2

Generative Theory of Shape

2.1 New Foundations to Geometry

In my book, *A Generative Theory of Shape* (Springer-Verlag, 550 pages), I give New Foundations to Geometry. The present chapter gives a brief summary of the New Foundations because the Process Grammar is based on these foundations.

These New Foundations to Geometry directly oppose the Standard Foundations to Geometry that have existed for almost three thousand years. The central proposal in my New Foundations to Geometry is this:

NEW FOUNDATIONS TO GEOMETRY

Leyton 1992

According to the New Foundations to Geometry:
Shape is equivalent to memory storage.

Therefore, in the New Foundations:
Geometry is the mathematical theory of memory storage,
invented by the New Foundations.

Let us see how this opposes the Standard Foundations to Geometry. In the Standard Foundations, a geometric object consists of those properties of a figure that do not change under actions. These unchanged properties are called the *invariants* of the actions. Geometry began with the study of invariance, in the form of Euclid's concern with *congruence*, which is really a concern with invariance (properties that do not change). The full invariance program was explicitly defined in the 19th century by Felix Klein who defined geometry as the study of invariants under groups of transformations. Klein's invariance program became the basis of 20th century mathematics and physics.

M. Leyton, *Process Grammar: The Basis of Morphology*,
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My argument is that the problem with invariants is that they are *memoryless*. That is, if a property is invariant (unchanged) under an action, then one cannot infer, from the property, that the action has taken place. Thus I argue: *Invariants cannot act as memory stores*. In consequence, I conclude that the Standard Foundations to Geometry are concerned with *memorylessness*. In fact, since the Standard Foundations try to maximize the discovery of invariants, those foundations essentially try to maximize memorylessness.

A basic argument of mine is this: The Standard Foundations to Geometry are inappropriate for the computational age, because the computational age is based on memory storage, archival systems, etc., whereas the Standard Foundations are based on the invariants program, which tries to maximize memorylessness.

As a consequence, I embarked on a 30-year project to build up entirely New Foundations to Geometry. Rather than basing geometry on the *maximization of memorylessness* (the aim of the Standard Foundations), I base geometry on the *maximization of memory storage*. The result is a theoretical system that is profoundly different, both on a conceptual level and on a detailed mathematical level. Everything in the Standard Foundations is *inverted* in the New Foundations. For example, whereas in the Standard Foundations, groups are used to describe symmetries, in the New Foundations, groups are used to describe asymmetries. Again, whereas in the Standard Foundations, the goal is reference-frame independence, in the New Foundations, the goal is reference-frame dependence. Again, the relationship between hierarchical levels in the Standard Foundations, e.g., Euclidean to Affine to Projective, are entirely the opposite relationships in the New Foundations. These are all consequences of the fact that the Standard Foundations try to maximize memorylessness, and the New Foundations try to maximize memory storage.

The conceptual structure of the New Foundations is elaborated in my book *Symmetry, Causality, Mind* (MIT Press, 630 pages); and the mathematical structure is elaborated in my book *A Generative Theory of Shape* (Springer-Verlag, 550 pages). The latter book also gives extensive applications of the mathematics to computer-aided design, software engineering, computer vision, gestalt psychology, robotics, music, architecture, and physics. Besides these applications, parts of my mathematical theory have been applied by scientists in over 40 disciplines, such as chemical engineering, meteorology, radiology, geology, botany, structural engineering, mathematical control theory, etc.

The remainder of the present chapter will briefly describe some of the basic concepts and mathematical structures of my New Foundations to Geometry. This will give some useful background to the Process-Grammar.

First, being a generative theory of shape, the New Foundations define any shape by a sequence of operations needed to create it. Next, the New Foundations require that this sequence be *intelligent*. In fact:

The New Foundations to Geometry elaborate a mathematical theory of intelligence, and base the entire New Foundations on this mathematical theory.

The two most basic principles of this mathematical theory of intelligence are:

(1) Maximization of Transfer. Any agent is regarded as displaying intelligence and insight when it is able to *transfer* actions used in previous situations to new situations. In fact, the agent must maximize the transfer of parts of a generative sequence onto other parts of a generative sequence. Thus a basic part of the New Foundations to Geometry is giving a *Mathematical Theory of Transfer*.

(2) Maximization of Recoverability. Any intelligent agent must be able to infer the causes of its own current state, in order to identify why it failed or succeeded, and thereby edit its behavior. Notice that this is part of a still larger problem, which we call the problem of recoverability: Given the present state of an object, recover the sequence of operations which generated that current state. Thus a basic part of the New Foundations to Geometry is giving a *Mathematical Theory of Recoverability*.

2.2 Mathematical Theory of Transfer

Now let us begin to understand the Mathematical Theory of Transfer given by my New Foundations to Geometry. According to this theory, a situation of transfer involves two levels as illustrated in Fig 2.1: a group which the New Foundations call a **fiber group**, which is the group of actions *to be transferred*; and a group which the New Foundations call a **control group**, whose group elements will *transfer the fiber group*. The transferred versions of the fiber group are illustrated as the vertical copies in Fig 2.1, and the New Foundations calls these the **fiber-group copies**. The control group acts from above, and transfers the fiber-group copies onto each other, as illustrated by the arrow in the figure.

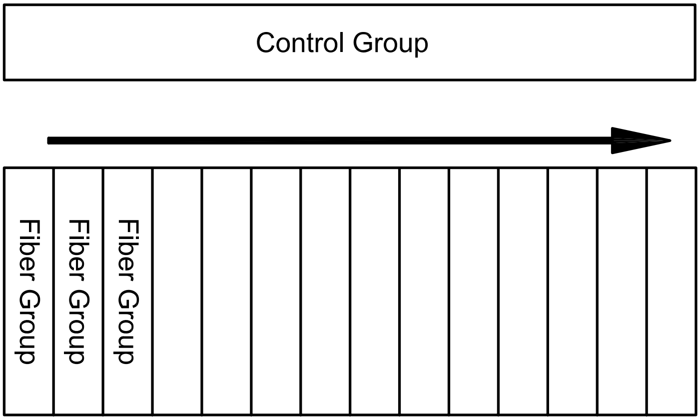

Fig. 2.1. The control group transferring the fiber-group copies onto each other.

Let us now describe the Mathematical Theory of Transfer in more detail. A claim in the theory is that a situation of transfer is built from two *group actions*. On the lower level, there is a group action of the fiber group $G(F)$, on a set F, and the New Foundations calls F the **fiber set**. On the upper level, there is a group action of the control group $G(C)$, on a set C, and the New Foundations calls C the **control set**.

Next, to model transfer, make the transferred versions of the fiber group, as follows: For each member c in the control set, make a copy $G(F)_c$ of the fiber group $G(F)$. These will be the transferred versions, i.e., the fiber-group copies, which again are illustrated by the columns in Fig 2.1. Most crucially, the action of the control group $G(C)$ on the control set C can therefore be *imitated* by an action of the control group $G(C)$ on the collection of fiber-group copies. It is this imitating action that will be regarded as the transferring action that the control group has on the fiber-group copies, i.e., this will be regarded as sending the fiber-group copies onto each other.

A fundamental property of the Mathematical Theory of Transfer is that it puts all these components together in a single encompassing structure. According to the theory, the encompassing structure is best given by a group-theoretic construct called a *wreath product*, which is defined as follows: Intuitively, a wreath-product is a group that contains the entire structure shown in Fig 2.1. The structure of this total group is as follows: In Fig 2.1, the entire lower block shown is the direct product $\prod_{c\in C} G(F)_c$ of the fiber-group copies. In the New Foundations, this is called the **fiber-group product**. Accordingly, the wreath product group is then built up by adding, to the fiber-group product, the control group $G(C)$ by a group-theoretic *semi-direct product*, explained as follows:

In any semi-direct product, the upper group (here the control group) sends the lower group (here the fiber-group product) to itself, by rearrangements that preserve the latter's group structure. Such rearrangements are called automorphisms. In a wreath product, this automorphic action is one in which the control group sends the fiber-group copies onto each other in a way that exactly imitates the action of the control group on the control set.

To state this rigorously: One constructs an **automorphism representation**

$$\tau : G(C) \longrightarrow Aut\{\prod_{c\in C} G(F)_c\}$$

such that, given an element g in the control group, its effect on the fiber-group product is defined thus:

$$\tau(g) : \prod_{c\in C} G(F)_c \longrightarrow \prod_{c\in C} G(F)_{g^{-1}c}$$

From this automorphism representation, one can then construct the corresponding external semi-direct product:

$$\{\prod_{c\in C} G(F)_c\} \circledS_{\tau} G(C)$$

To understand this notation, notice that, to the left of the semi-direct product symbol Ⓢ is the fiber-group product, i.e., the entire bottom block we diagramed in Fig 2.1. To the right of the Ⓢ symbol is the control group $G(C)$, the upper block diagramed in Fig 2.1. Notice also that the subscript τ on the symbol Ⓢ is the automorphism representation which defines what I call the *transfer* effect of the control group on the fiber-group copies.

It is this semi-direct product that is the *wreath product* of the fiber group and the control group, written like this:

$$\begin{aligned}\textbf{Fiber Group} \circledw \textbf{ Control Group} &= G(F) \circledw G(C) \\ &= \{\prod_{c \in C} G(F)_c\} \circledS_{\tau} G(C) \qquad (2.1)\end{aligned}$$

Let us now understand how the New Foundations to Geometry model transfer within the wreath product. The claim is that transfer corresponds to what is algebraically called conjugation, $g\phi g^{-1}$ where g is a member of the control group and ϕ is a member of the fiber-group product. Most crucially, let us understand its effect on the fiber-group copies. Notice that each fiber-group copy has an embedded version within the wreath product. We can call this, an *embedded fiber-group copy*; and, often, for convenience we will simply call it, a *fiber-group copy*. Similarly, the control group has an embedded version within the wreath product. Again, we can call this, the *embedded control group*; and, often, for convenience we will simply call it, the *control group*. The crucial fact is that, within the wreath product, the members of the control group send the fiber-group copies onto each other via *conjugation*. Therefore, we conclude:

Transfer of the fiber-group copies is modeled by their *algebraic conjugation,* **within the wreath product, by the members of the control group.**

Now a crucial role of the New Foundations to Geometry is that they give a comprehensive theory of **raw data representation** such that the data set is maximally useful throughout large-scale scientific and engineering systems. My book *A Generative Theory of Shape* [22] elaborates this theory in detail and comprehensively explains the use of this theory to represent **scientific data** and **manufacturing data**. The present chapter will give a summary of some parts of this theory, and the present book will show how the Process Grammar realizes this for morphology in both biology and manufacturing.

First we should note the following. Given each member c of the control set C, let us call the set-theoretic Cartesian product $F \times \{c\}$ the corresponding **fiber-set copy**, which will also be notated as F_c. By the principle of the Maximization of Transfer, any data set will actually be the union $F \times C$ of the fiber-set copies given by a transfer structure, i.e., a wreath product $G(F) \circledw G(C)$. Thus, given a wreath product, we will refer to the union of the fiber-set copies as the **data set**. A crucial fact is that there is a group action of the wreath-product group $G(F) \circledw G(C)$ on the data set $F \times C$ as follows: Given an element in the wreath-product group, i.e., an ordered pair $\langle\, \phi \mid g \,\rangle$ where $\phi \in \prod_{c \in C} G(F)_c, g \in G(C)$, and given an element (f, c) in the data set, define the effect of the former element on the latter, thus:

$$\langle\, \phi \mid g \,\rangle (f, c) \;=\; (\phi(gc)f, gc) \;\in\; F \times C \qquad (2.2)$$

Notice that this relates the **data-set element** (f, c) in the fiber-set copy F_c to the **data-set element** $(\phi(gc)f, gc)$ in the fiber-set copy F_{gc}.

Now let us examine an example to illustrate the Mathematical Theory of Transfer. Later in this chapter, and the Process Grammar, we will consider much more complex examples. But to enable the reader to begin to understand the mathematical theory, we will initially study a simple example. This example is the way the theory structures a square. We will model the typical way in which a person draws a square on a sheet of paper – i.e., drawing the sides sequentially around the square. Notice that, in fact, this involves a crucial transfer structure as follows:

The first side is generated by starting with a corner point, and applying translations to trace out the side, as shown in Fig 2.2.

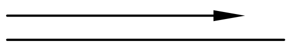

Fig. 2.2. The generation of a side, using translations.

Next, this translational structure is *transferred* from one side to the next – rotationally around the square. In other words, there is *transfer of translations by rotations*. This is illustrated in Fig 2.3.

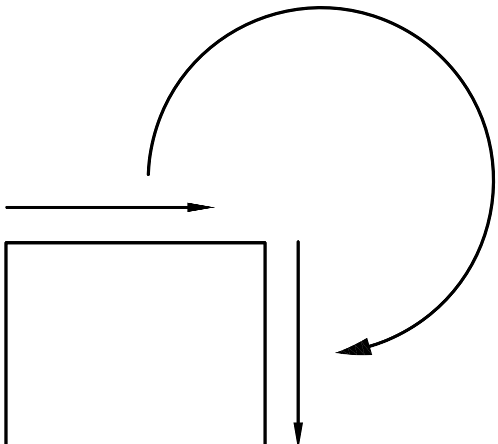

Fig. 2.3. Transfer of translation by rotation.

Therefore, according to our theory, the transfer structure, in drawing a square, is defined by the wreath product:

$$\text{Translations} \circledR{w} \text{Rotations}$$

where Translations is the fiber group (generating the side) and Rotations is the control group *transferring* the translation operations that generated the first side, in order to generate the next side, and so on, around the square. This will now be defined rigorously, as follows:

The translation group will be denoted by the additive group $\mathbb{R}$. The rotation group is $\mathbb{Z}_4$, the cyclic group of order 4, which will be represented as

$$\mathbb{Z}_4 \;=\; \{\; e, \quad r_{90}, \quad r_{180}, \quad r_{270}\; \}$$

where r_θ means clockwise rotation by θ degrees. We now construct our wreath product of these two groups.

The control group $G(C)$ will be $\mathbb{Z}_4$, and the control set C will be the set of four side-positions around the square:

$$c_1 = \text{top}, \quad c_2 = \text{right}, \quad c_3 = \text{bottom}, \quad c_4 = \text{left}. \tag{2.3}$$

The effect of the control group $\mathbb{Z}_4$ on the control set $\{c_1, c_2, c_3, c_4\}$ will be the clockwise rotation of the four side-positions onto each other.

The fiber group $G(F)$ will be the translation group $\mathbb{R}$, and the fiber set F will be the infinite line containing the finite side as a subset. The relationship between the infinite line F and the finite side, that it contains, will be defined in our mathematical theory in a crucial way to be described later. First, however, we note that the action of the fiber group $\mathbb{R}$ on the fiber set F will be the obvious translation of the infinite line along itself.

The fact that there are four elements in the control set $\{c_1, c_2, c_3, c_4\}$ implies that there are four fiber-group copies, which will be denoted as $\mathbb{R}_{c_1}, \mathbb{R}_{c_2}, \mathbb{R}_{c_3}, \mathbb{R}_{c_4}$. Also, it implies that there are four fiber-set copies, which will be denoted $F_{c_1}, F_{c_2}, F_{c_3}, F_{c_4}$. These are the four infinite lines that contain the four finite sides as subsets.

It is crucial to understand that each fiber-group copy (translation group $\mathbb{R}_{c_i}$) will act on its own "personal" copy of the fiber set (infinite line F_{c_i}). That is, for each member c_i of the control set, we have the corresponding group action

$$\mathbb{R}_{c_i} \times F_{c_i} \longrightarrow F_{c_i}$$

Based on this, we can now define the wreath product:

$$\mathbb{R} \circledR{w} \mathbb{Z}_4 \tag{2.4}$$

First observe that this is the semi-direct product:

$$[\mathbb{R}_{c_1} \times \mathbb{R}_{c_2} \times \mathbb{R}_{c_3} \times \mathbb{R}_{c_4}] \; \circledS_\tau \; \mathbb{Z}_4 \tag{2.5}$$

where τ, the automorphism representation,

$$\tau : \mathbb{Z}_4 \longrightarrow Aut\{\mathbb{R}_{c_1} \times \mathbb{R}_{c_2} \times \mathbb{R}_{c_3} \times \mathbb{R}_{c_4}\}$$

is such that, given any element in the control group, i.e., a rotation r_θ, its automorphic effect $\tau(r_\theta)$ on the fiber-group product, $\mathbb{R}_{c_1} \times \mathbb{R}_{c_2} \times \mathbb{R}_{c_3} \times \mathbb{R}_{c_4}$, corresponds to the effect of that rotation on the control set $\{c_1, c_2, c_3, c_4\}$. Therefore, the fiber-group copies are rotated around the square.

Now let us understand the data set $F \times C$, in this example. It is the *disjoint union* of the four fiber-set copies; i.e., the four infinite lines containing the four finite sides. Therefore it is important to understand that the fiber-set copies are independent sets, i.e., the four infinite lines do not intersect but *overlap*. To help understand this, one can think of them as *four infinite wires* overlapping each other.

Let us now model their relationship to the finite sides. According to the New Foundations two fundamental principles, Maximization of Transfer, and Maximization of Recoverability, the relationship is this: First, using the Theory of Recoverability of the generative operations (to be described later), the four finite sides are generated by cutting down the visibility of the four infinite lines at the end-points of the finite sides, by an extra generative operation that can switch visibility on or off. Furthermore, using the Theory of Transfer, the switching operation is incorporated as follows: First, it is defined by what our theory calls the **occupancy group**, $\mathbb{Z}_2$ (a cyclic group of order 2). The group switches between two states, "occupied" and "non-occupied", which, in the current example, determines whether a point is visible or not visible. Also, by the Maximization of Transfer principle, this group is *transferred* to each point along the infinite line, because the option of switching on and off the side-drawing program is available at any point along the infinite line.

Therefore, by the Mathematical Theory of Transfer, the group $\mathbb{Z}_2$ is placed as a fiber group below the group given in expression (2.4), thus:

$$\mathbb{Z}_2 \circledw \mathbb{R} \circledw \mathbb{Z}_4 \tag{2.6}$$

Notice that, with respect to the left wreath product symbol $\circledw$, the occupancy group $\mathbb{Z}_2$ is the fiber group, and the subsequence $\mathbb{R} \circledw \mathbb{Z}_4$ is the control group. Therefore, the subsequence $\mathbb{R} \circledw \mathbb{Z}_4$ has the effect of *transferring* the occupancy group. Our theory states that transfer maps the fiber-group copies onto each other. In the present case, the fiber-group copies are the copies of the occupancy group, i.e., one copy at each point in the data set $F \times C$ of the group $\mathbb{R} \circledw \mathbb{Z}_4$. Therefore, the copies of the occupancy group can be identified with the points in the data set $F \times C$. Furthermore, since the group $\mathbb{R} \circledw \mathbb{Z}_4$ transfers the copies of the occupancy group onto each other, we can understand the group $\mathbb{R} \circledw \mathbb{Z}_4$ as *transferring* the points in the data set onto each other. Therefore, we have this crucial conclusion: There is only one point. The remaining points have been created by *transferring* that point. Therefore, the square was created purely from a single point. The other points are merely transfers of that single point. This is an example of the principle of the Maximization of Transfer.

Furthermore, as seen in expression (2.6), the transfer is hierarchical; i.e., it is *transfer of transfer*. The occupancy group $\mathbb{Z}_2$ is transferred by the translation group $\mathbb{R}$, and this *transfer is transferred* by the rotation group $\mathbb{Z}_4$. This illustrates the following statement:

TRANSFER OF TRANSFER

An important property of the New Foundations to Geometry is hierarchical transfer, i.e., transfer is transferred.

To give the reader another illustration of my Mathematical Theory of Transfer, we will see how the theory represents another object, a cylinder.

Note first that, in computer vision and graphics, cylinders are described as the sweeping of a circular cross-section in the direction of the axis. The group of this sweeping structure has never been given. In contrast, my Mathematical Theory of Transfer creates the following group theory of the structure of a cylinder.

By the principle of the Maximization of Transfer, we proceed as follows:

First consider the cross-section. This is given generatively by a circular rotation of a point, as illustrated in Fig 2.4. That is, it is given by the following structure of *transfer*:

$$\mathbb{Z}_2 \textcircled{w} SO(2) \tag{2.7}$$

i.e., a *single point*, given by the occupancy group $\mathbb{Z}_2$, is transferred by the group $SO(2)$ which is the continuous rotation group in a plane.

Next, the sweeping of the cross-section, in the direction of the rotation axis, is given by the *transfer*, by translation, of the generative structure of the cross-section, as illustrated in Fig 2.5. Therefore, the wreath product in expression (2.7) is given as the fiber-group to which one adds, via an additional wreath product $\textcircled{w}$, the translation group $\mathbb{R}$ as the control group, thus:

$$\mathbb{Z}_2 \textcircled{w} SO(2) \textcircled{w} \mathbb{R} \tag{2.8}$$

Notice therefore that, as a result of this, the cylinder is decomposed into a structure of rotational fibers, as illustrated in Fig 2.6.

With respect to notation, we now make the following comment: When we need to help the reader concentrate on *spatial* levels of a structure, the occupancy level will be omitted from the notation. Thus for example, the structure of the square will be given as

$$\mathbb{R} \textcircled{w} \mathbb{Z}_4 \tag{2.9}$$

and the structure of the cylinder will be given as

$$SO(2) \textcircled{w} \mathbb{R} \tag{2.10}$$

Notice that, in both these cases, the fiber group is the movement of a point, and our theory defines this as the *transfer* of the generation of a point, and this implies the existence of the occupancy group as a fiber of the fiber. Thus, even though expressions (2.9) and (2.10) help one concentrate on the *spatial* levels of the generative structure, the presence of the occupancy group as an additional fiber is implied, because, by the Maximization of Transfer, the lowest *spatial* fiber must be transferring the occupancy group that generates the initial point.

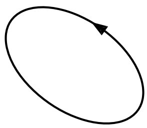

Fig. 2.4. A point is transferred by rotations, producing a circle.

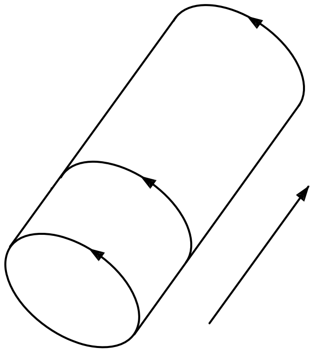

Fig. 2.5. The rotation is then transferred by translations, producing a straight cylinder.

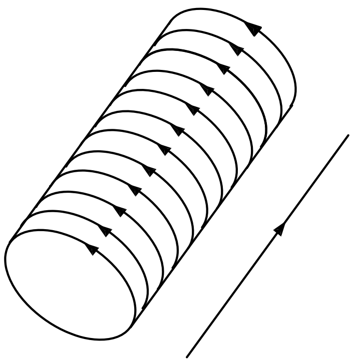

Fig. 2.6. As a result of transfer, a cylinder decomposes into rotational fibers.

2.3 Mathematical Theory of Recoverability: 1

Recall from section 2.1 that the New Foundations to Geometry are based on two fundamental principles: Maximization of Transfer and Maximization of Recoverability. Furthermore, the New Foundations give a Mathematical Theory of Transfer and a Mathematical Theory of Recoverability. So far, we have been looking at the Mathematical Theory of Transfer. We will now turn to the Mathematical Theory of Recoverability. This will take a number of sections to elaborate.

First, note that by *recovery*, we mean the following:

THE RECOVERY PROBLEM

Given a data set $\mathcal{D}$, infer from $\mathcal{D}$ the generative history that produced it.

According to the New Foundations, when one has recovered, from a data set $\mathcal{D}$, its generative history, one has *converted the data set into a memory store*. The New Foundations give a massive mathematical theory of memory storage. This theory begins with proposing the following two laws:

FIRST FUNDAMENTAL LAW OF MEMORY STORAGE
(Leyton, 1992)
Memory is stored only in asymmetries.

SECOND FUNDAMENTAL LAW OF MEMORY STORAGE
(Leyton, 1992)
Memory is erased by symmetries.

Let us begin with a simple example. Consider the sheet of paper shown on the left in Fig 2.7. Even if one had never seen that sheet before, one would conclude that it had undergone twisting. I claim that this is because the asymmetry in the sheet yields the information about the generative history. In other words, from the asymmetry, one can *recover* the past history. That is, the *asymmetry acts as a memory store for the past action* – as stated in my First Fundamental Law of Memory Storage (above).

Now let us un-twist the paper, thus obtaining the straight sheet given on the right in Fig 2.7. Suppose we show this straight sheet to any person on the street. They would not be able to infer from it the fact that it had once been twisted. I claim that this is because the symmetry of the straight sheet has wiped out the ability to recover the preceding history. This means that the *symmetry erases the memory store* – as stated in my Second Fundamental Law of Memory Storage (above).

In fact, from the symmetry, one concludes that the straight sheet had always been like this. For example, when you take a sheet of paper from a box of sheets of paper

Fig. 2.7. A twisted sheet is a source of information about the previous deformation history. A non-twisted sheet is not.

you have just bought, you do not assume that the sheet of paper had once been twisted or crumpled. Its very straightness (symmetry) leads you to conclude that it had always been straight.

The two diagrams in Fig 2.7 illustrate the two fundamental laws of memory storage given above. These two laws are the basis of the theory of recoverability in the New Foundations to Geometry. As *inference rules*, the two laws are stated in the following way:

ASYMMETRY PRINCIPLE
(Leyton, 1992)

Given a data set $\mathcal{D}$, a sequence of operations for generating $\mathcal{D}$ is *recoverable from* $\mathcal{D}$ **only if asymmetries in $\mathcal{D}$ go back to symmetries in the previously generated states; i.e., only if, in the forward-time direction, the sequence of operations was symmetry-breaking on each of the successively generated states.**

SYMMETRY PRINCIPLE
(Leyton, 1992)

Given a data set $\mathcal{D}$, a sequence of operations for generating $\mathcal{D}$ is *recoverable from* $\mathcal{D}$ **only if symmetries in $\mathcal{D}$ are preserved backwards through the previously generated states.**

At first it might seem as if there are exceptions to these two principles. However, my books show that the apparent exceptions are due to the incorrect description of data sets. For example, the Asymmetry Principle states that the only recoverable operations are symmetry-breaking ones. The reader might think that there are exceptions to this law because one is aware of many processes in the world that are not symmetry-breaking, but are symmetry-increasing; e.g., a tank of gas settling to equilibrium under the standard entropy-increasing process. Concerning such situations, my theory says this:

SYMMETRY-INCREASING PROCESSES. *A symmetry-increasing process is recoverable only if it is symmetry-breaking on successive data sets.*

So, for example, you can recover the fact that the tank of gas was entropy-increasing over time, if you kept a set of records (e.g., photographs) and the records are linearly ordered, e.g., they are laid out from left to right on a table, in which case the sequence of photographs breaks the left-right symmetry of the table. That is, the increase in spatial symmetry in the tank of gas is made to correspond to a decrease in spatial symmetry of the record structure. This issue is related to the concept of external and internal inference to be described in section 2.6. Most crucially, one must explicitly define the data set, and understand the detailed theory of how these laws are applied to data sets. To understand this theory, the reader should read Chapter 2 of my book *A Generative Theory of Shape*.

The Asymmetry Principle and Symmetry Principle lead to very powerful mathematics, for example a new algebraic theory of symmetry-breaking, and an extensive mathematical theory of memory storage.

However, before going into some of this mathematics, let us begin to develop a familiarity with the Asymmetry Principle and Symmetry Principle. The use of the two principles requires that we go through the following procedure: First partition the presented situation into its asymmetries and its symmetries. Then use the Asymmetry Principle on the asymmetries, and the Symmetry Principle on the symmetries. Note that the application of the Asymmetry Principle will return the asymmetries to symmetries. And the application of the Symmetry Principle will preserve the symmetries.

What does one obtain when one applies this procedure to a situation? The answer is this: One obtains the *past*!

Since this procedure is a basic component of the New Foundations, it will be stated succinctly as follows:

PROCEDURE FOR RECOVERING THE PAST

(1) Partition the situation into its asymmetries and symmetries.
(2) Apply the Asymmetry Principle to the asymmetries.
(3) Apply the Symmetry Principle to the symmetries.

An extended example will now be considered that will illustrate the power of this procedure, as follows: In a set of psychological experiments that I carried out in the psychology department in Berkeley in 1982, I found that, when subjects are presented with a rotated parallelogram, as shown in Fig 2.8a, they refer it in their minds to a non-rotated parallelogram, Fig 2.8b, which they then refer in their minds to a rectangle, Fig 2.8c, which they then refer in their minds to a square, Fig 2.8d. It is important to

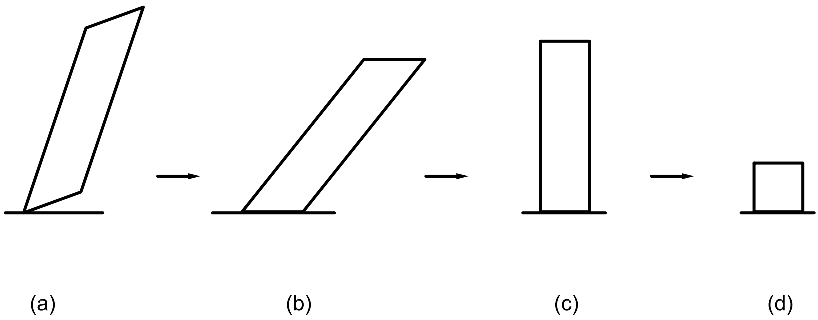

Fig. 2.8. The history inferred from a rotated parallelogram.

understand that the subjects are presented with only the first shape. The rest of the shapes are actually created by their minds, as a response to the presented shape.

Close examination reveals that what the subjects are doing is *recovering the history* of the rotated parallelogram. That is, they are saying that, prior to its current state, the rotated parallelogram, Fig 2.8a, was non-rotated, Fig 2.8b, and prior to this, it was a rectangle, Fig 2.8c, and prior to this, it was a square, Fig 2.8d.

The following should be noted about this sequence. The sequence from *right to left* – that is, going from the square to the rotated parallelogram – represents the direction of *forward time*; i.e., the history starts in the *past* (the square) and ends with the *present* (the rotated parallelogram). Conversely, the sequence from *left to right* – that is, going from the rotated parallelogram to the square – represents the direction of *backward time*. Thus, what the subjects are doing, when their minds create the sequence of shapes from the rotated parallelogram to the square, is this: They are **running time backwards!**

We shall now see that the subjects create this sequence by using the Asymmetry Principle and the Symmetry Principle. Recall that the way one uses the two principles is to apply the simple three-stage Procedure for Recovering the Past, given above: (1) Partition the presented situation into its asymmetries and symmetries, (2) apply the Asymmetry Principle to the asymmetries, and (3) apply the Symmetry Principle to the symmetries.

Thus to use this procedure on the rotated parallelogram, let us begin by identifying the asymmetries in that figure. It is important first to note that *asymmetries are the same as distinguishabilities*. In the rotated parallelogram, there are three distinguishabilities:

(1) The distinguishability between the orientation of the shape and the orientation of the environment – indicated by the difference between the bottom edge of the shape and the horizontal line which it touches.

(2) The distinguishability between adjacent angles in the shape: they are different sizes.

(3) The distinguishability between adjacent sides in the shape: they are different lengths.

It is clear that what happens in the sequence, from the rotated parallelogram to the square, is that these three distinguishabilities are removed successively backwards in time. The removal of the first distinguishability, that between the orientation of the shape and the orientation of the environment, results in the transition from the rotated parallelogram to the non-rotated one. The removal of the second distinguishability, that between adjacent angles, results in the transition from the non-rotated parallelogram to the rectangle, where the angles are equalized. The removal of the third distinguishability, that between adjacent sides, results in the transition from the rectangle to the square, where the sides are equalized.

Therefore, each successive step in the sequence is a use of the Asymmetry Principle, which says that an asymmetry must be returned to a symmetry backwards in time.

Having identified the asymmetries in the rotated parallelogram and applied the Asymmetry Principle to each of these, we now identify the symmetries in the rotated parallelogram and apply the Symmetry Principle to each of these. First we need to note that *symmetries are the same as indistinguishabilities*. In the rotated parallelogram, there are two indistinguishabilities:

(1) The opposite angles are indistinguishable in size.

(2) The opposite sides are indistinguishable in length.

The Symmetry Principle requires that these two symmetries in the rotated parallelogram must be preserved backwards in time. And indeed, this turns out to be the case. That is, the first symmetry, the equal size of the opposite angles in the rotated parallelogram, is preserved backwards through the entire sequence; i.e., each subsequent shape, from left to right, has the property that opposite angles are equal in size. Similarly, the other symmetry, the equal length of the opposite sides in the rotated parallelogram, is preserved backwards through the entire sequence; i.e., each subsequent shape, from left to right, has the property that opposite sides are equal in length.

Furthermore, what can also be seen is that, after any symmetry has been recovered, it is also preserved backward through the remaining sequence. For example, in going from Fig 2.8b to Fig 2.8c, the adjacent angles, which are initially different, are made the same size. This symmetry is then preserved in all the remaining figures. Again, this is a use of the Symmetry Principle.

Thus what we have seen in this entire example is this: The sequence from the rotated parallelogram to the square is determined by two rules: the Asymmetry Principle which returns asymmetries to symmetries, and the Symmetry Principle which preserves the symmetries. These two rules allow us to *recover the past*, i.e., *run time backwards*.

We have just given an illustration of the use of the two basic inference rules, in the New Foundations, for recovering generative history: the Asymmetry Principle and Symmetry Principle. My books also instantiate these two basic rules as an enormous number of inference rules for recovering history in many scientific and artistic domains; from general relativity and software engineering to painting and music.

Now recall that, according to the New Foundations, a data set becomes a memory store when one has *inferred* its generative history. Thus, the theory says that the inference has *converted* the data set into a memory store.

As an example, return to the psychological results in Fig 2.8. What these results show is that the human mind *converts* the first shape, the rotated parallelogram, into a memory store, by recovering its generative history backwards in time, via the fundamental inference rules, the Asymmetry Principle and Symmetry Principle. The crucial fact is that the first shape is then *defined* by its generative history.

This fundamentally contrasts with the Standard Foundations to Geometry, in which a *geometric object* is defined as an *invariant* with respect to its history; i.e., a property that is *unchanged* through the history. In the present example, a geometric object, i.e., an invariant, would be the existence of four sides and their straightness. But this would not include the lengths of the sides. That is, the invariant would be blind to the side-lengths. Consequently, as our theory argues generally, the invariant is *memoryless* with respect to the history.

This illustrates the fact that a fundamental difference between the Standard Foundations to Geometry and the New Foundations to Geometry is that the two systems define a *geometric object* in the opposite way, as follows:

DEFINITION OF A GEOMETRIC OBJECT

STANDARD FOUNDATIONS TO GEOMETRY
(Klein)
A geometric object is an *invariant;* **i.e.,** *memoryless.*

NEW FOUNDATIONS TO GEOMETRY
(Leyton)
A geometric object is a *memory store.*

It is important to recall that the above definition given by the New Foundations arises from the Mathematical Theory of Intelligence elaborated in the New Foundations, as follows: Recall from page 7 that one of the two fundamental principles of this theory of intelligence is the Maximization of Recoverability. In fact, let us see how recoverability is related to intelligence, as follows: First recall from page 15 that recovery means this: Given a data set $\mathcal{D}$, infer from $\mathcal{D}$ the generative history that produced it. Thus, recovery means inferring, from the data set, an *explanation* of the data set. The next important thing to understand is that, according to the New Foundations, there should be no separation between the representation of the data set and the explanation given of the data set. This is a fundamental principle introduced in my book *Symmetry, Causality, Mind* (MIT Press), where it is called the *Representation is Explanation Principle*. In fact, in that book, there is an entire chapter called *Representation is Explanation*.

REPRESENTATION IS EXPLANATION
(Leyton, 1992)

In the New Foundations to Geometry:
The representation of a data set is an explanation of the data set.

In fact, in the New Foundations, I argue that it is the explanation that defines the data set as a *shape*. That is:

SHAPE $\equiv$ EXPLANATION

According to the New Foundations to Geometry:
All explanation is shape and all shape is explanation.
That is: shape and explanation are equivalent terms.

Thus, putting the above two claims together, it should be understood that, according to the New Foundations, we have this:

DEFINITION OF ANY OBJECT

Since, according to the New Foundations to Geometry, the representation of any object is an explanation of the object, the New Foundations conclude this:

Every object is defined as a shape; i.e., an explanation. That is:

Every object is a geometric object.

Again, this all relates to the issue of intelligence. The following contrasts intelligence and stupidity:

INTELLIGENCE vs. STUPIDITY

An intelligent representation of an object is one that explains it.

A stupid representation of an object is one that does not explain it.

The reader should understand that, as a result of the fact that the New Foundations are an extensive mathematical theory of explanation, the terms *intelligence* and *stupidity* are given a highly technical meaning in the New Foundations. As a result of this technical meaning, we have:

DEFINITION OF SHAPE

Definition of shape in the Standard Foundations to Geometry:

Shape = Configuration

This definition is independent of an explanation of the shape. Therefore:

The Standard Foundations to Geometry are *stupid.*

Definition of shape in the New Foundations to Geometry:

Shape = Explanation

This definition equates shape with its explanation. Therefore:

The New Foundations to Geometry are *intelligent.*

2.4 Combining Transfer and Recoverability

Now let us look further at the mathematics of the New Foundations to Geometry. Recall, from page 7, that the two fundamental principles of the New Foundations are (1) Maximization of Transfer, and (2) Maximization of Recoverability. It is important to understand that these two principles are deeply integrated in the theory, conceptually as well as mathematically.

First, conceptually, a fundamental argument in the New Foundations is that the *recovered past state* is *transferred* onto the *present state*. For example, in Fig 2.8 (page 18), the recovered past state, the square, is understood as *transferred* onto the present state, the rotated parallelogram. That is, having *recovered* the square from the parallelogram, via the inference rules, we then represent the parallelogram as a deformed and rotated version of the square.

Mathematically, this relationship between transfer and recoverability is realized in the following way. Recall from section 2.2 that the New Foundations model transfer by the group

$$\textbf{Fiber Group} \ ⓦ \ \textbf{Control Group}$$

where the fiber group is the group that is being transferred, the control group is the group that is doing the transfer, and the wreath product ⓦ models the relation *is transferred by*. That is, the above expression should be understood as saying *the fiber group is transferred by the control group*.

Now, combining this with the theory of recoverability, the fiber group gives the recovered past state, and the control group gives the recovered generative history that transfers the recovered past state onto the present state. In this way, the wreath product of the fiber group and control group gives the structure of the *present state*.

For example, in the case of the rotated parallelogram (Fig 2.8 page 18), the fiber group and control group must be as follows: The fiber group is the structure of the past state, the square. On page 11, we defined the structure of the square as $\mathbb{R}$ ⓦ $\mathbb{Z}_4$. Thus the fiber group must be $\mathbb{R}$ ⓦ $\mathbb{Z}_4$. The control group, which gives the generative operations that created the parallelogram from the square, must be the group of linear transformations, i.e., the general linear group $GL(2, \mathbb{R})$. Therefore, taking the wreath product of the fiber group $\mathbb{R}$ ⓦ $\mathbb{Z}_4$ and the control group $GL(2, \mathbb{R})$, we get:

$$\mathbb{R} \ ⓦ \ \mathbb{Z}_4 \ ⓦ \ GL(2, \mathbb{R})$$

This group produces, from the square, a space of parallelograms, each of which is defined as a *transferred version* of the square. Thus the group captures the psychological result that, given a parallelogram, one sees it as a distorted version of a square. That is, the group captures the fact that the parallelogram acts as a *memory store* for the past state, the square.

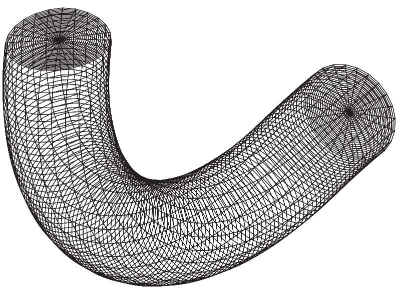

Fig. 2.9. A bent cylinder.

As another example, consider a bent cylinder, e.g., as shown in Fig 2.9. It is clear that, by describing this as *bent*, we are inferring that, originally, it must have been a straight cylinder. Furthermore, we see the past state, the straight cylinder, as *transferred* onto the present state, the bent cylinder. On page 13, the structure of the straight cylinder was defined to be the group $SO(2) \circledw \mathbb{R}$. Consequently, $SO(2) \circledw \mathbb{R}$ must be the fiber group that is transferred onto the bent cylinder by the control group, which we can take to be the group of diffeomorphisms $Diff$. Therefore, the present state, the bent cylinder, is given by the wreath product of the fiber group $SO(2) \circledw \mathbb{R}$ and the control group $Diff$, thus:

$$SO(2) \circledw \mathbb{R} \circledw Diff$$

The wreath product captures the psychological effect that, given a bent cylinder, one sees it as a distorted version of a straight cylinder; i.e., the bent cylinder acts as a *memory store* for the past state, the straight cylinder.

2.5 New Theory of Symmetry-Breaking

Our way of combining the theory of transfer and the theory of recoverability leads to a profound set of conclusions, as follows: Recall first that our theory of recoverability says that the only recoverable operations are symmetry-breaking ones. Thus, combining this with the fact that the past state must be transferred by the generative history onto the present state, we are lead to the following conclusion: *The control group must be symmetry-breaking on its fiber*. This, in turn, leads the New Foundations to the invention of a crucial structure which I call *symmetry-breaking wreath products*.

Close examination reveals that this gives a far more powerful theory of symmetry-breaking than the conventional one that underlies physics and chemistry.

CONVENTIONAL THEORY OF SYMMETRY-BREAKING

Symmetry-breaking is a reduction of symmetry group.

As an example, consider the transition from a square to a parallelogram, which is a symmetry-breaking one. The conventional theory says that the symmetry group of a square is D_4 which consists of the eight Euclidean transformations that map the square to itself: four rotations and four reflections. In contrast, a parallelogram is given by the symmetry group $\mathbb{Z}_2$ which consists of the only two Euclidean transformations that map a parallelogram to itself: rotation by 0^0 and rotation by 180^0. Thus, the transition from a square to a parallelogram is given by the following transition of groups:

$$D_4 \quad \longrightarrow \quad \mathbb{Z}_2 \tag{2.11}$$

In fact, the group $\mathbb{Z}_2$ is a subgroup of D_4, which means that the transition is given by reduction of symmetry group.

This theory of symmetry-breaking has dominated physics and chemistry for nearly a century. However, according to our theory, this is inherently weak because it means a loss of algebraic structure. That is, in the conventional theory, as one goes from a simpler object (such as a square) to a more complex object (such as a parallelogram), one is reducing the size of the description – which is logically absurd.

In contrast, we have just illustrated our theory of symmetry-breaking using the same example, the transition from a square to a parallelogram. We modeled this transition as follows: One takes our symmetry group of the square, and *adds* the group of linear transformations, *via a wreath product*. Therefore, in our approach, symmetry-breaking actually preserves the original group, and in fact increases it. Thus generally:

NEW THEORY OF SYMMETRY-BREAKING

The breaking of a symmetry group G_1 is given by its extension by another group G_2 via a wreath product thus: $G_1 \circledS G_2$, where G_2 is the symmetry group of the asymmetrizing action.

The fundamentally important feature of this theory is as follows: In the conventional theory of symmetry-breaking, the past symmetry is *lost* in the present broken symmetry. In contrast, in the new theory, the past symmetry is *transferred* onto the present broken symmetry. That is, one actually sees the past symmetry in the broken symmetry; i.e., the past symmetry is *stored* in the broken symmetry.

Let us give another illustrating example of this concept: Consider a bent pipe that one comes across in the street. Merely by the fact that one understands it as a bent pipe means that one sees the past symmetric version as transferred onto the present asymmetric version. That is, the past symmetric version is stored in the present asymmetric version.

This illustrates what the New Foundations mean by saying that *shape is equivalent to memory storage*. That is, the present shape of the pipe equals the history recovered from it.

The approach we have been describing, in fact, leads to a general theory of the mathematical structure of memory stores:

FUNDAMENTAL STRUCTURE OF MEMORY STORES

According to the New Foundations to Geometry:
Any memory store is structured as a symmetry-breaking wreath product.

Since, according to the New Foundations, any intelligent description of an object is the representation of the object as a memory store, the above principle gives the structure of objects in all scientific, computational, and artistic domains, when they are *intelligently* represented.

2.6 External vs. Internal Inference

This section gives more concepts from the theory of recoverability in the New Foundations, and therefore more of the theory of memory storage. First, the theory makes a fundamental distinction between two types of inference, as follows:

Definition 2.1. *In* **external inference***, also called the* **single-state assumption***, the observer assumes that the data set contains a record of only a single state of the generative process (i.e., a single snap-shot). Any inferred previous state therefore does not have a record in the data set. Therefore, in this case, we say that any inferred previous state is* **external** *to the data set.*

An example of external inference is the psychological experiment where subjects were presented with a rotated parallelogram, and inferred its past state to be a square (Fig 2.8 page 18). Although the square is inferred from the data set, the rotated parallelogram, it is not present in the data set. Thus the subjects have inferred a past state, the square, that is *external* to the data set. Therefore, by the above definition, we say that the inference is *external*; i.e., the inference goes outside the present data set.

Another example of external inference is the fact that, when one comes across a bent pipe in the road, one understands it to have originated as a straight pipe. This inference is made despite the fact that a straight pipe is not visible in the present data set. That is, the inferred past state, a straight pipe, is *external* to the present data set. Thus, by the above definition, we say that the inference is *external*; i.e., the inference goes outside the present data set.

The next definition is this:

Definition 2.2. *In* **internal inference***, also called the* **multiple-state assumption***, the observer assumes that the data set contains records of multiple states of the generative process (i.e., multiple snap-shots taken over time). A state, recorded in the data set, can therefore have a past state that also has a record* **internal** *to the data set.*

An example of internal inference is where one is presented with a trace of states, e.g., a scratch on a table. Each point along the trace was created at a different moment in time. Therefore, the present data set, the trace, contains records from all those previous moments in time. Thus, according to the above definition, we say that the past states have records that are *internal* to the present data set. Correspondingly, we say that the inference from the present data set, the complete trace, to the past states, the points inside the trace, is *internal inference*.

A detailed theory of external and internal inference is given in my book *Symmetry, Causality, Mind* (MIT Press, 630 pages). Remarkably, although external and internal inference are used in very different types of situations, they are both carried out by using the Asymmetry Principle. The difference is as follows:

APPLICATION OF RULES. *In external inference, the Asymmetry Principle is applied to* **intra-record** *asymmetries (asymmetries within the record of a state). In internal inference, the Asymmetry Principle is applied to* **inter-record** *asymmetries (asymmetries between the records of states).*

We are now going to describe one of the most powerful laws in the New Foundations. To do so, we first have to introduce one of the basic types of groups that were invented in the New Foundations:

Definition 2.3. *The New Foundations define an* **iso-regular group** *to be a group that satisfies the following three conditions:*

(1) It is an n-level wreath product, $G_1 \circledS G_2 \circledS \ldots \circledS G_n$; i.e., a structure of hierarchical transfer.
(2) Each level G_i is generated by a single generator (i.e., it is either a cyclic group or a connected 1-parameter Lie group).
(3) Each level G_i is an isometry group on its space of action.

In fact, two examples of iso-regular groups were given earlier in this chapter:

Square: $\mathbb{R} \circledS \mathbb{Z}_4$

Cylinder: $SO(2) \circledS \mathbb{R}$

The reader can easily see that they are iso-regular groups by comparing their structure with each of the three properties listed in Definition 2.3.

Given the sequence of definitions in this section, we are now ready to state a fundamental law from the New Foundations, a law that turns out to be one of the most powerful laws of memory storage:

EXTERNALIZATION PRINCIPLE

To maximize recoverability, any generative sequence, inferred by external inference, must lead back to a starting state whose internal structure is given by an iso-regular group.

As an example, consider the psychological experiment where subjects were presented with a rotated parallelogram, and inferred its past state to be a square (Fig 2.8 page 18). We have seen that this is an example of *external* inference. Observe that the inferred starting state, the square, is given by the iso-regular group $\mathbb{R} \circledS \mathbb{Z}_4$. Thus the inference, made by the subjects in the experiment, accords with the Externalization Principle; i.e., the starting state, inferred by the external inference, is given by an iso-regular group.

As another example, consider the case of the bent cylinder. Again, we note that the fact that it is called "bent" means that one infers its starting state to have been a straight cylinder. We have seen that this is an example of *external* inference. Observe that the inferred starting state, the straight cylinder, is structured by the iso-regular group $SO(2) \circledS \mathbb{R}$. Therefore, the inference again accords with the Externalization Principle; i.e., the starting state, inferred by the external inference, is given by an iso-regular group.

2.7 Algebraic Theory of Inheritance

The term *inheritance*, in object-oriented programming, refers to the passing of properties from a parent to a child. The child incorporates these parent properties, but also adds its own [30].

This kind of structure covers two types of situation. The first is class inheritance, which is a static software concept. The second is a type of dynamic linking created at run-time. My book *A Generative Theory of Shape* (Springer-Verlag) gives an algebraic theory of both types of inheritance; but in the present book we will require the theory of only the second type of inheritance.

This *run-time* created inheritance is fundamental to all computer-aided design, assembly, robotics, animation, etc. A typical example is a child object inheriting the transform of a parent object, and adding its own. For example, in architectural CAD, a door is defined as a child of a wall, and moves with the wall if the designer decides to change the position of the wall. However, the door can also open and close with respect to its attached position in the wall. This means that the door inherits the movement of the wall, but adds its personal movement with respect to the latter. In Leyton [24], I refer to this type of inheritance as *object-linked inheritance*.

The mathematical theory of object-linked inheritance, in the New Foundations, states this:

ALGEBRAIC THEORY OF OBJECT-LINKED INHERITANCE. *Object-linked inheritance arises from a wreath product:*

Parent $\longleftrightarrow$ *Control group*
Child $\longleftrightarrow$ *Fiber group*

Notice that this means that the basis of inheritance is *transfer*. The enormous power of this theory is that it explains inheritance in all of CAD, robotics, assembly, animation, etc. This is fully elaborated in my book *A Generative Theory of Shape* (Springer-Verlag).

Now let us return to the architectural example, where the door is a child of the wall. This example can be extended further: The designer can design a door-handle and attach it to the door. Standardly, the designer would define the door-handle as a child of the door. Thus, the door-handle would move with the door when the latter is opened or closed. However, the door-handle could also rotate with respect to its attached position in the door. This means that the door-handle would inherit the movement of the door, but also add its personal movement with respect to the latter. Notice that, because the door-handle is a child of the door, and the door is a child of the wall, the door-handle would also be inheriting the movement of the wall, if the designer decided to move the wall.

It will be helpful for the reader to consider Fig 2.10, which illustrates the diagrammatic method used in the animation program *3D Studio Max* to represent an inheritance hierarchy. Here, inheritance is represented by indentation – i.e., an indented object is a child of the next object above with respect to which it is indented. Each object, except the World object, has a transform shown just below it. The transform relates the coordinate

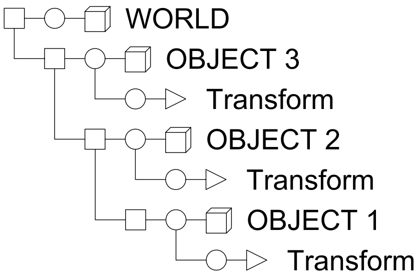

Fig. 2.10. The representation of parent-child relations in a program like *3D Studio Max*.

frame of the object to the coordinate frame of its parent. This transform is the "personal" transform of the object. In addition, the object inherits the transform of its parent. The object therefore adds its personal transform to its inherited transform. This means, of course, that via its parent, the object inherits the transform of its parent's parent, and so on.

Thus, return to the architectural example. In this case, Object 3 in Fig 2.10 would be the wall; Object 2 would be the door in the wall; and Object 1 would be the door-handle in the door. Thus, in the diagram, each of these three objects is shown with its personal transform below it; and the inheritance hierarchy is given by the successive indentations.

The New Foundations define the full group of such structures as follows:

GROUP OF ENTIRE TRANSFORM STRUCTURE. *Consider a set of $n+1$ objects: Object 1 to n, and the World. Suppose that they are linked such that Object i is the child of Object $i+1$, and Object n is the child of the World. Then the group of the entire transform structure is the wreath product:*

$$_{F_1}G_1^{F_2} \; Ⓦ \; _{F_2}G_2^{F_3} \; Ⓦ \; \ldots \; Ⓦ \; _{F_n}G_n^{W}$$

where:

(1) Object i has personal transform group G_i and frame F_i.
(2) Personal transform group G_i relates frame F_{i+1} of the parent, upper index, to the personal frame F_i, lower index. (The world frame F_{n+1} is written as W.)

Clearly this statement gives a new algebraic theory of *relative motion*, which is explained in detail in my book *A Generative Theory of Shape* (Springer-Verlag).

2.8 Complex Shape

The New Foundations are fundamentally a theory of *complex* shape. It is now necessary to understand a major way in which the New Foundations represent complex shape.

Consider the main problem in providing a generative theory of complex structure. As stated in section 2.3, according to the New Foundations, recoverability is possible only if the generative operations are symmetry-breaking. Notice that, in the Standard Foundations to Geometry, this would mean that, as one proceeds forward in the generative sequence, the symmetry group of the structure would quickly reduce to nothing. Thus, there would be a loss of algebraic information. In the New Foundations, this problem is solved using an entirely opposite theory of symmetry-breaking, as described in section 2.5. In this theory, the group describing the symmetric past state is actually increased in the symmetry-broken state. This is done by making it the fiber group of a wreath product in which the control group is the group of the asymmetrizing action. Thus, in using this wreath product, the group of the past state is *transferred* onto the symmetry-broken state.

In the previous sections, we have seen how this theory of symmetry-breaking gives a formulation of *deformation* that does not have the problems of the Standard Foundations to Geometry. That is, as illustrated by the examples of deforming a square into a parallelogram and deforming a straight cylinder into a bent one, we saw that, according to the New Foundations, the inferred symmetric past state is given by what the New Foundations call an iso-regular group, and the deformation is modeled by extending the iso-regular group, as a fiber group, by the deforming group, as a control group, in a wreath product.

Having shown how our theory of symmetry-breaking solves the problems of the Standard Foundations in modeling *deformation*, we now need to show how our theory solves the problems in modeling *concatenation*. Consider Fig 2.11. Each of the two objects *individually* has a high-degree of symmetry. However, the *combined* structure, shown, looses much of this symmetry. Thus, the Standard Foundations would encode the concatenated structure by a reduced group. In contrast, the New Foundations develops the opposite kind of group theory. In this, the group of the concatenated structure not only preserves the symmetry groups of the individual objects, but adds the extra information of the concatenation.

This is how the New Foundations proceed: The generative history starts with the two independent objects, and therefore the symmetry of this starting situation is given thus:

$$G_{cylinder} \times G_{cube}$$

which is the *direct product* of the *iso-regular groups*, $G_{cylinder}$ and G_{cube}, that describe the two independent objects.

Now, by the Maximization of Transfer, the starting group, i.e., this direct product group, must be transferred onto subsequent states in the generative history, and therefore it must be the fiber of the wreath product in which the control group creates the subsequent generative process.

Let us take the control group to be the *affine group* $AGL(3, \mathbb{R})$ on three-dimensional real space.The full structure, fiber plus control, is therefore the following wreath product:

$$[G_{cylinder} \times G_{cube}] \circledw AGL(3, \mathbb{R})$$

Now, it is necessary to fix the *group representation* of this wreath product. First, by our theory of recoverability, the control group must have an asymmetrizing action with respect to the fiber. Thus proceed as follows: The particular fiber-group copy

$$[G_{cylinder} \times G_{cube}]_e$$

corresponding to the identity element e in the affine control group, i.e., the starting state, must be the most symmetrical configuration possible. This exists only when the cube and the cylinder are coincident, with their symmetry structures (axes, etc.) *maximally aligned*. The New Foundations call this configuration the **alignment kernel**.

Next, choose one of the two objects to be a reference object. This will remain fixed at the origin of the world coordinate frame. Let us choose the cube as the referent. Given this, now describe the action of the affine control group as providing an affine motion of the cylinder *relative* to the cube. Each fiber-group copy

$$[G_{cylinder} \times G_{cube}]_g$$

for some member g, of the control affine group, is therefore an arrangement of this system. In fact, any fiber copy will be called a **configuration** of the system. For example, Fig 2.11 corresponds to a configuration. The crucial concept is this: The role of the affine control group is to **transfer configurations onto configurations**.

The wreath product we have presented:

$$[G_{cylinder} \times G_{cube}] \ (w) \ AGL(3, \mathbb{R})$$

gives the *complete* symmetry group of the concatenated situation. It has all the internal symmetries of the objects individually, as well as their relationships.

Let us now understand how to add a further object, for example a sphere. First of all, the fiber becomes the following, with the added *iso-regular group* G_{sphere} of the sphere:

$$G_{sphere} \times G_{cylinder} \times G_{cube}$$

Let us define the cube is the referent for the cylinder-sphere pair, and the cylinder is the referent for the sphere.

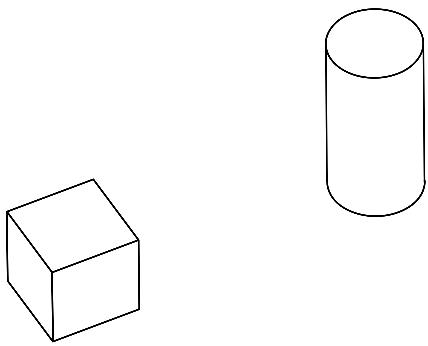

Fig. 2.11. Concatenation of cylinder and cube.

Accordingly, there are now two levels of control, each of which is the affine group $AGL(3, \mathbb{R})$, and each of which is added via a wreath product. Thus we obtain the 3-level wreath product:

$$[G_{sphere} \times G_{cylinder} \times G_{cube}] \circledR AGL(3, \mathbb{R}) \circledR AGL(3, \mathbb{R})$$

This is interpreted in the following way: Initially, the three objects (sphere, cylinder, cube) are coincident with their symmetry structures maximally aligned. This corresponds to the fiber-group copy that the New Foundations call the *alignment kernel*. The higher affine group moves the cylinder-sphere pair in relation to the cube. The lower affine group moves the sphere in relation to the cylinder.

Recall that, in the above situation, the cube is fixed at the origin of the world-frame. In fact, in our theory, its symmetries are maximally aligned with the symmetries of the world frame. Now, if we also allow the cube to move with respect to the world-frame, then we add the group G_W, defining the symmetries of world frame, into the alignment kernel, and add a third level of control above the two control levels that have already been included, thus:

$$[G_{sphere} \times G_{cylinder} \times G_{cube} \times G_W] \circledR AGL(3, \mathbb{R}) \circledR AGL(3, \mathbb{R}) \circledR AGL(3, \mathbb{R})$$

The new top control level will move the cube-cylinder-sphere triple with respect to the world-frame. Notice that the full control group corresponds to the group of the entire transform structure given on page 29 for our algebraic theory of inheritance. In my book *A Generative Theory of Shape*, there is a chapter devoted to giving an algebraic theory of reference frames, and I show that the appropriate group G_W, for the world frame, is the iso-regular group that is the maximal normal subgroup of the hyperoctahedral group.

The crucial point here is that, initially, the four objects (sphere, cylinder, cube, world-frame) are coincident with their symmetry structures maximally aligned. This corresponds to the fiber-group copy that I call the *alignment kernel*. The hierarchy of control groups move these objects hierarchically out of alignment with each other, in correspondence with the inheritance hierarchy.

The above discussion has been illustrating a class of groups which were invented in the New Foundations, called **unfolding groups**. The basic idea is that any complex structure, such as a design in CAD, or a musical composition, is *unfolded* from a maximally collapsed form, which I call the alignment kernel. Two main properties characterize unfolding groups:

Selection: The control group acts selectively on only part of its fiber.

Misalignment. The control group is symmetry-breaking by misalignment.

The New Foundations show that unfolding groups are a fundamental structure in all complex shape generation. The remainder of this section will give a brief introduction to this. First, it is necessary to give more of the algebraic theory of object-oriented software developed by the New Foundations. Within that theory, there is an analysis of class structure which says that each geometric class consists of (1) an *internal symmetry* *group*, which gives the correct structure of what are conventionally, but incorrectly, called the invariants clauses of the software text for the class; and (2) an *external group* consisting of command operations, such as deformations, specified in the feature clauses of the class text.

A principal claim of the theory of software, in the New Foundations, is that the relation between the internal symmetry group and the command structure, in the software text, is a wreath product, thus:

$$G_{sym} \circledS G(C)$$

where G_{sym} is the internal symmetry group and $G(C)$ is the group of command operations. Thus *transfer* is made to be the basic structure of a class.

This solves major well-known problems in object-oriented software. For example, what is standardly called an invariance clause of the class which is known as the *square* – the specification that it has four equal sides – is violated by the stretching deformation operator applied to the square. This illustrates the fact that object-oriented software does not accord with the Standard Foundations to Geometry, but instead accords with my New Foundations to Geometry: That is, as illustrated earlier in this chapter, according to the New Foundations, the internal group of a square is *transferred* onto the stretched version, a rectangle. The way that this corrects major problems in software-engineering is shown in detail in my paper *Interoperability and Objects* [24].

Now let us turn to *cloning* in object-oriented programming. It is important to notice that, when one clones an object, one is producing a copy with the same instance values. This means that one is essentially creating a copy that is *aligned* with the original, as can be seen in such programs such as 3D Studio Max and Viz. The copy can then be manipulated via its command operations, which will pull the clone out of alignment, i.e., break the symmetry of the object-clone pair. For example, a designer might wish to clone a cylinder that already exists in the design. The clone command creates an exact copy of the cylinder that is completely coincident with the existing one, and, after its creation, the designer will then move the copy out of alignment with the existing one.

Now let us consider a fundamental process in design. Consider first mechanical CAD, which is the design process in mechanical engineering – forming the basis, for example, of the aerospace and automotive industries. It is generally accepted that mechanical CAD proceeds by a process called *feature attachment*. This is the process of the successive addition of structural units and components.

The New Foundations give a mathematical theory of feature attachment in mechanical design, and also shows that it is mathematically equivalent to other design processes such as musical composition. The basic proposal is this:

THEORY OF FEATURE ATTACHMENT

When one creates objects and attaches them in the design structure, one is entering new instances into the alignment kernel, and positioning the command group for each new instance in the appropriate wreath level within the unfolding group corresponding to the inheritance hierarchy of the structure.

If the reader wishes to see how this shows that mechanical CAD is equivalent to musical composition, then the reader should consult my paper *Musical Works are Maximal Memory Stores* [23], which fully explains this equivalence.

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