Chapter 2
Manifolds and Varieties via Sheaves

In rough terms, a manifold is a "space" that looks locally like Euclidean space. An algebraic variety can be defined similarly as a "space" that looks locally like the zero set of a collection of polynomials. Point set topology alone would not be sufficient to capture this notion of space. These examples come with distinguished classes of functions (C^∞ functions in the first case, and polynomials in the second), and we want these classes to be preserved under the above local identifications. Sheaf theory provides a natural language in which to make these ideas precise. A sheaf on a topological space X is essentially a distinguished class of functions, or things that behave like functions, on open subsets of X. The main requirement is that the condition to be distinguished be *local*, which means that it can be checked in a neighborhood of every point of X. For a sheaf of rings, we have an additional requirement, that the distinguished functions on $U \subseteq X$ should form a commutative ring. With these definitions, the somewhat vague idea of a space can be replaced by the precise notion of a *concrete ringed space*, which consists of a topological space together with a sheaf of rings of functions. Both manifolds and varieties are concrete ringed spaces.

Sheaves were first defined by Leray in the late 1940s. They played a key role in the development of algebraic and complex analytic geometry, in the pioneering works of Cartan, Grothendieck, Kodaira, Serre, Spencer, and others in the following decade. Although it is rarely presented in this way in introductory texts (e.g., [110, 111, 117]), basic manifold theory can also be developed quite naturally in this framework. In this chapter we want to lay the basic foundation for the rest of the book. The goal here is to introduce the language of sheaves, and then to carry out a uniform treatment of real and complex manifolds and algebraic varieties from this point of view. This approach allows us to highlight the similarities, as well as the differences, among these spaces.

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2.1 Sheaves of Functions

As we said above, we need to define sheaves in order eventually to define manifolds and varieties. We start with a more primitive notion. In many parts of mathematics, we encounter topological spaces with distinguished classes of functions on them: continuous functions on topological spaces, C^∞-functions on $\mathbb{R}^n$, holomorphic functions on $\mathbb{C}^n$, and so on. These functions may have singularities, so they may be defined only over subsets of the space; we will be interested primarily in the case that these subsets are open. We say that such a collection of functions is a presheaf if it is closed under restriction. Given sets X and T, let $\mathrm{Map}_T(X)$ denote the set of maps from X to T. Here is the precise definition of a presheaf, or rather of the kind of presheaf we need at the moment.

Definition 2.1.1. Suppose that X is a topological space and T a nonempty set. A presheaf of T-valued functions on X is a collection of subsets $\mathscr{P}(U) \subseteq \mathrm{Map}_T(U)$, for each open $U \subseteq X$, such that the restriction $f|_V$ belongs to $\mathscr{P}(V)$ whenever $f \in \mathscr{P}(U)$ and $V \subset U$.

The collection of all functions $\mathrm{Map}_T(U)$ is of course a presheaf. Less trivially:

Example 2.1.2. Let T be a topological space. Then the set of continuous functions $\mathrm{Cont}_{X,T}(U)$ from $U \subseteq X$ to T is a presheaf.

Example 2.1.3. Let X be a topological space and let T be a set. The set $T^P(U)$ of constant functions from U to T is a presheaf called the constant presheaf.

Example 2.1.4. Let $X = \mathbb{R}^n$. The sets $C^\infty(U)$ of C^∞ real-valued functions form a presheaf.

Example 2.1.5. Let $X = \mathbb{C}^n$. The sets $\mathscr{O}(U)$ of holomorphic functions on U form a presheaf. (A function of several variables is holomorphic if it is C^∞ and holomorphic in each variable.)

Example 2.1.6. Let L be a linear differential operator on $\mathbb{R}^n$ with C^∞ coefficients (e.g., $\sum \partial^2/\partial x_i^2$). Let $S(U)$ denote the space of C^∞ solutions to $Lf = 0$ in U. This is a presheaf with values in $\mathbb{R}$.

Example 2.1.7. Let $X = \mathbb{R}^n$. The sets $L^p(U)$ of measurable functions $f : U \to \mathbb{R}$ satisfying $\int_U |f|^p < \infty$ form a presheaf.

Upon comparing these examples, we see some qualitative differences. The continuity, C^∞, and holomorphic conditions are *local* conditions, which means that they can be checked in a neighborhood of a point. The other conditions such as constancy or L^p-ness, by contrast, are not. A presheaf is called a sheaf if the defining property is local. More precisely:

Definition 2.1.8. A presheaf of functions $\mathscr{P}$ is called a sheaf if given any open set U with an open cover $\{U_i\}$, a function f on U lies in $\mathscr{P}(U)$ if $f|_{U_i} \in \mathscr{P}(U_i)$ for all i.

Examples 2.1.2, 2.1.4, 2.1.5, and 2.1.6 are sheaves, while the other examples are not, except in trivial cases. More explicitly, suppose that T has at least two elements t_1, t_2, and that X contains a disconnected open set U. Then we can write $U = U_1 \cup U_2$ as a union of two disjoint open sets. The function τ taking the value of t_i on U_i is not in $T^P(U)$, but $\tau|_{U_i} \in T^P(U_i)$. Therefore T^P is not a sheaf. Similarly, L^p is not a sheaf for $0 < p < \infty$ because the constant function 1 is not in $L^p(\mathbb{R}^n)$, even though $1 \in L^p(B)$ for any ball B of finite radius.

However, there is a simple remedy.

Example 2.1.9. A function is locally constant if it is constant in a neighborhood of a point. For instance, the function τ constructed above is locally constant but not constant. The set of locally constant functions, denoted by $T(U)$ or $T_X(U)$, is now a sheaf, precisely because the condition can be checked locally. A sheaf of this form is called a constant sheaf.

We can always create a sheaf from a presheaf by the following construction.

Example 2.1.10. Given a presheaf $\mathscr{P}$ of functions from X to T. Define

$$\mathscr{P}^s(U) = \{f : U \to T \mid \forall x \in U, \exists \text{ a neighborhood } U_x \text{ of } x \text{ such that } f|_{U_x} \in \mathscr{P}(U_x)\}.$$

This is a sheaf called the sheafification of $\mathscr{P}$.

When $\mathscr{P}$ is a presheaf of constant functions, $\mathscr{P}^s$ is exactly the sheaf of locally constant functions. When this construction is applied to the presheaf L^p, we obtain the sheaf of locally L^p functions.

Exercises

2.1.11. Check that $\mathscr{P}^s$ is a sheaf.

2.1.12. Let $\mathscr{B}$ be the presheaf of bounded continuous real-valued functions on $\mathbb{R}$. Describe $\mathscr{B}^s$ in explicit terms.

2.1.13. Let $\pi : B \to X$ be a surjective continuous map of topological spaces. Prove that the presheaf of sections

$$B(U) = \{\sigma : U \to B \mid \sigma \text{ continuous}, \forall x \in U, \pi \circ \sigma(x) = x\}$$

is a sheaf.

2.1.14. Given a sheaf $\mathscr{P}$ on X and an open set $U \subset X$, let $\mathscr{P}|_U$ denote the presheaf on U defined by $V \mapsto \mathscr{P}(V)$ for each $V \subseteq U$. Check that $\mathscr{P}_U$ is a sheaf when $\mathscr{P}$ is.

2.1.15. Let $Y \subset X$ be a closed subset of a topological space. Let $\mathscr{P}$ be a sheaf of T-valued functions on X. For each open $U \subset Y$, let $\mathscr{P}_Y(U)$ be the set of functions

$f : U \to T$ locally extendible to an element of $\mathscr{P}$, i.e., $f \in \mathscr{P}_Y(U)$ if and only if for each $y \in U$, there exist a neighborhood $V \subset X$ and an element of $\mathscr{P}(V)$ restricting to $f|_{V\cap U}$. Show that $\mathscr{P}_Y$ is a sheaf.

2.1.16. Let $F : X \to Y$ be surjective continuous map. Suppose that $\mathscr{P}$ is a sheaf of T-valued functions on X. Define $f \in \mathscr{Q}(U) \subset \mathrm{Map}_T(U)$ if and only if its *pullback* $F^*f = f \circ F|_{f^{-1}U}$ belongs to $\mathscr{P}(F^{-1}(U))$. Show that $\mathscr{Q}$ is a sheaf on Y.

2.2 Manifolds

As explained in the introduction, a manifold consists of a topological space with a distinguished class of functions that looks locally like $\mathbb{R}^n$. We now set up the language necessary to give a precise definition. Let k be a field. Then $\mathrm{Map}_k(X)$ is a commutative k-algebra with pointwise addition and multplication.

Definition 2.2.1. Let $\mathscr{R}$ be a sheaf of k-valued functions on X. We say that $\mathscr{R}$ is a sheaf of algebras if each $R(U) \subseteq \mathrm{Map}_k(U)$ is a subalgebra when U is nonempty. We call the pair $(X,\mathscr{R})$ a concrete ringed space over k or simply a concrete k-space. We will sometimes refer to elements of $\mathscr{R}(U)$ as distinguished functions.

The sheaf $\mathscr{R}$ is called the structure sheaf of X. In this chapter, we usually omit the modifier "concrete," but we will use it later on after we introduce a more general notion. Basic examples of $\mathbb{R}$-spaces are $(\mathbb{R}^n, \mathrm{Cont}_{\mathbb{R}^n,\mathbb{R}})$ and $(\mathbb{R}^n, C^\infty)$, while $(\mathbb{C}^n, \mathscr{O})$ is an example of a $\mathbb{C}$-space.

We also need to consider maps $F : X \to Y$ between such spaces. We will certainly insist on continuity, but in addition we require that when a distinguished function is precomposed with F, or "pulled back" along F, it remain distinguished.

Definition 2.2.2. A morphism of k-spaces $(X,\mathscr{R}) \to (Y,\mathscr{S})$ is a continuous map $F\colon X \to Y$ such that if $f \in \mathscr{S}(U)$, then $F^*f \in \mathscr{R}(F^{-1}U)$, where $F^*f = f \circ F|_{f^{-1}U}$.

It is worthwhile noting that this completely captures the notion of a C^∞, or holomorphic, map between Euclidean spaces.

Example 2.2.3. A C^∞ map $F : \mathbb{R}^n \to \mathbb{R}^m$ induces a morphism $(\mathbb{R}^n, C^\infty) \to (\mathbb{R}^m, C^\infty)$ of $\mathbb{R}$-spaces, since C^∞ functions are closed under composition. Conversely, if F is a morphism, then the coordinate functions on $\mathbb{R}^m$ are expressible as C^∞ functions of the coordinates of $\mathbb{R}^n$, which implies that F is C^∞.

Example 2.2.4. Similarly, a continuous map $F : \mathbb{C}^n \to \mathbb{C}^m$ induces a morphism of $\mathbb{C}$-spaces if and only if it is holomorphic.

This is a good place to introduce, or perhaps remind the reader of, the notion of a *category* [82]. A category $\mathscr{C}$ consists of a set (or class) of objects $\mathrm{Obj}\mathscr{C}$ and for each pair $A,B \in \mathscr{C}$, a set $\mathrm{Hom}_{\mathscr{C}}(A,B)$ of morphisms from A to B. There is a composition law

$$\circ : \mathrm{Hom}_{\mathscr{C}}(B,C) \times \mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{C}}(A,C),$$

and distinguished elements $id_A \in \mathrm{Hom}_{\mathscr{C}}(A,A)$ that satisfy

(C1) associativity: $f \circ (g \circ h) = (f \circ g) \circ h$,
(C2) identity: $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_A \circ g = g$,

whenever these are defined. Categories abound in mathematics. A basic example is the category of Sets. The objects are sets, $\mathrm{Hom}_{\mathrm{Sets}}(A,B)$ is just the set of maps from A to B, and composition and id_A have the usual meanings. Similarly, we can form the category of groups and group homomorphisms, the category of rings and ring homomorphisms, and the category of topological spaces and continuous maps. We have essentially constructed another example. We can take the class of objects to be k-spaces, and morphisms as above. These can be seen to constitute a category once we observe that the identity is a morphism and the composition of morphisms is a morphism.

The notion of an isomorphism makes sense in any category. We will spell this out for k-spaces.

Definition 2.2.5. An isomorphism of k-spaces $(X,\mathscr{R}) \cong (Y,\mathscr{S})$ is a homeomorphism $F : X \to Y$ such that $f \in \mathscr{S}(U)$ if and only if $F^*f \in \mathscr{R}(F^{-1}U)$.

Given a sheaf $\mathscr{S}$ on X and an open set $U \subset X$, let $\mathscr{S}|_U$ denote the sheaf on U defined by $V \mapsto \mathscr{S}(V)$ for each $V \subseteq U$.

Definition 2.2.6. An n-dimensional C^∞ manifold is an $\mathbb{R}$-space (X, C^∞_X) such that

1. The topology of X is given by a metric.
2. X admits an open cover $\{U_i\}$ such that each $(U_i, C^\infty_X|_{U_i})$ is isomorphic to $(B_i, C^\infty|_{B_i})$ for some open balls $B_i \subset \mathbb{R}^n$.

Remark 2.2.7. It is equivalent and perhaps more standard to require that the topology be Hausdorff and paracompact rather than metrizable. The equivalence can be seen as follows. The paracompactness of metric spaces is a theorem of Stone [112, 70]. In the opposite direction, a Riemannian metric can be constructed using a partition of unity [110]. The associated Riemannian distance function, which is the infimum of the lengths of curves joining two points, then provides a metric in the sense of point set topology.

The isomorphisms $(U_i, C^\infty|_{U_i}) \cong (B_i, C^\infty|_{B_i})$ correspond to coordinate charts in more conventional treatments. The collection of all such charts is called an atlas. Given a coordinate chart, we can pull back the standard coordinates from the ball to U_i. So we always have the option of writing expressions locally in these coordinates.

There are a number of variations on this idea:

Definition 2.2.8.
1. An n-dimensional topological manifold is defined as above but with $(\mathbb{R}^n, C^\infty)$ replaced by $(\mathbb{R}^n, \mathrm{Cont}_{\mathbb{R}^n,\mathbb{R}})$.
2. An n-dimensional complex manifold can be defined by replacing $(\mathbb{R}^n, C^\infty)$ by $(\mathbb{C}^n, \mathscr{O})$.

The one-dimensional complex manifolds are usually called *Riemann surfaces*.

Definition 2.2.9. A C^∞ map from one C^∞ manifold to another is just a morphism of $\mathbb{R}$-spaces. A holomorphic map between complex manifolds is defined as a morphism of $\mathbb{C}$-spaces.

The class of C^∞ manifolds and maps form a category; an isomorphism in this category is called a *diffeomorphism*. Likewise, the class of complex manifolds and holomorphic maps forms a category, with isomorphisms called *biholomorphisms*. By definition, any point of a manifold has a neighborhood, called a coordinate neighborhood, diffeomorphic or biholomorphic to a ball. Given a complex manifold $(X, \mathcal{O}_X)$, we say that $f : X \to \mathbb{R}$ is C^∞ if and only if $f \circ g$ is C^∞ for each holomorphic map $g : B \to X$ from a coordinate ball $B \subset \mathbb{C}^n$. We state for the record the following:

Lemma 2.2.10. *An n-dimensional complex manifold together with its sheaf of C^∞ functions is a $2n$-dimensional C^∞ manifold.*

Proof. An n-dimensional complex manifold $(X, \mathcal{O}_X)$ is locally biholomorphic to a ball in $\mathbb{C}^n$, and hence (X, C^∞_X) is locally diffeomorphic to the same ball regarded as a subset of $\mathbb{R}^{2n}$. $\square$

Later on, we will need to write things in coordinates. The pullbacks of the standard coordinates on a ball $B \subset \mathbb{C}^n$ under local biholomorphism from $X \supset B' \cong B$, are referred to as local *analytic coordinates* on X. We typically denote these by $z_1, \dots, z_n$. Then the real and imaginary parts $x_1 = \mathrm{Re}(z_1), y_1 = \mathrm{Im}(z_1), \dots$ give local coordinates for the underlying C^∞-manifold.

Let us consider some examples of manifolds. Certainly any open subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a (complex) manifold in an obvious fashion. To get less trivial examples, we need one more definition.

Definition 2.2.11. Given an n-dimensional C^∞ manifold X, a closed subset $Y \subset X$ is called a closed m-dimensional submanifold if for any point $x \in Y$, there exist a neighborhood U of x in X and a diffeomorphism to a ball $B \subset \mathbb{R}^n$ containing 0 such that $Y \cap U$ maps to the intersection of B with an m-dimensional linear subspace. A similar definition holds for complex manifolds.

When we use the word "submanifold" without qualification, we will always mean "closed submanifold." Given a closed submanifold $Y \subset X$, define C^∞_Y to be the sheaf of continuous functions that are locally extendible to C^∞ functions on X (see Exercise 2.1.15). This means that $f \in C^\infty_Y(U)$ if every point of U possesses a neighborhood $V \subset X$ such $f|_{V\cap Y} = \tilde{f}|_{V\cap Y}$ for some $\tilde{f} \in C^\infty(V)$. For a complex submanifold $Y \subset X$, we define $\mathcal{O}_Y$ to be the sheaf of functions that locally extend to holomorphic functions.

Lemma 2.2.12. *If $Y \subset X$ is a closed submanifold of a C^∞ (respectively complex) manifold, then (Y, C^∞_Y) (respectively $(Y, \mathcal{O}_Y)$) is also a C^∞ (respectively complex) manifold.*

Proof. We treat the C^∞ case; the holomorphic case is similar. Choose a local diffeomorphism (X, C^∞_X) to a ball $B \subset \mathbb{R}^n$ such that Y correponds to $B \cap \mathbb{R}^m$. Then any C^∞ function $f(x_1, \ldots, x_m)$ on $B \cap \mathbb{R}^m$ extends trivially to a C^∞ function on B and conversely. Thus (Y, C^∞_Y) is locally diffeomorphic to a ball in $\mathbb{R}^m$. □

With this lemma in hand, it is possible to produce many interesting examples of manifolds starting from $\mathbb{R}^n$. For example, the unit sphere $S^{n-1} \subset \mathbb{R}^n$, which is the set of solutions to $\sum x_i^2 = 1$, is an $(n-1)$-dimensional manifold. Further examples are given in the exercises.

The following example, which was touched upon earlier, is of fundamental importance in algebraic geometry.

Example 2.2.13. Complex projective space $\mathbb{P}^n_{\mathbb{C}} = \mathbb{CP}^n$ is the set of one-dimensional subspaces of $\mathbb{C}^{n+1}$. (We will usually drop the $\mathbb{C}$ and simply write $\mathbb{P}^n$ unless there is danger of confusion.) Let $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ be the natural projection that sends a vector to its span. In the sequel, we usually denote $\pi(x_0, \ldots, x_n)$ by $[x_0, \ldots, x_n]$. Then $\mathbb{P}^n$ is given the quotient topology, which is defined so that $U \subset \mathbb{P}^n$ is open if and only if $\pi^{-1}U$ is open. Define a function $f : U \to \mathbb{C}$ to be holomorphic exactly when $f \circ \pi$ is holomorphic. Then the presheaf of holomorphic functions $\mathscr{O}_{\mathbb{P}^n}$ is a sheaf (Exercise 2.1.16), and the pair $(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n})$ is a complex manifold. In fact, if we set

$$U_i = \{[x_0, \ldots, x_n] \mid x_i \neq 0\},$$

then the map

$$[x_0, \ldots, x_n] \mapsto (x_0/x_i, \ldots, \widehat{x_i/x_i}, \ldots, x_n/x_i)$$

induces an isomorphism $U_i \cong \mathbb{C}^n$. The notation $\ldots, \widehat{x}, \ldots$ means skip x in the list.

Exercises

2.2.14. Given k-spaces X, Y, prove that morphisms from X to Y can be patched, i.e., that the set of morphisms from open subsets of X to Y is a sheaf.

2.2.15. Show that the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a C^∞ morphism and a homeomorphism, but that it is not a diffeomorphism.

2.2.16. Let $f_1, \ldots, f_r$ be C^∞ functions on $\mathbb{R}^n$, and let X be the set of common zeros of these functions. Suppose that the rank of the Jacobian $(\partial f_i / \partial x_j)$ is $n - m$ at every point of X. Then show that X is an m-dimensional submanifold using the implicit function theorem [109, p. 41]. In particular, show that the sphere $x_1^2 + \cdots + x_n^2 = 1$ is a closed $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.

2.2.17. Apply the previous exercise to show that the set $\mathrm{O}(n)$ (respectively $\mathrm{U}(n)$) of orthogonal (respectively unitary) $n \times n$ matrices is a submanifold of $\mathbb{R}^{n^2}$ (respectively $\mathbb{C}^{n^2}$).

2.2.18. A manifold that is also a group with C^∞ group operations is called a *Lie group*. Show that $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{O}(n)$, and $\mathrm{U}(n)$ are examples.

2.2.19. Suppose that Γ is a group of diffeomorphisms of a manifold X. Suppose that the action of Γ is fixed-point-free and properly discontinuous in the sense that every point possesses a neighborhood N such that $\gamma(N)\cap N=\emptyset$ unless $\gamma=\mathrm{id}$. Give $Y=X/\Gamma$ the quotient topology and let $\pi : X\to Y$ denote the projection. Define $f\in C^\infty_Y(U)$ if and only if the pullback $f\circ\pi$ is C^∞ in the usual sense. Show that (Y,C^∞_Y) is a C^∞ manifold. Deduce that the torus $T=\mathbb{R}^n/\mathbb{Z}^n$ is a manifold, and in fact a Lie group.

2.2.20. Check that the previous exercise applies to complex manifolds, with the appropriate modifications. In particular, show that $E=\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, $\mathrm{Im}(\tau)>0$, is Riemann surface (called an elliptic curve).

2.2.21. The complex Grassmannian $G=\mathbb{G}(2,n)$ is the set of 2-dimensional subspaces of $\mathbb{C}^n$. Let $M\subset\mathbb{C}^{2n}$ be the open set of $2\times n$ matrices of rank 2. Let $\pi : M\to G$ be the surjective map that sends a matrix to the span of its rows. Give G the quotient topology induced from M, and define $f\in\mathscr{O}_G(U)$ if and only if $\pi\circ f\in\mathscr{O}_M(\pi^{-1}U)$. For $i\neq j$, let $U_{ij}\subset M$ be the set of matrices with $(1,0)^t$ and $(0,1)^t$ for the ith and jth columns. Show that

$$\mathbb{C}^{2n-4}\cong U_{ij}\cong\pi(U_{ij})$$

and conclude that G is a $(2n-4)$-dimensional complex manifold.

2.2.22. Generalize the previous exercise to the Grassmannian $\mathbb{G}(r,n)$ of r-dimensional subspaces of $\mathbb{C}^n$.

2.3 Affine Varieties

Algebraic varieties are spaces that are defined by polynomial equations. Unlike the case of manifolds, algebraic varieties can be quite complicated even locally. We first study the local building blocks in this section, before turning to arbitrary algebraic varieties. Standard references for the material of this section and the next are Eisenbud–Harris [34], Harris [58], Hartshorne [60], Mumford [92], and Shafarevich [104].

Let k be an algebraically closed field. Affine space of dimension n over k is defined as $\mathbb{A}^n_k=k^n$. When $k=\mathbb{C}$, we can endow this space with the standard topology induced by the Euclidean metric, and we will refer to this as the *classical topology*. At the other extreme is the *Zariski topology*, which will be defined below. It makes sense for any k, and it is useful even for $k=\mathbb{C}$. Unless stated otherwise, topological notions are with respect to the Zariski topology for the remainder of this section. On $\mathbb{A}^1_k=k$, the open sets consist of complements of finite sets together with the empty set. In general, this topology can be defined to be the weakest topology for which the polynomials $\mathbb{A}^n_k \to k$ are continuous functions. The closed sets of $\mathbb{A}^n_k$ are precisely the sets of zeros

$$Z(S) = \{a \in \mathbb{A}^n \mid f(a) = 0, \forall f \in S\}$$

of sets of polynomials $S \subset R = k[x_1, \dots, x_n]$. Sets of this form are also called *algebraic*. The Zariski topology has a basis given by open sets of the form $D(g) = X - Z(g)$, $g \in R$. Given a subset $X \subset \mathbb{A}^n_k$, the set of polynomials

$$I(X) = \{f \in R \mid f(a) = 0, \forall a \in X\}$$

is an ideal that is radical in the sense that $f \in (X)$ whenever a power of it lies in $I(X)$. Since k is algebraically closed, Hilbert's Nullstellensatz [8, 33] gives a correspondence:

Theorem 2.3.1 (Hilbert). *Let $R = k[x_1, \dots, x_n]$ with k algebraically closed. There is a bijection between the collection of algebraic subsets of $\mathbb{A}^n_k$ and radical ideals of R given by $X \mapsto I(X)$ with inverse $I \mapsto Z(I)$.*

This allows us to translate geometry into algebra and back. For example, an algebraic subset X is called *irreducible* if it cannot be written as a union of two proper algebraic sets. This implies that $I(X)$ is *prime* or equivalently that $R/I(X)$ has no zero divisors. We summarize the correspondence below:

Theorem 2.3.2 (Hilbert). *With R as above, the map $I \mapsto Z(I)$ gives a one-to-one correspondence between the objects in the left- and right-hand columns below:*

Algebra	*Geometry*
maximal ideals of R	*points of $\mathbb{A}^n$*
maximal ideals of R/J	*points of $Z(J)$*
prime ideals in R	*irreducible algebraic subsets of $\mathbb{A}^n$*
radical ideals in R	*algebraic subsets of $\mathbb{A}^n$*

If $U \subseteq \mathbb{A}^n_k$ is open, a function $F : U \to k$ is called *regular* if it can be expressed as a ratio of polynomials $F(x) = f(x)/g(x)$ such that g has no zeros on U.

Lemma 2.3.3. *Let $\mathscr{O}_{\mathbb{A}^n}(U)$ denote the set of regular functions on U. Then $U \mapsto \mathscr{O}_{\mathbb{A}^n}(U)$ is a sheaf of k-algebras. Thus $(\mathbb{A}^n_k, \mathscr{O}_{\mathbb{A}^n})$ is a k-space.*

Proof. It is clearly a presheaf. Suppose that $F : U \to k$ is represented by a ratio of polynomials f_i/g_i on $U_i \subseteq U$, where $\bigcup U_i = U$. Since $k[x_1, \dots, x_n]$ has unique factorization, we can assume that these are reduced fractions. Since $f_i(a)/g_i(a) = f_j(a)/g_j(a)$ for all $a \in U_i \cap U_j$, equality holds as elements of $k(x_1, \dots, x_n)$. Therefore, we can assume that $f_i = f_j$ and $g_i = g_j$. Thus $F \in \mathscr{O}_X(U)$. $\square$

An *affine algebraic variety* is an irreducible subset of some $\mathbb{A}^n_k$. We give X the topology induced from the Zariski topology of affine space. This is called the Zariski topology of X. Suppose that $X \subset \mathbb{A}^n_k$ is an algebraic variety. Given an open set

$U \subset X$, a function $F : U \to k$ is regular if it is locally extendible to a regular function on an open set of $\mathbb{A}^n$ as defined above, that is, if every point of U has an open neighborhood $V \subset \mathbb{A}^n_k$ with a regular function $G : V \to k$ for which $F = G|_{V \cap U}$.

Lemma 2.3.4. *Let X be an affine variety, and let $\mathscr{O}_X(U)$ denote the set of regular functions on U. Then $U \to \mathscr{O}_X(U)$ is a sheaf of k-algebras and $\mathscr{O}_X(X) \cong k[x_0,\ldots,x_n]/I(X)$.*

Proof. The sheaf property of $\mathscr{O}_X$ is clear from Exercise 2.1.15 and the previous lemma. So it is enough to prove the last statement. Let $S = k[x_0,\ldots,x_n]/I(X)$. Clearly there is an injection of $S \to \mathscr{O}_X(X)$ given by sending a polynomial to the corresponding regular function. Suppose that $F \in \mathscr{O}_X(X)$. Let $J = \{g \in S \mid gF \in S\}$. This is an ideal, and it suffices to show that $1 \in J$. By the Nullstellensatz, it is enough to check that $Z(J') = \emptyset$, where $J' \subset k[x_0,\ldots,x_n]$ is the preimage of J. By assumption, for any $a \in X$ there exist polynomials f, g such that $g(a) \neq 0$ and $F(x) = f(x)/g(x)$ for all x in a neighborhood of a. We have $\bar{g} \in J$, where $\bar{g}$ is the image of g in S. Therefore $a \notin Z(J')$. □

Thus an affine variety gives rise to a k-space $(X, \mathscr{O}_X)$. The ring of global regular functions $\mathscr{O}(X) = \mathscr{O}_X(X)$ is an integral domain called the *coordinate ring* of X. Its field of fractions $k(X)$ is called the *function field* of X. An element f/g of this field that determines a regular function on the open subset $D(g)$ is called a *rational function* on X.

As we will explain later, $\mathscr{O}(X)$ is a complete invariant for an affine variety, that is, it is possible to reconstruct X from its coordinate ring. For now, we will be content to recover the underlying topological space. Given a ring R, we define the maximal ideal spectrum $\mathrm{Spec}_m R$ as the set of maximal ideals of R. For any ideal $I \subset R$, let

$$V(I) = \{p \in \mathrm{Spec}_m R \,|\, I \subseteq p\}.$$

The verification of the following standard properties will be left as an exercise.

Lemma 2.3.5.

(a) $V(IJ) = V(I) \cup V(J)$.
(b) $V(\sum I_i) = \bigcap_i V(I_i)$.

As a corollary, it follows that the collection of sets of the form $V(I)$ constitutes the closed sets of a topology on $\mathrm{Spec}_m R$ called the Zariski topology once again. A basis of the Zariski topology on $\mathrm{Spec}_m R$ is given by $D(f) = X - V(f)$.

Lemma 2.3.6. *Suppose that X is an affine variety. Given $a \in X$, let*

$$m_a = \{f \in \mathscr{O}(X) \mid f(a) = 0\}.$$

Then m_a is a maximal ideal, and the map $a \mapsto m_a$ induces a homeomorphism

$$X \cong \mathrm{Spec}_m \mathscr{O}(X).$$

Proof. The set m_a is clearly an ideal. It is maximal because evaluation at a induces an isomorphism $\mathscr{O}(X)/m_a \cong k$. The bijectivity of $a \mapsto m_a$ follows from the Nullstellensatz. The pullback of $V(I)$ is precisely $Z(I)$. □

In the sequel, we will use the symbols Z and V interchangeably.

Exercises

2.3.7. Identify $\mathbb{A}_k^{n^2}$ with the space of square matrices. Determine the closures in the Zariski topology of

(a) The set of matrices of rank r.
(b) The set of diagonalizable matrices.
(c) The set of matrices A of finite order in the sense that $A^N = I$ for some N.

2.3.8. Prove Lemma 2.3.5.

2.3.9. Given affine varieties $X \subset \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$. Define a map $F : X \to Y$ to be a morphism if

$$F(a_1, \ldots, a_n) = (f_1(a_1, \ldots, a_n), \ldots, f_m(a_1, \ldots, a_n))$$

for polynomials $f_i \in k[x_1, \ldots, x_n]$. Show that a morphism F is continuous and F^*f is regular whenever f is regular function defined on $U \subset Y$. Conversely, show that any map with this property is a morphism. Finally, show that morphisms are closed under composition, so that they form a category.

2.3.10. Show that the map $X \to \mathscr{O}(X)$ determines a contravariant functor from the category of affine varieties to the category of affine domains, i.e., finitely generated k-algebras that are domains, and algebra homomorphisms. Show that this determines an antiequivalence of categories, which means

1. $\mathrm{Hom}(X,Y) \cong \mathrm{Hom}(\mathscr{O}(Y), \mathscr{O}(X))$.
2. Every affine domain is isomorphic to some $\mathscr{O}(X)$.

2.3.11. Given closed (irreducible) subsets $X \subset \mathbb{A}_k^n$ and $Y \subset \mathbb{A}_k^m$, show that $X \times Y \subset \mathbb{A}_k^{n+m}$ is closed (and irreducible). This makes $X \times Y$ into an affine variety called the product. Prove that $\mathscr{O}(X \times Y) \cong \mathscr{O}(X) \otimes_k \mathscr{O}(Y)$ as algebras, and deduce that $X \times Y \cong X' \times Y'$ if $X \cong X'$ and $Y \cong Y'$. So it does not depend on the embedding.

2.3.12. An (affine) algebraic group is an algebraic geometer's version of a Lie group. It is an affine variety G that is also a group such that the group multiplication $G \times G \to G$ and inversion $G \to G$ are morphisms. Show that the set $G = \mathrm{GL}_n(k)$ of $n \times n$ invertible matrices is an algebraic group (embed this into $\mathbb{A}_k^{n^2+1}$ by $A \mapsto (A, \det A)$).

2.4 Algebraic Varieties

Fix an algebraically closed field k once again. In analogy with manifolds, we can define an (abstract) algebraic variety as a k-space that is locally isomorphic to an affine variety and that satisfies some version of the Hausdorff condition. It will be convenient to ignore this last condition for the moment. The resulting objects are dubbed prevarieties.

Definition 2.4.1. A prevariety over k is a k-space $(X, \mathscr{O}_X)$ such that X is connected and there exists a finite open cover $\{U_i\}$, called an affine open cover, such that each $(U_i, \mathscr{O}_X|_{U_i})$ is isomorphic, as a k-space, to an affine variety. A morphism of prevarieties is a morphism of the underlying k-spaces.

Before going further, let us consider the most important nonaffine example.

Example 2.4.2. Let $\mathbb{P}^n_k$ be the set of one-dimensional subspaces of k^{n+1}. Using the natural projection $\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n_k$, give $\mathbb{P}^n_k$ the quotient topology ($U \subset \mathbb{P}^n_k$ is open if and only if $\pi^{-1}U$ is open). Equivalently, the closed sets of $\mathbb{P}^n_k$ are zeros of sets of homogeneous polynomials in $k[x_0, \ldots, x_n]$. Define a function $f : U \to k$ to be regular exactly when $f \circ \pi$ is regular. Such a function can be represented as the ratio

$$f \circ \pi(x_0, \ldots, x_n) = \frac{p(x_0, \ldots, x_n)}{q(x_0, \ldots, x_n)}$$

of two homogeneous polynomials of the same degree such that q has no zeros on $\pi^{-1}U$. Then the presheaf of regular functions $\mathscr{O}_{\mathbb{P}^n}$ is a sheaf, and the pair $(\mathbb{P}^n_k, \mathscr{O}_{\mathbb{P}^n})$ is easily seen to be a prevariety with affine open cover $\{U_i\}$ as in Example 2.2.13.

The Zariski topology is never actually Hausdorff except in the most trivial cases. However, there is a good substitute called the separation axiom. To motivate it, we make the following observation:

Lemma 2.4.3. *Let X be a topological space. Then the following statements are equivalent:*

(a) X is Hausdorff.
(b) If $f, g : Y \to X$ is a pair of continuous functions, the set $\{y \in Y \mid f(y) = g(y)\}$ is closed.
(c) The diagonal $\Delta = \{(x,x) \mid x \in X\}$ is closed in $X \times X$ with its product topology.

Proof. Suppose that X is Hausdorff. If $f, g : Y \to X$ are continuous functions with $f(y_0) \neq g(y_0)$, then $f(y) \neq g(y)$ for y in a neighborhood of y_0, so (b) holds. Assuming that (b), (c) is obtained by applying (b) to the projections $p_1, p_2 : X \times X \to X$. The final implication from (c) to (a) is clear. □

We use (b) of the previous lemma as the model for our separation axiom.

Definition 2.4.4. A prevariety $(X, \mathscr{O}_X)$ is said to be a variety, or is called separated, if for any prevariety Y and pair of morphisms $f, g : Y \to X$, the set $\{y \in Y \mid f(y) = g(y)\}$ is closed. A morphism of varieties is simply a morphism of prevarieties.

Example 2.4.5. Let $X = \mathbb{A}^1 \cup \mathbb{A}^1$ glued along $U = \mathbb{A}^1 - \{0\}$ via the identity, but with the origins not identified. Then X is a prevariety, but it is not a variety, because the identity $U \to X$ extends to $\mathbb{A}^1$ in two different ways by sending 0 to the first or second copy of $\mathbb{A}^1$.

Item (c) of Lemma 2.4.3 gives a more usable criterion for separation. Before we can formulate the analogous condition for varieties, we need products. These were constructed for affine varieties in Exercise 2.3.11, and the general case can be reduced to this. Full details can be found in [92, I§6].

Proposition 2.4.6. *Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be prevarieties. Then the Cartesian product $X \times Y$ carries the structure of a prevariety such that the projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are morphisms, and if $(Z, \mathscr{O}_Z)$ is any prevariety that maps via morphisms f and g to X and Y, then the map $f \times g : Z \to X \times Y$ is a morphism of prevarieties.*

Lemma 2.4.7. *A prevariety is a variety if and only if the diagonal Δ is closed in $X \times X$.*

Proof. If X is a variety, then Δ is closed because it is the locus where p_1 and p_2 coincide. Conversely, if Δ is closed, then so is $\{y \mid f(y) = g(y)\} = (f \times g)^{-1}\Delta$. $\square$

Here are some basic examples.

Example 2.4.8. Affine spaces are varieties, since the diagonal $\Delta \subset \mathbb{A}_k^{2n} = \mathbb{A}_k^n \times \mathbb{A}_k^n$ is the closed set defined by $x_i = x_{i+n}$.

Example 2.4.9. Projective spaces are also varieties. The product can be realized as the image of the Segre map $\mathbb{P}_k^n \times \mathbb{P}_k^n \subset \mathbb{P}_k^{(n+1)(n+1)-1}$ given by

$$([x_0, \dots, x_n], [y_0, \dots, y_n]) \mapsto [x_0y_0, x_0y_1, \dots, x_ny_n].$$

The diagonal is given by explicit equations. See Exercise 2.4.14.

Further examples can be produced by taking suitable subsets. Let $(X, \mathscr{O}_X)$ be an algebraic variety over k. A closed irreducible subset $Y \subset X$ is called a *closed subvariety*. Given an open set $U \subset Y$, define $\mathscr{O}_Y(U)$ to be the set functions that are locally extendible to regular functions on X.

Proposition 2.4.10. *Suppose that $Y \subset X$ is a closed subvariety of an algebraic variety. Then $(Y, \mathscr{O}_Y)$ is an algebraic variety.*

Proof. Let $\{U_i\}$ be an open cover of X by affine varieties. Choose an embedding $U_i \subset \mathbb{A}_k^N$ as a closed subset. Then $Y \cap U_i \subset \mathbb{A}_k^N$ is also embedded as a closed set, and the restriction $\mathscr{O}_Y|_{Y \cap U_i}$ is the sheaf of functions on $Y \cap U_i$ that are locally extendible to regular functions on $\mathbb{A}_k^N$. Thus $(Y \cap U_i, \mathscr{O}_Y|_{Y \cap U_i})$ is an affine variety. This implies that Y is a prevariety. Denoting the diagonal of X and Y by Δ_X and Δ_Y respectively, we see that $\Delta_Y = \Delta_X \cap Y \times Y$ is closed in $X \times X$, and therefore in $Y \times Y$. $\square$

It is worth making the description of projective varieties, or closed subvarieties of projective space, more explicit. A nonempty subset of $\mathbb{A}_k^{n+1}$ is conical if it contains 0 and is stable under the action of $\lambda \in k^*$ given by $v \mapsto \lambda v$. Given $X \subset \mathbb{P}_k^n$, $CX = \pi^{-1}X \cup \{0\} \subset \mathbb{A}_k^{n+1}$ is conical, and all conical sets arise in this way. If $I \subseteq S = k[x_0,\ldots,x_n]$ is a homogeneous ideal, then $Z(I)$ is conical and so corresponds to a closed subset of $\mathbb{P}_k^n$. From the Nullstellensatz, we obtain a dictionary similar to the earlier one.

Theorem 2.4.11. *Let $S_+ = (x_0,\ldots,x_n)$ be the maximal ideal of the origin. Then there is a one-to-one correspondence as shown in the table below:*

Algebra	*Geometry*
homogeneous radical ideals in S other than S_+	*algebraic subsets of $\mathbb{P}^n$*
homogeneous prime ideals in S other than S_+	*algebraic subvarieties of $\mathbb{P}^n$*

Given a subvariety $X \subseteq \mathbb{P}_k^n$, the elements of $\mathcal{O}_X(U)$ are functions $f : U \to k$ such that $f \circ \pi$ is regular. Such a function can be represented locally as the ratio of two homogeneous polynomials of the same degree.

When $k = \mathbb{C}$, we can use the stronger classical topology on $\mathbb{P}_{\mathbb{C}}^n$ introduced in Example 2.2.13. This is inherited by subvarieties, and is also called the classical topology. When there is danger of confusion, we write X^{an} to indicate a variety X with its classical topology. (The superscript an stands for "analytic.")

Exercises

2.4.12. Given an open subset U of an algebraic variety X, let $\mathcal{O}_U = \mathcal{O}_X|_U$. Prove that $(U, \mathcal{O}_U)$ is a variety. An open subvariety of a projective (respectively affine) variety is called quasiprojective (respectively quasiaffine).

2.4.13. Let $X = \mathbb{A}_k^n - \{0\}$ with $n > 2$. Show that $\mathcal{O}(X) \cong k[x_1,\ldots,x_n]$. Deduce that X is not isomorphic to an affine variety with the help of Exercise 2.3.10.

2.4.14. Verify that the image of Segre's embedding $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{(n+1)(m+1)-1}$ is Zariski closed, and the diagonal Δ is closed in the product when $m = n$.

2.4.15. Prove that $\mathcal{O}(\mathbb{P}_k^n) = k$. Deduce that $\mathbb{P}_k^n$ is not affine unless $n = 0$.

2.4.16. Fix an integer $d > 0$ and let $N = \binom{n+d}{d} - 1$. The dth Veronese map $v_d : \mathbb{P}_k^n \to \mathbb{P}_k^N$ is given by sending $[x_0,\ldots,x_n]$ to $[v]$, where v is the vector of degree-d monomials listed in some order. Show that this map is a morphism and that the image is Zariski closed.

2.4.17. Given a nonconstant homogeneous polynomial $f \in k[x_0,\ldots,x_n]$, define $D(f)$ to be the complement of the hypersurface in $\mathbb{P}_k^n$ defined by $f = 0$. Prove that $(D(f), \mathcal{O}_{\mathbb{P}^n}|_{D(f)})$ is an affine variety. (Use the Veronese map to reduce to the case of a linear polynomial.)

2.4.18. Suppose that X is a prevariety such that any pair of points is contained in an affine open set. Prove that X is a variety.

2.4.19. Make the Grassmannian $\mathbb{G}_k(r,n)$, which is the set of r-dimensional subspaces of k^n, into a prevariety by imitating the constructions of Exercise 2.2.22. Check that $\mathbb{G}_k(r,n)$ is in fact a variety.

2.4.20. After identifying $k^6 \cong \wedge^2 k^4$, $\mathbb{G}_k(2,4)$ can be embedded in $\mathbb{P}^5_k$ by sending the span of $v,w \in k^4$ to the line spanned by $\omega = v \wedge w$. Check that this is a morphism and that the image is a subvariety given by the Plücker equation $\omega \wedge \omega = 0$. Write this out as a homogeneous quadratic polynomial equation in the coordinates of ω.

2.4.21. Given an algebraic group G (Exercise 2.3.12), an action on a variety X is a morphism $G \times X \to X$ denoted by "$\cdot$" such that $(gh)\cdot x = g \cdot (h \cdot x)$. A variety is called homogeneous if an algebraic group acts transitively on it. Check that affine spaces, projective spaces, and Grassmannians are homogeneous.

2.4.22. The blowup of the origin of $\mathbb{A}^n_k$ is the set

$$B = \mathrm{Bl}_0\mathbb{A}^n = \{(v,\ell) \in \mathbb{A}^n_k \times \mathbb{P}^{n-1}_k \mid v \in \ell\}.$$

Show that this is Zariski closed and irreducible. When $k = \mathbb{C}$, show that B is a complex submanifold of $\mathbb{C}^n \times \mathbb{P}^{n-1}_{\mathbb{C}}$. Show that the morphism $\pi : B \to \mathbb{A}^n_k$ given by projection is an isomorphism over $\mathbb{A}^n_k - \{0\}$.

2.4.23. Given an affine variety $X \subset \mathbb{A}^n_k$ containing 0, its blowup is given by

$$\mathrm{Bl}_0 X = \overline{\pi^{-1}(X - \{0\})} \subset \mathrm{Bl}_0\mathbb{A}^n \subset \mathbb{A}^n_k \times \mathbb{P}^{n-1}_k.$$

Given a variety $x \in X$ with affine open cover $\{U_i\}$, show that there exists a variety $\mathrm{Bl}_x X$ locally isomorphic to $\mathrm{Bl}_x(U_i)$.

2.5 Stalks and Tangent Spaces

Given two functions defined in possibly different neighborhoods of a point $x \in X$, we say they have the same *germ* at x if their restrictions to some common neighborhood agree. This is is an equivalence relation. The germ at x of a function f defined near X is the equivalence class containing f. We denote this by f_x.

Definition 2.5.1. Given a presheaf of functions $\mathscr{P}$, its stalk $\mathscr{P}_x$ at x is the set of germs of functions contained in some $\mathscr{P}(U)$ with $x \in U$.

It will be useful to give a more abstract characterization of the stalk using *direct limits* (which are also called inductive limits, or filtered colimits). We explain direct limits in the present context, and refer to [33, Appendix 6] or [76] for a more complete discussion. Suppose that a set L is equipped with a family of maps $\mathscr{P}(U) \to L$, where U ranges over open neighborhoods of x. We will say that the family is a compatible family if $\mathscr{P}(U) \to L$ factors through $\mathscr{P}(V)$ whenever $V \subset U$. For instance, the maps $\mathscr{P}(U) \to \mathscr{P}_x$ given by $f \mapsto f_x$ form a compatible family. A set L equipped with a compatible family of maps is called a direct limit of $\mathscr{P}(U)$ if for any M with a compatible family $\mathscr{P}(U) \to M$, there is a unique map $L \to M$ making the obvious diagrams commute. This property characterizes L up to isomorphism, so we may speak of *the* direct limit

$$\varinjlim_{x \in U} \mathscr{P}(U).$$

Lemma 2.5.2. $\mathscr{P}_x = \varinjlim_{x \in U} \mathscr{P}(U)$.

Proof. Suppose that $\phi : \mathscr{P}(U) \to M$ is a compatible family. Then $\phi(f) = \phi(f|_V)$ whenever $f \in \mathscr{P}(U)$ and $x \in V \subset U$. Therefore $\phi(f)$ depends only on the germ f_x. Thus ϕ induces a map $\mathscr{P}_x \to M$ as required. □

All the examples of k-spaces encountered so far (C^∞-manifolds, complex manifolds, and algebraic varieties) satisfy the following additional property.

Definition 2.5.3. We will say that a concrete k-space $(X, \mathscr{R})$ is *locally ringed* if $1/f \in \mathscr{R}(U)$ when $f \in \mathscr{R}(U)$ is nowhere zero.

Recall that a ring R is *local* if it has a unique maximal ideal, say m. The quotient R/m is called the *residue field*. We will often convey all this by referring to the triple $(R, m, R/m)$ as a local ring.

Lemma 2.5.4. *If $(X, \mathscr{R})$ is locally ringed, then for any $x \in X$, $\mathscr{R}_x$ is a local ring with residue field isomorphic to k.*

Proof. Let m_x be the set of germs of functions vanishing at x. For $\mathscr{R}_x$ to be local with maximal ideal m_x, it is necessary and sufficient that each $f \in \mathscr{R}_x - m_x$ be invertible. This is clear, since $1/f|_U \in R(U)$ for some $x \in U$.

To see that $\mathscr{R}_x/m_x = k$, it is enough to observe that the ideal m_x is the kernel of the evaluation map $\mathrm{ev} : \mathscr{R}_x \to k$ given by $\mathrm{ev}(f) = f(x)$, and the map is surjective, because $\mathrm{ev}(a) = a$ when $a \in k$. □

When $(X, \mathscr{O}_X)$ is an n-dimensional complex manifold, the local ring $\mathscr{O}_{X,x}$ can be identified with ring of convergent power series in n variables. When X is a variety, the local ring $\mathscr{O}_{X,x}$ is also well understood. We may replace X by an affine variety with coordinate ring $R = \mathscr{O}_X(X)$. Consider the maximal ideal

$$m_x = \{f \in R \mid f(x) = 0\}.$$

Lemma 2.5.5. *$\mathscr{O}_{X,x}$ is isomorphic to the localization*

$$R_{m_x} = \left\{ \frac{g}{f} \mid f, g \in R, f \notin m_x \right\}.$$

Proof. Let K be the field of fractions of R. A germ in $\mathcal{O}_{X,x}$ is represented by a regular function defined in a neighborhood of x, but this is the fraction $f/g \in K$ with $g \notin m_x$. □

By standard commutative algebra [8, Corollary 7.4], the local rings of algebraic varieties are Noetherian, since they are localizations of Noetherian rings. This is also true for complex manifolds, although the argument is a bit more delicate [46, p. 12]. By contrast, when X is a C^∞-manifold, the stalks are non-Noetherian local rings. This is easy to check by a theorem of Krull [8, pp. 110–111] that implies that a Noetherian local ring R with maximal ideal m satisfies $\cap_n m^n = 0$. When R is the ring of germs of C^∞ functions on $\mathbb{R}$, then the intersection $\cap_n m^n$ contains nonzero functions such as

$$\begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Nevertheless, the maximal ideals are finitely generated.

Proposition 2.5.6. *If R is the ring of germs at 0 of C^∞ functions on $\mathbb{R}^n$, then its maximal ideal m is generated by the coordinate functions $x_1, \ldots, x_n$.*

Proof. See the exercises. □

In order to talk about tangent spaces in this generallity, it will be convenient to introduce the following axioms:

Definition 2.5.7. We will say that a local ring R with maximal ideal m and residue field k satisfies the tangent space conditions if

1. There is an inclusion $k \subset R$ that gives a splitting of the natural map $R \to k$.
2. The ideal m is finitely generated.

For stalks of C^∞ and complex manifolds and algebraic varieties over k, the residue fields are respectively $\mathbb{R}, \mathbb{C}$, and k. The inclusion of germs of constant functions gives the first condition in these examples, and the second was discussed above.

Definition 2.5.8. When (R, m, k) is a local ring satisfying the tangent space conditions, we define its cotangent space as $T_R^* = m/m^2 = m \otimes_R k$, and its tangent space as $T_R = \mathrm{Hom}(T_R^*, k)$. When X is a manifold or variety, we write $T_x = T_{X,x}$ (respectively $T_x^* = T_{X,x}^*$) for $T_{\mathcal{O}_{X,x}}$ (respectively $T_{\mathcal{O}_{X,x}}^*$).

When (R, m, k) satisfies the tangent space conditions, R/m^2 splits canonically as $k \oplus T_x^*$.

Definition 2.5.9. Let (R, m, k) satisfy the tangent space conditions. Given $f \in R$, define its differential df as the projection of $(f \mod m^2)$ to T_x^* under the above decomposition.

To see why this terminology is justified, we compute the differential when R is the ring of germs of C^∞ functions on $\mathbb{R}^n$ at 0. Then $f \in R$ can be expanded using Taylor's formula,

$$f(x_1,\ldots,x_n) = f(0) + \sum \frac{\partial f}{\partial x_i}\bigg|_0 x_i + r(x_1,\ldots,x_n),$$

where the remainder r lies in m^2. Therefore df coincides with the image of the second term on the right, which is the usual expression

$$df = \sum \frac{\partial f}{\partial x_i}\bigg|_0 dx_i.$$

Lemma 2.5.10. *$d : R \to T_R^*$ is a k-linear derivation, i.e., it satisfies the Leibniz rule $d(fg) = f(x)dg + g(x)df$.*

Proof. See the exercises. $\square$

As a corollary, it follows that a tangent vector $v \in T_R = T_R^{**}$ gives rise to a derivation $\delta_v = v \circ d : R \to k$.

Lemma 2.5.11. *The map $v \mapsto \delta_v$ yields an isomorphism between T_R and the vector space $\mathrm{Der}_k(R,k)$ of k-linear derivations from R to k.*

Proof. Given $\delta \in \mathrm{Der}_k(R,k)$, we can see that $\delta(m^2) \subseteq m$. Therefore it induces a map $v : m/m^2 \to R/m = k$, and we can check that $\delta = \delta_v$. $\square$

Lemma 2.5.12. *When (R,m,k) is the ring of germs at 0 of C^∞ functions on $\mathbb{R}^n$ (or holomorphic functions on $\mathbb{C}^n$, or regular functions on $\mathbb{A}_k^n$). Then a basis for $\mathrm{Der}_k(R,k)$ is given by*

$$D_i = \frac{\partial}{\partial x_i}\bigg|_0, \quad i = 1,\ldots,n.$$

A homomorphism $F : S \to R$ of local rings is called *local* if it takes the maximal ideal of S to the maximal ideal of R. Under these conditions, we get map of cotangent spaces $T_S^* \to T_R^*$ called the codifferential of F. When residue fields coincide, we can dualize this to get a map $dF : T_R \to T_S$. We study this further in the exercises.

A big difference between algebraic varieties and manifolds is that the former can be very complicated, even locally. We want to say that a variety over an algebraically closed field k is nonsingular or smooth if it looks like affine space (very) locally, and in particular if it is a manifold when $k = \mathbb{C}$. The implicit function suggests a way to make this condition more precise. Suppose that $X \subseteq \mathbb{A}_k^N$ is a closed subvariety defined by the ideal $(f_1,\ldots,f_r)$ and let $x \in X$. Then $x \in X$ should be nonsingular if the Jacobian matrix $(\frac{\partial f_i}{\partial x_j}|_x)$ has the expected rank $N - \dim X$, where $\dim X$ can be defined as the transcendence degree of the function field $k(X)$ over k. We can reformulate this in a more intrinsic fashion thanks to the following:

Lemma 2.5.13. *The vector space $T_{X,x}$ is isomorphic to the kernel of $(\frac{\partial f_i}{\partial x_j}|_x)$.*

Proof. Let $R = \mathscr{O}_{\mathbb{A}^N,x}$, $S = \mathscr{O}_{X,x} \cong R/(f_1,\ldots,f_r)$ and $\pi : R \to S$ be the natural map. We also set $J = (\frac{\partial f_i}{\partial x_j}|_x)$. Then any element $\delta' \in \mathrm{Der}_k(S,k)$ gives a derivation $\delta' \circ \pi \in \mathrm{Der}_k(R,k)$, which vanishes only if δ' vanishes. A derivation in $\delta \in \mathrm{Der}_k(R,k)$ comes from S if and only if $\delta(f_i) = 0$ for all i. We can use the basis $\partial/\partial x_j|_x$ to identify $\mathrm{Der}_k(R,k)$ with k^N. Putting all of this together gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}(S,k) \arrow[r] & \mathrm{Der}(R,k) \arrow[r, "\delta\mapsto\delta(f_i)"] \arrow[d, "\cong"] & k^r \arrow[d, "="] \\
 & & k^N \arrow[r, "J"] & k^r
\end{tikzcd}$$

from which the lemma follows. $\square$

Definition 2.5.14. A point x on a (not necessarily affine) variety X is called a nonsingular or smooth point if $\dim T_{X,x} = \dim X$; otherwise, x is called singular; X is nonsingular or smooth if every point is nonsingular.

The condition for nonsingularity of x is usually formulated as saying that the local ring $\mathscr{O}_{X,x}$ is a regular local ring, which means that $\dim \mathscr{O}_{X,x} = \dim T_x$ [8, 33]. But this is equivalent to what was given above, since $\dim X$ coincides with the Krull dimension [8, 33] of the ring $\mathscr{O}_{X,x}$. Affine and projective spaces are examples of nonsingular varieties.

Over $\mathbb{C}$, we have the following characterization.

Proposition 2.5.15. *Given a subvariety $X \subset \mathbb{C}^N$ and a point $x \in X$, the point x is nonsingular if and only if there exists a neighborhood U of x in $\mathbb{C}^N$ for the classical topology such that $X \cap U$ is a closed complex submanifold of $\mathbb{C}^N$, with dimension equal to* $\dim X$.

Proof. This follows from the holomorphic implicit function theorem [66, Theorem 2.1.2]. $\square$

Corollary 2.5.16. *Given a nonsingular algebraic subvariety X of $\mathbb{A}^n_{\mathbb{C}}$ or $\mathbb{P}^n_{\mathbb{C}}$, the space X^{an} is a complex submanifold of $\mathbb{C}^n$ or $\mathbb{P}^n_C$.*

Finally, we note the following result:

Proposition 2.5.17. *The set of nonsingular points of an algebraic variety forms an open dense subset.*

Proof. See [60, II Corollary 81.6]. $\square$

Exercises

2.5.18. Prove Proposition 2.5.6. (Hint: given $f \in m$, let

$$f_i = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_n)\, dt.$$

Show that $f = \sum f_i x_i$.)

2.5.19. Prove Lemma 2.5.10.

2.5.20. Let $F : (X, \mathscr{R}) \to (Y, \mathscr{S})$ be a morphism of locally ringed k-spaces. If $x \in X$ and $y = F(x)$, check that the homomorphism $F^* : \mathscr{S}_y \to \mathscr{R}_x$ taking a germ of f to the germ of $f \circ F$ is well defined and is local. Conclude that there is an induced linear map $dF : T_x \to T_y$, called the differential or derivative.

2.5.21. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ map taking 0 to 0. Calculate $dF : T_0 \to T_0$, constructed above, and show that this is given by a matrix of partial derivatives.

2.5.22. Check that with the appropriate identification given a C^∞ function on X viewed as a C^∞ map from $f : X \to \mathbb{R}$, df in the sense of Definition 2.5.9 and in the sense of the previous exercise coincide.

2.5.23. Check that the operation $(X, x) \mapsto T_x$ determines a functor on the category of C^∞-manifolds and base-point-preserving maps. (The definition of functor can be found in §3.1.) Interpret this as the chain rule.

2.5.24. Given a Lie group G with identity e, an element $g \in G$ acts on G by $h \mapsto ghg^{-1}$. Let $\mathrm{Ad}(g) : T_e \to T_e$ be the differential of this map. Show that Ad defines a homomorphism from G to $\mathrm{GL}(T_e)$ called the adjoint representation.

2.5.25. The ring of dual numbers D is defined as $k[\varepsilon]/(\varepsilon^2)$. Let (R, m, k) be a local ring satisfying the tangent space conditions. Show that T_R is isomorphic to the space of k-algebra homomorphisms $\mathrm{Hom}_{k\text{-alg}}(R, D)$.

2.5.26. Prove the identity $\det(I + \varepsilon A) = 1 + \mathrm{trace}(A)\varepsilon$ for square matrices over D. Use this to prove that the tangent space T_I to $\mathrm{SL}_n(k)$ is isomorphic to the space of trace-zero $n \times n$ matrices, where $\mathrm{SL} - n(k)$ is the group of matrices with determinant 1.

2.5.27. If $f(x_0, \dots, x_n)$ is a homogeneous polynomial of degree d, prove Euler's formula $\sum x_i \frac{\partial f}{\partial x_i} = d \cdot f(x_0, \dots, x_n)$. Use this to show that the point p on the projective hypersurface defined by f is singular if and only if all the partials of f vanish at p. Determine the set of singular points defined by $x_0^5 + \cdots + x_4^5 - 5x_0 \cdots x_4$ in $\mathbb{P}^4_{\mathbb{C}}$.

2.5.28. Prove that if $X \subset \mathbb{A}^N_k$ is a variety, then $\dim T_x \leq N$. Give an example of a curve in $\mathbb{A}^N_k$ for which equality is attained, for $N = 2, 3, \dots$.

2.5.29. Give a direct proof of Proposition 2.5.17 for hypersurfaces in $\mathbb{A}^n_k$.

2.5.30. Show that a homogeneous variety is nonsingular.

2.5.31. Let $f(x_0,\ldots,x_n) = 0$ define a nonsingular hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$. Show that there exists a hyperplane H such that $X \cap H$ is nonsingular. This is a special case of Bertini's theorem.

2.6 1-Forms, Vector Fields, and Bundles

A C^∞ vector field on a manifold X is essentially a choice $v_x \in T_x$, for each $x \in X$, that varies in a C^∞ fashion. The dual notion, called a covector field, a differential form of degree 1, or simply a 1-form, is easier to make precise. So we start with this. Given a C^∞ function f on X, we can define df as the collection of local derivatives $df_x \in T^*_x$. This is the basic example of a 1-form.

Definition 2.6.1. A C^∞ 1-form on X is a finite linear combination $\sum g_i df_i$ with $f_i, g_i \in C^\infty(X)$. Let $\mathscr{E}^1(X)$ denote the space of these.

Clearly, $\mathscr{E}^1(X)$ is a module over the ring $C^\infty(X)$. Also, given an open set $U \subset X$, a 1-form can be restricted to U as a $\bigcup T^*_x$-valued function. In this way, $U \mapsto \mathscr{E}^1(U)$ becomes a presheaf and in fact a sheaf. If U is a coordinate neighborhood with coordinates $x_1,\ldots,x_n$, then any 1-form on U can be expanded uniquely as $\sum f_i(x_1,\ldots,x_n)dx_i$ with C^∞ coefficients. In other words, $\mathscr{E}^1(U)$ is a free module with basis dx_i. The module $\mathscr{E}^1(X)$ is generally not free.

Now we can define vector fields as the dual in the appropriate sense. Let $\langle , \rangle$ denote the pairing between T_x and T^*_x.

Definition 2.6.2. A C^∞ vector field on X is a collection of vectors $v_x \in T_x$, $x \in X$, such that the map $x \mapsto \langle v_x, df_x \rangle$ lies in $C^\infty(U)$ for each open $U \subseteq X$ and $f \in C^\infty(U)$. Let $\mathscr{T}(X)$ denote the set of vector fields.

The definition is rigged to ensure that any $D \in \mathscr{T}(X)$ defines a derivation $C^\infty(U) \to C^\infty(U)$ by $f \mapsto \langle D, df \rangle$. It can be seen that $\mathscr{T}$ is a sheaf of $\bigcup T_x$-valued functions. If U is a coordinate neighborhood with coordinates $x_1,\ldots,x_n$, then any vector fields on U are given by $\sum f_i \partial/\partial x_i$.

There is another standard approach to defining vector fields on a manifold X. The disjoint union of the tangent spaces $T_X = \bigcup_x T_x$ can be assembled into a manifold called the tangent bundle T_X, which comes with a projection $\pi : T_X \to X$ such that $T_x = \pi^{-1}(x)$. We define the manifold structure on T_X in such a way that the vector fields correspond to C^∞ cross sections. The tangent bundle is an example of a structure called a vector bundle. In order to give the general definition simultaneously in several different categories, we will fix a choice of:

(a) a C^∞-manifold X and the standard C^∞-manifold structure on $k = \mathbb{R}$,
(b) a C^∞-manifold X and the standard C^∞-manifold structure on $k = \mathbb{C}$,
(c) a complex manifold X and the standard complex manifold structure on $k = \mathbb{C}$,
(d) an algebraic variety X with an identification $k \cong \mathbb{A}^1_k$.

A rank-n vector bundle is a map $\pi : V \to X$ that is locally a product $X \times k^n \to X$. Here is the precise definition.

Definition 2.6.3. A rank-n vector bundle on X is a morphism $\pi : V \to X$ such that there exist an open cover $\{U_i\}$ of X and commutative diagrams

$$\begin{tikzcd}
\pi^{-1}U_i \arrow[rr, "\cong", "\phi_i"'] \arrow[dr] & & U_i \times k^n \arrow[dl] \\
& U_i &
\end{tikzcd}$$

such that the isomorphisms

$$\phi_i \circ \phi_j^{-1} : U_i \cap U_j \times k^n \cong U_i \cap U_j \times k^n$$

are k-linear on each fiber. A bundle is called C^∞ real in case (a), C^∞ complex in case (b), holomorphic in case (c), and algebraic in case (d). A rank-1 vector bundle will also be called a line bundle.

The product $X \times k^n$ is an example of a vector bundle, called the *trivial bundle* of rank n. A simple nontrivial example to keep in mind is the Möbius strip, which is a real line bundle over the circle. The datum $\{(U_i, \phi_i)\}$ is called a local trivialization. Given a C^∞ real vector bundle $\pi : V \to X$, define the presheaf of sections

$$\mathscr{V}(U) = \{s : U \to \pi^{-1}U \mid s \text{ is } C^\infty,\ \pi \circ s = \mathrm{id}_U\}.$$

This is easily seen to be a sheaf. When $V = X \times \mathbb{R}^n$ is the trivial vector bundle, a section is given by $(x, f(x))$, where $f : X \to \mathbb{R}^n$, so that $\mathscr{V}(X)$ is isomorphic to the space of vector-valued C^∞ functions on X. In general, a section $s \in \mathscr{V}(U)$ is determined by the collection of vector-valued functions on $U_i \cap U$ given by projecting $\phi_i \circ s$ to $\mathbb{R}^n$. Thus $\mathscr{V}(U)$ has a natural $\mathbb{R}$-vector space structure. Later on, we will characterize the sheaves, called locally free sheaves, that arise from vector bundles in this way.

Theorem 2.6.4. *Given an n-dimensional manifold X, there exists a C^∞ real vector bundle T_X of rank n, called the tangent bundle, whose sheaf of sections is exactly $\mathscr{T}_X$.*

Proof. Complete details for the construction of T_X can be found in [110, 111, 117]. We outline a construction when $X \subset \mathbb{R}^N$ is a submanifold. (According to Whitney's embedding theorem [111], every manifold embeds into a Euclidean space. So in fact, this is no restriction at all.) Fix standard coordinates $y_1, \dots, y_N$ on $\mathbb{R}^N$. We define $T_X \subset X \times \mathbb{R}^N$ such that $(p; v_1, \dots, v_N) \in T_X$ if and only if

$$\sum v_i \left.\frac{\partial f}{\partial y_j}\right|_p = 0$$

whenever f is a C^∞ function defined in a neighborhood of p in $\mathbb{R}^N$ such that $X \cap U \subseteq f^{-1}(0)$. We have the obvious projection $\pi : T_X \to X$. A sum $v = \sum_j g_j \partial/\partial y_j$, with

$g_j \in C^\infty(U)$, defines a vector field on $U \subseteq X$ precisely when $(p; g_1(p), \dots, g_N(p)) \in T_X$ for a $p \in U$. In other words, vector fields are sections of T_X.

It remains to find a local trivialization. We can find an open cover $\{U_i\}$ of X by coordinate neighborhoods. Choose local coordinates $x_1^{(i)}, \dots, x_n^{(i)}$ in each U_i. Then the map

$$(p; w) \mapsto \left(p; \left(\frac{\partial y_j}{\partial x_k^{(i)}} \bigg|_p \right) w \right)$$

identifies $U_i \times \mathbb{R}^n$ with $\pi^{-1} U_i$. □

Tangent bundles also exist for complex manifolds and nonsingular algebraic varieties. However, we will postpone the construction. An example of an algebraic vector bundle of fundamental importance is given below.

Example 2.6.5. Projective space $\mathbb{P}^n_k$ is the set of lines $\{\ell\}$ in k^{n+1} through 0, and we can choose each line as a fiber of L. That is,

$$L = \{(x, \ell) \in k^{n+1} \times \mathbb{P}^n_k \,|\, x \in \ell\}.$$

Let $\pi : L \to \mathbb{P}^n_k$ be given by projection onto the second factor. Then L is a rank-one algebraic vector bundle, or line bundle, over $\mathbb{P}^n_k$. It is called the tautological line bundle. When $k = \mathbb{C}$, this can also be regarded as a holomorphic line bundle or a C^∞ complex line bundle.

L is often called the universal line bundle for the following reason:

Theorem 2.6.6. *If X is a compact C^∞ manifold with a C^∞ complex line bundle $p : M \to X$, then for $n \gg 0$, there exists a C^∞ map $f : X \to \mathbb{P}^n_{\mathbb{C}}$, called a classifying map, such that M is isomorphic (as a bundle) to the pullback*

$$f^* L = \{(v, x) \in L \times X \,|\, \pi(v) = f(x)\} \to X.$$

Proof. We just sketch the proof. Here we consider the dual line bundle M^* (see Exercise 2.6.14). Sections of this correspond to $\mathbb{C}$-valued functions on M that are linear on the fibers. Choose a local trivialization $\phi_i : M|_{U_i} \cong U_i \times \mathbb{C}$. A section of $M^*(U_i)$ can be identified with a function by $M^*(U_i) = C^\infty(U_i)\phi_i^{-1}(1)$. For each point $x \in U_i$, we can choose a C^∞ function f with compact support in U_i such that $f(x) \neq 0$ (which exists by Exercise 2.6.9). This can be extended by 0 to a global section. Thus by compactness, we can find finitely many sections $f_0, \dots, f_n \in M^*(X)$ that do not simultaneously vanish at any point $x \in X$. Therefore we get an injective bundle map $M \hookrightarrow X \times \mathbb{C}^n$ given by $v \mapsto (f_0(v), \dots, f_n(v))$. Or in more explicit terms, if we view $f_j|_{U_i}$ as functions, $M|_{U_i}$ can be identified with the span of $(f_0(x), \dots, f_n(x))$ in $U_i \times \mathbb{C}^n$.

The maps

$$x \mapsto [f_0(x), \dots, f_n(x)] \in \mathbb{P}^n, \quad x \in U_i,$$

are independent of the choice of trivialization. So this gives a map $f : X \to \mathbb{P}^n$. The pullback f^*L can also be described as the sub-line bundle of $X \times \mathbb{C}^n$ spanned by $(f_0(x), \dots, f_n(x))$. So this coincides with M. $\square$

Remark 2.6.7. When $M \to X$ is a holomorphic bundle on a complex manifold, the map $f : X \to \mathbb{P}^n_{\mathbb{C}}$ need not be holomorphic. This will follow from Exercise 2.7.13.

Exercises

2.6.8. Show that $v = \sum f_i(x) \frac{\partial}{\partial x_i}$ is a C^∞ vector field on $\mathbb{R}^n$ in the sense of Definition 2.6.2 if and only if the coefficients f_i are C^∞.

2.6.9. Show that the function

$$f(x) = \begin{cases} \exp\left(\frac{1}{\|x\|^2 - 1}\right) & \text{if } \|x\| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

defines a nonzero C^∞-function on $\mathbb{R}^n$ with support in the unit ball. Conclude that any C^∞-manifold possesses nonconstant C^∞-functions.

2.6.10. Check that L is an algebraic line bundle.

2.6.11. Given a vector bundle $\pi : V \to X$ over a manifold and a C^∞ map $f : Y \to X$, show that the set

$$f^*V = \{(y, v) \in Y \times V \mid \pi(v) = f(y)\}$$

with its first projection to Y determines a vector bundle.

2.6.12. Given a vector bundle $V \to X$ with local trivialization $\phi_i : V|_{U_i} \xrightarrow{\sim} U_i \times k^n$, check that the matrix-valued functions $g_{ij} = \phi_i^{-1} \circ \phi_j$ on $U_i \cap U_j$ satisfy the cocycle identity $g_{ik} = g_{ij} g_{jk}$ on $U_i \cap U_j \cap U_k$. Conversely, check that any collection g_{ij} satisfying this identity arises from a vector bundle.

2.6.13. Given a vector bundle V with cocycle g_{ij} (as in the previous exercise), show that a section can be identified with a collection of vector-valued functions f_i on U_i satisfying $f_i = g_{ij} f_j$.

2.6.14. Suppose we are given a vector bundle $V = \bigcup V_x \to X$ with cocycle g_{ij}. Show that the union of dual spaces $V^* = \bigcup V_x^*$ can be made into a vector bundle with cocycle $(g_{ij}^{-1})^T$. Show that the sections of the dual of the tangent bundle T_X^*, called the cotangent bundle, are exactly the 1-forms.

2.6.15. Let $G = \mathbb{G}(r, n)$ be the Grassmannian of r-dimensional subspaces of k^n. This is an algebraic variety by Exercise 2.4.19. Let $S = \{(x, V) \in k^n \times G \mid x \in V\}$. Show that the projection $S \to G$ is an algebraic vector bundle of rank 2. This is called the universal bundle of rank r on G.

2.6.16. Prove the analogue of Theorem 2.6.6 for rank-r vector bundles: Any rank-two C^∞ complex vector bundle on a compact C^∞ manifold X is isomorphic to the pullback of the universal bundle for some C^∞ map $X \to \mathbb{G}(r,n)$ with $n \gg 0$. You may assume without proof that for any vector bundle $\pi : V \to X$, there exist finitely many sections f_i that span the fibers $V_x = \pi^{-1}(x)$.

2.7 Compact Complex Manifolds and Varieties

Up to this point, we have been treating C^∞ and complex manifolds in parallel. However, there are big differences, owing to the fact that holomorphic functions are much more rigid than C^∞ functions. We illustrate this in a couple of ways. In particular, we will see that the (most obvious) holomorphic analogue to Theorem 2.6.6 would fail. We start by proving some basic facts about holomorphic functions in many variables.

Theorem 2.7.1. *Let Δ^n be an open polydisk, that is, a product of disks, in $\mathbb{C}^n$.*

(1) If two holomorphic functions on Δ^n agree on a nonempty open set, then they agree on all of Δ^n.
(2) The maximum principle: If the absolute value of a holomorphic function f on Δ^n attains a maximum in Δ^n, then f is constant on Δ^n.

Proof. This can be reduced to the corresponding statements in one variable [1]. We leave the first statement as an exercise and we do the second. Suppose that $|f|$ attains a maximum at $(a_1,\dots,a_n) \in \Delta$. The maximum principle in one variable implies that $f(z,a_2,\dots,a_n)$ is constant. Fixing $z_1 \in \Delta$, we see that $f(z_1,z,a_3,\dots)$ is constant, and so on. □

We saw in the exercises that all C^∞-manifolds carry nonconstant global C^∞-functions. By contrast we have the following:

Proposition 2.7.2. *If X is a compact connected complex manifold, then all holomorphic functions on X are constant.*

Proof. Let $f : X \to \mathbb{C}$ be holomorphic. Since X is compact, $|f|$ attains a maximum somewhere, say at $x_0 \in X$. The set $S = f^{-1}(f(x_0))$ is closed by continuity. It is also open by the maximum principle. So $S = X$. □

Corollary 2.7.3. *A holomorphic function is constant on a nonsingular complex projective variety.*

Proof. $\mathbb{P}^n_{\mathbb{C}}$ with its classical topology is compact, since the unit sphere in $\mathbb{C}^{n+1}$ maps onto it. Therefore any submanifold of it is also compact. □

We want to prove a version of Corollary 2.7.3 for algebraic varieties over arbitrary fields. We first need a good substitute for compactness. To motivate it, we make the following observation:

Lemma 2.7.4. *If X is a compact metric space, then for any metric space Y, the projection $p : X \times Y \to Y$ is closed.*

Proof. Given a closed set $Z \subset X \times Y$ and a sequence $y_i \in p(Z)$ converging to $y \in Y$, we have to show that y lies in $p(Z)$. By assumption, we have a sequence $x_i \in X$ such that $(x_i, y_i) \in Z$. Since X is compact, we can assume that x_i converges to say $x \in X$ after passing to a subsequence. Then we see that (x, y) is the limit of (x_i, y_i), so it must lie in Z because it is closed. Therefore $y \in p(Z)$. □

Definition 2.7.5. An algebraic variety X is called complete or proper if for any variety Y, the projection $p : X \times Y \to Y$ is closed, i.e., p takes closed sets to closed sets.

Theorem 2.7.6. *Projective varieties are complete.*

Proof. Complete proofs can be found in [60, 92, 104]. We give an outline, leaving the details for the exercises. First, we can reduce the theorem to showing that $\pi : \mathbb{P}^n_k \times \mathbb{A}^m_k \to \mathbb{A}^m_k$ is closed for each m, n. This case can be handled by classical elimination theory [22, §8.5]. A closed set of the product is defined by a collection of polynomials $f_i(x_0, \ldots, x_n, y_1, \ldots, y_m)$ homogeneous in the x's. Let I be the ideal in $k[x_0, \ldots, x_n, y_1, \ldots, y_m]$ generated by the f_i. The elimination ideal is defined by

$$J = \{g \in k[y_1, \ldots, y_m] \mid \forall i \exists e_i, x_i^{e_i} g \in I\}.$$

Then the image of the projection $\pi(V(I))$ is $V(J)$, hence closed. □

There is an analogue of Proposition 2.7.2.

Proposition 2.7.7. *If X is a complete algebraic variety, then all regular functions on X are constant.*

Proof. Let f be a regular function on X. We can view it as a morphism $f : X \to \mathbb{A}^1_k$. Since $\mathbb{A}^1_k \subset \mathbb{P}^1_k$, we also have a morphism $F : X \to \mathbb{P}^1_k$ given by composition. The image $f(X)$ is closed, since it coincides with the image of the graph $\{(x, f(x)) \mid x \in X\}$ under projection. Similarly, $F(X)$ is closed. Since X is irreducible, the same is true of $f(X)$. The only irreducible closed subsets of $\mathbb{A}^1_k$ are points and $\mathbb{A}^1_k$ itself. If $f(X) = \mathbb{A}^1_k$, then we would be forced to conclude that $F(X)$ was not closed. Therefore $f(X)$ must be a point. □

The converse is false (Exercise 2.7.15).

Corollary 2.7.8. *An affine variety is not complete, unless it is a point.*

Proof. If $p, q \in X$ are distinct, there exists a regular function such that $f(p) = 1$ and $f(q) = 0$. □

We can form the sheaf of regular sections of the tautological line bundle L (Example 2.6.5) on $\mathbb{P}^n_k$, which we denote by $\mathscr{O}_{\mathbb{P}^n_k}(-1)$ or often simply by $\mathscr{O}(-1)$. We write $\mathscr{O}_{\mathbb{P}^n_{an}}(-1)$, or simply $\mathscr{O}_{\mathbb{P}^n}(-1)$ when there is no danger of confusion, for the sheaf of holomorphic sections over the complex manifold $\mathbb{P}^n_{\mathbb{C}}$. These examples are central to algebraic geometry. For any open set $U \subset \mathbb{P}^n_k$ viewed as a set of lines, we have

$$\mathcal{O}(-1)(U) = \{f : U \to k^{n+1} \mid f \text{ regular and } f(\ell) \in \ell\}.$$

We saw in the course of proving Theorem 2.6.6 that any line has many C^∞-sections. The corresponding statements in the analytic and algebraic worlds are false:

Lemma 2.7.9. *The spaces of global sections $\mathcal{O}(-1)(\mathbb{P}^n_k)$ and $\mathcal{O}_{\mathbb{P}^n_{an}}(-1)(\mathbb{P}^n_{\mathbb{C}})$ are both equal to 0.*

Proof. We prove the first statement. The second is similar. A global section is given by a regular function $f : \mathbb{P}^n \to k^{n+1}$ satisfying $f(\ell) \in \ell$. However, f is constant with value, say, v. Thus $v \in \bigcap \ell = \{0\}$. □

Exercises

2.7.10. Finish the proof of Theorem 2.7.1 (1).

2.7.11. Check that Theorem 2.7.6 can be reduced to showing that $\pi : \mathbb{P}^n_k \times \mathbb{A}^m_k \to \mathbb{A}^m_k$ is closed.

2.7.12. In the notation of the proof of Theorem 2.7.6, show that the elimination ideal J is an ideal and that $V(J) = \pi(V(I))$. For the "hard" direction use the projective Nullstellensatz that $V(K) \subset \mathbb{P}^n_k$ is empty if and only if K contains a power of $(x_0, \dots, x_n)$.

2.7.13. If $f : X \to \mathbb{P}^n_{\mathbb{C}}$ is a nonconstant holomorphic function between compact manifolds, prove that $f^* \mathcal{O}_{\mathbb{P}^n}(-1)$ has no nonzero holomorphic sections.

2.7.14. Let $\mathcal{O}(1)$ denote the sheaf of holomorphic sections of the dual L^* (Exercise 2.6.14) of the tautological bundle on $\mathbb{P}^n_{\mathbb{C}}$. Show that $\mathcal{O}(1)$ has at least $n+1$ independent nonzero sections. Using the previous exercise, deduce that Theorem 2.6.6 will fail for holomorphic maps and line bundles.

2.7.15. Let $X = \mathbb{P}^n_k - \{p\}$ with $n > 1$. Show that X has no nonconstant regular functions, and that it is not complete.

2.7.16. Given a variety X, the collection of *constructible* subsets of X is the Boolean algebra generated by Zariski open sets. In other words, it is the smallest collection containing open sets and closed under finite unions, intersections, and complements. Prove Chevalley's theorem that a projection $p : \mathbb{A}^n_k \to \mathbb{A}^m_k$ takes constructible sets to constructible sets.

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