\}$. Consequently, \mathcal{G} has the finite intersection property, so the common part C of all the elements of \mathcal{G} is a Hausdorff continuum.

We now show that $C = G_\eta$. If $x \in G_\eta$ and $G'(\beta_1, \beta_2, \dots, \beta_n) \in \mathcal{G}$ then $x \in G'(\beta_1, \beta_2, \dots, \beta_n)$ so $G_\eta \subseteq C$. If $x \notin G_\eta$ then there exist α and β in D such that $\alpha \preceq \beta \preceq \eta$ and $x_\alpha \notin f_{\alpha\beta}(x_\beta)$. Then, $x \notin G(\alpha, \beta)$ so $x \notin C$ and we have that $C \subseteq G_\eta$. \square

As a consequence of the preceding theorems, in the next theorem we have the result we sought in this subsection.

Theorem 125 Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is a consistent inverse limit system such that, for each $\alpha \in D$, X_α is a Hausdorff continuum and D is totally ordered. If, for each α and β in D such that $\alpha \preceq \beta$, $f_{\alpha\beta}$ is Hausdorff continuum-valued or $f_{\alpha\beta}(X_\beta)$ is connected with $f_{\alpha\beta}^{-1} : f_{\alpha\beta}(X_\beta) \rightarrow 2^{X_\beta}$ Hausdorff continuum-valued, then $\varprojlim f$ is a Hausdorff continuum.

Proof. This is immediate from Theorems 124 and 116. \square

Because of its importance, we state the following special case of Theorem 125. Recall that inverse limit sequences are consistent (Theorem 108).

Theorem 126 *Suppose $\{X_i, f_i\}$ is an inverse limit sequence on Hausdorff continua with upper semi-continuous bonding functions such that, if i is a positive integer, then f_i is Hausdorff continuum-valued or $f_i(X_{i+1})$ is connected with $f_i^{-1} : f_i(X_{i+1}) \rightarrow 2^{X_{i+1}}$ Hausdorff continuum-valued. Then $\varprojlim f$ is a Hausdorff continuum.*

If $f : X \rightarrow X$ is a mapping of a Hausdorff continuum into itself, then $f(X)$ is connected and $f^{-1} : f(X) \rightarrow 2^X$ is an upper semi-continuous function whose inverse is continuum valued (in fact each value is degenerate). Thus, we have the following corollary to Theorem 126.

Corollary 127 *Suppose $\{X, f^{-1}\}$ is an inverse limit sequence where X is a Hausdorff continuum and f is a mapping from X into X . Then $\varprojlim f^{-1}$ is a Hausdorff continuum homeomorphic to $f(X)$.*

That $\varprojlim f^{-1}$ is a Hausdorff continuum follows directly from Theorems 121 and 126. Moreover, it is not difficult to show that $\varprojlim f^{-1}$ is homeomorphic to the Hausdorff continuum $f(X)$ because $h : f(X) \rightarrow \varprojlim f^{-1}$ given by $h(x) = (x, f(x), f^2(x), \dots)$ is a homeomorphism.

Theorem 125 gives a sufficient condition that inverse limits on Hausdorff continua with upper semi-continuous bonding functions be Hausdorff continua. However, in general the bonding functions do not have to satisfy the conditions of Theorem 125 in order to produce a connected inverse limit. For instance, see Example 132. Theorem 156 of Section 2.9 provides a different sufficient condition for an inverse limit with upper semi-continuous bonding functions on $[0, 1]$ to be connected. Deciding in general when an inverse limit system with upper semi-continuous bonding functions produces a connected inverse limit remains an interesting problem.

2.7 Examples in the special case that each factor space is $[0, 1]$

In this section we assume that the directed set is the set of positive integers and each factor space is $[0, 1]$. We give some examples of some upper semi-continuous bonding functions that illustrate additional aspects of the nature of inverse limits with upper semi-continuous bonding functions. Each example

is produced by a constant sequence of bonding functions. In most of these examples we specify the graph of the bonding function rather than a formula for the function. We denote the Hilbert cube $[0, 1]^\infty$ by \mathcal{Q} . A point p of a Hausdorff continuum M is a *separating point* of M provided $M - \{p\}$ is not connected. Separating points are also called *cut points*. An *arc* is a continuum with only two nonseparating points.

Our first example shows that inverse limits with upper semi-continuous bonding functions can produce the Hilbert cube.

Example 128 (The Hilbert cube) Let $f : [0, 1] \rightarrow 2^{[0, 1]}$ be given by $f(x) = [0, 1]$ for each x in $[0, 1]$; that is, $G(f) = [0, 1] \times [0, 1]$. Then $\varprojlim \mathbf{f}$ is the Hilbert cube.

In our next example the inverse limit is a closed set in the Hilbert cube \mathcal{Q} consisting of a convergent sequence of points.

Example 129 (A convergent sequence) Let $G(f)$ be the union of the point $\{(1, 1)\}$ and $[0, 1] \times \{0\}$ (see Figure 2.2).

Proof. Let $K = \varprojlim \mathbf{f}$. No point of K has a coordinate between 0 and 1 inasmuch as the range of f is $\{0, 1\}$. If \mathbf{p} is in K and $p_i = 1$ then $p_{i+1} = 1$ whereas if $p_i = 0$ then p_{i+1} is 0 or 1. Thus, the points of K are $(0, 0, 0, \dots)$ and the set of all points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots$ where, for each positive integer n , $\pi_i(\mathbf{p}_n)$ is 1 if $i \geq n$ and 0 if $i < n$. \square

A Cantor set can be the result of an inverse limit on $[0, 1]$ with an upper semi-continuous bonding function.

Example 130 (A Cantor set) Let $G(f)$ be the union of $[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$ (see Figure 2.3).

Proof. Note that if \mathbf{x} is in $\varprojlim \mathbf{f}$ and i is a positive integer x_i cannot be strictly between 0 and 1. Moreover, if $x_i \in \{0, 1\}$ then $x_{i+1} \in \{0, 1\}$. It follows that $\varprojlim \mathbf{f}$ is a Cantor set. \square

Example 114 showed that the inverse limit may fail to be connected even if the graph of the bonding function is connected. Theorem 125 provided a sufficient condition that the inverse limit be a Hausdorff continuum and one of its consequences is that the inverse limit in the next example is a continuum. As we show in the remaining examples in this section, quite a variety of continua can be obtained as an inverse limit with upper semi-continuous bonding functions that generally do not satisfy the conditions from Theorem 125. The inverse limit in our next example is a *fan* (i.e., a continuum that is the union of a collection of arcs such that the intersection of each two of them is a point v , called the vertex, that is an endpoint of each of the arcs in the collection).

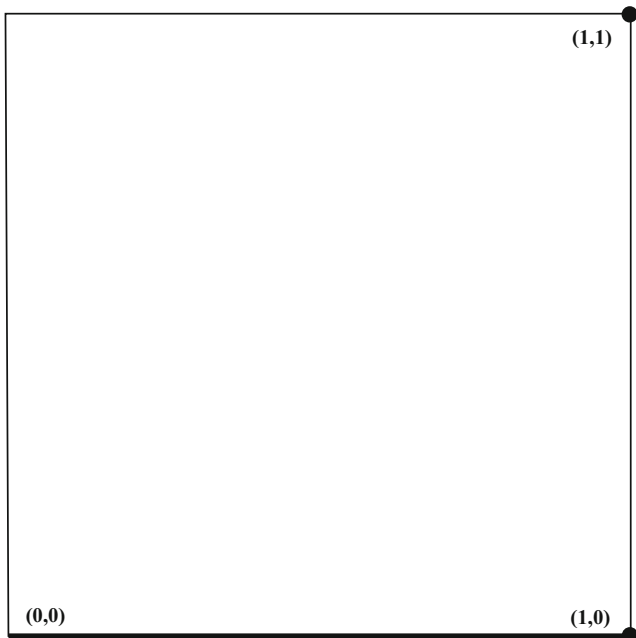


Fig. 2.2 The function from Example 129

Example 131 (A simple fan) Let $G(f)$ be the union of the graph of the identity function and the interval $I \times \{0\}$ (see Figures 2.4 and 2.5).

Proof. Inasmuch as f^{-1} is continuum-valued, $\varprojlim \mathbf{f}$ is connected by Theorem 126. Let $\mathbf{v} = (0, 0, 0, \dots)$. Note that if $\mathbf{p} \in \varprojlim \mathbf{f}$ and for some positive integer n , $p_n > 0$, then $p_j = p_n$ for all $j > n$. For each positive integer n , let K_n be the set of all points $\mathbf{p} \in \varprojlim \mathbf{f}$ such that $p_j = 0$ for $j < n$ and $p_j = p_n > 0$ for $j \geq n$. The closure of K_n is an arc of length $1/2^{n-1}$ having one endpoint \mathbf{v} . Moreover no two of these arcs intersect except at \mathbf{v} and $\varprojlim \mathbf{f} = \bigcup_{i>0} K_i$. \square

In the following example we see that a simple upper semi-continuous function can produce a Cantor fan. The reader will also note that the inverse limit is a continuum even though the bonding function does not satisfy the hypothesis of Theorem 125. Although here we show directly that the inverse limit in this example is a continuum, this fact is also a consequence of Theorem 156 of Section 2.9.

Example 132 (The Cantor fan) Let $G(f)$ be the union of the identity, Id , and the map $1 - Id$ on $[0, 1]$ (see Figures 2.6 and 2.7).

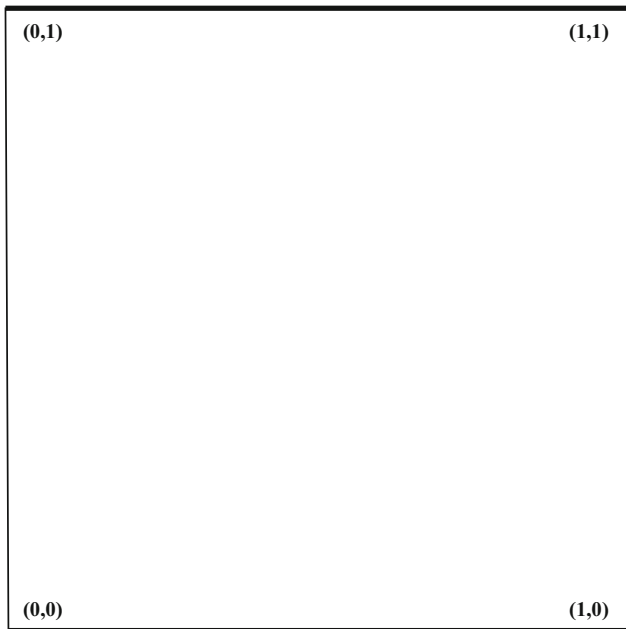


Fig. 2.3 The function from Example 130

Proof. Let $K = \varprojlim \mathbf{f}$. The vertex \mathbf{v} of K is the point $(1/2, 1/2, 1/2, \dots)$ and the Cantor set C at the base of the fan is $\varprojlim \{\{0, 1\}, f|_{\{0, 1\}}\}$. If \mathbf{c} is a point of C , the arc joining \mathbf{v} and \mathbf{c} is the inverse limit of the inverse sequence $\{J_i, g_i\}$ where, for each i , J_i is the interval joining c_i and $1/2$ and g_i is a homeomorphism that fixes $1/2$ and whose graph is a subset of the graph of f . (It could be of help for the reader to observe that the graph of f is the union of four intervals having only the point $(1/2, 1/2)$ in common. For each i , the graph of g_i is one of these four intervals.) \square

Next we consider three examples produced by similar upper semi-continuous functions. Each graph consists of the union of the horizontal line $I \times \{0\}$ and a vertical line. We produce the examples by choosing the vertical line that intersects this horizontal line in three places: $(1, 0)$, $(0, 0)$, and $(1/2, 0)$. The resulting inverse limits are, respectively, an arc (i.e., a continuum with only two nonseparating points), infinite-dimensional, and an arc with a sequence of stickers.

Example 133 (An arc) Let $G(f)$ be the union of $[0, 1] \times \{0\}$ and $\{1\} \times [0, 1]$ (see Figure 2.8).

Proof. Let $M = \varprojlim \mathbf{f}$. We show that if $\mathbf{x} \in M - \{(0, 0, 0, \dots), (1, 1, 1, \dots)\}$ then \mathbf{x} is a separating point of M thus showing that M is an arc. If $\mathbf{x} \in$

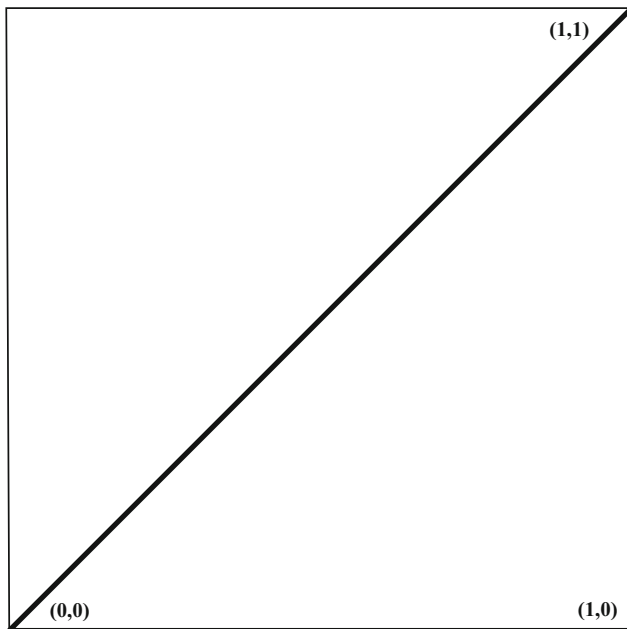


Fig. 2.4 The function from Example 131

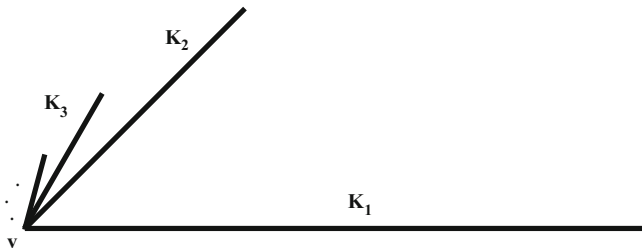


Fig. 2.5 A depiction of the inverse limit from Example 131

$M - \{(0,0,0,\dots), (1,1,1,\dots)\}$ then there is a positive integer k such that $x_k \neq 0$ but $x_{k-1} = 0$ if $k > 1$. There are two cases: $x_k = 1$ and $x_k < 1$.

Suppose $x_k = 1$. Then, $k \geq 2$ so $x_{k-1} = 0$. Let $A = \{\mathbf{y} \in M \mid y_{k-1} \in (0, 1]\}$ and $B = \{\mathbf{y} \in M \mid y_k \in [0, 1)\}$. Then A and B are open in M . If $\mathbf{z} \in M - \{\mathbf{x}\}$ and $z_k \neq 1$ then $\mathbf{z} \in B$ whereas if $z_k = 1$ then $z_{k-1} \neq 0$ so $\mathbf{z} \in A$. Thus, $M - \{\mathbf{x}\} = A \cup B$. If $\mathbf{z} \in B$, then $z_k \in [0, 1)$ and $z_{k-1} = 0$. Thus, $\mathbf{z} \notin A$ so $A \cap B = \emptyset$. Therefore, \mathbf{x} is a separating point of M .

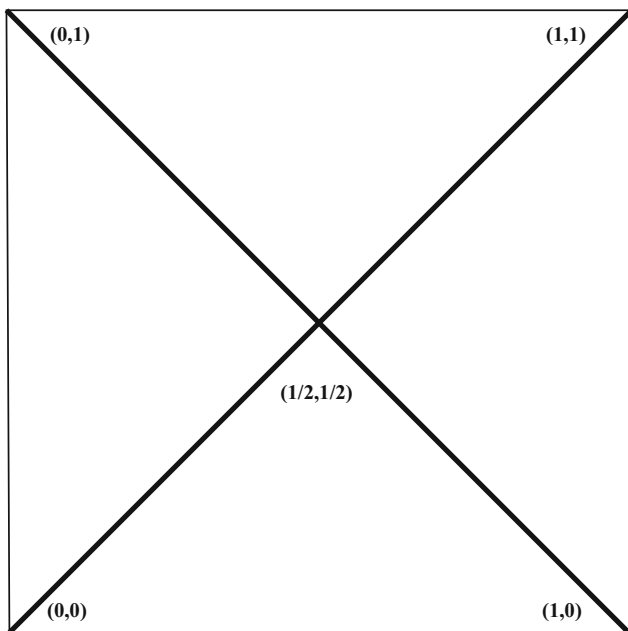


Fig. 2.6 *The function from Example 132*

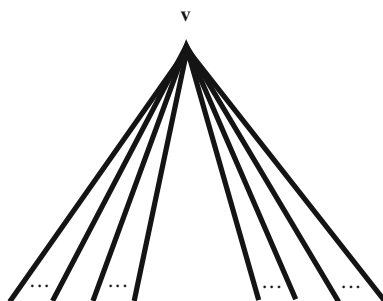


Fig. 2.7 *A depiction of the inverse limit from Example 132*

Suppose $x_k < 1$. Let $A = \{\mathbf{y} \in M \mid y_k < x_k\}$ and $B = \{\mathbf{y} \in M \mid y_k > x_k\}$ and note that A and B are mutually exclusive open sets in M . If $\mathbf{y} \in M$ and $y_k = x_k$ it follows that $\mathbf{y} = \mathbf{x}$ so $M - \{\mathbf{x}\} = A \cup B$ and, again, we have that \mathbf{x} is a separating point of M .

In either case, we have shown M has only two nonseparating points so M is an arc. \square

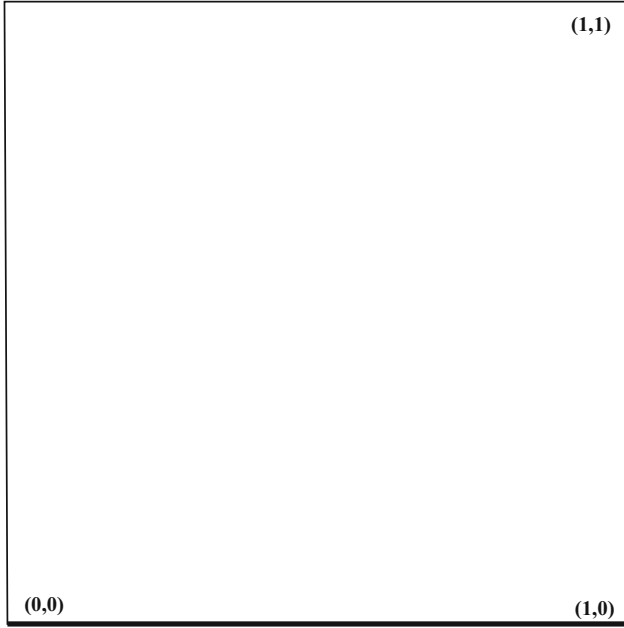


Fig. 2.8 The function from Example 133

Let f be the function from Example 133 and $M = \varprojlim \mathbf{f}$. The following is another way to look at M . Let $A_1 = \{\mathbf{x} \in M \mid x_2 = 1\}$. Note that if $\mathbf{x} \in A_1$ then $x_k = 1$ for $k \geq 2$. For each positive integer $i > 1$, let $A_i = \{\mathbf{x} \in \varprojlim \mathbf{f} \mid x_k = 0 \text{ for } k < i \text{ and } x_k = 1 \text{ for } k > i\}$. Then, $\varprojlim \mathbf{f} = A_1 \cup A_2 \cup A_3 \cup \dots \cup \{(0, 0, 0, \dots)\}$. To see that A is an arc observe that $A_i \cap A_j = \emptyset$ if $|i - j| > 1$ and $A_i \cap A_{i+1} = \{P_i\}$ where the first i coordinates of P_i are 0 and all of the remaining coordinates are 1. This arc plays a crucial role later in analyzing Examples 136 and 137.

Example 134 (An infinite-dimensional continuum) Let $G(f)$ be the union of $[0, 1] \times \{0\}$ and $\{0\} \times [0, 1]$ (see [Figure 2.9](#)).

Let $M = \varprojlim \mathbf{f}$. Observe that M contains $[0, 1] \times \{0\} \times [0, 1] \times \{0\} \times [0, 1] \times \{0\} \times \dots$ and this set is homeomorphic to the Hilbert cube, so M is infinite-dimensional.

Example 135 (An arc with stickers) Let $G(f)$ be the union of $[0, 1] \times \{0\}$ and $\{1/2\} \times [0, 1]$ (see [Figure 2.10](#)).

Proof. Let $M = \varprojlim \mathbf{f}$. For each positive integer n , let $A_n = \{\mathbf{x} \in M \mid x_n \in [0, 1], x_j = 1/2 \text{ for } j > n \text{ and if } n > 1 \text{ then } x_i = 0 \text{ for } i < n\}$. Note that for

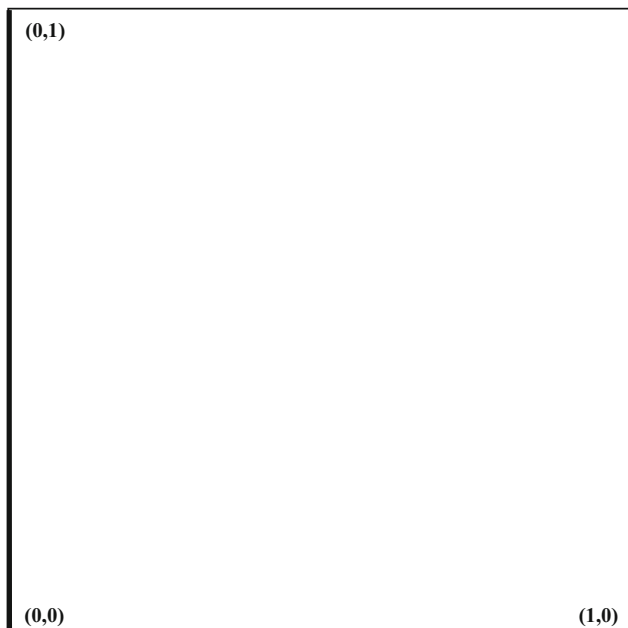


Fig. 2.9 The function from Example 134

each positive integer i , $A_{i+1} \cap A_i = \{(0, 0, \dots, 0, 1/2, 1/2, 1/2, \dots)\}$ where the last 0 occurs in the i th coordinate. Moreover, $A_i \cap A_j = \emptyset$ if $|i - j| > 1$. Finally, $(0, 0, 0, \dots)$ is a limit point of $A_1 \cup A_2 \cup A_3 \cup \dots = M - \{(0, 0, 0, \dots)\}$. For a picture of M see [Figure 2.11](#). \square

Example 136 (An arc with a fan and spines) Let $G(f)$ be the union of $[0, 1] \times \{0\}$ and $\{1\} \times [0, 1]$ together with the portion of the identity above $[0, 1/4]$ (see [Figures 2.12](#) and [2.13](#)).

Let $M = \varprojlim \mathbf{f}$. Let $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup \{(0, 0, 0, \dots)\}$ where $A_1 = \{\mathbf{x} \in M \mid x_2 = 1\}$ and if i is an integer greater than one then $A_i = \{\mathbf{x} \in \varprojlim \mathbf{f} \mid x_k = 0 \text{ for } k < i \text{ and } x_k = 1 \text{ for } k > i\}$. Then, A is an arc lying in M . The arc A is the inverse limit in Example 133. Let $B_1 = \{\mathbf{x} \in M \mid x_j \in [0, 1/4] \text{ for all } j\}$ and, for $i > 1$, let $B_i = \{\mathbf{x} \in M \mid x_j = 0 \text{ for } j < i, x_i \in [0, 1/4] \text{ and } x_j = x_i \text{ for } j \geq i\}$. Let $B = B_1 \cup B_2 \cup B_3 \cup \dots \cup \{(0, 0, 0, \dots)\}$ and note that B is a fan lying in M with vertex $(0, 0, 0, \dots)$. Suppose each of i and j is a positive integer. Let $C_{1j} = \{\mathbf{x} \in M \mid x_k = 1 \text{ for } k \geq j + 1 \text{ and } x_k \in [0, 1/4] \text{ for } 1 \leq k \leq j\}$ and, for $i > 1$, let $C_{ij} = \{\mathbf{x} \in M \mid x_k = 1 \text{ for } k \geq i + j, x_k = 0$

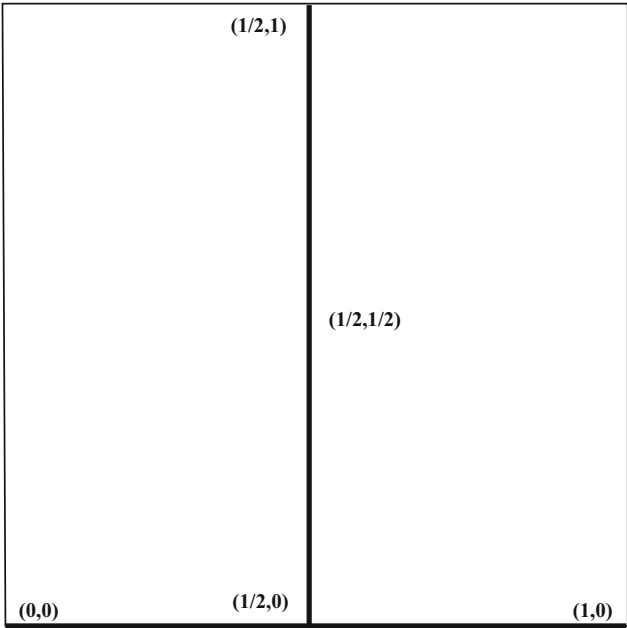


Fig. 2.10 The function from Example 135

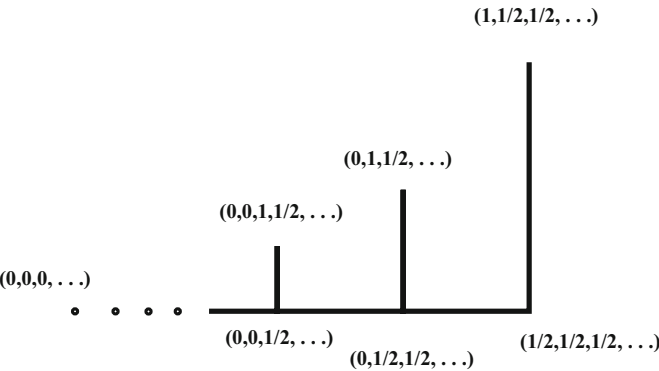


Fig. 2.11 A depiction of the inverse limit from Example 135

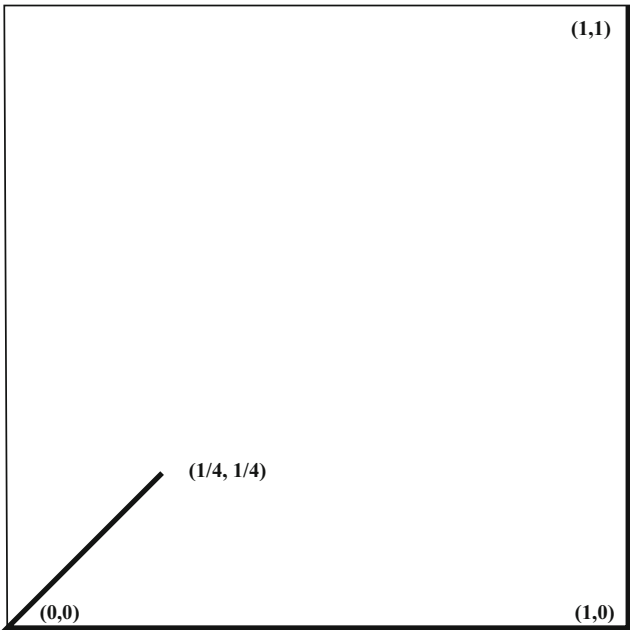


Fig. 2.12 The function from Example 136

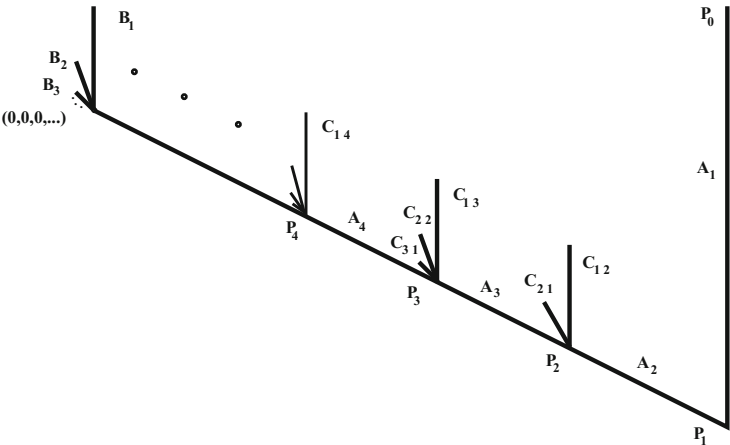


Fig. 2.13 A depiction of the inverse limit from Example 136

for $k < i$, and $x_k \in [0, 1/4]$ for $i \leq k < i + j$. Note that if \mathbf{x} is a point of M not in $A \cup B$ then $\mathbf{x} \in C_{i,j}$ for some i and j . Also, $C_{1,1} \subset A_1$. Because $A \cup B$ is connected and $C_{i,j}$ intersects A for each i and j with $j \leq i$, M is connected. In Figure 2.13, we depict M . In the picture, $P_0 = (1, 1, 1, \dots)$ and, if i is a positive integer, $P_i = (0, \dots, 0, 1, 1, 1, \dots)$ has its first i coordinates 0 followed by all 1s.

Example 137 (An arc plus a sequence of fans) Let $G(f)$ be the union of $[0, 1] \times \{0\}$ and $\{1\} \times [0, 1]$ together with the straight line interval from $(3/4, 1/4)$ to $(1, 1)$ (see Figures 2.14 and 2.15).

Let $M = \varprojlim \mathbf{f}$. Let $A = A_1 \cup A_2 \cup A_3 \cup \dots \cup \{(0, 0, 0, \dots)\}$ where $A_1 = \{\mathbf{x} \in M \mid x_2 = 1\}$ and if i is an integer greater than one then $A_i = \{\mathbf{x} \in \varprojlim \mathbf{f} \mid x_k = 0 \text{ for } k < i \text{ and } x_k = 1 \text{ for } k > i\}$. Then, A is an arc lying in M . Let $D_0 = \{\mathbf{x} \in M \mid x_1 \in [0, 1]\}$ and for each positive integer k , $x_{k+1} = (x_k + 2)/3$ and, for $i \geq 1$, let $D_i = \{\mathbf{x} \in M \mid x_k = 0 \text{ for } k \leq i, x_{i+1} \in [1/4, 1], \text{ and } x_{k+1} = (x_k + 2)/3 \text{ for each } k > i\}$. For each positive integer j , let $E_{0,j} = \{\mathbf{x} \in M \mid x_1 \in [1/4, 1], x_k = 1 \text{ if } k > j, \text{ and if } 1 < k < j \text{ then } x_{k+1} = (x_k + 2)/3\}$. If each of i and j is a positive integer, let $E_{i,j} = \{\mathbf{x} \in M \mid x_k = 1 \text{ for } k > j, x_k = 0 \text{ for } k < i, x_i \in [1/4, 1] \text{ and if } i \leq k < j \text{ then } x_{k+1} = (x_k + 2)/3\}$. For each nonnegative integer i let $F_i = E_{i,1} \cup E_{i,2} \cup E_{i,3} \cup \dots \cup D_i$. Note that each F_i is a fan with vertex P_i where $P_0 = (1, 1, 1, \dots)$ and, for $i > 0$, P_i is the point of M having k th coordinate 0 for $k \leq i$ and all remaining coordinates 1. Furthermore, if $\mathbf{x} \in M$ and $\mathbf{x} \notin A$ then $\mathbf{x} \in F_i$ for some $i \geq 0$. It follows that M is a continuum. In Figure 2.15 we depict the continuum M .

At this point we observe that the set-theoretic union of the graphs of the functions from Examples 136 and 137 is the graph of the function from Example 114. Although the inverse limit in Example 136 is a continuum and the inverse limit in Example 137 is a continuum and the arc A lies in the intersection of these two continua, the inverse limit of the union of the two graphs is not connected. In Section 2.9 we study inverse limits of upper semi-continuous functions that are graphs of unions of mappings. Among other things we show for maps of intervals that if at least one of the mappings is surjective then the inverse limit is connected.

A consequence of Theorem 166 of Section 2.10 is that if f is a mapping from I into I , then $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{f}^2$. In the case where an upper semi-continuous bonding function f is not a mapping, $\varprojlim \mathbf{f}$ and $\varprojlim \mathbf{f} \circ \mathbf{f}$ may not be homeomorphic. The following example shows this. In this example we specify the function instead of the graph of the function. This example is a minor modification of an example in [7].

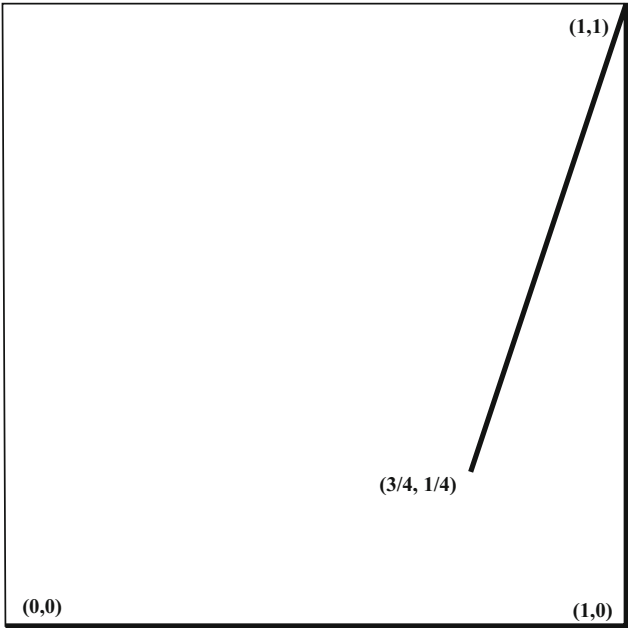


Fig. 2.14 The function from Example 137

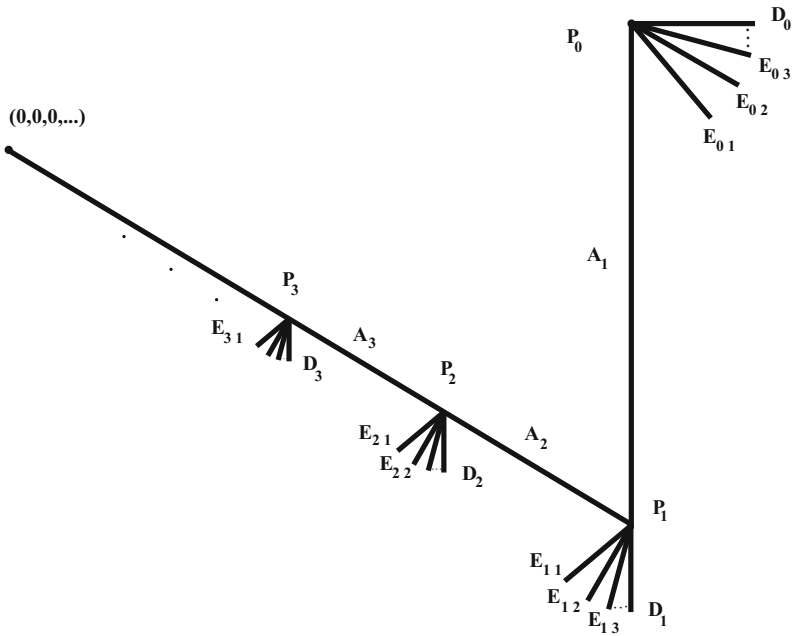


Fig. 2.15 A depiction of the inverse limit from Example 137

Example 138 ($\varprojlim \mathbf{f}$ and $\varprojlim \mathbf{f} \circ \mathbf{f}$ are not necessarily homeomorphic) Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f(x) = \{1/2, 1 - x\}$ for $0 \leq x < 1/2$, $f(x) = \{1/2\}$ for $1/2 \leq x < 1$ and $f(1) = [0, 1/2]$ (see Figure 2.16).

Proof. Note that $K = \varprojlim \mathbf{f}$ contains a triod which is the union of the three arcs described below. Let A_1 be the set of all points of K whose first coordinate is in the half-open interval $(1/2, 1]$. The closure of A_1 is an arc from $(1, 0, 1, 0, \dots)$ to $(1/2, 1/2, 1, 0, 1, 0, \dots)$. Let A_2 be the set of all points of K whose first two coordinates are $1/2$ and whose third coordinate is in the half-open interval $(1/2, 1]$. The closure of A_2 is an arc from $(1/2, 1/2, 1/2, 1/2, 1, 0, 1, 0, \dots)$ to $(1/2, 1/2, 1, 0, 1, 0, \dots)$. Finally let A_3 be the set of all points of K whose first coordinate is $1/2$ and whose second coordinate is in the half-open interval $[0, 1/2)$. The closure of A_3 is an arc from $(1/2, 0, 1, 0, 1, 0, \dots)$ to $(1/2, 1/2, 1, 0, 1, 0, \dots)$. The union of the closures of A_1 , A_2 , and A_3 is a simple triod contained in K .

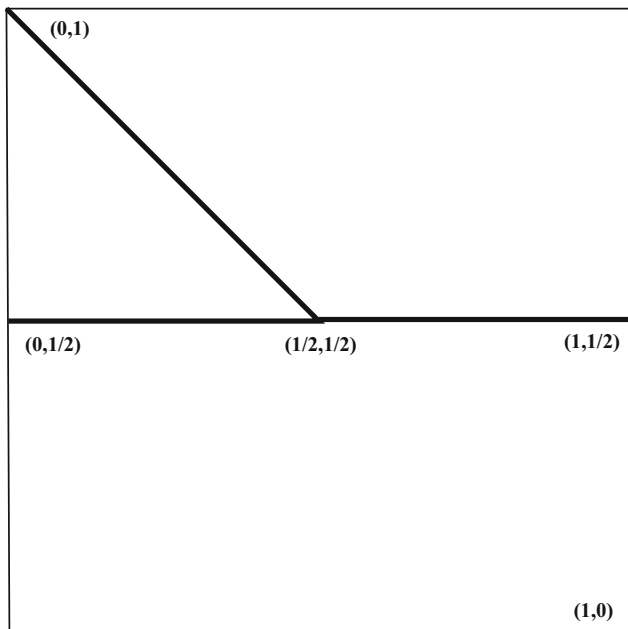


Fig. 2.16 The function from Example 138

On the other hand, $f \circ f$ is the union of the three straight line intervals, $\{0\} \times [0, 1/2]$, $I \times \{1/2\}$, and $\{1\} \times [1/2, 1]$ (see Figure 2.17). $H = \varprojlim \mathbf{f} \circ \mathbf{f}$ is a continuum by Theorem 126. We show that H is an arc with endpoints $\mathbf{a} = (0, 0, 0, \dots)$ and $\mathbf{b} = (1, 1, 1, \dots)$. Let \mathbf{p} be a point of H different from

a and **b**. There is an n such that p_n is neither 0 nor 1. If $p_n \neq 1/2$ then $H \cap \pi_n^{-1}(p_n)$ is degenerate and separates H into the two mutually separated sets $H \cap \pi_n^{-1}([0, p_n))$ and $H \cap \pi_n^{-1}((p_n, 1])$. If $p_n = 1/2$ and $p_{n+1} \in \{0, 1\}$ then $H \cap \pi_n^{-1}(p_n)$ is degenerate and a separating point (i.e., cut point) of H . Thus we may assume that p_{n+1} is neither 0 nor 1 and again conclude that $H \cap \pi_{n+1}^{-1}(p_{n+1})$ is a separating point unless $p_{n+1} = 1/2$. Continuing this process, the only point remaining to consider is the constant sequence $(1/2, 1/2, 1/2, \dots)$. But this point also is clearly a separating point of H . Thus H is a continuum having at most two nonseparating points and is an arc. See [4, Theorem 2-1, p. 49] and [4, Theorem 2-27, p. 54]. \square

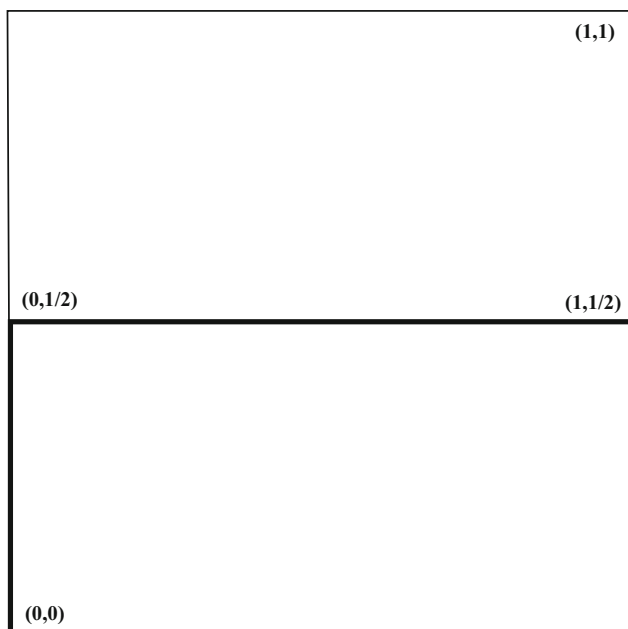


Fig. 2.17 The graph of f^2 where f is the function from Example 138

The next example provides an upper semi-continuous function whose inverse limit is the union of a 2-cell and an arc with a single point in common. With a sequence of upper semi-continuous functions it is possible to get a 2-cell as the inverse limit. One only needs to use the sequence f_1, f_2, f_3, \dots where $G(f_1) = [0, 1] \times [0, 1]$ and $G(f_i)$ is the identity on $[0, 1]$ for $i > 1$. However, Van Nall has shown [10] that if $f : [0, 1] \rightarrow 2^{[0, 1]}$ is an upper semi-continuous function then $\varprojlim f$ is not a 2-cell (see Theorem 186 of Section 2.12).

Example 139 (A two-dimensional example) Let $G(f)$ consist of the union of the four straight line intervals, $[0, 1/2] \times \{0\}$, $\{1/2\} \times [0, 1/2]$, $[1/2, 1] \times \{1/2\}$, and $\{1\} \times [1/2, 1]$ (see Figures 2.18 and 2.19).

Proof. Let $K = \varprojlim \mathbf{f}$. Here K is the union of a 2-cell D and an arc A . To identify D , let i and j be positive integers with $j > i + 1$ and let $D_{i,j}$ be the 2-cell, $\{\mathbf{p} \in K \mid p_i \in [0, 1/2], p_j \in [1/2, 1], p_k = 0 \text{ if } k < i, p_k = 1/2 \text{ if } i < k < j, p_k = 1 \text{ if } k > j\}$. In Figure 2.19 we provide a schematic picture to assist the reader. In this picture we have labeled a few of the disks $D_{i,j}$. In the figure the disks $D_{i,j}$ and $D_{i,j+1}$ share a common horizontal border and the disks $D_{i,j}$ and $D_{i+1,j}$ share a common vertical border as long as $i + 1 < j - 1$. Let D be the closure of the union of all the disks $D_{i,j}$ where $i \geq 1$ and $j > i + 1$.

Suppose k is a positive integer. Let $\alpha_k = \{\mathbf{p} \in K \mid p_k \in [1/2, 1], p_m = 1/2 \text{ for } m < k, \text{ and } p_m = 1 \text{ for } m > k\}$ and $\beta_k = \{\mathbf{p} \in K \mid p_k \in [0, 1/2], p_m = 0 \text{ for } m < k, \text{ and } p_m = 1/2 \text{ for } m > k\}$. The arc α_k forms the right-hand vertical edge of the disk $D_{k-2,k}$ for $k = 3, 4, 5, \dots$ in the figure and the arc β_k lies directly below all of the disks $D_{k,k+n}$ for $n = 2, 3, 4, \dots$. The closure of $\beta_1 \cup \beta_2 \cup \beta_3 \cup \dots$ is an arc from $(0, 0, 0, \dots)$ to $(1/2, 1/2, 1/2, \dots)$ forming the bottom edge of the disk D and the closure of $\alpha_3 \cup \alpha_4 \cup \alpha_5 \cup \dots$ is an arc from $(1/2, 1/2, 1/2, \dots)$ to $(1/2, 1/2, 1, 1, \dots)$ forming the right-hand edge of D .

Let $A = \alpha_1 \cup \alpha_2$. Then, A is an arc, D is a 2-cell, $K = D \cup A$, and $D \cap A = \{(1/2, 1/2, 1, 1, \dots)\}$. \square

One can modify Example 139 to produce an inverse limit of dimension n for any choice of n . For example, to produce an inverse limit of dimension 3 add a second stairstep between $1/4$ and $1/2$. That is, let M be the union of the intervals $[0, 1/4] \times \{0\}$, $\{1/4\} \times [0, 1/4]$, $[1/4, 1/2] \times \{1/4\}$, $\{1/2\} \times [1/4, 1/2]$, $[1/2, 1] \times \{1/2\}$, and $\{1\} \times [1/2, 1]$. Additional stairsteps can be added to produce higher-dimensional inverse limits.

2.8 Mapping theorems

The inverse limit of an inverse limit system $\{X_\alpha, f_{\alpha\beta}, D\}$ is a subset of the product of the factor spaces, therefore a number of mapping theorems for inverse limits derive from mapping theorems for product spaces. One fundamental mapping property of product spaces is that a function into a product

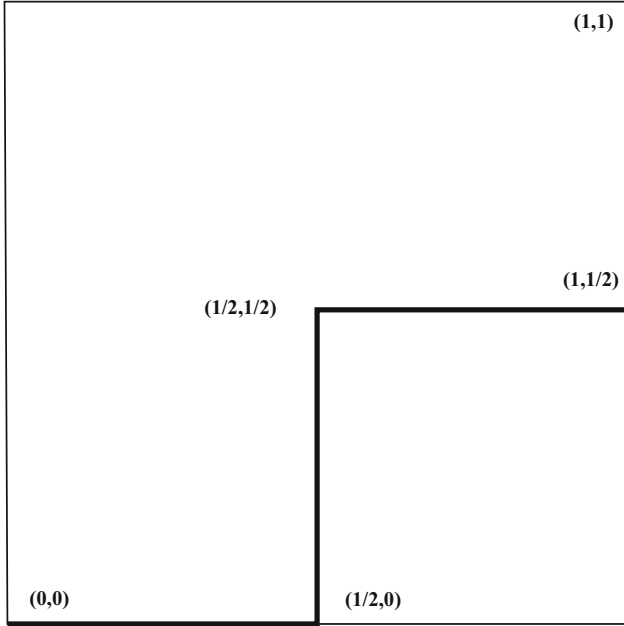


Fig. 2.18 The function from Example 139

is continuous if and only if the composition of the function with each of the projection maps is continuous. A mapping f is called *one-to-one* (or *reversible* or 1-1) provided if $f(x) = f(y)$ then $x = y$.

Suppose D and E are sets and $\sigma : E \rightarrow D$ is a one-to-one function from E into D . If $\{X_\alpha \mid \alpha \in D\}$ and $\{Y_\beta \mid \beta \in E\}$ are collections of sets and for each $\beta \in E$ there is a function $\varphi_\beta : X_{\sigma(\beta)} \rightarrow Y_\beta$ then the collection $\{\varphi_\beta \mid \beta \in E\}$ induces a function $\Phi : \prod_{\alpha \in D} X_\alpha \rightarrow \prod_{\beta \in E} Y_\beta$ defined by $\pi_\beta(\Phi(\mathbf{x})) = \varphi_\beta(\pi_{\sigma(\beta)}(\mathbf{x}))$ (i.e., $\pi_\beta(\Phi(\mathbf{x})) = \varphi_\beta(x_{\sigma(\beta)})$) for each β in E .

Theorem 140 Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ and $\{Y_\alpha, g_{\alpha\beta}, E\}$ are inverse limit systems with upper semi-continuous bonding functions and $\sigma : E \rightarrow D$ is a one-to-one function such that if $\alpha \preceq \beta$ in E then $\sigma(\alpha) \preceq \sigma(\beta)$ in D . Suppose further that for each $\alpha \in E$ there is a function $\varphi_\alpha : X_{\sigma(\alpha)} \rightarrow Y_\alpha$ such that if $\alpha \preceq \beta$ in E then $\varphi_\alpha \circ f_{\sigma(\alpha)\sigma(\beta)} = g_{\alpha\beta} \circ \varphi_\beta$. Then, if \mathbf{x} is in $\varprojlim \mathbf{f}$, $\Phi(\mathbf{x})$ is in $\varprojlim \mathbf{g}$.

Proof. Suppose $\mathbf{x} \in \varprojlim \mathbf{f}$. If α and β are in E and $\alpha \preceq \beta$, then $\sigma(\alpha) \preceq \sigma(\beta)$, so $x_{\sigma(\alpha)} \in f_{\sigma(\alpha)\sigma(\beta)}(x_{\sigma(\beta)})$. Thus, $\varphi_\alpha(x_{\sigma(\alpha)}) \in \varphi_\alpha(f_{\sigma(\alpha)\sigma(\beta)}(x_{\sigma(\beta)}))$. Because $\pi_\alpha(\Phi(\mathbf{x})) = \varphi_\alpha(x_{\sigma(\alpha)})$ and $\varphi_\alpha \circ f_{\sigma(\alpha)\sigma(\beta)} = g_{\alpha\beta} \circ \varphi_\beta$, we have $\pi_\alpha(\Phi(\mathbf{x})) \in g_{\alpha\beta}(\varphi_\beta(x_{\sigma(\beta)}))$. But, $\varphi_\beta(x_{\sigma(\beta)}) = \pi_\beta(\Phi(\mathbf{x}))$. It follows that $\pi_\alpha(\Phi(\mathbf{x})) \in g_{\alpha\beta}(\pi_\beta(\Phi(\mathbf{x})))$ and thus $\Phi(\mathbf{x}) \in \varprojlim \mathbf{g}$. \square

there is an element β of E such that $\alpha \preceq \beta$. It is not difficult to show that if E is cofinal in D then E is a directed set. If $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system and E is a cofinal subset of D , there is an associated inverse limit system $\{Y_\beta, g_{\alpha\beta}, E\}$ such that $Y_\beta = X_\beta$ for each $\beta \in E$ and $g_{\alpha\beta} = f_{\alpha\beta}$ for each α and β in E such that $\alpha \preceq \beta$. We refer to this system over E as the *restriction* of $\{X_\alpha, f_{\alpha\beta}, D\}$ to E .

Theorem 142 *Suppose $D = \{1, 2, 3, \dots\}$ and $\{X_n, f_{nm}, D\}$ is an inverse limit system with upper semi-continuous bonding functions such that $\varprojlim \mathbf{f}$ is nonempty, E is a cofinal subset of D , and $\{Y_m, g_{nm}, E\}$ is the restriction of $\{X_n, f_{nm}, D\}$ to E . For each $m \in E$, let φ_m denote the identity on X_m . Then, the function φ induced by $\{\varphi_m \mid m \in E\}$ is a mapping from $\varprojlim \mathbf{f}$ onto $\varprojlim \mathbf{g}$.*

Proof. By Theorem 141, $\varphi : \varprojlim \mathbf{f} \rightarrow \varprojlim \mathbf{g}$ is a mapping. Suppose \mathbf{y} is in $\varprojlim \mathbf{g}$. Let \mathbf{x} be a point of $\varprojlim \mathbf{f}$ chosen as follows. For $m \in E$ let $x_m = y_m$. For n and m in E with $n < m - 1$ and no element of E between them, choose $x_{n+1} \in f_{n+1m}(x_m)$ such that $x_n \in f_{nn+1}(x_{n+1})$. (Such a choice is possible because $f_{nm} = f_{nn+1} \circ f_{n+1m}$.) Next, choose $x_{n+2} \in f_{n+2m}(x_m)$ such that $x_{n+1} \in f_{n+1n+2}(x_{n+2})$. Continuing, we may choose x_i for each i , $n < i < m$, so that $x_i \in f_{ik}(x_k)$ for $i \leq k \leq m$ and $x_n \in f_{ni}(x_i)$. This process determines a point \mathbf{x} of $\varprojlim \mathbf{f}$ such that $\varphi(\mathbf{x}) = \mathbf{y}$. \square

In the case where the bonding functions are mappings, we can draw the stronger conclusion in Theorem 142 that the induced mapping is a homeomorphism even for directed sets D other than the set of positive integers. (See Theorem 165 of Section 2.10.) One crucial difference between inverse limits with upper semi-continuous bonding functions and those with (ordinary) bonding maps is that the map induced by identities (i.e., each map in the collection $\{\varphi_\beta \mid \beta \in E\}$ is the identity) need not be a homeomorphism. In Example 138 we saw that $\varprojlim \mathbf{f}$ and $\varprojlim \mathbf{f} \circ \mathbf{f}$ may not be homeomorphic. Example 143 not only produces another example demonstrating this phenomenon but also shows that the induced map need not be a homeomorphism when the bonding functions have values that are closed sets. In this example, $E = \{1, 3, 5, \dots\}$ and $\varphi_n = Id$ for each $n \in E$.

Example 143 (The Hurewicz continuum) *Let $G(\mathbf{f})$ be the union of the four straight line intervals joining the points $(0, 1/2)$ to $(1/2, 1)$, $(1/2, 1)$ to $(1, 1/2)$, $(1, 1/2)$ to $(1/2, 0)$, and $(1/2, 0)$ to $(0, 1/2)$. Then, for $\mathbf{g} = \mathbf{f} \circ \mathbf{f}$, $\varprojlim \mathbf{f}$ and $\varprojlim \mathbf{g}$ are not homeomorphic (see Figure 2.20).*

Proof. Note that the four arcs whose union is $G(\mathbf{f})$ form a diamond in I^2 . Label these arcs A_i for $i \in \{1, 2, 3, 4\}$ in a clockwise direction so that $A_1 \subseteq [0, 1/2] \times [1/2, 1]$ and $A_4 \subseteq [0, 1/2] \times [0, 1/2]$. Let $K = \varprojlim \mathbf{f}$. The set K contains a simple closed curve that is the union of the four arcs B_i for $i \in \{1, 2, 3, 4\}$ determined as follows. If $i = 2$ or $i = 4$, B_i is the set of all points

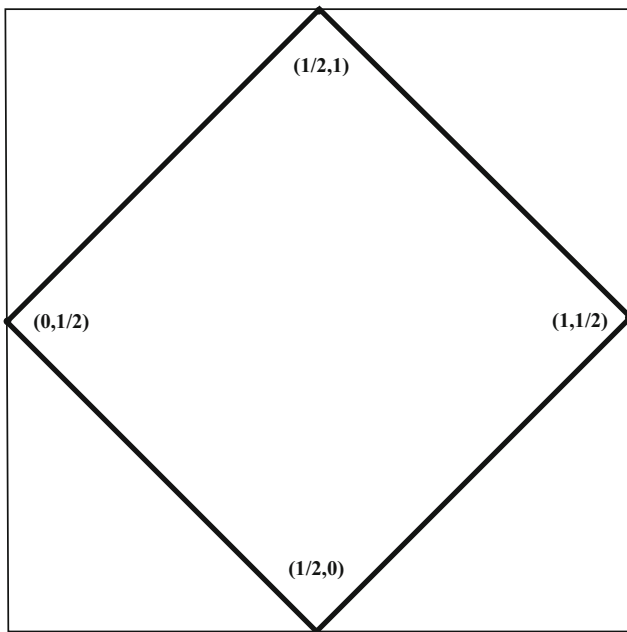


Fig. 2.20 *The function from Example 143*

$\mathbf{p} \in K$ such that for each n , $(p_{n+1}, p_n) \in A_i$. If $i = 1$ or 3 , then B_i is the set of all points $\mathbf{p} \in K$ such that, for each odd n , $(p_{n+1}, p_n) \in A_i$ and, for each even n , $(p_{n+1}, p_n) \in A_{i+2(mod 4)}$.

On the other hand, the graph of $g = f \circ f$ is the union of two arcs, one from $(0, 0)$ to $(1, 1)$ and the other from $(0, 1)$ to $(1, 0)$ and $\varprojlim g$ is homeomorphic to the cone over a Cantor set (see Example 132) so $\varprojlim f$ and $\varprojlim g$ are not homeomorphic. (See also Example 138 in Section 2.7 for another such example.)

Finally, let D denote the set of positive integers, E the set of odd positive integers, and φ_n the identity on $[0, 1]$ for each $n \in E$. The surjective induced map $\varphi : \varprojlim f \rightarrow \varprojlim f^2$ from Theorem 142 cannot be a homeomorphism. \square

Perhaps also of interest is the continuum $K = \varprojlim f$. K contains two mutually exclusive Cantor sets: C_0 consisting of all points \mathbf{p} of K such that $p_n = 1/2$ if n is even and C_1 consisting of all points \mathbf{p} of K such that $p_n = 1/2$ if n is odd. If \mathbf{a} is a point of C_0 and \mathbf{b} is a point of C_1 , then, for each n , there is an integer i_n where $1 \leq i_n \leq 4$ such that (a_{n+1}, a_n) and (b_{n+1}, b_n) are endpoints of the arc A_{i_n} . It can be shown that the set of all points \mathbf{x} of K

such that (x_{n+1}, x_n) is in A_{i_n} is an arc joining \mathbf{a} and \mathbf{b} and K is the union of all these arcs, no two of which have a point in common that is not an endpoint. This is the example of a universal continuum given by Hurewicz in [5]. In fact, Hurewicz showed that if C is a (metric) continuum then there exist a subcontinuum H of K and a monotone map of H onto C .

In the case where $D = E$, we are able to draw some stronger conclusions regarding the nature of the induced map from Theorem 141.

Theorem 144 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ and $\{Y_\alpha, g_{\alpha\beta}, D\}$ are inverse limit systems with upper semi-continuous bonding functions such that $\varprojlim \mathbf{f}$ is nonempty. Suppose further that for each $\alpha \in D$ there is a one-to-one mapping $\varphi_\alpha : X_\alpha \rightarrow Y_\alpha$ such that if $\alpha \preceq \beta$ in D then $\varphi_\alpha \circ f_{\alpha\beta} = g_{\alpha\beta} \circ \varphi_\beta$. Then, $\varphi = \Phi| \varprojlim \mathbf{f}$ is a one-to-one mapping of $\varprojlim \mathbf{f}$ into $\varprojlim \mathbf{g}$.*

Proof. By Theorem 141, φ is a mapping from $\varprojlim \mathbf{f}$ into $\varprojlim \mathbf{g}$ so we need only show that φ is one-to-one. Suppose $\varphi(\mathbf{x}) = \varphi(\mathbf{t})$. Then, if $\alpha \in D$, $\pi_\alpha(\varphi(\mathbf{x})) = \pi_\alpha(\varphi(\mathbf{t}))$. From this we conclude that $\varphi_\alpha(x_\alpha) = \varphi_\alpha(t_\alpha)$ for each $\alpha \in D$. Each φ_α is 1-1, thus $x_\alpha = t_\alpha$ for each $\alpha \in D$, consequently $\mathbf{x} = \mathbf{t}$. \square

Theorem 145 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ and $\{Y_\alpha, g_{\alpha\beta}, D\}$ are inverse limit systems with upper semi-continuous bonding functions such that $\varprojlim \mathbf{f}$ is nonempty. Suppose further that, for each $\alpha \in D$, $\varphi_\alpha : X_\alpha \rightarrow Y_\alpha$ is a one-to-one and surjective mapping such that if β is in D and $\alpha \preceq \beta$ then $\varphi_\alpha \circ f_{\alpha\beta} = g_{\alpha\beta} \circ \varphi_\beta$ and $\varphi = \Phi| \varprojlim \mathbf{f}$. Then, $\varphi : \varprojlim \mathbf{f} \rightarrow \varprojlim \mathbf{g}$ is one-to-one and surjective.*

Proof. By Theorem 141 with $E = D$ and for each $\alpha \in D$, $\sigma(\alpha) = \alpha$, $\varprojlim \mathbf{g}$ is nonempty. By Theorem 144 φ is a 1-1 mapping of $\varprojlim \mathbf{f}$ into $\varprojlim \mathbf{g}$, so that we only have to check that φ is surjective. Let \mathbf{y} be the function from $\varprojlim \mathbf{g}$ to $\varprojlim \mathbf{f}$ induced by $\{\varphi_\alpha^{-1} | \alpha \in D\}$ as given by Theorem 140. Suppose that \mathbf{y} is in $\varprojlim \mathbf{g}$ and let $\mathbf{x} = \psi(\mathbf{y})$. Then $\mathbf{x} \in \varprojlim \mathbf{f}$ and $\varphi(\mathbf{x}) = \mathbf{y}$ because if $\alpha \in D$, $\pi_\alpha(\mathbf{x}) = \varphi_\alpha(x_\alpha) = \varphi_\alpha(\varphi_\alpha^{-1}(y_\alpha))$. \square

Corollary 146 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ and $\{Y_\alpha, g_{\alpha\beta}, D\}$ are inverse limit systems on compact Hausdorff spaces with upper semi-continuous bonding functions. Suppose further that, for each $\alpha \in D$, $\varphi_\alpha : X_\alpha \rightarrow Y_\alpha$ is a homeomorphism and if β is in D and $\alpha \preceq \beta$ then $\varphi_\alpha \circ f_{\alpha\beta} = g_{\alpha\beta} \circ \varphi_\beta$ and $\varphi = \Phi| \varprojlim \mathbf{f}$. Then, $\varphi : \varprojlim \mathbf{f} \rightarrow \varprojlim \mathbf{g}$ is a homeomorphism.*

As an application of Corollary 146 we present the following example. We begin with a lemma.

Lemma 147 *Let g be the mapping from $[0, 1]$ onto $[0, 1]$ whose graph consists of two straight line intervals, one from $(0, 0)$ to $(1/2, 1)$ and the other from $(1/2, 1)$ to $(1, 1/2)$. Suppose $f : [a, b] \rightarrow [a, b]$ is a mapping of the interval*

$[a, b]$ onto itself and c is a point of the open interval (a, b) such that $f(a) = a, f(c) = b, f(b) = c$, and $h = f|_{[a, c]}$ and $k = f|_{[c, b]}$ are homeomorphisms. If φ is a homeomorphism of $[0, 1]$ onto $[a, b]$ such that $\varphi(0) = a, \varphi(1/2) = c$, and $\varphi(1) = b$ then there is a homeomorphism $\psi : [0, 1] \rightarrow [a, b]$ such that $\psi(0) = a, \psi(1/2) = c, \psi(1) = b$, and $f \circ \psi = \varphi \circ g$.

Proof. Let $\psi(x) = h^{-1}(\varphi(g(x)))$ if $0 \leq x \leq 1/2$ and $\psi(x) = k^{-1}(\varphi(g(x)))$ if $1/2 \leq x \leq 1$. Because $g(1/2) = 1, \varphi(1) = 1$, and $h^{-1}(1) = k^{-1}(1) = 1/2$, we see that ψ is a mapping. Suppose x and y are in $[0, 1]$ and $x \neq y$. If one of x and y is in $[0, 1/2]$ and the other is in $(1/2, 1]$, then by definition $\psi(x) \neq \psi(y)$. If both are in $[0, 1/2]$ or both are in $(1/2, 1]$ then again $\psi(x) \neq \psi(y)$ inasmuch as $g(x) \neq g(y)$. Thus, ψ is one-to-one and therefore is a homeomorphism. \square

Example 148 Suppose $f : [a, b] \rightarrow [a, b]$ is a mapping of the interval $[a, b]$ onto itself and c is a point of the open interval (a, b) such that $f(a) = a, f(c) = b$, and $f(b) = c$ and $f|_{[a, c]}$ is a homeomorphism as is $f|_{[c, b]}$. Then, $\varprojlim \mathbf{f}$ is homeomorphic to the closure of the graph of $y = \sin(1/x)$ on $(0, 1]$.

Proof. Let $g : [0, 1] \rightarrow [0, 1]$ be the map whose graph consists of two straight line intervals, one from $(0, 0)$ to $(1/2, 1)$ and the other from $(1/2, 1)$ to $(1, 1/2)$. In Chapter 1 we showed that $\varprojlim \mathbf{g}$ is homeomorphic to the closure of the graph of $y = \sin(1/x)$ on $(0, 1]$. We now show that $\varprojlim \mathbf{f}$ and $\varprojlim \mathbf{g}$ are homeomorphic.

Let φ_1 be the homeomorphism of $[0, 1]$ onto $[a, b]$ such that the graph of φ consists of two straight line intervals, one from $(0, a)$ to $(1/2, c)$ and the other from $(1/2, c)$ to $(1, b)$. Inductively, suppose $\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n$ have been defined so that for $1 \leq i \leq n$ we have $\varphi_i(0) = a, \varphi_i(1/2) = c, \varphi_i(1) = b$, and, if $i > 1, \varphi_{i-1} \circ g = f \circ \varphi_i$. By Lemma 147 there is a homeomorphism φ_{n+1} such that $\varphi_{n+1}(0) = a, \varphi_{n+1}(1/2) = c, \varphi_{n+1}(1) = b$, and $\varphi_n \circ g = f \circ \varphi_{n+1}$. With the sequence φ thus defined it follows from Corollary 146 with D as the set of positive integers that $\varprojlim \mathbf{f}$ and $\varprojlim \mathbf{g}$ are homeomorphic. \square

Suppose X is a compact Hausdorff space. If $f : X \rightarrow 2^X$ and $g : X \rightarrow 2^X$ are upper semi-continuous functions, f and g are *topologically conjugate* provided there is a homeomorphism h such that $h(X) = X$ and $h \circ f = g \circ h$. We conclude this section with a theorem that provides sufficient conditions under which inverse limit sequences with topologically conjugate bonding functions produce homeomorphic inverse limits.

Theorem 149 Suppose D is the set of positive integers and X is a compact Hausdorff space such that for each i in $D, X_i = X$. If $f : X \rightarrow 2^X$ and $g : X \rightarrow 2^X$ are topologically conjugate upper semi-continuous functions, then $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{g}$.

Proof. X is a compact Hausdorff space and D is the set of positive integers, thus Theorem 111 gives that $\varprojlim \mathbf{f}$ and $\varprojlim \mathbf{g}$ are not empty. There is a homeomorphism $h : X \rightarrow X$ such that $h(X) = X$ and $h \circ f = g \circ h$. Let each map $\varphi_i = h$ and let $\varphi : \varprojlim \mathbf{f} \rightarrow \varprojlim \mathbf{g}$ be the mapping induced by $\varphi_1, \varphi_2, \varphi_3, \dots$. By Theorem 145, φ is 1-1 and surjective. Because φ is a 1-1 mapping from a compact space onto a Hausdorff space, φ is a homeomorphism. \square

2.9 Upper semi-continuous functions that are unions of functions

In this section we consider an interesting class of upper semi-continuous functions, those whose graphs are unions of the graphs of (set-valued) functions. We are primarily concerned with conditions that ensure that inverse limits of inverse limit sequences with such functions as bonding functions are connected. Of course, without some conditions on the functions, the inverse limit may not be connected because the union of the mapping that is identically 0 on $[0, 1]$ with the mapping that is identically 1 on $[0, 1]$ yields a Cantor set for its inverse limit; see Example 2.3. Although the subject of this section is of interest in and of itself, some who are looking at applications of inverse limits in economics have asked about the nature of inverse limits with upper semi-continuous functions that are unions of mappings.

If $f : X \rightarrow 2^Y$ and $g : X \rightarrow 2^Y$ are set-valued functions, we say that f and g have a *coincidence point* provided there is a point x of X such that $f(x) \cap g(x) \neq \emptyset$.

Lemma 150 *Suppose X_1, X_2, X_3, \dots is a sequence of compact Hausdorff spaces and $f_i : X_{i+1} \rightarrow 2^{X_i}$ is an upper semi-continuous function for each positive integer i . If n is a positive integer, $g : X_{n+1} \rightarrow 2^{X_n}$ is an upper semi-continuous function such that f_n and g have a coincidence point, f_i is surjective for each $i \geq n$, and φ is a sequence of functions such that $\varphi_i = f_i$ for $i \neq n$ and $\varphi_n = g$, then $\varprojlim \mathbf{f}$ and $\varprojlim \varphi$ have a point in common.*

Proof. Inasmuch as f_n and g have a coincidence point, there are points t of X_{n+1} and z of X_n such that $z \in f_n(t) \cap g(t)$. Because f_i is surjective for each $i > n$, by Theorem 112 there is a point \mathbf{x} of $\varprojlim \mathbf{f}$ such that $x_{n+1} = t$ and $x_n = z$. Because $z \in g(t)$, \mathbf{x} is in $\varprojlim \varphi$. \square

Suppose X and Y are compact Hausdorff spaces and \mathcal{F} is a collection of set-valued functions from X into 2^Y . A function $f \in \mathcal{F}$ is said to be *universal with respect to \mathcal{F}* provided f has a coincidence point with each member of \mathcal{F} . Recall that $C(X)$ denotes the connected elements of 2^X .

Theorem 151 *If \mathcal{F} is a collection of upper semi-continuous functions of a continuum X into $C(X)$ one of which is surjective and universal with respect to \mathcal{F} and f is a closed subset of $X \times X$ that is the set-theoretic union of the graphs of the functions in the collection \mathcal{F} , then $f : X \rightarrow 2^X$ is an upper semi-continuous function such that, if $f_i = f$ for each positive integer i , then $\varprojlim f$ is a continuum.*

Proof. f is a closed subset of $X \times X$ and each point of X is a first coordinate of some point of f , therefore f is upper semi-continuous. Because $\varprojlim f$ is compact, we only need to show that this inverse limit is connected. Suppose $f_1 \in \mathcal{F}$ and f_1 is surjective and universal with respect to \mathcal{F} . Choose a point $\mathbf{x} \in \varprojlim f_1$ and let $\mathbf{y} \in \varprojlim f$. There exists a sequence $\varphi_1, \varphi_2, \varphi_3, \dots$ such that $\varphi_i \in \mathcal{F}$ and $y_i \in \varphi_i(y_{i+1})$ for each positive integer i . Let $C_1 = \varprojlim f_1$, and, if n is an integer with $n > 1$, let C_n be the inverse limit of the sequence $\varphi_1, \varphi_2, \dots, \varphi_{n-1}, f_1, f_1, f_1, \dots$. For each n , C_n is a continuum by Theorem 126 and, by Lemma 150, $C_n \cap C_{n+1} \neq \emptyset$. Thus, $\bigcup_{i>0} C_i$ is connected. Moreover, for each n , because f_1 is surjective there is a point \mathbf{p}^n of C_n such that $\pi_i(\mathbf{p}^n) = y_i$ for $i \leq n$. It follows that $\mathbf{y} \in \overline{C}$ and because $\mathbf{x} \in C_1$, $\varprojlim f$ is the union of a collection of continua all containing \mathbf{x} . Thus $\varprojlim f$ is connected. \square

The set-theoretic union of a finite collection of mappings of $[0, 1]$ into itself is a closed subset of $[0, 1] \times [0, 1]$, thus we have the following corollary to Theorem 151. In Section 2.12 we show that the continuum that results in Theorem 152 is one-dimensional (see Theorem 185).

Theorem 152 *If \mathcal{F} is a finite collection of mappings from $[0, 1]$ into itself one of which is surjective, f is the set-theoretic union of the maps in \mathcal{F} , and $f_i = f$ for each positive integer i , then $\varprojlim f$ is a one-dimensional continuum.*

Just after Example 137 we observed that the preceding theorem does not hold if \mathcal{F} is allowed to contain upper semi-continuous functions. In fact, as we show in our next example, there is a two-element collection \mathcal{F} consisting of one upper semi-continuous function having a connected inverse limit and one mapping that has a union with a nonconnected inverse limit.

Example 153 (An upper semi-continuous function and a map whose union produces a nonconnected inverse limit) *Let g_1 be the function from Example 136 and g_2 be the piecewise linear mapping passing through $(0, 1)$, $(3/4, 1/4)$, $(7/8, 1/2)$, and $(1, 0)$ and let $\mathcal{F} = \{g_1, g_2\}$. If $f = G(g_1) \cup g_2$, $\varprojlim f$ is not connected (see Figure 2.21 for the graph of f).*

By an argument virtually identical to that provided in Example 114 it may be shown that if f is the upper semi-continuous function whose graph is the

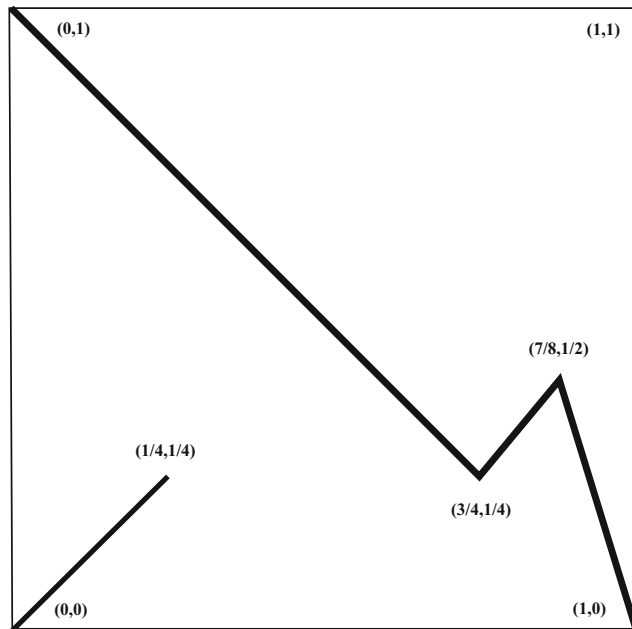


Fig. 2.21 The function from Example 153

set-theoretic union of the graphs of g_1 and g_2 , then $\varprojlim \mathbf{f}$ is not connected. In fact $\{\mathbf{x} \in \varprojlim \mathbf{f} \mid x_1 = x_2 = 1/4, x_3 = 3/4\}$ is both open and closed in the inverse limit. This inverse limit contains $(I \times \{1\} \times \{0\})^\infty$.

For the remainder of this section we consider unions of mappings. We are interested in inverse limits with upper semi-continuous functions on $[0, 1]$ that are unions of finitely many mappings that are not necessarily surjective such as the function in Example 143 (Figure 2.20). First we prove a lemma that is of use in the proof of Theorem 155.

Lemma 154 *If $f : [0, 1] \rightarrow [0, 1]$ is a mapping from $[0, 1]$ into itself such that $f^2([0, 1]) = f([0, 1])$, $t \in f([0, 1])$, and $f_i = f$ for each positive integer i , then there is a point $\mathbf{x} \in \varprojlim \mathbf{f}$ such that $x_1 = t$.*

Proof. Let t be a point of $f([0, 1])$ and let $x_1 = t$. Because $f(f([0, 1])) = f([0, 1])$ and $t \in f([0, 1])$ there is a point x_2 of $f([0, 1])$ such that $f(x_2) = x_1$. Similarly, because x_2 is in $f([0, 1]) = f^2([0, 1])$, there is a point $x_3 \in f([0, 1])$ such that $f(x_3) = x_2$. Continuing in this manner we obtain a point $\mathbf{x} \in \varprojlim \mathbf{f}$ such that $x_1 = t$. \square

Theorem 155 *Suppose \mathcal{F} is a finite collection of mappings from $[0, 1]$ into itself that contains a mapping f_1 with the following properties.*

1. $f_1([0, 1])$ is nondegenerate
2. If $g \in \mathcal{F}$ there is a point $p_g \in f_1([0, 1])$ such that $f_1(p_g) = g(p_g)$
3. If $g \in \mathcal{F}$ then $g(f_1([0, 1])) = g([0, 1])$.

If f is the set-theoretic union of all the elements of \mathcal{F} and $f_i = f$ for each positive integer i , then $\varprojlim \mathbf{f}$ is a one-dimensional continuum.

Proof. Choose a point \mathbf{y} in $\varprojlim \mathbf{f}$. There exists a sequence $\varphi_1, \varphi_2, \varphi_3, \dots$ such that φ_i is in \mathcal{F} and $\varphi_i(y_{i+1}) = y_i$ for each positive integer i . Let C_1 be the inverse limit of the sequence f_1, f_1, f_1, \dots and if n is an integer greater than one, let C_n be the inverse limit of the sequence $\varphi_1, \varphi_2, \dots, \varphi_{n-1}, f_1, f_1, \dots$. Using condition (2), it follows from Lemma 150 that C_i and C_{i+1} have a point in common for each positive integer i . Thus, $C_1 \cup C_2 \cup C_3 \cup \dots$ is connected. Using Lemma 154 with $t = y_n$ there is a point \mathbf{x} of $\varprojlim \mathbf{f}_1$ such that $x_1 = y_n$. The point $\mathbf{p}_n = (y_1, y_2, \dots, y_n, x_2, x_3, \dots)$ belongs to C_n and the distance from \mathbf{y} to \mathbf{p}_n is less than $1/2^n$. Thus, \mathbf{y} belongs to $\text{Cl}(C_1 \cup C_2 \cup C_3 \cup \dots)$. Each point of $\varprojlim \mathbf{f}$ belongs to a continuum lying in $\varprojlim \mathbf{f}$ that contains the continuum $\varprojlim \mathbf{f}_1$, therefore $\varprojlim \mathbf{f}$ is a continuum.

From condition (1) and Lemma 154 it follows that $\varprojlim \mathbf{f}_1$ is nondegenerate. Thus, the dimension of $\varprojlim \mathbf{f}$ is one by Theorem 185. \square

Theorem 156 *If $f : [0, 1] \rightarrow 2^{[0, 1]}$ is an upper semi-continuous function that is the union of a finite collection \mathcal{F} of mappings from $[0, 1]$ into itself one of which is surjective and $f_i = f$ for each positive integer i , then $\varprojlim \mathbf{f}$ is a one-dimensional continuum that contains a copy of every inverse limit $\varprojlim \mathbf{g}$ where $g_i \in \mathcal{F}$ for each i .*

Proof. Theorem 152 yields that $\varprojlim \mathbf{f}$ is a continuum. It is easy to see that $\varprojlim \mathbf{f}$ contains a copy of every inverse limit $\varprojlim \mathbf{g}$ where $g_i \in \mathcal{F}$ for each i . That the dimension of the inverse limit is one follows from Theorem 185 from the final section of this chapter. \square

Richard M. Schori [11] constructed a chainable continuum that contains a copy of every chainable continuum. Although Schori's result is stronger, we still observe the following.

Corollary 157 *There exists an upper semi-continuous function f such that if $f_i = f$ for each positive integer i , then $\varprojlim \mathbf{f}$ is a one-dimensional continuum that contains a copy of every chainable continuum.*

Proof. There exist two mappings $\varphi : [0, 1] \rightarrow [0, 1]$ and $\psi : [0, 1] \rightarrow [0, 1]$ such that if M is a chainable continuum then there exists a sequence \mathbf{k} such that $k_i \in \{\varphi, \psi\}$ for each i and M is homeomorphic to $\varprojlim \mathbf{k}$, [2] or [12]. Let $f = \varphi \cup \psi$ and apply Theorem 156. \square

2.10 Theorems for inverse limit systems with bonding functions that are mappings

In this section we consider inverse limit systems in which the bonding functions are mappings and contrast some of these results with the fact that they fail for systems with upper semi-continuous bonding functions. Earlier in this chapter we established the inverse limit is a nonempty compact Hausdorff space for any inverse limit system $\{X_\alpha, f_{\alpha\beta}, D\}$ where each X_α is a nonempty compact Hausdorff space and each f_α is a mapping; see Theorems 107 and 111. Furthermore, in Theorem 117 we showed that the inverse limit of a system with mappings is a Hausdorff continuum when each factor space is a Hausdorff continuum.

2.10.1 A basis for the topology

One useful feature of inverse limits of systems of mappings lies in the fact that a collection of open sets that looks as if it were only a subbasis for the topology of the inverse limit is, in fact, a basis for the topology of the inverse limit. We see this in our next theorem. In its proof we employ the convention that the domain of π_α is the inverse limit space.

Theorem 158 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system where each $f_{\alpha\beta}$ is a mapping and $M = \varprojlim \mathbf{f}$ is nonempty. Then, $\mathcal{B} = \{\pi_\alpha^{-1}(O) \mid \alpha \in D \text{ and } O \text{ is an open subset of } X_\alpha\}$ is a basis for the topology for $\varprojlim \mathbf{f}$.*

Proof. Choose an element R of the usual basis for the topology of the product space \mathbf{I} and let \mathbf{x} be an element of $R \cap M$. Then $R = \prod_{\alpha \in D} U_\alpha$ where each factor is open and $U_\alpha = X_\alpha$ except for finitely many elements of D , say $\alpha_1, \alpha_2, \dots, \alpha_n$. There is a member α of D such that $\alpha_i \preceq \alpha$ for $1 \leq i \leq n$. Because $f_{\alpha_i \alpha}$ is a mapping for each i , there is an open subset O of X_α containing x_α such that $f_{\alpha_i \alpha}(O) \subseteq U_{\alpha_i}$ for $1 \leq i \leq n$. Then, $\pi_\alpha^{-1}(O)$ contains \mathbf{x} and is a subset of $R \cap M$. It follows that \mathcal{B} is a basis for the topology of $\varprojlim \mathbf{f}$. \square

2.10.2 Closed subsets

Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system and $M = \varprojlim \mathbf{f}$ is nonempty. If H is a closed subset of M , we denote $\pi_\alpha(H)$ by H_α . For convenience, if

$f : X \rightarrow 2^Y$ and $A \subset X$ and no confusion should arise, we may denote $f|_A$ by f .

Theorem 159 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system such that each $f_{\alpha\beta}$ is a mapping and $M = \varprojlim \mathbf{f}$ is nonempty. Then, if H is a closed subset of M , H is the inverse limit of the inverse limit system $\{H_\alpha, g_{\alpha\beta}, D\}$ where $g_{\alpha\beta} = f_{\alpha\beta}|_{H_\beta}$.*

Proof. It is immediate that $H \subseteq \varprojlim \{H_\alpha, g_{\alpha\beta}, D\}$. On the other hand, suppose $\mathbf{x} \in \varprojlim \{H_\alpha, g_{\alpha\beta}, D\}$. Then, $x_\alpha \in H_\alpha$ for each $\alpha \in D$. Therefore, if $\alpha \in D$ and O_α is an open subset of X_α such that $\mathbf{x} \in \pi_\alpha^{-1}(O_\alpha)$, there is a point \mathbf{p} of H such that $p_\alpha = x_\alpha$. Thus, $\mathbf{p} \in \pi_\alpha^{-1}(O_\alpha)$. Each basis element containing \mathbf{x} contains a point of H , therefore $\mathbf{x} \in \overline{H}$. Because H is closed, $\mathbf{x} \in H$. \square

Corollary 160 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system such that each $f_{\alpha\beta}$ is a mapping and $M = \varprojlim \mathbf{f}$ is nonempty. Then, if H is a closed subset of M such that $H_\alpha = X_\alpha$ for each $\alpha \in D$, we have $H = M$.*

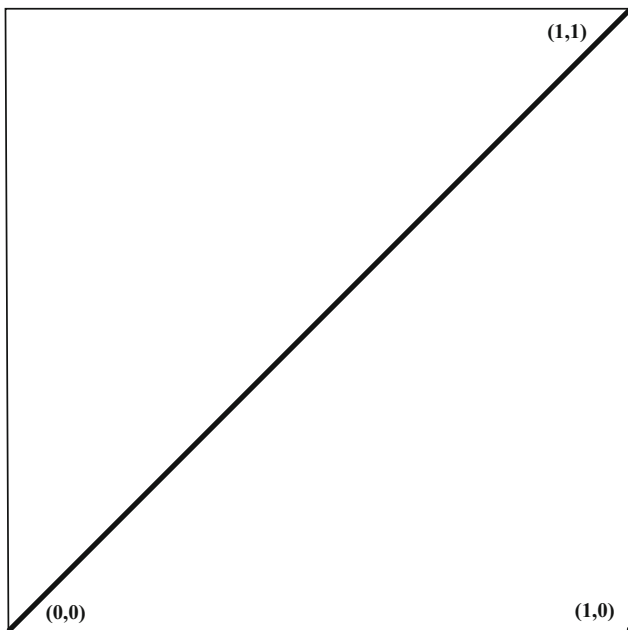


Fig. 2.22 *The function from Example 161*

2.10.3 Closed subsets of a system with upper semi-continuous bonding functions

That each of Theorem 158, Theorem 159, and Corollary 160 fail to hold for inverse limit sequences using upper semi-continuous bonding functions may be seen from the following example. We specify the upper semi-continuous bonding function by means of its graph.

Example 161 Let D be the set of positive integers and let $G(f)$ be the subset of $[0, 1] \times [0, 1]$ containing the point $(1, 0)$ and the line joining $(0, 0)$ and $(1, 1)$ (see Figure 2.22).

Proof. Let $K = \varprojlim \mathbf{f}$. Without proof, we note that K is the union of an arc $A = \{(t, t, t, \dots) \mid t \in [0, 1]\}$ and a sequence of points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots$ not in A converging to $(0, 0, 0, \dots)$ where \mathbf{p}_i has its first i coordinates 0 and all other coordinates 1.

To see that Theorem 158 fails for K , one needs only to note that the point $(0, 1, 1, 1, \dots)$ of K belongs to $R = [0, 1/4] \times (3/4, 1] \times \mathcal{Q}$ but R fails to contain any set of the form $\pi_i^{-1}(O)$ where O is open in $[0, 1]$. Indeed, if O is an open subset of $[0, 1]$, O contains a point t of $[0, 1]$ not in both $[0, 1/4]$ and $(3/4, 1]$. If i is a positive integer (t, t, t, \dots) is a point of $\pi_i^{-1}(O)$ that is not in R .

Furthermore, $H = \{(t, t, t, \dots) \in M \mid t \in [0, 1]\}$ is a closed proper subset of K such that $H_i = [0, 1]$ for each positive integer i so Theorem 159 and Corollary 160 fail for f . \square

2.10.4 Intersections of closed subsets of the inverse limit

Another interesting property of inverse limit systems with mappings is found in the following theorem. As before, if H is a closed subset of $\varprojlim \mathbf{f}$, we denote $\pi_\alpha(H)$ by H_α .

Theorem 162 Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system such that each $f_{\alpha\beta}$ is a mapping, and H and K are closed subsets of $\varprojlim \mathbf{f}$. Then, $H \cap K = \varprojlim \{H_\alpha \cap K_\alpha, f_{\alpha\beta}[(H_\beta \cap K_\beta), D]\}$.

Proof. Because $\pi_\alpha(H \cap K) \subseteq H_\alpha \cap K_\alpha$ for each $\alpha \in D$, $H \cap K \subseteq \varprojlim \{H_\alpha \cap K_\alpha, f_{\alpha\beta}[(H_\beta \cap K_\beta), D]\}$. On the other hand, if $\mathbf{x} \in \varprojlim \{H_\alpha \cap K_\alpha, f_{\alpha\beta}[(H_\beta \cap K_\beta), D]\}$ then $x_\alpha \in H_\alpha \cap K_\alpha$ for each $\alpha \in D$ so $\mathbf{x} \in \varprojlim \{H_\alpha, f_{\alpha\beta}[(H_\beta), D]\} = H$. Similarly, $\varprojlim \{H_\alpha \cap K_\alpha, f_{\alpha\beta}[(H_\beta \cap K_\beta), D]\} \subseteq K$. \square

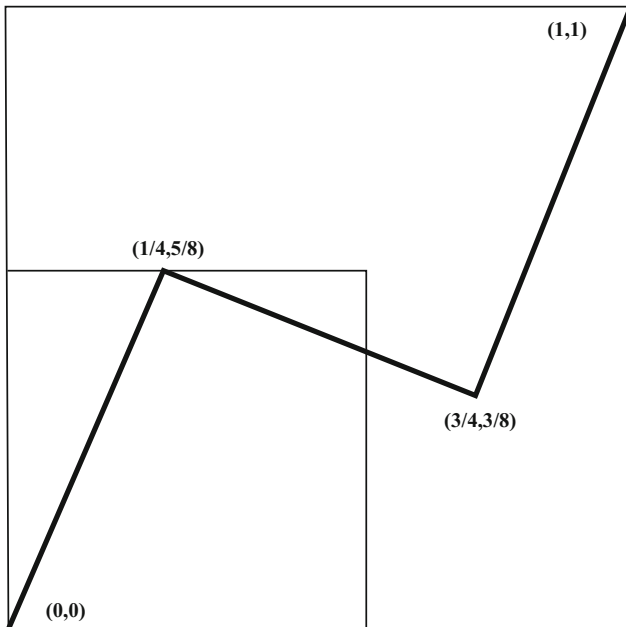


Fig. 2.23 The mapping from Example 163

In the previous theorem, $\pi_\alpha(H \cap K)$ is not necessarily equal to $H_\alpha \cap K_\alpha$ for any $\alpha \in D$ even if D is the set of positive integers. An example of this phenomenon follows.

Example 163 Let $f : [0, 1] \rightarrow [0, 1]$ be the piecewise linear map passing through the points $(0, 0)$, $(1/4, 5/8)$, $(3/4, 3/8)$, and $(1, 1)$. Then, $\varprojlim \mathbf{f}$ is the union of two arcs $H = \varprojlim \{[0, 5/8], f|_{[0, 5/8]}\}$ and $K = \varprojlim \{[3/8, 1], f|_{[3/8, 1]}\}$. Then, $H \cap K = (1/2, 1/2, 1/2, \dots)$ but $H_i \cap K_i = [7/16, 9/16]$ for each positive integer i (see Figure 2.23).

Proof. The maps $f|_{[0, 5/8]}$ and $f|_{[3/8, 1]}$ are topologically conjugate, so H and K are homeomorphic by Theorem 149. To see that H is an arc, we employ Theorem 39 from Chapter 1 to observe that H is the closure of a ray with remainder $\varprojlim \{[1/4, 5/8], f|_{[1/4, 5/8]}\}$. The slope of f on $[1/4, 5/8]$ is $-1/2$, thus $\bigcap_{n>0} \tilde{f}^n([1/4, 5/8]) = \{1/2\}$. It follows from Theorem 113 that the remainder is a single point. \square

2.10.5 The subsequence theorem

One of the fundamental tools in analyzing the nature of inverse limits of inverse limit systems using mappings as bonding functions is the subsequence theorem, Theorem 166. Unfortunately, this theorem does not hold for general inverse limit systems with upper semi-continuous bonding functions under study earlier in the present chapter (see Example 143). Here we state and prove a general version of the subsequence theorem for inverse limit systems in which the bonding functions are mappings.

Theorem 164 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system over a directed set D such that the inverse limit is nonempty, each $f_{\alpha\beta}$ is a mapping, and E is a cofinal subset of D . If $\{Y_\beta, f_{\alpha\beta}, E\}$ is the restriction of $\{X_\alpha, f_{\alpha\beta}, E\}$ to E , then there is a one-to-one mapping from $\varprojlim \mathbf{f}$ onto $\varprojlim \mathbf{g}$.*

Proof. For each α in E , let φ_α denote the identity on X_α . By Theorem 141 the function φ induced by $\{\varphi_\alpha \mid \alpha \in E\}$ is a mapping of $\varprojlim \mathbf{f}$ into $\varprojlim \mathbf{g}$. Suppose $\mathbf{y} \in \varprojlim \mathbf{g}$ and $\gamma \in D$. We now construct a point \mathbf{x} of $\varprojlim \mathbf{f}$ so that $\varphi(\mathbf{x}) = \mathbf{y}$. If $\gamma \in E$ let $x_\gamma = y_\gamma$. Suppose γ is not in E . There exists an element δ of E such that $\gamma \preceq \delta$. Let $x_\gamma = f_{\gamma\delta}(y_\delta)$. Every bonding function is a mapping, thus the choice of x_γ is independent of the choice of δ and it is not difficult to check that \mathbf{x} is in $\varprojlim \mathbf{f}$ so φ is surjective. To see that φ is 1-1, suppose each of \mathbf{x} and \mathbf{t} is in $\varprojlim \mathbf{f}$ and $\varphi(\mathbf{x}) = \varphi(\mathbf{t})$. Let α be a member of D . There is a member β of E such that $\alpha \preceq \beta$. Because $\beta \in E$, $x_\beta = t_\beta$, so $f_{\alpha\beta}(x_\beta) = f_{\alpha\beta}(t_\beta)$. Thus, $x_\alpha = t_\alpha$ for each $\alpha \in D$, so $\mathbf{x} = \mathbf{t}$. \square

Theorem 165 *Suppose D is a directed set and X_α is a compact Hausdorff space for each $\alpha \in D$. Suppose further $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system where each $f_{\alpha\beta}$ is a mapping, and E is a cofinal subset of D . Then $\varprojlim \{X_\alpha, f_{\alpha\beta}, D\}$ and $\varprojlim \{X_\alpha, f_{\alpha\beta}, E\}$ are homeomorphic.*

Proof. The 1-1 mapping φ from Theorem 164 is a homeomorphism because $\varprojlim \{X_\alpha, f_{\alpha\beta}, D\}$ is compact and $\varprojlim \{X_\alpha, f_{\alpha\beta}, E\}$ is Hausdorff. \square

Of course, if D is the set of positive integers, any increasing sequence of positive integers n_1, n_2, n_3, \dots is cofinal in D . This observation leads to a restatement of Theorem 165. Although it is merely a restatement of the previous theorem in the specific case that D is the set of positive integers, it is of enough value to the theory of inverse limits to merit a separate statement. We recall the following standard notation. If \mathbf{X} is a sequence of spaces, \mathbf{f} is a sequence of mappings such that $f_i : X_{i+1} \rightarrow X_i$, and $i < j$, then $f_{ij} : X_j \rightarrow X_i = f_i \circ f_{i+1} \circ \dots \circ f_{j-1}$ and f_{ii} is the identity on X_i .

Theorem 166 (The subsequence theorem) *Suppose D is the set of positive integers and n_1, n_2, n_3, \dots is an increasing sequence of positive integers. Suppose further that \mathbf{X} is a sequence of compact Hausdorff spaces and \mathbf{f} is*

a sequence of mappings such that $f_i : X_{i+1} \rightarrow X_i$ for each i . Let \mathbf{g} be the sequence of maps such that $g_i = f_{n_i n_{i+1}}$ for each i . Then, $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{g}$.

We end this subsection with an application of Theorem 166. We employ the following terminology in the next theorem. If $f : X \rightarrow Y$ is a mapping, we say that f *factors* through Z if there are maps $g : X \rightarrow Z$ and $h : Z \rightarrow Y$ such that $f = h \circ g$.

Theorem 167 *Suppose $\{X_i, f_i\}$ is an inverse limit sequence such that, for each positive integer i , f_i is a mapping and n_1, n_2, n_3, \dots is an increasing sequence of positive integers such that $f_{n_i n_{i+1}}$ factors through $[0, 1]$ for each i . Then $M = \varprojlim \mathbf{f}$ is homeomorphic to an inverse limit on $[0, 1]$; that is, M is a chainable continuum.*

Proof. By Theorem 166, M is homeomorphic to $\varprojlim \mathbf{g}$ where $g_i = f_{n_i n_{i+1}}$ for each i . Because $f_{n_i n_{i+1}}$ factors through $[0, 1]$, there exist maps $\psi_i : [0, 1] \rightarrow X_{n_i}$ and $\varphi_i : X_{n_{i+1}} \rightarrow [0, 1]$ such that $f_{n_i n_{i+1}} = \psi_i \circ \varphi_i$. It follows from Theorem 166 that M is homeomorphic to $\varprojlim \{Y_i, h_i\}$ where $Y_i = X_{n_{(i+1)/2}}$ and $h_i = \psi_{(i+1)/2}$ if i is odd, and $Y_i = [0, 1]$ and $h_i = \varphi_{i/2}$ if i is even. One final application of Theorem 166 yields that $\varprojlim \{Y_i, h_i\}$ is homeomorphic to $\varprojlim \mathbf{k}$ where $k_i = \varphi_i \circ \psi_{i+1}$, a map from $[0, 1]$ to $[0, 1]$. \square

Of course, a more general theorem than the one stated here for $[0, 1]$ holds, but this theorem illustrates a way that the subsequence theorem can be used.

2.10.6 Other induced homeomorphisms

Other important consequences of Theorem 165 are found in the next two theorems.

Theorem 168 (The shift homeomorphism) *Suppose $\{X_i, f_i, D\}$ is an inverse limit sequence over the set of positive integers where, for each i , X_i is a compact Hausdorff space and f_i is a mapping. Let $E = D - \{1\}$. Then, $h : \varprojlim \{X_i, f_i, D\} \rightarrow \varprojlim \{X_i, f_i, E\}$ given by $h(\mathbf{x}) = (x_2, x_3, x_4, \dots)$ is a homeomorphism.*

The homeomorphism h from Theorem 168 is called the *shift* homeomorphism. In dynamics, one reason for interest in inverse limits is that by passing to the inverse limit, one is able to replace a dynamical system consisting of a topological space and a continuous function with a (possibly more complicated) space (the inverse limit) and a homeomorphism (the shift). In the case

where each factor space X_i is the same topological space X and each bonding map f_i is the same map f , the inverse of the shift homeomorphism is given by $h^{-1}(\mathbf{x}) = (f(x_1), x_1, x_2, \dots)$. The map h^{-1} is also called the shift homeomorphism by many authors (consequently, in each instance we have tried to make it clear which of these “shifts” we are using as we did in Theorem 20 in Chapter 1). It is interesting to note that, in this case, h^{-1} is induced by a sequence of mappings that are not necessarily 1-1, $\varphi_i = f$ for each i .

Our next theorem is another valuable tool in analyzing inverse limits with a constant sequence of factor spaces and a constant sequence of bonding maps, the so-called inverse limits with a single bonding map. In the case where we have an inverse limit with a single bonding map, we denote the inverse limit sequence by $\{X, f\}$ and the inverse limit by $\varprojlim \mathbf{f}$. If $f : X \rightarrow X$ is a mapping, $f^n : X \rightarrow X$ denotes the n -fold composition of f with itself.

Theorem 169 *Suppose X is a compact Hausdorff space and $f : X \rightarrow X$ is a mapping. If n is a positive integer, $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{f}^n$.*

Proof. Let $n_1 = 1$ and $n_{k+1} = n_k + n$ for $k = 1, 2, 3, \dots$ and apply Theorem 166. \square

2.10.7 Inverse limits as sequential limiting sets

Suppose \mathbf{M} is a sequence of sets in a topological space. By the *limiting set* (or *lim sup*) of the sequence is meant the set to which the point P belongs if and only if it is true that if U is an open set containing P then U contains a point of M_i for infinitely many integers i . By the *sequential limiting set* of the sequence is meant the set that is the limiting set of every subsequence of \mathbf{M} .

Theorem 170 *Suppose D is the set of positive integers and $\{X_i, f_{ij}, D\}$ is an inverse limit sequence where each X_i is a compact Hausdorff space and each map f_{ij} is surjective. Let \mathbf{p} be a point of $\mathbf{\Pi} = \prod_{i>0} X_i$ and, for each positive integer n , let $h_n : X_n \rightarrow \mathbf{\Pi}$ be given by $\pi_i(h_n(x)) = f_{in}(x)$ if $i \leq n$ and $\pi_i(h_n(x)) = p_i$ if $n < i$. Let $Y_n = h_n(X_n)$. Then, for each n , h_n is a homeomorphism and $\varprojlim \mathbf{f}$ is the sequential limiting set of the sequence \mathbf{Y} .*

Proof. That each h_n is a homeomorphism is an immediate consequence of the fact that h_n is 1-1 and $\pi_i \circ h_n$ is continuous for each i . Let $K = \varprojlim \mathbf{f}$ and let \mathbf{x} be a point of K . If $Y_{n_1}, Y_{n_2}, Y_{n_3}, \dots$ is a subsequence of \mathbf{Y} and $O = \prod_{i>0} O_i$ is a basic open set containing \mathbf{x} , then, inasmuch as there is a positive integer j such that $O_i = X_i$ for $i \geq j$, O contains a point of Y_{n_i} for each $n_i \geq j$ so \mathbf{x} is in the limiting set of $Y_{n_1}, Y_{n_2}, Y_{n_3}, \dots$. On the other hand

if \mathbf{x} is in the limiting set of $Y_{n_1}, Y_{n_2}, Y_{n_3}, \dots$ and O is a basic open set in the product space containing \mathbf{x} there is a positive integer n such that if $j \geq n$ then $O_j = X_j$. Because O contains a point y of Y_{n_k} for some $n_k > n$ and there is a point t of K such that $t_i = y_i$ for $i \leq n_k$, O contains a point of K . Because K is closed, \mathbf{x} is in K . \square

2.10.8 Inverse limits as intersections of closed sets

We close Section 2.10 with a theorem relating inverse limits and intersections of monotonic collections of closed sets.

Theorem 171 *Suppose D is a directed set and $\{X_\alpha \mid \alpha \in D\}$ is a collection of topological spaces such that if $\alpha \preceq \beta$ in D then X_β is a subset of X_α . Furthermore, if $\alpha \preceq \beta$ in D , let $f_{\alpha\beta}$ be the identity on X_β . Then, $\bigcap_{\alpha \in D} X_\alpha$ is homeomorphic to $\varprojlim \mathbf{f}$. Moreover, $\bigcap_{\alpha \in D} X_\alpha \neq \emptyset$ if and only if $\varprojlim \mathbf{f} \neq \emptyset$.*

Proof. The point \mathbf{x} of $\bigcap_{\alpha \in D} X_\alpha$ is in $\varprojlim \mathbf{f}$ if and only if there is a point p of $\bigcap_{\alpha \in D} X_\alpha$ such that $x_\alpha = p$ for each $\alpha \in D$. Let $h : \varprojlim \mathbf{f} \rightarrow \bigcap_{\alpha \in D} X_\alpha$ be given by $h(\mathbf{x})$ is the point p such that $x_\alpha = p$ for each $\alpha \in D$. That h^{-1} is a homeomorphism follows from the observation that the composition of h^{-1} with each projection is the identity on $\bigcap_{\alpha \in D} X_\alpha$. It is clear that $\bigcap_{\alpha \in D} X_\alpha \neq \emptyset$ if and only if $\varprojlim \mathbf{f} \neq \emptyset$. \square

2.11 Some theorems for inverse limit systems with metric factor spaces

In this section we present some theorems for inverse limits that require a metric on the factor spaces. Every metric space has an equivalent metric bounded by 1, thus we assume that all of our metric spaces have a metric bounded by 1. If $(X_1, d_1), (X_2, d_2), (X_3, d_3), \dots$ is a sequence of metric spaces each with a metric bounded by 1, then a metric for $\prod_{i>0} X_i$ is given by $d(\mathbf{x}, \mathbf{y}) = \sum_{i>0} d_i(x_i, y_i)/2^i$. This is the metric that we use for the inverse limit.

Theorem 172 *Suppose $\{X_\alpha, f_{\alpha\beta}, D\}$ is an inverse limit system such that X_α is a compact metric space for each $\alpha \in D$ and each bonding function is a mapping. If D has a countable cofinal subset then $\varprojlim \mathbf{f}$ is a metric space.*

Proof. Suppose E is a countable cofinal subset of D . By Theorem 165, $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \{X_\alpha, f_{\alpha\beta}, E\}$. The latter is a metric space because it is a subset of the metric space $\prod_{\alpha \in E} X_\alpha$. \square

For the remainder of this chapter we deal only with inverse limit sequences in metric spaces.

Our next theorem, although easy to prove, is of fundamental importance to inverse limits in continuum theory. By the *diameter* of a subset A of a metric space with metric d we mean the least upper bound of $\{d(x, y) \mid x, y \in A\}$. We denote the diameter of A by $\text{diam } A$.

Theorem 173 *Suppose $\{X_i, f_i\}$ is an inverse limit sequence where, for each i , X_i is a compact metric space. If $\varepsilon > 0$ there exist a positive integer n and a positive number δ such that if A is a subset of X_n and $\text{diam } A < \delta$ then $\text{diam } \pi_n^{-1}(A) < \varepsilon$.*

Proof. Let $\varepsilon > 0$. There is a positive integer n such that $\sum_{j \geq n} 2^{-j} < \varepsilon/3$. For each $i < n$, $f_{i,n}$ is uniformly continuous so there is a positive number $\delta < \varepsilon/3$ such that if p and q are points of X_n and $d_n(p, q) < \delta$ then $d_i(f_{i,n}(p), f_{i,n}(q)) < \varepsilon/3$. Suppose A is a subset of X_n and the diameter of A is less than δ . If \mathbf{x} and \mathbf{y} are points of $\pi_n^{-1}(A)$ then $d_n(x_n, y_n) < \delta$ so $d(\mathbf{x}, \mathbf{y}) < 2\varepsilon/3$. \square

One consequence of Theorem 173 is the following theorem that generalizes the theorem from the first chapter that inverse limits on $[0, 1]$ are chainable.

Theorem 174 *If M_1, M_2, M_3, \dots is a sequence of chainable continua and f_1, f_2, f_3, \dots is a sequence of mappings such that $f_i : M_{i+1} \rightarrow M_i$, then $\varprojlim \mathbf{f}$ is a chainable continuum.*

If $f : X \rightarrow Y$ is a mapping of a metric space X onto a topological space Y , then f is called an ε -map provided $\text{diam } f^{-1}(y) < \varepsilon$ for each point y of Y . We end this section with a theorem similar to Theorem 173. Its proof is left to the reader.

Theorem 175 *Suppose $\{X_i, f_i\}$ is an inverse limit sequence where X_i is a compact metric space and f_i is a mapping for each positive integer i . Then, if $\varepsilon > 0$, there is a positive integer n such that if $i \geq n$ then π_i is an ε -map.*

Mardešić and Segal, [9, Theorem 1*, p. 148], have shown that a converse of Theorem 175 holds in the case where the factor spaces are connected polyhedra (triangulable continua). We state this result without proof, instead referring the reader to their paper. If Π is a class of polyhedra, a compact metric space is said to be Π -like provided for each $\varepsilon > 0$ there exist a polyhedron $P \in \Pi$ and an ε -mapping $f : X \rightarrow P$ from X onto P .

Theorem 176 (Mardešić and Segal) *Let Π be a class of connected polyhedra. Then, the class of Π -like continua coincides with the class of inverse limits of inverse sequences $\{P_i, f_i\}$ where $P_i \in \Pi$ and f_i is surjective for each positive integer i .*

2.12 Dimension

If G is a finite collection of sets and n is a positive integer, we say that the *order* of G is n provided n is the largest of the integers i such that there are $i + 1$ members of G with a common element. Recall that the *mesh* of a finite collection G of sets is the largest of the diameters of the elements of G . If G and H are collections of sets we say that H *refines* G provided for each element h of H there is an element g of G such that $h \subseteq g$. If n is a positive integer, the compact metric space X is said to have dimension not greater than n , written $\dim(X) \leq n$, provided, for each positive number ε , there is a finite collection of open sets covering X that has mesh less than ε and order not greater than n . We say the dimension of X is n , written $\dim(X) = n$, provided $\dim(X) \leq n$ and $\dim(X) \not\leq n - 1$. It is convenient to use this definition of dimension (sometimes called *covering dimension*) in the study of inverse limits. For compact metric spaces the property of having dimension n (respectively, not greater than n) is equivalent to the usual definition of having small inductive dimension n (respectively, not greater than n) [6, Theorem V 8, p. 67]. In [6, Theorem V 1, p. 54] it is shown that if X is a compact metric space with $\dim(X) \leq n$ and G is a finite collection of open sets covering X then there is a finite collection H of open sets covering X that refines G having order not greater than n .

We begin our look at dimension in inverse limits by presenting a theorem of Nall, Theorem 181, on the dimension of an inverse limit when the bonding functions have zero-dimensional values. Suppose $X_1, X_2, X_3, \dots, X_n$ is a finite collection of compact metric spaces and f_1, f_2, \dots, f_{n-1} is a finite collection of upper semi-continuous functions such that $f_i : X_{i+1} \rightarrow 2^{X_i}$ for $1 \leq i < n$. Let $G'_n = \{(x_1, x_2, \dots, x_n) \in X_1 \times X_2 \times \dots \times X_n \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i < n\}$.

Our first lemma is a special case of Theorem 110. By using the directed set D in that theorem to be the set of integers $\{1, 2, \dots, n\}$ we have the following.

Lemma 177 *Suppose $X_1, X_2, X_3, \dots, X_n$ is a finite collection of compact Hausdorff spaces and $f_i : X_{i+1} \rightarrow 2^{X_i}$ is an upper semi-continuous function for $1 \leq i < n$. Then G'_n is compact.*

Suppose $X_1, X_2, X_3, \dots, X_n$ is a finite collection of compact metric spaces and f_1, f_2, \dots, f_{n-1} is a finite collection of upper semi-continuous functions such that $f_i : X_{i+1} \rightarrow 2^{X_i}$ for $1 \leq i < n$. Let $Y = X_1 \times X_2 \times \dots \times X_{n-1}$ and define $F_n : X_n \rightarrow 2^Y$ by $F_n(x) = \{(x_1, x_2, \dots, x_{n-1}) \in G'_{n-1} \mid x_{n-1} \in f_n(x)\}$. In [10], Nall makes the following useful observation.

Theorem 178 *Suppose $X_1, X_2, X_3, \dots, X_n$ is a finite collection of compact metric spaces and $f_i : X_{i+1} \rightarrow 2^{X_i}$ is an upper semi-continuous function for $1 \leq i < n$. Then, F_n is upper semi-continuous.*

Proof. The graph of F_n is homeomorphic to G'_n which is compact by Lemma 177. Theorem 105 yields that F_n is upper semi-continuous. \square

Lemma 179 *Suppose $X_1, X_2, X_3, \dots, X_n$ is a finite collection of compact metric spaces and $f_i : X_{i+1} \rightarrow 2^{X_i}$ is an upper semi-continuous function for $1 \leq i < n$. If x is a point of X_n such that $\dim(F_n(x)) > 0$ then there exist an integer j , $1 \leq j < n$, and a point z of X_{j+1} such that $\dim(f_j(z)) > 0$.*

Proof. $\dim(F_n(x)) > 0$, therefore it contains a nondegenerate continuum K [6, Theorem D, p.22]. Some projection of K into one of the factor spaces X_i is nondegenerate. Let j be the largest integer i so that the projection of K into X_i is nondegenerate. If $j = n - 1$, let $z = x$ and it follows that $f_j(z)$ contains a nondegenerate continuum so $\dim(f_j(z)) > 0$. If $j < n - 1$ then the projection of K into X_{j+1} is a single point z . It follows that $\dim(f_j(z)) > 0$. \square

If $(X_1, d_1), (X_2, d_2), \dots, (X_n, d_n)$ is a finite collection of compact metric spaces, there are numerous metrics that are compatible with the product topology on $X_1 \times X_2 \times \dots \times X_n$. One that is particularly convenient is $d(x, y) = \sum_{i=1}^{i=n} d_i(x_i, y_i)/2^i$.

Lemma 180 *Suppose $X_1, X_2, X_3, \dots, X_n$ is a finite collection of compact metric spaces, $f_i : X_{i+1} \rightarrow 2^{X_i}$ is an upper semi-continuous function for $1 \leq i < n$, and m is a positive integer. If $\dim(X_n) \leq m$ and for each i , $1 \leq i < n$ and each point x of X_{i+1} $\dim(f_i(x)) = 0$ then $\dim(G'_n) \leq m$.*

Proof. Suppose $\varepsilon > 0$ and x is a point of X_n . It follows from Lemma 179 that $\dim(F_n(x)) = 0$ so there exists a finite collection \mathcal{V}_x of mutually exclusive open sets covering $F_n(x)$ such that the mesh of \mathcal{V}_x is less than $\varepsilon/3$. Because F_n is upper semi-continuous, there is an open set u_x containing x of diameter less than ε such that $F_n(u_x) \subseteq \mathcal{V}_x^*$ (where \mathcal{V}_x^* denotes the union of all the sets in \mathcal{V}_x). The collection of open sets $\mathcal{U} = \{u_x \mid x \in X_n\}$ covers the compact set X_n so there is a finite subcollection \mathcal{U}' of \mathcal{U} that covers X_n . The dimension of X_n is not greater than m so there is a finite collection \mathcal{W} of open sets covering X_n such that the order of \mathcal{W} is not greater than m and \mathcal{W} refines \mathcal{U}' . The mesh of \mathcal{W} is less than $\varepsilon/2$. For each $w \in \mathcal{W}$ choose a point x_w of

X_n in w such that $w_{x_w} \subseteq u_{x_w} \in \mathcal{U}'$. Because $\mathcal{U}' \subseteq \mathcal{U}$, $F_n(w_{x_w}) \subseteq \mathcal{V}_{x_w}^*$. Then, $\{v \times w \mid w \in \mathcal{W} \text{ and there is a point } x \in X_n \text{ such that } w = w_x \text{ and } v \in \mathcal{V}_x\}$ is a collection of open sets covering G'_n of order not greater than m and mesh less than ε . Thus, $\dim(G'_n) \leq m$. \square

Theorem 181 (Nall) *Suppose $\{X_i, f_i\}$ is an inverse limit sequence with upper semi-continuous bonding functions such that X_i is a compact metric space for each positive integer i and m is a positive integer. Suppose further that $\dim(f_i(x)) = 0$ for each positive integer i and each point x of X_{i+1} . If there is an increasing sequence n_1, n_2, n_3, \dots of positive integers such that $\dim(X_{n_i}) \leq m$ for $i = 1, 2, 3, \dots$, then $\dim(\varprojlim \mathbf{f}) \leq m$.*

Proof. Recall that $\varprojlim \mathbf{f} = \bigcap_{n \geq 2} G_n$ where $G_n = \{x \in \prod_{i \geq 0} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i < n\}$. Observe $G_n = G'_n \times \prod_{i \geq n} X_i$. Let $\varepsilon > 0$. There is a positive integer N such that $\sum_{i \geq N} 2^{-i} < \varepsilon/2$. Let i be an integer such that $n_i > N$. By Lemma 180 $\dim(G'_{n_i}) \leq m$. Let \mathcal{U} be a collection of open sets of order not greater than m and mesh less than $\varepsilon/2$ that covers G'_{n_i} . Then, $\{\pi^{-1}(u) \mid u \in \mathcal{U}\}$ is a collection of open sets of mesh less than ε and order not greater than m that covers $\varprojlim \mathbf{f}$. \square

If each of X and Y is a compact metric space and $f : X \rightarrow 2^Y$ is a function that is the union of finitely many mappings of X into Y , then f is upper semi-continuous and $\dim(f(x)) = 0$ for each x in X . As a consequence, we have the following corollary to Theorem 181.

Corollary 182 *Suppose n is a positive integer and $\{X_i, f_i\}$ is an inverse limit sequence in which each X_i is a compact metric space and each f_i is the union of finitely many mappings. If for each j there is a positive integer $i \geq j$ such that $\dim(X_i) \leq n$ then $\dim(\varprojlim \mathbf{f}) \leq n$.*

Of course one consequence of Corollary 182 is that ordinary inverse limits do not raise dimension. On the other hand, dimension may be lowered by the ordinary inverse limit construction even if the bonding maps are surjective as may be seen from the following example.

Example 183 *Let φ denote the projection of the unit square $C = [0, 1] \times [0, 1]$ onto the interval $I = [0, 1]$. Let ψ denote a map of I onto C . Let $X_i = C$ and $f_i = \psi \circ \varphi$ for each i . Let $Y_i = I$ and $g_i = \varphi \circ \psi$ for each i . Let $Z_i = C$ for odd integers i and $Z_i = I$ for even integers i . Let $k_i = \psi$ for odd i and $k_i = \varphi$ for even i . Using $n_1 = 1, n_2 = 3, n_3 = 5, \dots$ in the subsequence theorem we see that $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{k}$. Using $n_1 = 2, n_2 = 4, n_3 = 6, \dots$ in the subsequence theorem we see that $\varprojlim \mathbf{k}$ is homeomorphic to $\varprojlim \mathbf{g}$. Therefore, $\varprojlim \mathbf{f}$ is homeomorphic to $\varprojlim \mathbf{g}$ so $\dim(\varprojlim \mathbf{f}) \leq 1$ even though each factor space is two-dimensional and each bonding map is surjective.*

By substituting an n -cell or the Hilbert cube for C in Example 183, we see that an inverse limit of n -dimensional or even infinite-dimensional continua can have dimension one.

Actually, the dimension of $\varprojlim \mathbf{f}$ in Example 183 is one as shown by the following theorem.

Theorem 184 *Suppose $\{X_i, f_i\}$ is an inverse limit sequence where X_i is a continuum of dimension one and f_i is a surjective mapping for each i . Then, $\dim(\varprojlim \mathbf{f}) = 1$.*

Proof. Let $M = \varprojlim \mathbf{f}$. Because $\dim(X_1) = 1$, X_1 is nondegenerate. Let p and q be two different points of X_1 . Each bonding map is surjective, thus there are points \mathbf{x} and \mathbf{y} of M such that $x_1 = p$ and $y_1 = q$. Because M is a nondegenerate continuum, $\dim(M) \not\leq 0$. By Theorem 181, $\dim(M) \leq 1$, so $\dim(M) = 1$. \square

In the case that the bonding functions are upper semi-continuous functions each of which is the union of finitely many mappings, the dimension of the inverse limit is not greater than one.

Theorem 185 *If $f_i : I \rightarrow 2^I$ is an upper semi-continuous function that is the union of finitely many mappings $f_1^i, f_2^i, \dots, f_{k_i}^i$ of $I = [0, 1]$ into itself for each i , then the dimension of $\varprojlim \mathbf{f}$ is not greater than one. Moreover, if there is a sequence \mathbf{g} such that $g_i \in \{f_1^i, f_2^i, \dots, f_{k_i}^i\}$ for each positive integer i and $\varprojlim \mathbf{g}$ is nondegenerate then the dimension of $\varprojlim \mathbf{f}$ is one.*

Proof. Inasmuch as f is the union of finitely many mappings, $\dim(f(t)) = 0$ for each $t \in [0, 1]$. By Nall's theorem (Theorem 181), the dimension of $\varprojlim \mathbf{f}$ is not greater than one. If there is a sequence \mathbf{g} such that $g_i \in \{f_1^i, f_2^i, \dots, f_{k_i}^i\}$ for each positive integer i and $\varprojlim \mathbf{g}$ is nondegenerate, then $\varprojlim \mathbf{f}$ contains a nondegenerate continuum so its dimension is one. \square

We close with a proof that one cannot get a 2-cell as an inverse limit with a single upper semi-continuous bonding function from $[0, 1]$ into $2^{[0,1]}$. Recall that, unless otherwise noted, if $f : [0, 1] \rightarrow 2^{[0,1]}$ is an upper semi-continuous function, we consider the domain of the projection π_i to be the inverse limit space, $\varprojlim \mathbf{f}$, whereas \hat{f} denotes the shift map on $\varprojlim \mathbf{f}$ given by $\hat{f}(\mathbf{x}) = (x_2, x_3, x_4, \dots)$. In general, the shift map on an inverse limit with upper semi-continuous bonding functions is not a homeomorphism. However, when it is restricted to a compact set on which it is one-to-one, its restriction is a homeomorphism. The following proof is based on work of Nall [10].

Theorem 186 (Nall) *Suppose $f : [0, 1] \rightarrow 2^{[0,1]}$ is an upper semi-continuous function such that if $y \in [0, 1]$ there exists a point $x \in [0, 1]$ such that $y \in f(x)$. Then $\varprojlim \mathbf{f}$ is not a 2-cell.*

Proof. Suppose $M = \varprojlim \mathbf{f}$ is a 2-cell. If $0 < t < 1$, then $\pi_1^{-1}(t)$ separates M . Because M is a 2-cell, $\pi_1^{-1}(t)$ is not zero-dimensional [6, Corollary 2, p. 48], so it contains a nondegenerate continuum H . There is a positive integer $m \geq 2$ such that $\pi_m(H)$ is nondegenerate but $\pi_i(H)$ is a single point for $1 \leq i < m$. Suppose J is an interval such that $\pi_m(H) = J$ and if $1 \leq i < m$, let $\pi_i(H) = \{t_i\}$ where $t_1 = t$. Let $K = \{\mathbf{x} \in M \mid x_i = t_i \text{ for } 1 \leq i < m \text{ and } x_m \in J\}$. Note that $H \subseteq K \subseteq \pi_1^{-1}(t)$.

By [6, Theorem IV 3, p. 44], $\pi_1^{-1}(J)$ is 2-dimensional being a closed set with interior lying in a 2-cell. Let \mathbf{z} be a point of $\pi_1^{-1}(J)$. Because $z_1 \in J$ and $J = \pi_m(H)$, there is a point \mathbf{w} of H such that $w_m = z_1$. Let \mathbf{y} be the point of $[0, 1]^\infty$ such that $y_i = t_i$ for $1 \leq i < m$ and $y_{m+i} = z_{i+1}$ for $i = 0, 1, 2, \dots$. Because $\mathbf{w} \in H \subseteq M$ and $\mathbf{z} \in M$, it follows that $\mathbf{y} \in K$. Moreover, $\hat{f}^{m-1}(\mathbf{y}) = \mathbf{z}$. Thus, $\mathbf{z} \in \hat{f}^{m-1}(K)$ and we have established that $\pi_1^{-1}(J) \subseteq \hat{f}^{m-1}(K)$. Note that \hat{f} is 1-1 on K because $\pi_1(K)$ is degenerate so \hat{f} is a homeomorphism on K . In fact, \hat{f}^{m-1} is a homeomorphism on K and $\hat{f}^{m-1}(K)$ contains a two-dimensional subset so K contains a two-dimensional subset. But, K is a subset of $\pi_1^{-1}(t)$ so it follows that $\pi_1^{-1}(t)$ contains an open set.

Thus, we have for each t in $(0, 1)$, $\pi_1^{-1}(t)$ contains an open set. But, if $s \neq t$ and $0 < s, t < 1$ then $\pi_1^{-1}(s)$ and $\pi_1^{-1}(t)$ have no point in common so the 2-cell M contains uncountably many mutually exclusive open sets, a contradiction. \square

Nall actually proves more than we state in Theorem 186. He shows that a continuum that is the union of a countable collection of n -cells and compact n -dimensional manifolds is not homeomorphic to an inverse limit on $[0, 1]$ with a single upper semi-continuous bonding function. His proof is similar to the one we present for Theorem 186.

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