

Chapter 2

Berry-Esseen Inequality for Unbounded Exchangeable Pairs

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Abstract The Berry-Esseen inequality is well-established by the Stein method of exchangeable pair approach when the difference of the pair is bounded. In this paper we obtain a general result which can achieve the optimal bound under some moment assumptions. As an application, a Berry-Esseen bound of $O(1/\sqrt{n})$ is derived for an independence test based on the sum of squared sample correlation coefficients.

2.1 Introduction and Main Result

Let W be the random variable of interest. Our goal is to prove a Berry-Esseen bound for W . One powerful approach in establishing a Berry-Esseen bound is the Stein method of exchangeable pairs. Let (W, W^*) be an exchangeable pair. Assume that

$$E(W - W^*|W) = \lambda(W - R) \quad (2.1)$$

for some $0 < \lambda < 1$, where R is a random variable. Put $\Delta = W - W^*$. The Berry-Esseen bounds are extensively studied under assumption (2.1). Rinott and Rotar [19] proved that (see e.g. [4, Theorem 5.2]) if $|\Delta| \leq \delta$ for some constant δ , $EW = 0$ and $EW^2 = 1$, then

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$$\sup_z |P(W \leq z) - \Phi(z)| \leq \delta \left(1.1 + \frac{1}{2\lambda} E|W|\Delta^2 \right) + 2.7E \left| 1 - \frac{1}{2\lambda} E(\Delta^2|W) \right| + 0.63E|R|. \quad (2.2)$$

For the unbounded case, Stein [23] proved that [see e.g. 4, (Theorem 5.5)] if $EW = 0$ and $EW^2 = 1$, then under (2.1),

$$\sup_z |P(W \leq z) - \Phi(z)| \leq E \left| 1 - \frac{1}{2\lambda} E(\Delta^2|W) \right| + E|R| + 0.64(E|\Delta|^3/\lambda)^{1/2}. \quad (2.3)$$

The bound (2.2) is usually optimal, however, the contribution of the last term in (2.3) may result in non-optimality. For example, when W is the standardized sum of n i.i.d. random variables, the last term in (2.3) is of order $n^{-1/4}$ instead of the optimal $n^{-1/2}$. The main purpose of this note is to develop a general result which can achieve the optimal bound under some moment assumptions.

Theorem 2.1. *Assume that (2.1) is satisfied. Then*

$$\begin{aligned} \sup_z |P(W \leq z) - \Phi(z)| &\leq E|R| + \frac{1}{4\lambda} E(|W| + 1)|\Delta|^3 \\ &+ (1 + \tau^2) \left(4\lambda^{1/2} + 4\tau\lambda^{1/2} + 6E \left| 1 - \frac{1}{2\lambda} E(\Delta^2|W) \right| + \frac{2}{E\Lambda} E|\Lambda - E\Lambda| \right) \end{aligned} \quad (2.4)$$

where Λ is any variable such that $\Lambda \geq E(\Delta^4|W)$ and $\tau = \sqrt{E\Lambda}/\lambda$.

We remark that if $|\Delta| \leq \delta$ for some constant δ , then one can choose Λ equal to δ^4 and hence the bound on the right hand side of (2.4) reduces to $C(E|R| + \delta^3/\lambda + \lambda^{1/2} + E|1 - E(\Delta^2|W)/(2\lambda)|)$, which is of the similar order as (2.2).

The paper is organized as follows. The proof of Theorem 2.1 will be given in the next section. In Sect. 2.3, as an application, a Berry-Esseen bound is derived for an independence test based on the sum of squared sample correlation coefficients.

2.2 Proof of the Main Result

Let f be absolutely continuous. Following the discussion in [4, Sect. 2.3], observing that $E[(W - W^*)(f(W) + f(W^*))] = 0$, we have

$$E(Wf(W)) = \frac{1}{2\lambda} E[(W - W^*)(f(W) - f(W^*))] + E(Rf(W)). \quad (2.5)$$

Recall $\Delta = W - W^*$ and define $\hat{K}(t) = \frac{\Delta}{2\lambda} (I_{(-\Delta \leq t \leq 0)} - I_{(0 \leq t \leq -\Delta)})$. Then

$$\int_{-\infty}^{\infty} \hat{K}(t) dt = \frac{\Delta^2}{2\lambda}, \quad \int_{-\infty}^{\infty} |t| \hat{K}(t) dt = \frac{1}{4\lambda} |\Delta|^3. \quad (2.6)$$

By (2.5),

$$\begin{aligned} E(Wf(W)) &= \frac{1}{2\lambda} E\left(\int_{-\Delta}^0 f'(W+t)(W-W^*) dt\right) + E(Rf(W)) \\ &= E\left(\int_{-\infty}^{\infty} f'(W+t)\hat{K}(t) dt\right) + E(Rf(W)). \end{aligned} \quad (2.7)$$

and by (2.6),

$$\begin{aligned} Ef'(W) &= E\left(f'(W)\left(1 - \frac{1}{2\lambda}\Delta^2\right)\right) + E\left(\int_{-\infty}^{\infty} f'(W)\hat{K}(t) dt\right) \\ &= E\left(f'(W)\left(1 - \frac{1}{2\lambda}E(\Delta^2|W)\right)\right) + E\left(\int_{-\infty}^{\infty} f'(W)\hat{K}(t) dt\right). \end{aligned}$$

To prove Theorem 2.1, we first need to show the following concentration inequality.

Lemma. 2.1 *Assume that (2.1) is satisfied and that $E|W| \leq 1$, $E|R| \leq 1$. Let Λ be a random variable such that $\Lambda \geq E(\Delta^4|W)$. Then*

$$\begin{aligned} P(a \leq W \leq b) \\ \leq 2(b-a) + 2(E\Lambda/\lambda)^{1/2} + 2E|1 - E(\Delta^2|W)/(2\lambda)| + E|\Lambda - E\Lambda|/E\Lambda. \end{aligned} \quad (2.8)$$

Proof Let

$$f(w) = \begin{cases} -\frac{1}{2}(b-a) - \delta & \text{for } w < a - \delta, \\ w - \frac{1}{2}(b+a) & \text{for } a - \delta \leq w \leq b + \delta, \\ \frac{1}{2}(b-a) + \delta & \text{for } w > b + \delta, \end{cases}$$

where $\delta = (4E\Lambda/(27\lambda))^{1/2}$. Clearly, $f' \geq 0$ and $\hat{K} \geq 0$. By (2.7).

$$\begin{aligned} 2\lambda E\{(W-R)f(W)\} &= 2\lambda E \int_{-\infty}^{\infty} f'(W+t)\hat{K}(t) dt \\ &\geq 2\lambda E \int_{-\delta}^{\delta} f'(W+t)\hat{K}(t) dt \\ &\geq 2\lambda E I_{(a \leq W \leq b)} \int_{|t| \leq \delta} \hat{K}(t) dt \\ &= E I_{(a \leq W \leq b)} |\Delta| \min(\delta, |\Delta|) \\ &\geq E I_{(a \leq W \leq b)} \left(\Delta^2 - \frac{4\Delta^4}{27\delta^2}\right), \end{aligned} \quad (2.9)$$

where in the last inequality we use the fact that $\min(x, y) \geq x - 4x^3/(27y^2)$ for $x > 0, y > 0$. The right hand side of (2.9) is equal to

$$\begin{aligned}
& E \left\{ I_{(a \leq W \leq b)} \left(\Delta^2 - \frac{4\Delta^4}{27\delta^2} \right) \right\} \\
&= E \left\{ I_{(a \leq W \leq b)} \left(E(\Delta^2 | W) - \frac{4E(\Delta^4 | W)}{27\delta^2} \right) \right\} \\
&\geq E \left\{ I_{(a \leq W \leq b)} \left(E(\Delta^2 | W) - \frac{4\Lambda}{27\delta^2} \right) \right\} [\text{for } \Lambda \geq E(\Delta^4 | W)] \\
&\geq P(a \leq W \leq b) (2\lambda - 4E\Lambda / (27\delta^2)) - E|E(\Delta^2 | W) - 2\lambda| - 427\delta^2 E|\Lambda - E\Lambda| \\
&= \lambda P(a \leq W \leq b) - E|E(\Delta^2 | W) - 2\lambda| - \frac{4}{27\delta^2} E|\Lambda - E\Lambda|,
\end{aligned} \tag{2.10}$$

recalling that $\delta = (4E\Lambda / (27\lambda))^{1/2}$. For the left hand side of (2.9), we have

$$2\lambda E\{(W - R)f(W)\} \leq 2\lambda \left(\frac{1}{2}(b - a) + \delta \right) (E|W| + E|R|) \leq 4\lambda \left(\frac{1}{2}(b - a) + \delta \right). \tag{2.11}$$

Combining (2.9)–(2.11) gives (2.8), as desired. $\square$

Proof of Theorem 2.1 For fixed z , let $f = f_z$ be the solution to the Stein equation

$$f'(w) - wf(w) = I_{\{w \leq z\}} - \Phi(z). \tag{2.12}$$

It is known that $\|f\| \leq 1$ and $\|f'\| \leq 1$ (see e.g., [6]). Hence by (2.12),

$$|P(W \leq z) - \Phi(z)| = |E(f'(W) - Wf(W))| \leq E \left| \left(1 - \frac{1}{2\lambda} E(\Delta^2 | W) \right) \right| + E|R| + H,$$

where

$$H = E \left| \int_{-\infty}^{\infty} (f'(W) - f'(W + t)) \hat{K}(t) dt \right|.$$

From the definition of f it follows that

$$\begin{aligned}
H &\leq E \left| \int_{-\infty}^{\infty} (Wf(W) - (W + t)f(W + t)) \hat{K}(t) dt \right| \\
&\quad + E \int_0^{\infty} I_{\{z-t \leq W \leq z\}} \hat{K}(t) dt + E \int_{-\infty}^0 I_{\{z \leq W \leq z-t\}} \hat{K}(t) dt \\
&\leq \frac{1}{4\lambda} E(|W| + 1) |\Delta|^3 + H_1 + H_2,
\end{aligned}$$

where

$$H_1 = E \int_0^{\infty} I_{\{z-t \leq W \leq z\}} \hat{K}(t) dt, \quad H_2 = E \int_{-\infty}^0 I_{\{z \leq W \leq z-t\}} \hat{K}(t) dt.$$

To evaluate H_1 , divide the integral into two parts: $\int_0^{\sqrt{\lambda}}$ and $\int_{\sqrt{\lambda}}^{\infty}$. For the first part, we have

$$\begin{aligned}
& E \int_0^{\sqrt{\lambda}} I_{\{z-t \leq W \leq z\}} \hat{K}(t) dt \\
&= \frac{1}{2\lambda} E \int_0^{\sqrt{\lambda}} I_{\{z-t \leq W \leq z\}} (-\Delta) I_{\{0 \leq t \leq -\Delta\}} dt \\
&\leq \frac{1}{2\lambda} E \int_0^{\sqrt{\lambda}} I_{\{z-\sqrt{\lambda} \leq W \leq z\}} (-\Delta) I_{\{0 \leq t \leq -\Delta\}} dt \\
&\leq \frac{1}{2\lambda} E I_{\{z-\sqrt{\lambda} \leq W \leq z\}} \Delta^2 \\
&\leq P(z - \sqrt{\lambda} \leq W \leq z) + E|1 - E(\Delta^2|W)/(2\lambda)| \\
&\leq 2\sqrt{\lambda} + 2(E\Lambda/\lambda)^{1/2} + 3E|1 - E(\Delta^2|W)/(2\lambda)| + E|\Lambda - E\Lambda|/E\Lambda,
\end{aligned} \tag{2.13}$$

by Lemma 2.1. For the second part, noting that $\hat{K}(t) \leq \Delta^4/(2\lambda t^3)$ for $t \geq 0$, we have

$$\begin{aligned}
& 2\lambda E \int_{\sqrt{\lambda}}^{\infty} I_{\{z-t \leq W \leq z\}} \hat{K}(t) dt \\
&\leq E \int_{\sqrt{\lambda}}^{\infty} I_{\{z-t \leq W \leq z\}} \Delta^4/t^3 dt \\
&= E \int_{\sqrt{\lambda}}^{\infty} I_{\{z-t \leq W \leq z\}} E(\Delta^4|W)/t^3 dt \\
&\leq E \int_{\sqrt{\lambda}}^{\infty} I_{\{z-t \leq W \leq z\}} \Lambda/t^3 dt \\
&\leq \frac{1}{2\lambda} E \int_{\sqrt{\lambda}}^{\infty} P(z-t \leq W \leq z) E(\Lambda)/t^3 dt + E|\Lambda - E\Lambda| \int_{\sqrt{\lambda}}^{\infty} 1/t^3 dt \\
&\leq E(\Lambda) E \int_{\sqrt{\lambda}}^{\infty} \left(2t + 2(E\Lambda/\lambda)^{1/2} + 2E|1 - E(\Delta^2|W)/(2\lambda)| \right. \\
&\quad \left. + E|\Lambda - E\Lambda|/E\Lambda \right) t^{-3} dt + E|\Lambda - E\Lambda|/(2\lambda) \quad [\text{by Lemma 2.1 again}] \\
&= E(\Lambda) \left(2/\sqrt{\lambda} + \{2(E\Lambda/\lambda)^{1/2} + 2E|1 - E(\Delta^2|W)/(2\lambda)| \right. \\
&\quad \left. + E|\Lambda - E\Lambda|/E\Lambda\} / (2\lambda) \right) + E|\Lambda - E\Lambda|/(2\lambda) \\
&= \frac{E\Lambda}{2\lambda} \left(4\sqrt{\lambda} + 2(E\Lambda/\lambda)^{1/2} + 2E|1 - E(\Delta^2|W)/(2\lambda)| \right. \\
&\quad \left. + 2E|\Lambda - E\Lambda|/E\Lambda \right).
\end{aligned} \tag{2.14}$$

Therefore, by (2.13) and (2.14),

$$\begin{aligned}
H_1 &\leq 2\sqrt{\lambda} + 2(E\Lambda/\lambda)^{1/2} + 3E|1 - E(\Delta^2|W)/(2\lambda)| + E|\Lambda - E\Lambda|/E\Lambda \\
&\quad + \frac{E\Lambda}{4\lambda^2} \left(4\sqrt{\lambda} + 2(E\Lambda/\lambda)^{1/2} + 2E|1 - E(\Delta^2|W)/(2\lambda)| \right. \\
&\quad \left. + 2E|\Lambda - E\Lambda|/E\Lambda \right).
\end{aligned} \tag{2.15}$$

Similarly, (2.15) holds with H_1 replaced by H_2 . This proves (2.4) by combining the above inequalities. $\square$

2.3 An Application to an Independence Test

Consider a m -variable population represented by a random vector $X = (X_1, \dots, X_m)'$ with covariance matrix Σ and let $\{X_1, \dots, X_n\}$ be a random sample of size n from the population. In applications of multivariate analysis, m is usually large and even larger than the sample size n . However, most inference procedures in classical multivariate analysis are based on asymptotic theory which has the sample size n going to infinity while m is fixed. Therefore, these procedures may not be very accurate when m is of the same order of magnitude as n . More and more attentions have been paid to the asymptotic theory for large m , see for example, [2, 9, 10, 12, 14, 20, 21] and references therein.

For concreteness, we focus on a common testing problem: complete independence. Let R be the sample correlation matrix of $\{X_1, \dots, X_n\}$. When the population has a multivariate normal distribution, testing for complete independence is equivalent to testing $\Sigma = I_m$, and many test statistics have been developed in the literature. The likelihood ratio test statistic

$$-\left(n - \frac{2m+5}{6}\right) \log |R|$$

is commonly used for $m < n$; however, it is degenerate when m exceeds n since $|R| = 0$ for $m > n$. Nagao [18] proposed to use $\frac{1}{m} \text{tr}[(R - I)^2]$, where tr denotes the trace. However, this test is not consistent against the alternative when m goes to infinity with n . Leboit and Wolf [14] introduced a revised statistic $\frac{1}{m} \text{tr}[(R - I)^2] - \frac{m}{n} [\frac{1}{m} \text{tr}(R)]^2 + \frac{m}{n}$, which is robust against m large, and even larger than n . Schott [21] considered a test based on the sum of squared sample correlation coefficients. Let $R = (r_{ij}, 1 \leq i, j \leq m)$ be the sample correlation matrix, where

$$r_{ij} = \frac{\sum_{k=1}^n (X_{ik} - \bar{X}_i)(X_{jk} - \bar{X}_j)}{\sqrt{\sum_{k=1}^n (X_{ik} - \bar{X}_i)^2} \sqrt{\sum_{k=1}^n (X_{jk} - \bar{X}_j)^2}},$$

$X_i = (X_{1i}, \dots, X_{mi})'$ and $\bar{X}_i = \frac{1}{n} \sum_{k=1}^n X_{ik}$. Let $t_{n,m}$ be the sum of squared r_{ij} 's for $i > j$,

$$t_{n,m} = \sum_{i=2}^m \sum_{j=1}^{i-1} r_{ij}^2,$$

and

$$W_{n,m} = c_{n,m} \left(t_{n,m} - \frac{m(m-1)}{2(n-1)} \right), \text{ where } c_{n,m} = \frac{n\sqrt{n+2}}{\sqrt{m(m-1)(n-1)}}.$$

Under the assumption $0 < \lim_{n \rightarrow \infty} m/n < \infty$ and complete independence, Schott [21] proved the central limit theorem

$$W_{n,m} \rightarrow N(0, 1),$$

and then uses $W_{n,m}$ to test the complete independence. When the population is not necessarily normally distributed, Jiang [13], Liu, Lin and Shao [15] introduced new statistics based on the maximum of absolute values of the sample correlation coefficients and prove that the limiting distribution is an extreme distribution of type I.

As an application of Theorem 2.1, we establish the following Berry-Essen bound for $W_{n,m}$ with an optimal rate of $O(m^{-1/2})$.

Theorem 2.2. *Let $\{X_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$ be i.i.d. random variables, and let Z be a standard normally distributed random variable. Assume $m = O(n)$. If $E(X_{11}^{24}) < \infty$, then*

$$\sup_z |P(W_{n,m} < z) - \Phi(z)| = O(m^{-1/2}). \quad (2.16)$$

To apply Theorem 2.1, we first construct W^* so that (W, W^*) is an exchangeable pair. Let X_i^* , $1 \leq i \leq n$ be an independent copy of X_i , $1 \leq i \leq n$, and let I be a random index uniformly distributed over $\{1, 2, \dots, m\}$, independent of $\{X_i^*, X_i, 1 \leq i \leq n\}$. Define $t_{n,m}^* = t_{n,m} - \sum_{j=1}^m r_{Ij}^2 + \sum_{j=1}^m r_{I^*j}^2$, where

$$r_{i^*j} = \frac{\sum_{k=1}^n (X_{ik}^* - \bar{X}_{i^*})(X_{jk} - \bar{X}_j)}{\sqrt{\sum_{k=1}^n (X_{ik}^* - \bar{X}_{i^*})^2} \sqrt{\sum_{k=1}^n (X_{jk} - \bar{X}_j)^2}}.$$

It is easy to see that $(t_{n,m}, t_{n,m}^*)$ is an exchangeable pair. Write $W = W_{n,m}$ and define

$$W^* = c_{n,m} \left(t_{n,m}^* - \frac{m(m-1)}{2(n-1)} \right).$$

Clearly, (W, W^*) is also an exchangeable pair. To prove (2.16), it suffices to show by Theorem 2.1 that there exists Λ such that $\Lambda \geq \lambda^2 + E(\Delta^4 | X_1, \dots, X_n)$ with $\lambda = 2/m$,

$$E(W - W^*|W) = \lambda W, \quad (2.17)$$

$$E\left|1 - \frac{1}{2\lambda}E(\Delta^2|W)\right| = O(m^{-1/2}), \quad (2.18)$$

$$E|\Delta|^3 = O(m^{-3/2}), \quad (2.19)$$

$$m^{-2} \leq E(\Lambda) = O(m^{-2}), \quad (2.20)$$

$$\text{var}(\Lambda) = O(m^{-5}), \quad (2.21)$$

$$E(|W| + 1)|\Delta|^3 = O(m^{-3/2}). \quad (2.22)$$

To prove (2.17)–(2.22) above, we start with some preliminary properties on moments of r_{ij} . Let $u_i = (u_{i1}, \dots, u_{in})'$, where $u_{ik} = \frac{X_{ik} - \bar{X}_i}{\sqrt{\sum_{l=1}^n (X_{il} - \bar{X}_i)^2}}$. Clearly, we have

$$\sum_{k=1}^n u_{ik} = 0, \sum_{k=1}^n u_{ik}^2 = 1, \quad (2.23)$$

and $r_{ij} = u_i' u_j$. It follows from (2.23) and the symmetry of the variables involved that

$$E(u_{ik}) = 0, E(u_{ik}^2) = \frac{1}{n}, \quad (2.24)$$

$$E(u_{ij}u_{ik}) = -\frac{1}{(n-1)n} \quad \text{for } j \neq k. \quad (2.25)$$

By (2.24) and (2.25), we have for $i \neq j$

$$\begin{aligned} E(r_{ij}^2) &= E(u_i' u_j u_j' u_i) = E(E(u_i' u_j u_j' u_i | u_i)) \\ &= E\left(u_i' \left(\frac{1}{n-1} I_n - \frac{1}{n(n-1)} \mathbf{1}_n\right) u_i\right) = \frac{1}{n-1}, \end{aligned} \quad (2.26)$$

where I_n is the $n \times n$ identity matrix and $\mathbf{1}_n$ denotes a $n \times n$ matrix with all entries 1.

If $EX_{11}^6 < \infty$, it is easy to see that (see e.g., [16])

$$E(u_{ik}^4) = \frac{K}{n^2} + O\left(\frac{1}{n^3}\right), \text{ where } K = \frac{E(X_{11} - \mu)^4}{\sigma^4}. \quad (2.27)$$

From (2.23), we also have

$$\begin{aligned}
E(u_{ik_1}^3 u_{ik_2}) &= -\frac{E(u_{11}^4)}{n-1}, \\
E(u_{ik_1}^2 u_{ik_2}^2) &= -\frac{E(u_{11}^4)}{n-1} + \frac{1}{n(n-1)}, \\
E(u_{ik_1}^2 u_{ik_2} u_{ik_3}) &= \frac{2E(u_{11}^4)}{(n-1)(n-2)} - \frac{1}{n(n-1)(n-2)}, \\
E(u_{ik_1} u_{ik_2} u_{ik_3} u_{ik_4}) &= \frac{3}{n(n-1)(n-2)(n-3)} - \frac{6E(u_{11}^4)}{(n-1)(n-2)(n-3)},
\end{aligned}$$

for distinct k_1, k_2, k_3, k_4 . Hence for $i \neq j$,

$$\begin{aligned}
E(r_{ij}^4) &= E\left(\sum_{k_1=1}^n \sum_{k_2=1}^n \sum_{k_3=1}^n \sum_{k_4=1}^n u_{ik_1} u_{ik_2} u_{ik_3} u_{ik_4} u_{jk_1} u_{jk_2} u_{jk_3} u_{jk_4}\right) \\
&= nE(u_{ik}^4 u_{jk}^4) + 4n(n-1)E(u_{ik_1}^3 u_{ik_2} u_{jk_1}^3 u_{jk_2}) \\
&\quad + 3n(n-1)E(u_{ik_1}^2 u_{ik_2}^2 u_{jk_1}^2 u_{jk_2}^2) \\
&\quad + 6n(n-1)(n-2)E(u_{ik_1}^2 u_{ik_2} u_{ik_3} u_{jk_1}^2 u_{jk_2} u_{jk_3}) \\
&\quad + n(n-1)(n-2)(n-3)E(u_{ik_1} u_{ik_2} u_{ik_3} u_{ik_4} u_{jk_1} u_{jk_2} u_{jk_3} u_{jk_4}) \\
&= n\left(\frac{K}{n^2} + O\left(\frac{1}{n^3}\right)\right)^2 + 4n(n-1)\left(-\frac{K}{(n-1)n^2} + O\left(\frac{1}{n^4}\right)\right)^2 \\
&\quad + 3n(n-1)\left(\frac{1}{n(n-1)} - \frac{K}{(n-1)n^2} + O\left(\frac{1}{n^4}\right)\right)^2 \\
&\quad + 6n(n-1)(n-2)\left(\frac{2K}{n^2(n-1)(n-2)} - \frac{1}{n(n-1)(n-2)} + O\left(\frac{1}{n^5}\right)\right)^2 \\
&\quad + n(n-1)(n-2)(n-3)\left(\frac{3}{n(n-1)(n-2)(n-3)}\right. \\
&\quad \left.- \frac{K}{n^2(n-1)(n-2)(n-3)} + O\left(\frac{1}{n^6}\right)\right)^2 \\
&= \frac{3}{n^2} + O\left(\frac{1}{n^3}\right).
\end{aligned} \tag{2.28}$$

Similarly to (2.26), for $j_1 \neq j_2$,

$$E(r_{ij_1}^2 r_{ij_2}^2) = E(E(r_{ij_1}^2 r_{ij_2}^2 | X_i)) = E\left(\frac{1}{(n-1)^2}\right) = \frac{1}{(n-1)^2}, \tag{2.29}$$

$$\begin{aligned}
E(r_{ij}^4 | X_j) &= E(u_{ik}^4) \sum_{k=1}^n u_{jk}^4 + 4E(u_{ik_1}^3 u_{ik_2}) \sum_{k_1=1}^n \sum_{\substack{k_2=1 \\ k_2 \neq k_1}}^n u_{jk_1}^3 u_{jk_2} \\
&\quad + 3E(u_{ik_1}^2 u_{ik_2}^2) \sum_{k_1=1}^n \sum_{\substack{k_2=1 \\ k_2 \neq k_1}}^n u_{jk_1}^2 u_{jk_2}^2
\end{aligned}$$

$$\begin{aligned}
& + 6E(u_{ik_1}^2 u_{ik_2} u_{ik_3}) \sum_{k_1=1}^n \sum_{\substack{k_2=1 \\ k_2 \neq k_1}}^n \sum_{\substack{k_3=1 \\ k_3 \neq k_1 \\ k_3 \neq k_2}}^n u_{jk_1}^2 u_{jk_2} u_{jk_3} \\
& + E(u_{ik_1} u_{ik_2} u_{ik_3} u_{ik_4}) \sum_{k_1=1}^n \sum_{\substack{k_2=1 \\ k_2 \neq k_1}}^n \sum_{\substack{k_3=1 \\ k_3 \neq k_1 \\ k_3 \neq k_2}}^n \sum_{\substack{k_4=1 \\ k_4 \neq k_1 \\ k_4 \neq k_2 \\ k_4 \neq k_3}}^n u_{jk_1} u_{jk_2} u_{jk_3} u_{jk_4}.
\end{aligned}$$

Since $\sum_{k=1}^n u_{ik} = 0$ and $\sum_{k=1}^n u_{ik}^2 = 1$, we have:

$$\begin{aligned}
E(r_{ij}^4 | X_j) &= \left(E(u_{ik}^4) - 4E(u_{ik_1}^3 u_{ik_2}) - 3E(u_{ik_1}^2 u_{ik_2}^2) + 12E(u_{ik_1}^2 u_{ik_2} u_{ik_3}) \right. \\
&\quad \left. - 6E(u_{ik_1} u_{ik_2} u_{ik_3} u_{ik_4}) \right) \sum_{k=1}^n u_{jk}^4 \\
&\quad + 3E(u_{ik_1}^2 u_{ik_2}^2) - 6E(u_{ik_1}^2 u_{ik_2} u_{ik_3}) + 3E(u_{ik_1} u_{ik_2} u_{ik_3} u_{ik_4}) \\
&= \frac{3}{n^2} + O\left(\frac{1}{n^3}\right) + \left(\frac{K-3}{n^2} + O\left(\frac{1}{n^3}\right)\right) \sum_{k=1}^n u_{jk}^4.
\end{aligned}$$

Thus,

$$\begin{aligned}
E(r_{ij_1}^4 r_{ij_2}^4) &= E(E(r_{ij_1}^4 r_{ij_2}^4 | X_i)) \\
&= E\left(\left(\frac{3}{n^2} + O\left(\frac{1}{n^3}\right) + \left(\frac{K-3}{n^2} + O\left(\frac{1}{n^3}\right)\right) \sum_{k=1}^n u_{ik}^4\right)^2\right) \\
&= \frac{9}{n^4} + O\left(\frac{1}{n^5}\right).
\end{aligned} \tag{2.30}$$

Similarly, if $EX_{11}^{12} < \infty$,

$$E(r_{ij}^8) = O(m^{-4}), \tag{2.31}$$

and if $EX_{11}^{24} < \infty$,

$$E(r_{ij}^{16}) = O(m^{-4}),$$

By (2.26), (2.28) and (2.29), we have

$$\begin{aligned}
E(t_{n,m}) &= \frac{m(m-1)}{2} E(r_{ij}^2) = \frac{m(m-1)}{2(n-1)}, \\
\text{var}(t_{n,m}) &= E(t_{n,m}^2) - (E(t_{n,m}))^2 \\
&= \frac{m(m-1)}{2} E(r_{ij}^4) + \frac{m(m-1)(m-2)}{2} E(r_{ij_1}^2 r_{ij_2}^2)
\end{aligned}$$

$$\begin{aligned}
E(t_{n,m}) &= \frac{m(m-1)}{2} E(r_{ij}^2) = \frac{m(m-1)}{2(n-1)}, \\
\text{var}(t_{n,m}) &= E(t_{n,m}^2) - (E(t_{n,m}))^2 \\
&= \frac{m(m-1)}{2} E(r_{ij}^4) + \frac{m(m-1)(m-2)}{2} E(r_{i_1 j_1}^2 r_{i_2 j_2}^2) \\
&\quad + \frac{m(m-1)^2(m-2)}{4} E(r_{i_1 j_1}^2 r_{i_2 j_2}^2) - (E(t_{n,m}))^2 \\
&= \frac{m(m-1)}{n^2} + O\left(\frac{m^2}{n^3}\right).
\end{aligned}$$

Proof of (2.17) Let $\mathcal{X}_n = \{X_1, \dots, X_n\}$ and define u_i^* similarly as u_i by using $\{X_{ij}^*\}$ instead of $\{X_{ij}\}$. As in (2.26), we have

$$\begin{aligned}
E(r_{i^* j}^2 | \mathcal{X}_n) &= E(E(u'_j u_i^* u_i^{*'} u_j | \mathcal{X}_n)) \\
&= \frac{1}{n-1} u'_j E(u_i^* u_i^{*'}) u_j = \frac{1}{n-1},
\end{aligned}$$

and hence

$$E(W - W^* | W) = \frac{c_{n,m}}{m} \sum_{i=1}^m E\left(\sum_{\substack{j=1 \\ j \neq i}}^m r_{ij}^2 - \sum_{\substack{j=1 \\ j \neq i}}^m r_{i^* j}^2 \middle| W\right) = \frac{2}{m} W$$

as desired. $\square$

Proof of (2.18) Recall $\lambda = 2/m$ and the definition of Δ and observe that

$$\begin{aligned}
&\left| \frac{E(\Delta^2 | \mathcal{X}_n)}{2\lambda} - 1 \right| \\
&= \frac{c_{n,m}^2 m}{4} \left| \frac{1}{m} \sum_{i=1}^m E\left(\left(\sum_{\substack{j=1 \\ j \neq i}}^m r_{ij}^2 - r_{i^* j}^2\right)^2 \middle| \mathcal{X}_n\right) - \frac{4(m-1)(n-1)}{n^2(n+2)} \right| \\
&= \frac{c_{n,m}^2 m}{4} \left| \frac{1}{m} \sum_{i=1}^m \left(\sum_{\substack{j=1 \\ j \neq i}}^m \left(r_{ij}^2 - \frac{1}{n-1}\right)^2\right) - \frac{2(m-1)(n-1)}{n^2(n+2)} \right| \\
&\quad + \frac{1}{m} \sum_{i=1}^m E\left(\left(\sum_{\substack{j=1 \\ j \neq i}}^m \left(r_{i^* j}^2 - \frac{1}{n-1}\right)\right)^2 \middle| \mathcal{X}_n\right) - \frac{2(m-1)(n-1)}{n^2(n+2)} \right| \\
&\leq \frac{c_{n,m}^2 m}{4} (J_1 + J_2) + O(1/n),
\end{aligned} \tag{2.32}$$

where

$$J_1 = \left| \frac{1}{m} \sum_{i=1}^m \left\{ \sum_{\substack{j=1 \\ j \neq i}}^m \left(r_{ij}^2 - \frac{1}{n-1} \right) \right\}^2 - \frac{2(m-1)}{n^2} \right|,$$

$$J_2 = \left| \frac{1}{m} \sum_{i=1}^m E \left(\left\{ \sum_{\substack{j=1 \\ j \neq i}}^m \left(r_{ij}^2 - \frac{1}{n-1} \right) \right\}^2 \middle| \mathcal{X}_n \right) - \frac{2(m-1)}{n^2} \right|.$$

Instead of estimate $E J_1$, we look at the second moment of J_1 . From (2.28) and (2.29) we obtain

$$\frac{1}{m} \sum_{i=1}^m E \left\{ \sum_{\substack{j=1 \\ j \neq i}}^m \left(r_{ij}^2 - \frac{1}{n-1} \right) \right\}^2 = \frac{2(m-1)}{n^2} + O(m/n^3),$$

and hence

$$\begin{aligned} E J_1^2 &= E \left(\frac{1}{m} \sum_{i=1}^m \left\{ \sum_{\substack{j=1 \\ j \neq i}}^m \left(r_{ij}^2 - \frac{1}{n-1} \right) \right\}^2 - \frac{2(m-1)}{n^2} \right)^2 \\ &= E \left(\frac{1}{m} \sum_{i=1}^m \left\{ \sum_{\substack{j=1 \\ j \neq i}}^m \left(r_{ij}^2 - \frac{1}{n-1} \right) \right\}^2 \right)^2 - \frac{4(m-1)^2}{n^4} + O(m^2/n^5) \\ &= m^{-2} \sum_{i=1}^m E \left\{ \sum_{\substack{j=1 \\ j \neq i}}^m \left(r_{ij}^2 - \frac{1}{n-1} \right) \right\}^4 \\ &\quad + m^{-2} \sum_{1 \leq i_1 \neq i_2 \leq m} E \left\{ \sum_{\substack{j=1 \\ j \neq i_1}}^m \left(r_{i_1 j}^2 - \frac{1}{n-1} \right) \right\}^2 \left\{ \sum_{\substack{l=1 \\ l \neq i_2}}^m \left(r_{i_2 l}^2 - \frac{1}{n-1} \right) \right\}^2 \right. \\ &\quad \left. - \frac{4m^2}{n^4} + O(m^2/n^5) \right. \\ &= m^{-2} \left\{ \sum_{i=1}^m \sum_{\substack{j=1 \\ j \neq i}}^m E \left(r_{ij}^2 - \frac{1}{n-1} \right)^4 \right. \\ &\quad + 4 \sum_{i=1}^m \sum_{\substack{j_1=1 \\ j_1 \neq i}}^m \sum_{\substack{j_2=1 \\ j_2 \neq i \\ j_2 \neq j_1}}^m E \left(\left(r_{ij_1}^2 - \frac{1}{n-1} \right)^3 \left(r_{ij_2}^2 - \frac{1}{n-1} \right) \right) \\ &\quad \left. + 3 \sum_{i=1}^m \sum_{\substack{j_1=1 \\ j_1 \neq i}}^m \sum_{\substack{j_2=1 \\ j_2 \neq i \\ j_2 \neq j_1}}^m E \left(\left(r_{ij_1}^2 - \frac{1}{n-1} \right)^2 \left(r_{ij_2}^2 - \frac{1}{n-1} \right)^2 \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + 6 \sum_{i=1}^m \sum_{\substack{j_1=1 \\ j_1 \neq i}}^m \sum_{\substack{j_2=1 \\ j_2 \neq i \\ j_2 \neq j_1}}^m \sum_{\substack{j_3=1 \\ j_3 \neq i \\ j_3 \neq j_1 \\ j_3 \neq j_2}}^m E\left(\left(r_{ij_1}^2 - \frac{1}{n-1}\right)^2 \left(r_{ij_2}^2 - \frac{1}{n-1}\right) \left(r_{ij_3}^2 - \frac{1}{n-1}\right)\right) \\
& + \sum_{i=1}^m \sum_{\substack{j_1=1 \\ j_1 \neq i}}^m \sum_{\substack{j_2=1 \\ j_2 \neq i \\ j_2 \neq j_1}}^m \sum_{\substack{j_3=1 \\ j_3 \neq i \\ j_3 \neq j_1 \\ j_3 \neq j_2}}^m \sum_{\substack{j_4=1 \\ j_4 \neq i \\ j_4 \neq j_1 \\ j_4 \neq j_2 \\ j_4 \neq j_3}}^m E\left(\left(r_{ij_1}^2 - \frac{1}{n-1}\right) \left(r_{ij_2}^2 - \frac{1}{n-1}\right) \right. \\
& \times \left. \left(r_{ij_3}^2 - \frac{1}{n-1}\right) \left(r_{ij_4}^2 - \frac{1}{n-1}\right)\right) + \sum_{i_1=1}^m \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^m E\left(r_{i_1 i_2}^2 - \frac{1}{n-1}\right)^4 \\
& + \sum_{i_1=1}^m \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^m \sum_{j_1=1}^m \sum_{j_2=1}^m I(j_1 \neq i_2 \text{ or } j_2 \neq i_1) \\
& E\left(\left(r_{i_1 j_1}^2 - \frac{1}{n-1}\right)^2 \left(r_{i_2 j_2}^2 - \frac{1}{n-1}\right)^2\right) \\
& + 4 \sum_{i_1=1}^m \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^m \sum_{\substack{j=1 \\ j \neq i_1 \\ j \neq i_2}}^m \left(E\left(\left(r_{i_1 i_2}^2 - \frac{1}{n-1}\right)^2 \left(r_{i_1 j}^2 - \frac{1}{n-1}\right) \left(r_{i_2 j}^2 - \frac{1}{n-1}\right)\right) \right. \\
& \left. + E\left(\left(r_{i_1 i_2}^2 - \frac{1}{n-1}\right) \left(r_{i_1 j}^2 - \frac{1}{n-1}\right)^2 \left(r_{i_2 j}^2 - \frac{1}{n-1}\right)\right) \right) \Big\} - \frac{4m^2}{n^4} + O(m^2/n^5) \\
& = O(n^{-4}) + 0 + 3m(4n^{-4} + O(n^{-5})) + 0 + 0 + O(n^{-4}) \\
& + m^2(4n^{-4} + O(n^{-5})) + mO(n^{-4}) - \frac{4m^2}{n^4} + O(m^2/n^5) \\
& = O(m/n^4).
\end{aligned} \tag{2.33}$$

We get the last equality by (2.30), (2.31) and the method in (2.26). Thus

$$E J_1 = O(m^{1/2}/n^2). \tag{2.34}$$

Similarly, we have

$$\begin{aligned}
E J_2^2 & = E\left(\frac{1}{m} \sum_{i=1}^m \left\{ \sum_{\substack{j=1 \\ j \neq i}}^m \left(r_{ij}^2 - \frac{1}{n-1}\right) \right\}^2 - \frac{2(m-1)}{n^2}\right) \\
& \times \left(\frac{1}{m} \sum_{i=1}^m \left\{ \sum_{\substack{j=1 \\ j \neq i}}^m \left(r_{i^*j}^2 - \frac{1}{n-1}\right) \right\}^2 - \frac{2(m-1)}{n^2}\right) \\
& = O(m/n^4).
\end{aligned}$$

Thus

$$E J_2 = O(m^{1/2}/n^2). \quad (2.35)$$

Noting that $c_{n,m} = O(n/m)$, we prove (2.18) by (2.32), (2.34) and (2.35). $\square$

Proof of (2.19) We estimate the forth moment of Δ . We have

$$\begin{aligned} E \Delta^4 &= c_{n,m}^4 \frac{1}{m} \sum_{i=1}^m E \left(\sum_{\substack{j=1 \\ j \neq i}}^m (r_{ij}^2 - r_{i^*j}^2) \right)^4 \\ &= O((n/m)^4) E \left(\sum_{j=2}^m \left(r_{1j}^2 - \frac{1}{n-1} \right) \right)^4 \\ &\quad + O((n/m)^4) E \left(\sum_{j=2}^m \left(r_{1^*j}^2 - \frac{1}{n-1} \right) \right)^4. \end{aligned} \quad (2.36)$$

Following the same argument as in (2.33), we have

$$\begin{aligned} &E \left(\sum_{j=2}^m \left(r_{1j}^2 - \frac{1}{n-1} \right) \right)^4 \\ &= \sum_{j=2}^m E \left(r_{1j}^2 - \frac{1}{n-1} \right)^4 \\ &\quad + 4 \sum_{j_1=2}^m \sum_{\substack{j_2=2 \\ j_2 \neq j_1}}^m E \left(\left(r_{1j_1}^2 - \frac{1}{n-1} \right)^3 \left(r_{1j_2}^2 - \frac{1}{n-1} \right) \right) \\ &\quad + 3 \sum_{j_1=2}^m \sum_{\substack{j_2=2 \\ j_2 \neq j_1}}^m E \left(\left(r_{1j_1}^2 - \frac{1}{n-1} \right)^2 \left(r_{1j_2}^2 - \frac{1}{n-1} \right)^2 \right) \\ &\quad + 6 \sum_{j_1=2}^m \sum_{\substack{j_2=2 \\ j_2 \neq j_1}}^m \sum_{\substack{j_3=2 \\ j_3 \neq j_1 \\ j_3 \neq j_2}}^m E \left(\left(r_{1j_1}^2 - \frac{1}{n-1} \right)^2 \left(r_{1j_2}^2 - \frac{1}{n-1} \right) \left(r_{1j_3}^2 - \frac{1}{n-1} \right) \right) \\ &\quad + \sum_{j_1=2}^m \sum_{\substack{j_2=2 \\ j_2 \neq j_1}}^m \sum_{\substack{j_3=2 \\ j_3 \neq j_1 \\ j_3 \neq j_2}}^m \sum_{\substack{j_4=2 \\ j_4 \neq j_1 \\ j_4 \neq j_2 \\ j_4 \neq j_3}}^m E \left(\left(r_{1j_1}^2 - \frac{1}{n-1} \right) \left(r_{1j_2}^2 - \frac{1}{n-1} \right) \right. \\ &\quad \times \left. \left(r_{1j_3}^2 - \frac{1}{n-1} \right) \left(r_{1j_4}^2 - \frac{1}{n-1} \right) \right) \end{aligned} \quad (2.37)$$

$$\begin{aligned}
&= m O(n^{-4}) + 0 + O(m^2) O(n^{-4}) + 0 + 0 \\
&= O(m^2 n^{-4}).
\end{aligned} \tag{2.38}$$

Similarly,

$$E\left(\sum_{j=2}^m \left(r_{1^*j}^2 - \frac{1}{n-1}\right)\right)^4 = O(m^2 n^{-4}). \tag{2.39}$$

This proves $E\Delta^4 = O(m^{-2})$ by (2.36)–(2.39), and hence (2.19) holds. $\square$

Proof of (2.20) and (2.21) Similarly to (2.36), we have

$$\begin{aligned}
E(\Delta^4 | \mathcal{X}_n) &= O((n/m)^4) E\left(\left(\sum_{\substack{j=1 \\ j \neq l}}^m (r_{lj}^2 - \frac{1}{n-1})\right)^4 \middle| \mathcal{X}_n\right) \\
&\quad + O((n/m)^4) E\left(\left(\sum_{\substack{j=1 \\ j \neq l}}^m (r_{l^*j}^2 - \frac{1}{n-1})\right)^4 \middle| \mathcal{X}_n\right).
\end{aligned} \tag{2.40}$$

Let the right hand side of (2.40) be Λ . From the proof of (2.19) we see that (2.21) holds.

Consider the variance of the first term of Λ . Note that

$$\begin{aligned}
&E\left(E\left(\left(\sum_{\substack{j=1 \\ j \neq l}}^m (r_{lj}^2 - \frac{1}{n-1})\right)^4 \middle| \mathcal{X}_n\right) - E\left(\left(\sum_{\substack{j=1 \\ j \neq l}}^m (r_{lj}^2 - \frac{1}{n-1})\right)^4\right)\right)^2 \\
&= E\left(\left(E\left(\left(\sum_{\substack{j=1 \\ j \neq l}}^m (r_{lj}^2 - \frac{1}{n-1})\right)^4 \middle| \mathcal{X}_n\right)\right)^2 - \left(E\left(\sum_{\substack{j=1 \\ j \neq l}}^m (r_{lj}^2 - \frac{1}{n-1})\right)^4\right)^2\right) \\
&= \frac{1}{m^2} \sum_{i_1=1}^m \sum_{i_2=1}^m \left\{ E\left(\left(\sum_{\substack{j_1=1 \\ j_1 \neq i_1}}^m (r_{i_1 j_1}^2 - \frac{1}{n-1})\right)^4 \left(\sum_{\substack{j_2=1 \\ j_2 \neq i_2}}^m (r_{i_2 j_2}^2 - \frac{1}{n-1})\right)^4\right) \right. \\
&\quad \left. - E\left(\sum_{\substack{j_1=1 \\ j_1 \neq i_1}}^m (r_{i_1 j_1}^2 - \frac{1}{n-1})\right)^4 E\left(\sum_{\substack{j_2=1 \\ j_2 \neq i_2}}^m (r_{i_2 j_2}^2 - \frac{1}{n-1})\right)^4 \right\} \\
&= \frac{1}{m^2} \sum_{i_1=1}^m \left\{ E\left(\left(\sum_{\substack{j_1=1 \\ j_1 \neq i_1}}^m (r_{i_1 j_1}^2 - \frac{1}{n-1})\right)^4 \left(\sum_{\substack{j_2=1 \\ j_2 \neq i_1}}^m (r_{i_1 j_2}^2 - \frac{1}{n-1})\right)^4\right) \right. \\
&\quad \left. - E\left(\sum_{\substack{j_1=1 \\ j_1 \neq i_1}}^m (r_{i_1 j_1}^2 - \frac{1}{n-1})\right)^4 E\left(\sum_{\substack{j_2=1 \\ j_2 \neq i_1}}^m (r_{i_1 j_2}^2 - \frac{1}{n-1})\right)^4 \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{m^2} \sum_{i_1=1}^m \sum_{\substack{i_2=1 \\ i_2 \neq i_1}}^m \left\{ E \left(\left(\sum_{\substack{j_1=1 \\ j_1 \neq i_1}}^m \left(r_{i_1 j_1}^2 - \frac{1}{n-1} \right) \right)^4 \left(\sum_{\substack{j_2=1 \\ j_2 \neq i_2}}^m \left(r_{i_2 j_2}^2 - \frac{1}{n-1} \right) \right)^4 \right) \right. \\
& \left. - E \left(\sum_{\substack{j_1=1 \\ j_1 \neq i_1}}^m \left(r_{i_1 j_1}^2 - \frac{1}{n-1} \right) \right)^4 E \left(\sum_{\substack{j_2=1 \\ j_2 \neq i_2}}^m \left(r_{i_2 j_2}^2 - \frac{1}{n-1} \right) \right)^4 \right\} \\
& = O\left(\frac{1}{m}\right) \left(A_{4-4} + A_{4-3,1} + A_{4-2,2} + A_{4-2,1,1} + A_{4-1,1,1,1} \right. \\
& \quad + A_{3,1-3,1} + A_{3,1-2,2} + A_{3,1-2,1,1} + A_{3,1-1,1,1,1} + A_{2,2-2,2} + A_{2,2-2,1,1} \\
& \quad + A_{2,2-1,1,1,1} + A_{2,1,1-2,1,1} + A_{2,1,1-1,1,1,1} + A_{1,1,1,1-1,1,1,1} \Big) \\
& \quad + O(1) \left(B_{4-4} + B_{4-3,1} + B_{4-2,2} + B_{4-2,1,1} + B_{4-1,1,1,1} \right. \\
& \quad + B_{3,1-3,1} + B_{3,1-2,2} + B_{3,1-2,1,1} + B_{3,1-1,1,1,1} + B_{2,2-2,2} + B_{2,2-2,1,1} \\
& \quad + B_{2,2-1,1,1,1} + B_{2,1,1-2,1,1} + B_{2,1,1-1,1,1,1} + B_{1,1,1,1-1,1,1,1} \Big),
\end{aligned}$$

where

$$\begin{aligned}
& A_{a_{11}, a_{12}, \dots, a_{21}, a_{22}, \dots} \\
& = \sum_{\substack{j_{11}=1 \\ j_{11} \neq i_1}}^m \sum_{\substack{j_{12}=1 \\ j_{12} \neq i_1 \\ j_{12} \neq j_{11}}}^m \sum_{\substack{j_{21}=1 \\ j_{21} \neq i_2}}^m \sum_{\substack{j_{22}=1 \\ j_{22} \neq i_2 \\ j_{22} \neq j_{21}}}^m \left\{ E \left(\left(r_{i j_{11}}^2 - \frac{1}{n-1} \right)^{a_{11}} \left(r_{i j_{12}}^2 - \frac{1}{n-1} \right)^{a_{12}} \dots \right. \right. \\
& \quad \times \left(r_{i j_{21}}^2 - \frac{1}{n-1} \right)^{a_{21}} \left(r_{i j_{22}}^2 - \frac{1}{n-1} \right)^{a_{22}} \dots \Big) \\
& \quad - E \left(\left(r_{i j_{11}}^2 - \frac{1}{n-1} \right)^{a_{11}} \left(r_{i j_{12}}^2 - \frac{1}{n-1} \right)^{a_{12}} \dots \right) \\
& \quad \times E \left(\left(r_{i j_{21}}^2 - \frac{1}{n-1} \right)^{a_{21}} \left(r_{i j_{22}}^2 - \frac{1}{n-1} \right)^{a_{22}} \dots \right) \Big\}, \\
& B_{b_{11}, b_{12}, \dots, b_{21}, b_{22}, \dots} \\
& = \sum_{\substack{j_{11}=1 \\ j_{11} \neq i_1}}^m \sum_{\substack{j_{12}=1 \\ j_{12} \neq i_1 \\ j_{12} \neq j_{11}}}^m \sum_{\substack{j_{21}=1 \\ j_{21} \neq i_2 \\ j_{21} \neq j_{11}}}^m \sum_{\substack{j_{22}=1 \\ j_{22} \neq i_2 \\ j_{22} \neq j_{21}}}^m \left\{ E \left(\left(r_{i_1 j_{11}}^2 - \frac{1}{n-1} \right)^{b_{11}} \left(r_{i_1 j_{12}}^2 - \frac{1}{n-1} \right)^{b_{12}} \dots \right. \right. \\
& \quad \times \left(r_{i_2 j_{21}}^2 - \frac{1}{n-1} \right)^{b_{21}} \left(r_{i_2 j_{22}}^2 - \frac{1}{n-1} \right)^{b_{22}} \dots \Big) \\
& \quad - E \left(\left(r_{i_1 j_{11}}^2 - \frac{1}{n-1} \right)^{b_{11}} \left(r_{i_1 j_{12}}^2 - \frac{1}{n-1} \right)^{b_{12}} \dots \right) \\
& \quad \times E \left(\left(r_{i_2 j_{21}}^2 - \frac{1}{n-1} \right)^{b_{21}} \left(r_{i_2 j_{22}}^2 - \frac{1}{n-1} \right)^{b_{22}} \dots \right) \Big\},
\end{aligned}$$

for $i_1 \neq i_2$. Since there are m^2 terms in A_{4-4} and every term is of order $O(\frac{1}{n^8})$, $A_{4-4} = O(\frac{m^2}{n^8})$. Any term in $A_{4-3,1}$ is zero if j_{22} is not equal to either i or j_{11} , thus there are $O(m^2)$ nonzero terms in $A_{4-3,1}$ and $A_{4-3,1} = O(\frac{m^2}{n^8})$. By similar arguments, all the A-type terms are of order $O(\frac{m^4}{n^8})$ and all the B-type terms are of order $O(\frac{m^3}{n^8})$ except $B_{2,1,1-2,1,1}$ and $B_{1,1,1,1-1,1,1}$. Similarly,

$$\begin{aligned}
 B_{2,1,1-2,1,1} &= O\left(\frac{m^3}{n^8}\right) + O(m^4)E\left(\left(r_{13}^2 - \frac{1}{n-1}\right)\left(r_{23}^2 - \frac{1}{n-1}\right)\right. \\
 &\quad \times \left(r_{14}^2 - \frac{1}{n-1}\right)\left(r_{24}^2 - \frac{1}{n-1}\right)\left(r_{15}^2 - \frac{1}{n-1}\right)^2\left(r_{26}^2 - \frac{1}{n-1}\right)^2\Big) \\
 &= O\left(\frac{m^3}{n^8}\right), \\
 B_{1,1,1,1-1,1,1} &= O\left(\frac{m^3}{n^8}\right) + O(m^4)E\left(\left(r_{13} - \frac{1}{n-1}\right)\left(r_{14} - \frac{1}{n-1}\right)\left(r_{15} - \frac{1}{n-1}\right)\right. \\
 &\quad \times \left(r_{16} - \frac{1}{n-1}\right)\left(r_{23} - \frac{1}{n-1}\right)\left(r_{24} - \frac{1}{n-1}\right) \\
 &\quad \times \left(r_{25} - \frac{1}{n-1}\right)\left(r_{26} - \frac{1}{n-1}\right)\Big) \\
 &= O\left(\frac{m^3}{n^8}\right).
 \end{aligned}$$

Therefore, the variance of the first term of Λ is $O(\frac{1}{m^5})$ and so is the variance of the second term. Hence (2.20) holds. $\square$

Proof of (2.22) Following the proof of (2.20) gives $E\Delta^8 = O(m^{-4})$. Thus

$$E(|W| + 1)|\Delta|^3 \leq (E(1 + |W|)^2)^{1/2}(E\Delta^6)^{1/2} = O(m^{-3/2}),$$

as desired. $\square$

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