Preface

The main goal of our study is an attempt to understand and classify nonsmooth structures arising within the optimization setting,

$$P(f,F): \quad \min f(x) \text{ s.t. } x \in M[F],$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a smooth real-valued objective function, $F : \mathbb{R}^n \longrightarrow \mathbb{R}^l$ is a smooth vector-valued function, and $M[F] \subset \mathbb{R}^n$ is a feasible set defined by F in some structured way. The nonsmoothness is given by the structure that fits the smooth function F to define the feasible set $M[F]$. The following optimization problems with particular types of nonsmoothness are considered (Chapters 2–5):

- mathematical programming problems with complementarity constraints,
- general semi-infinite programming problems,
- mathematical programming problems with vanishing constraints,
- bilevel optimization.

The basis of our study is the topological approach introduced in detail in Chapter 1. It encompasses the following questions:

(a) Under which conditions on F is $M[F]$ a Lipschitz manifold of an appropriate dimension?
(b) Under which conditions on F is $M[F]$ stable (i.e., $M[F]$ remains invariant up to a homomorphism w.r.t. smooth perturbations of F)?
(c) How does the homotopy type of lower-level set

$$M[f,F]^a := \{x \in M[F] \,|\, f(x) \leq a\}$$

change (as $a \in \mathbb{R}$ varies)?

Questions (a) and (b) deal with topological invariants of $M[F]$ and, more precisely, its structure. They lead to suitable constraint qualifications. Topological changes of $M[f,F]^a$ give rise to defining stationary points and developing critical point theory for $P(f,F)$ in the sense of Morse. In so doing, we get new topologically relevant optimization notions in terms of derivatives of f and F. It is worth pointing out that the same topological questions provide different (analytical) optimization concepts when applied to the particular problems above. The difference between these analytically described optimization concepts is a key point in understanding and comparing different kinds of nonsmoothness.

In Chapter 6, we discuss the impact of the topological approach on nonsmooth analysis. Topologically regular points of a min-type nonsmooth mapping $F : \mathbb{R}^n \longrightarrow \mathbb{R}^l$ are introduced. The crucial property is that for a topologically regular value $y \in \mathbb{R}^l$ of F the nonempty set $F^{-1}(y)$ is an $(n-l)$-dimensional Lipschitz manifold. Corresponding nonsmooth versions of Sard's Theorem are given.

We point out that the topological approach in the optimization context was introduced by H. Th. Jongen in the early 1980s ([61], [62]). The introduction of topological issues turned out to be extremely fruitful for establishing an adequate optimization theory in the smooth setting ([63]). The present book sheds light on nonsmooth optimization from the topological point of view, continuing to exploit the ideas of H. Th. Jongen.

I would like to thank my teacher H. Th. Jongen for sharing with me his insights on optimization and steering my studies toward its topological nature. This book originated mainly from a collaboration with him. I also thank my other coauthors, D. Dorsch, F. Guerra-Vázquez, Jan-J. Rückmann, S. Steffensen, and O. Stein, for fruitful collaborations. I am very grateful to H. Günzel, A. Ioffe, D. Klatte, B. Kummer, B. Mordukhovich, Yu. Nesterov, and D. Pallaschke for many interesting and helpful discussions.

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