

THETA FUNCTIONS AND FACTORS OF AUTOMORPHY

4. Classical theta functions

4.1. We define the classical theta function θ and its modification φ by

$$(4.1) \quad \theta(u, z; r, s) = \sum_{g-r \in \mathbf{Z}^n} \mathbf{e}(2^{-1} \cdot {}^t g z g + {}^t g(u+s)),$$

$$(4.2) \quad \varphi(u, z; r, s) = \mathbf{e}(2^{-1} \cdot {}^t u(z - \bar{z})^{-1} u) \theta(u, z; r, s).$$

Here $u \in \mathbf{C}^n$, $z \in \mathfrak{H}_n$, and $r, s \in \mathbf{R}^n$. To prove the convergence of the infinite series of (4.1), put $g = h + r$ with $h \in \mathbf{Z}^n$ and $y = \text{Im}(z)$; take compact subsets U of $\mathbf{C}^n$ and Z of $\mathfrak{H}_n$. Then for fixed $r, s \in \mathbf{R}^n$, $u \in U$, and $z \in Z$ we easily see that

$$\text{Re}(\pi i \cdot {}^t g z g + 2\pi i \cdot {}^t g(u+s)) = -\pi \cdot {}^t h y h + {}^t h v + w$$

with $v \in \mathbf{R}^n$ and $w \in \mathbf{R}$ in some compact sets depending on r, s, U , and Z . Let λ be the smallest eigenvalue of πy . Then $\pi \cdot {}^t h y h \geq \lambda \cdot {}^t h h$, and so the sum of (4.1) is majorized by

$$\sum_{h \in \mathbf{Z}^n} \exp(-\lambda \cdot {}^t h h + {}^t h v + w) = e^w \prod_{j=1}^n \sum_{k \in \mathbf{Z}} \exp(-\lambda k^2 + k v_j),$$

where v_j is the j th component of v . Therefore we see that the right-hand side of (4.1) is locally uniformly convergent on $\mathbf{C}^n \times \mathfrak{H}_n$, and so defines a holomorphic function in (u, z) .

The function $\theta(u, z; r, s)$ is called **Riemann's theta function**. By purely formal calculations we can easily verify

$$(4.3) \quad \begin{aligned} \theta(u + za + b, z; r, s) \\ = \mathbf{e}(-2^{-1} \cdot {}^t a z a - {}^t a(u+b+s)) \theta(u, z; r+a, s+b) \end{aligned} \quad (a, b \in \mathbf{R}^n),$$

$$(4.4) \quad \begin{aligned} \theta(u + za + b, z; r, s) \\ = \mathbf{e}(-2^{-1} \cdot {}^t a z a - {}^t a u + {}^t r b - {}^t s a) \theta(u, z; r, s) \end{aligned} \quad (a, b \in \mathbf{Z}^n),$$

$$(4.5) \quad \begin{aligned} \theta(u, z; r+a, s+b) \\ = \mathbf{e}({}^t r b) \theta(u, z; r, s) = \theta(u+b, z; r, s) \end{aligned} \quad (a, b \in \mathbf{Z}^n).$$

The function $\theta(u, z; r, s)$ was introduced by Riemann for the purpose of studying abelian integrals on an algebraic curve. In fact, these theta functions are essential in the geometric investigation of abelian varieties. In the present book, however, we merely employ them as a technical tool for studying automorphic forms of several types. The reader who is interested in their geometric and other aspects may be referred to [S98] and earlier articles by various authors cited there.

4.2. Let us now put

$$(4.6) \quad \Gamma_n^\theta = \Gamma^\theta = \{\gamma \in \Gamma(1) \mid \{a_\gamma \cdot {}^t b_\gamma\} \equiv \{c_\gamma \cdot {}^t d_\gamma\} \equiv 0 \pmod{2\mathbf{Z}^n}\},$$

where $\{s\}$ is the column vector consisting of the diagonal elements of s . Notice that $a_\gamma \cdot {}^t b_\gamma$ and $c_\gamma \cdot {}^t d_\gamma$ belong to $S_n(\mathbf{Z})$ by (1.2a). We note an easy fact:

(4.7) For $s \in S_n(\mathbf{Z})$ we have

$$\begin{aligned} {}^t x s x \in 2\mathbf{Z} \text{ for every } x \in \mathbf{Z}^n &\iff \{s\} \in 2\mathbf{Z}^n \\ &\implies \{{}^t y s y\} \in 2\mathbf{Z}^n \text{ for every } y \in \mathbf{Z}_n^n. \end{aligned}$$

Clearly $\Gamma(2) \subset \Gamma^\theta$. Put $F(x, y) = x \cdot {}^t y$ for $x, y \in \mathbf{Z}_n^1$. By (1.2c) we have

$$F((x, y)\gamma) - F(x, y) = x a_\gamma \cdot {}^t b_\gamma \cdot {}^t x + y c_\gamma \cdot {}^t d_\gamma \cdot {}^t y + 2x b_\gamma \cdot {}^t c_\gamma \cdot {}^t y$$

for $\gamma \in \Gamma(1)$, and so we see that

$$(4.8a) \quad \Gamma^\theta = \{\gamma \in \Gamma(1) \mid F((x, y)\gamma) - F(x, y) \in 2\mathbf{Z} \text{ for every } x, y \in \mathbf{Z}_n^1\}.$$

This shows that Γ^θ is a subgroup of $\Gamma(1)$. Taking γ^{-1} in place of γ , from (1.2b) we see that

$$(4.8b) \quad \Gamma^\theta = \{\gamma \in \Gamma(1) \mid \{{}^t a_\gamma c_\gamma\} \equiv \{{}^t b_\gamma d_\gamma\} \equiv 0 \pmod{2\mathbf{Z}^n}\}.$$

We can let $Sp(n, \mathbf{R})$ act on $\mathbf{C}^n \times \mathfrak{H}_n$ by the rule

$$(4.9) \quad \gamma(u, z) = ({}^t \mu_\gamma(z)^{-1} u, \gamma z) \quad \text{for } \gamma \in Sp(n, \mathbf{R}), u \in \mathbf{C}^n, \text{ and } z \in \mathfrak{H}_n.$$

From (1.7) we obtain $(\beta\gamma)(u, z) = \beta(\gamma(u, z))$ for $\beta, \gamma \in Sp(n, \mathbf{R})$; also 1_{2n} gives the identity map on $\mathbf{C}^n \times \mathfrak{H}_n$.

Lemma 4.3. (i) $\mu_\gamma(z)^{-1} c_\gamma \in S_n(\mathbf{C})$ for every $\gamma \in Sp(n, \mathbf{R})$ and $z \in \mathfrak{H}_n$.

(ii) Let $\kappa(u, z) = {}^t u(z - \bar{z})^{-1} u$ for $u \in \mathbf{C}^n$ and $z \in \mathfrak{H}_n$. Then

$$(4.10) \quad \kappa(\gamma(u, z)) - \kappa(u, z) = -{}^t u \mu_\gamma(z)^{-1} c_\gamma u$$

for every $\gamma \in Sp(n, \mathbf{R})$.

PROOF. Put $p_z(u, v) = {}^t u(z - \bar{z})^{-1} v$ for $u, v \in \mathbf{C}^n$. Then $p_z(u, v) = p_z(v, u)$ and $\kappa(u, z) = p_z(u, u)$. From (1.16) we obtain

$$(4.11) \quad (\gamma z - \gamma \bar{z})^{-1} = \mu_\gamma(z)(z - \bar{z})^{-1} \cdot {}^t \mu_\gamma(\bar{z})$$

for every $\gamma \in Sp(n, \mathbf{R})$. Therefore

$$\begin{aligned} p_{\gamma z}({}^t\mu_{\gamma}(z)^{-1}u, {}^t\mu_{\gamma}(z)^{-1}v) &= {}^tu\mu_{\gamma}(z)^{-1}(\gamma z - \gamma \bar{z})^{-1} \cdot {}^t\mu_{\gamma}(z)^{-1}v \\ &= {}^tu(z - \bar{z})^{-1} \cdot {}^t\mu_{\gamma}(\bar{z}) \cdot {}^t\mu_{\gamma}(z)^{-1}v. \end{aligned}$$

Since $\mu_{\gamma}(\bar{z}) = \mu_{\gamma}(z) - c_{\gamma}(z - \bar{z})$, we have ${}^t\mu_{\gamma}(\bar{z}) \cdot {}^t\mu_{\gamma}(z)^{-1} = 1_n - (z - \bar{z}) \cdot {}^tc_{\gamma} \cdot {}^t\mu_{\gamma}(z)^{-1}$. Thus

$$p_{\gamma z}({}^t\mu_{\gamma}(z)^{-1}u, {}^t\mu_{\gamma}(z)^{-1}v) - p_z(u, v) = -{}^tu \cdot {}^tc_{\gamma} \cdot {}^t\mu_{\gamma}(z)^{-1}v.$$

Since the left-hand side is symmetric in (u, v) , we see that ${}^tc_{\gamma} \cdot {}^t\mu_{\gamma}(z)^{-1} \in S_n(\mathbf{C})$, which proves (i). Putting $u = v$, we obtain (ii). Here we add a more direct proof of (i). We have $c_{\gamma} \cdot {}^t\mu_{\gamma}(z) = c_{\gamma}z \cdot {}^tc_{\gamma} + c_{\gamma} \cdot {}^td_{\gamma} \in S_n(\mathbf{C})$, as can be seen from (1.2a). Thus $\mu_{\gamma}(z)^{-1}c_{\gamma} = \mu_{\gamma}(z)^{-1}c_{\gamma} \cdot {}^t\mu_{\gamma}(z) \cdot {}^t\mu_{\gamma}(z)^{-1} \in S_n(\mathbf{C})$ as expected.

Theorem 4.4. (i) *For every $\gamma \in \Gamma(1)$ we have*

$$(4.12) \quad \theta(\gamma(u, z); r, s) = \zeta \cdot j_{\gamma}(z)^{1/2} \mathbf{e}(2^{-1} \cdot {}^tu\mu_{\gamma}(z)^{-1}c_{\gamma}u) \theta(u, z; r'', s'')$$

with a constant $\zeta \in \mathbf{T}$ depending on r, s, γ , and a suitable choice of a branch of $j_{\gamma}(z)^{1/2}$, and

$$\begin{bmatrix} r'' \\ s'' \end{bmatrix} = {}^t\gamma \begin{bmatrix} r \\ s \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \{^tac\} \\ \{^tbd\} \end{bmatrix},$$

where the symbol $\{*\}$ is defined in §4.2.

(ii) *For every $\gamma \in \Gamma^{\theta}$ there is a holomorphic function $h_{\gamma}(z)$ in $z \in \mathfrak{H}_n$, written also $h(\gamma, z)$, such that $h_{\gamma}(z)^2 = \zeta \cdot j_{\gamma}(z)$ with a constant $\zeta \in \mathbf{T}$ and*

$$(4.13) \quad \begin{aligned} \theta(\gamma(u, z); r, s) \\ = \mathbf{e}(2^{-1}({}^trs - {}^tr's')) h_{\gamma}(z) \mathbf{e}(2^{-1} \cdot {}^tu\mu_{\gamma}(z)^{-1}c_{\gamma}u) \theta(u, z; r', s') \end{aligned}$$

$$\text{with } \begin{bmatrix} r' \\ s' \end{bmatrix} = {}^t\gamma \begin{bmatrix} r \\ s \end{bmatrix}.$$

$$(iii) \quad h_{\gamma}(z)^4 = j_{\gamma}(z)^2 \text{ if } \gamma \in \Gamma(2).$$

These formulas are classical (see [KP92], for example). We will give a shorter proof by proving (4.12) for some special γ in §4.9, and the remaining statements in Section A1 of the Appendix.

Putting $u = 0$, from (4.12) and (4.13) we obtain

$$(4.14) \quad \theta(0, \gamma z; r, s) = \zeta \cdot j_{\gamma}(z)^{1/2} \theta(0, z; r'', s'') \quad \text{if } \gamma \in \Gamma(1),$$

$$(4.15) \quad \theta(0, \gamma z; r, s) = \mathbf{e}(2^{-1}({}^trs - {}^tr's')) h_{\gamma}(z) \theta(0, z; r', s') \quad \text{if } \gamma \in \Gamma^{\theta},$$

where (r'', s'') is as in (4.12) and (r', s') is as in (4.13).

4.5. Put $q(u, z) = \mathbf{e}(2^{-1} \cdot {}^tu(z - \bar{z})^{-1}u)$. Then from (4.10) we obtain

$$(4.16) \quad q(\gamma(u, z)) = q(u, z) \mathbf{e}(-2^{-1} \cdot {}^tu\mu_{\gamma}(z)^{-1}c_{\gamma}u).$$

We now consider φ of (4.2). Since $\varphi(u, z; r, s) = q(u, z) \theta(u, z; r, s)$, we easily see that (4.12) and (4.13) are equivalent to

$$(4.17) \quad \varphi(\gamma(u, z); r, s) = \zeta \cdot j_\gamma(z)^{1/2} \varphi(u, z; r'', s''),$$

$$(4.18) \quad \varphi(\gamma(u, z); r, s) = \mathbf{e}(2^{-1}({}^t r s - {}^t r' s')) h_\gamma(z) \varphi(u, z; r', s').$$

Thus the transformation formulas for φ under $(u, z) \mapsto \gamma(u, z)$ are simpler than those for θ . It is mainly for this reason that we consider φ in addition to θ .

4.6. Let us now put

$$(4.19) \quad \theta(z) = \sum_{g \in \mathbf{Z}^n} \mathbf{e}(2^{-1} \cdot {}^t g z g) \quad (z \in \mathfrak{H}_n).$$

Then $\theta(z) = \theta(0, z; 0, 0) = \varphi(0, z; 0, 0)$, and so from (4.13) we obtain

$$(4.20) \quad \theta(\gamma z) = h_\gamma(z) \theta(z) \quad (\gamma \in \Gamma^\theta),$$

and consequently

$$(4.21) \quad h(\beta\gamma, z) = h_\beta(\gamma z) h_\gamma(z) \quad (\beta, \gamma \in \Gamma^\theta).$$

Clearly $\theta(az \cdot {}^t a) = \theta(z + b) = \theta(z)$ for $a \in GL_n(\mathbf{Z})$ and $b \in 2S_n(\mathbf{Z})$, and so

$$(4.22a) \quad h_\alpha = 1 \quad \text{for } \alpha = \text{diag}[a, {}^t a^{-1}], \quad a \in GL_n(\mathbf{Z}),$$

$$(4.22b) \quad h_\beta = 1 \quad \text{for } \beta = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad b \in 2S_n(\mathbf{Z}),$$

$$(4.22c) \quad h_{-\gamma} = h_\gamma \quad \text{for every } \gamma \in \Gamma^\theta.$$

In the following theorem we will see that h_γ^2 coincides with j_γ if $\gamma \in \Gamma(4)$. For every congruence subgroup Γ of Γ^θ we denote by $\mathcal{M}_{1/2}(\Gamma)$ the set of all holomorphic functions f on $\mathfrak{H}_n$ such that $f^2 \in \mathcal{M}_1$ and

$$(4.23) \quad f(\gamma z) = h_\gamma(z) f(z) \quad \text{for every } \gamma \in \Gamma.$$

In fact, the condition $f^2 \in \mathcal{M}_1$ follows from (4.23) if $n > 1$. Indeed, if (4.23) is satisfied, then $f(\gamma z)^2 = j_\gamma(z) f(z)^2$ for every $\gamma \in \Gamma \cap \Gamma(4)$, which implies that $f^2 \in \mathcal{M}_1$. If $f \in \mathcal{M}_{1/2}(\Gamma)$, then f has expansion (3.7a) with $c(h) \neq 0$ only for nonnegative h . Indeed, from (4.22a, b) we see that f satisfies (3.6a, b) for suitable M and U , and so we have at least an expansion of type (3.7a). If $n > 1$, Lemma 3.3 gives the desired result. If $n = 1$, the condition $f^2 \in \mathcal{M}_1$ implies condition (3.4d), as will be explained in §5.1. In fact, we will introduce in §5.1 modular forms of weight k for an arbitrary $k \in 2^{-1}\mathbf{Z}$ and will discuss (3.4d) in that context. We put

$$(4.24) \quad \mathcal{M}_{1/2} = \bigcup_{N=1}^{\infty} \mathcal{M}_{1/2}(\Gamma(2N)).$$

Theorem 4.7. (1) *If $\gamma \in \Gamma^\theta$ and $\det(d_\gamma) \neq 0$, then*

$$(4.25) \quad \lim_{z \rightarrow 0} h_\gamma(z) = \sum_{x \in A} \mathbf{e}(-{}^t x d_\gamma^{-1} c_\gamma x / 2) = \sum_{x \in B} \mathbf{e}({}^t x b_\gamma d_\gamma^{-1} x / 2),$$

where $A = \mathbf{Z}^n / {}^t d_\gamma \mathbf{Z}^n$ and $B = \mathbf{Z}^n / d_\gamma \mathbf{Z}^n$.

(2) If $\gamma \in \Gamma^\theta$ and $\det(d_\gamma)$ is odd, then $h_\gamma(z)^2 = \left(\frac{-1}{\det(d_\gamma)} \right) j_\gamma(z)$. In particular, $h_\gamma(z)^2 = j_\gamma(z)$ for every $\gamma \in \Gamma(4)$.

(3) $h_\iota(z) = \det(-iz)^{1/2}$ with the branch of the square root such that $h_\iota(z) > 0$ for $\operatorname{Re}(z) = 0$.

(4) Let $\alpha \in Gp^+(n, \mathbf{Q})$ and $r(z) = j_\alpha(z)^{1/2}$ with any choice of a branch. Then there is a congruence subgroup Δ of Γ^θ such that

$$h(\alpha\gamma\alpha^{-1}, \alpha z) = r(\gamma z)h(\gamma, z)r(z)^{-1}$$

for every $\gamma \in \Delta$.

We will prove the first two assertions in Section A1 of the Appendix; (3) will be proven in §4.9 and (4) in §5.3. Clearly (2) implies Theorem 4.4(iii). Once (2) of the above theorem is established, we see from (4.14) and (4.15) that $\theta(0, z; r, s)$ as a function of z belongs to $\mathcal{M}_{1/2}$ if $r, s \in \mathbf{Q}^n$. If $n = 1$, condition (3.4d) for $\theta(0, z; r, s)^2$ follows from (4.14).

We will often consider $\det(-iz)^{\pm 1/2}$ in our later treatment. We use the convention that it always means the function as in (3) above and its inverse.

4.8. Let us now recall some elementary facts on Fourier analysis. For $f \in L^1(\mathbf{R}^n)$ we define its **Fourier transform** $\hat{f}$ by

$$(4.26) \quad \hat{f}(x) = \int_{\mathbf{R}^n} f(y) \mathbf{e}(-{}^t xy) dy \quad (x \in \mathbf{R}^n),$$

where we consider x and y column vectors, so that ${}^t xy = \sum_{\nu=1}^n x_\nu y_\nu$, and dy is the standard volume element of $\mathbf{R}^n$. Then $\hat{f}$ is a continuous function. If f is continuous, then we have

$$(4.27) \quad \sum_{g \in \mathbf{Z}^n} f(r+g) = \sum_{h \in \mathbf{Z}^n} \hat{f}(h) \mathbf{e}({}^t hr) \quad (r \in \mathbf{R}^n),$$

provided both sides converge absolutely and uniformly. This is called the **Poisson summation formula** (see [S07, Theorem 2.3], for example). If we exchange f for $\hat{f}$, then we obtain

$$(4.27a) \quad \sum_{g \in \mathbf{Z}^n} \hat{f}(r+g) = \sum_{h \in \mathbf{Z}^n} f(h) \mathbf{e}(-{}^t hr) \quad (r \in \mathbf{R}^n).$$

It is well known that the function $\exp(-\pi x^2)$ is its own Fourier transform, that is,

$$(4.28) \quad \int_{\mathbf{R}} \exp(-\pi x^2) \mathbf{e}(-xt) dx = \exp(-\pi t^2).$$

For a short proof, see [S07, pp.14–15]. An n -dimensional version of (4.28) can be given as follows:

$$(4.29) \quad \int_{\mathbf{R}^n} \exp(-\pi h[x]) \mathbf{e}(-{}^t vx) dx = \det(h)^{-1/2} \exp(-\pi h^{-1}[v])$$

$$(v \in \mathbf{R}^n, 0 < h \in S_n(\mathbf{R})),$$

where $h[x] = {}^t x h x$. This can be proved by taking a real matrix α such that ${}^t \alpha h \alpha = 1_n$ and replacing x by $\alpha^{-1}x$, which reduces the problem to (4.28). From this we obtain

$$(4.30) \quad \int_{\mathbf{R}^n} \mathbf{e}(2^{-1}z[x])\mathbf{e}(-{}^t v x)dx = \det(-iz)^{-1/2}\mathbf{e}(-2^{-1}z^{-1}[v]) \\ (v \in \mathbf{R}^n, z \in \mathfrak{H}_n).$$

Indeed, if $z = ih$ with $0 < h \in S_n(\mathbf{R})$, this is exactly (4.29). Now the left-hand side of (4.30) is convergent and defines a holomorphic function in z ; the right-hand side is clearly holomorphic in z . Since they coincide on “the imaginary axis” of $\mathfrak{H}_n$, we obtain (4.30) on the whole $\mathfrak{H}_n$.

4.9. We now prove formula (4.12) for the elements γ of $\Gamma(1)$ of the forms

$$(4.31) \quad \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \quad \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

If $\gamma = \text{diag}[a, d] \in \Gamma(1)$, then $d = {}^t a^{-1} \in GL_n(\mathbf{Z})$ and $\gamma(z) = az \cdot {}^t a$, and so we easily see that

$$(4.32) \quad \theta(au, az \cdot {}^t a; r, s) = \theta(u, z; {}^t ar, a^{-1}s).$$

Next, if $\gamma = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in \Gamma(1)$, then ${}^t b = b$ and $\gamma(z) = z + b$. Observing that ${}^t x b x / 2 \equiv {}^t x \{b\} / 2 \pmod{\mathbf{Z}}$ if $x \in \mathbf{Z}^n$ (with $\{*\}$ defined as in §4.2), we obtain

$$(4.33) \quad \theta(u, z + b; r, s) = \mathbf{e}(-2^{-1}({}^t r b r + {}^t r \{b\}))\theta(u, z; r, s + br + 2^{-1}\{b\}),$$

which is (4.12) for the present γ . Finally, if $\gamma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then $\gamma(z) = -z^{-1}$. To discuss this case, we first put $v = y - u - s$ in (4.30) with real vectors y, u, s , and consider the Fourier transform $\hat{f}$ of the function $f(x) = \mathbf{e}(2^{-1}z[x] + {}^t x(u + s))$. Then (4.30) shows that

$$\hat{f}(y) = \det(-iz)^{-1/2}\mathbf{e}(-2^{-1}z^{-1}[y - u - s]).$$

Applying the Poisson summation formula (4.27) to the present case we obtain

$$(4.34) \quad \det(-iz)^{1/2}\mathbf{e}(2^{-1}z^{-1}[u])\theta(u, z; r, s) = \mathbf{e}({}^t r s)\theta(z^{-1}u, -z^{-1}; -s, r)$$

for real u , and so for all $u \in \mathbf{C}^n$, since both sides are holomorphic in u . Consequently, we know that (4.12) holds for γ of the forms of (4.31). Taking $u = r = s = 0$ in (4.34), we obtain

$$(4.34a) \quad \det(-iz)^{1/2}\theta(z) = \theta(-z^{-1}),$$

which gives Theorem 4.7(3).

4.10. Let us now look more closely at the special case $n = 1$. In this case our function takes the form

$$(4.35) \quad \theta(u, z; r, s) = \sum_{m \in \mathbf{Z}} \mathbf{e}(2^{-1}(m+r)^2 z + (m+r)(u+s)),$$

where $u \in \mathbf{C}$, $z \in \mathfrak{H}_1$, and $r, s \in \mathbf{R}$; in particular,

$$(4.36) \quad \theta(z) = \theta(0, z; 0, 0) = \sum_{m \in \mathbf{Z}} \mathbf{e}(m^2 z/2).$$

The function of (4.35) satisfies the differential equation

$$(4.37) \quad \frac{\partial^2 \theta(u, z; r, s)}{\partial u^2} = 4\pi i \cdot \frac{\partial \theta(u, z; r, s)}{\partial z}.$$

Also, we have

$$(4.38) \quad \Gamma^\theta = \Gamma(2) \cup \Gamma(2)\iota, \quad \iota = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Indeed, that $\Gamma(2) \cup \Gamma(2)\iota \subset \Gamma^\theta$ can easily be seen. Suppose $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^\theta$ and $a \notin 2\mathbf{Z}$. Then $c \in 2\mathbf{Z}$ and $ad - 1 \in 2\mathbf{Z}$. Thus $d \notin 2\mathbf{Z}$ and $b \in 2\mathbf{Z}$, and so $\gamma \in \Gamma(2)$. Suppose $\gamma \in \Gamma^\theta$ and $\gamma \notin \Gamma(2)$. Then $a \in 2\mathbf{Z}$, and so $b \notin 2\mathbf{Z}$. Since $b = a_{\gamma\iota}$, we see that $\gamma\iota \in \Gamma(2)$. This proves (4.38).

From Theorem 4.7(3) we obtain

$$(4.39) \quad h_\iota(z) = (-iz)^{1/2},$$

where the branch of $(-iz)^{1/2}$ is chosen so that it is a positive real number if $z = iy$ with $0 < y \in \mathbf{R}$. We use this convention throughout the present book.

Let us now show that

$$(4.40) \quad h_\gamma(z) = \varepsilon_d^{-1} \left(\frac{2c}{d} \right) (cz + d)^{1/2} \quad \text{if } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(2).$$

Here the branch of $(cz + d)^{1/2}$ is taken so that $-\pi/2 < \arg(cz + d)^{1/2} \leq \pi/2$, which means that

$$(4.41a) \quad \lim_{z \rightarrow 0} (cz + d)^{1/2} = \begin{cases} \sqrt{d} & \text{if } d > 0, \\ i\sqrt{|d|} & \text{if } d < 0 \text{ and } c \geq 0, \\ -i\sqrt{|d|} & \text{if } d < 0 \text{ and } c < 0; \end{cases}$$

$\left(\frac{2c}{d} \right)$ is the symbol defined in (0.4) and (0.5), and ε_d is defined by (as already done in (0.6))

$$(4.41b) \quad \varepsilon_d = \begin{cases} 1 & \text{if } d - 1 \in 4\mathbf{Z}, \\ i & \text{if } d + 1 \in 4\mathbf{Z}. \end{cases}$$

To prove (4.40), let us assume $c \neq 0$, since the case $c = 0$ is easy. Take the branch of $(cz + d)^{1/2}$ as specified, and let $q = \lim_{z \rightarrow 0} (cz + d)^{1/2}$. By Theorem 4.7(2), $h_\gamma(z)^2 = \pm j_\gamma(z)$, and so we can put $h_\gamma(z) = \eta(cz + d)^{1/2}$ with η such that $\eta^2 = \pm 1$. By (4.25) we obtain

$$\eta q = \sum_{x \in \mathbf{Z}/d\mathbf{Z}} \mathbf{e}(-cx^2/(2d)) = \sum_{y=1}^{|d|} \mathbf{e}(-2cy^2/d) = G(-2c, d)$$

with $G(*, *)$ of (2.5). Suppose $d > 0$; then by (2.9), $\eta d^{1/2} = \left(\frac{-2c}{d}\right) \varepsilon_d d^{1/2}$, and so $\eta = \varepsilon_d^{-1} \left(\frac{2c}{d}\right)$, which proves (4.40) when $d > 0$. If $d < 0$, employing (2.13), we obtain $q\eta = \left(\frac{2c}{-d}\right) \varepsilon_{-d} \sqrt{|d|}$. Comparing this with (4.41a), we obtain the desired result. We can also use the fact $h_\gamma = h_{-\gamma}$. However, for all practical purposes we need the formula only when $d > 0$; in fact, it is not advisable to use it when $d < 0$.

By (4.5), $\theta(u, z; r, s)$ up to a factor of $\mathbf{T}$ depends only on r, s modulo $\mathbf{Z}$. In particular, there are four functions determined by $r, s \in \{0, 1/2\}$. We can replace $1/2$ by $-1/2$ by multiplying the function by a root of unity. Now the explicit forms of these four functions are given as follows:

$$(4.42a) \quad \theta(u, z; 0, 0) = \sum_{m \in \mathbf{Z}} \mathbf{e}(m^2 z/2) \mathbf{e}(mu),$$

$$(4.42b) \quad \theta(u, z; 0, 1/2) = \sum_{m \in \mathbf{Z}} (-1)^m \mathbf{e}(m^2 z/2) \mathbf{e}(mu),$$

$$(4.42c) \quad \theta(u, z; 1/2, 0) = \sum_{m \in \mathbf{Z}} \mathbf{e}((2m+1)^2 z/8) \mathbf{e}((2m+1)u/2),$$

$$(4.42d) \quad \theta(u, z; 1/2, -1/2) = -i \sum_{m \in \mathbf{Z}} (-1)^m \mathbf{e}((2m+1)^2 z/8) \mathbf{e}((2m+1)u/2).$$

These were introduced by Jacobi, and so it is natural to call them **Jacobi's theta functions**. They are traditionally denoted by $\vartheta_\nu(u, z)$ with $0 \leq \nu \leq 3$, but the numbering depends on the author. Formulas (4.3), (4.4), (4.5), and (4.13) are of course valid for these, and the explicit transformation formulas for ϑ_ν are given in any textbook on elliptic functions. We do not need these ϑ_ν in this book, but we note them here for the purpose of giving a perspective that *the functions ϑ_ν and their transformation formulas are merely special cases obtained by taking n to be 1 and substituting 0 or $\pm 1/2$ for r or s , and this point is worthy of emphasis.*

We add here a few facts about $\partial\theta/\partial u$. Namely, put

$$(4.43) \quad \theta^*(z; r, s) = (2\pi i)^{-1} (\partial\theta/\partial u)(0, z; r, s).$$

Then

$$(4.44) \quad \theta^*(z; r, s) = \sum_{m \in \mathbf{Z}} (m+r) \mathbf{e}(2^{-1}(m+r)^2 z + (m+r)s),$$

and from (4.13) we immediately obtain

$$(4.45) \quad \theta^*(\gamma z; r, s) = \mathbf{e}((rs - r's')/2) j_\gamma(z) h_\gamma(z) \theta^*(z; r', s') \quad \text{if } \gamma \in \Gamma^\theta,$$

where $[r' \ s'] = [r \ s]\gamma$. In particular, put $\theta^*(z) = \theta^*(z; 0, 0)$; then

$$(4.46) \quad \theta^*(z) = \sum_{m \in \mathbf{Z}} m \mathbf{e}(m^2 z/2),$$

$$(4.47) \quad \theta^*(\gamma z) = j_\gamma(z) h_\gamma(z) \theta^*(z) \quad \text{if } \gamma \in \Gamma^\theta.$$

4.11. Returning to the case with $n \geq 1$, we consider the functions of the forms

$$(4.48a) \quad \varphi(u, z; \lambda) = \mathbf{e}(2^{-1} \cdot {}^t u(z - \bar{z})^{-1} u) \theta(u, z; \lambda),$$

$$(4.48b) \quad \theta(u, z; \lambda) = \sum_{\xi \in \mathbf{Q}^n} \lambda(\xi) \mathbf{e}(2^{-1} \cdot {}^t \xi z \xi + {}^t \xi u).$$

Here $u \in \mathbf{C}^n$ and $z \in \mathfrak{H}_n$ as before; λ is an element of the set $\mathcal{L}(\mathbf{Q}^n)$ defined in §0.3. By our explanation in §0.3, $\theta(u, z; \lambda)$ is a finite $\mathbf{C}$ -linear combination of $\theta(u, z; r, s)$ with $r, s \in \mathbf{Q}^n$, and such a $\theta(u, z; r, s)$ is a special case of (4.48b). We now reformulate Theorems 4.4 and 4.7 as follows.

Theorem 4.12. *Let P be the group defined in Lemma 2.2(iii). For $\alpha \in P\Gamma^\theta$ and $\lambda \in \mathcal{L}(\mathbf{Q}^n)$ we can define a holomorphic function $h_\alpha(z)$ on $\mathfrak{H}_n$ and an element λ^α of $\mathcal{L}(\mathbf{Q}^n)$ with the following properties:*

$$(4.49a) \quad \varphi(\alpha(u, z); \lambda) = h_\alpha(z) \varphi(u, z; \lambda^\alpha).$$

$$(4.49b) \quad h_\alpha(z)^2 = \zeta j_\alpha(z) \text{ with } \zeta \in \mathbf{T}.$$

$$(4.49c) \quad h_\alpha(z) = |\det(d_\alpha)|^{1/2} \text{ if } \alpha \in P.$$

$$(4.49d) \quad h_{\rho\alpha\tau}(z) = h_\rho(z) h_\alpha(\tau z) h_\tau(z) \text{ if } \rho \in P \text{ and } \tau \in \Gamma^\theta.$$

$$(4.49e) \quad \lambda^{\rho\alpha\tau} = ((\lambda^\rho)^\alpha)^\tau \text{ if } \rho \in P \text{ and } \tau \in \Gamma^\theta.$$

$$(4.49f) \text{ For each } \lambda \text{ the set } \{\alpha \in \Gamma^\theta \mid \lambda^\alpha = \lambda\} \text{ is a congruence subgroup.}$$

$$(4.49g) \text{ Let } \alpha \in Sp(n, \mathbf{Q}) \text{ and } r(z) = j_\alpha(z)^{1/2} \text{ with any choice of a branch. Then for every } \lambda \in \mathcal{L}(\mathbf{Q}^n) \text{ there exists an element } \mu \text{ of } \mathcal{L}(\mathbf{Q}^n) \text{ such that } \varphi(\alpha(u, z); \lambda) = r(z) \varphi(u, z; \mu).$$

PROOF. We already have $h_\gamma(z)$ for $\gamma \in \Gamma^\theta$. Formula (4.18) (or rather its $\mathbf{C}$ -linear combination) means that λ^γ can be determined by (4.49a). For $\beta = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in P$ we have $\beta(u, z) = ({}^t d^{-1} u, a z \cdot {}^t a + b d^{-1})$ and $q(\beta(u, z)) = q(u, z)$ by (4.16), and so $\varphi(\beta(u, z), \lambda) = \varphi(u, z; \lambda')$ with $\lambda'(\xi) = \mathbf{e}(2^{-1} \cdot {}^t \xi \cdot {}^t d b \xi) \lambda(d \xi)$. Thus putting $h_\beta(z) = |\det(d)|^{1/2}$ and $\lambda^\beta = |\det(d)|^{-1/2} \mathbf{e}(2^{-1} \cdot {}^t \xi \cdot {}^t d b \xi) \lambda(d \xi)$, we have (4.49a) with β in place of α . We can easily verify that $h_{\beta'\beta}(z) = h_{\beta'}(\beta z) h_\beta(z)$ and $\lambda^{\beta'\beta} = (\lambda^{\beta'})^\beta$ for $\beta', \beta \in P$. If $\gamma \in \Gamma^\theta \cap P$, then $h_\gamma = 1$ by (4.22a, b), and so it coincides with h_γ defined for γ as an element of P . The same is true for λ^γ , as it is determined by (4.49a). Now given $\alpha = \beta\gamma$ with $\beta \in P$ and $\gamma \in \Gamma^\theta$, we define h_α and λ^α by $h_\alpha(z) = h_\beta(\gamma z) h_\gamma(z)$ and $\lambda^\alpha = (\lambda^\beta)^\gamma$. These are well defined, since we have shown the consistency on

$\Gamma^\theta \cap P$. Then formulas (4.49a, b, c, d, e) can be verified in a straightforward way. To prove (4.49f), it is sufficient to show that given $r, s \in \mathbf{Q}^n$, there is a congruence subgroup Γ such that $\varphi(\gamma(u, z); r, s) = h_\gamma(z)\varphi(u, z; r, s)$ for every $\gamma \in \Gamma$. For that purpose take a positive integer m so that $mr, ms \in \mathbf{Z}^n$. Then for $\gamma \in \Gamma(2m^2)$ we have ${}^tr s - {}^tr' s' \in 2\mathbf{Z}$ in (4.13), and so (4.18) gives the desired result. As for (4.49g), since $Sp(n, \mathbf{Q})$ is generated by P and ι as noted in Lemma 2.2(iv), successive applications of (4.49a) with elements of P and ι as α establish (4.49g). This completes the proof.

Putting $u = 0$ in (4.48a, b) and (4.49a), we obtain

$$(4.50a) \quad \varphi(0, z; \lambda) = \theta(0, z; \lambda) = \sum_{\xi \in \mathbf{Q}^n} \lambda(\xi) e(2^{-1} \cdot {}^t \xi z \xi),$$

$$(4.50b) \quad \theta(0, \alpha z; \lambda) = h_\alpha(z) \theta(0, z; \lambda^\alpha) \quad \text{for every } \alpha \in P\Gamma^\theta.$$

The factors of automorphy $\mu_\gamma(z)$ and $j_\gamma(z)$ are defined for every $\gamma \in Sp(n, \mathbf{Q})$ and the “associativity” of the type (1.7) holds on the whole group $Sp(n, \mathbf{Q})$, but in the case of $h_\alpha(z)$ it is defined only for $\alpha \in P\Gamma^\theta$ and the associativity holds only to the extent given by (4.49d, e). However, these are sufficient for practical purposes, though we always have to be careful about our calculation involving $h_\alpha(z)$.

4.13 As generalizations of (4.43) and (4.47) when $n = 1$, we put

$$(4.51) \quad \theta^*(z, \lambda) = \sum_{\xi \in \mathbf{Q}} \lambda(\xi) \xi e(\xi^2 z/2) \quad (z \in \mathfrak{H}_1, \lambda \in \mathcal{L}(\mathbf{Q})).$$

Since $\theta^*(z, \lambda) = (2\pi i)^{-1}(\partial\varphi/\partial u)(0, z; \lambda)$, from (4.49a) we obtain

$$(4.52) \quad \theta^*(\alpha z, \lambda) = h_\alpha(z) j_\alpha(z) \theta^*(z, \lambda^\alpha) \quad \text{for every } \alpha \in P\Gamma^\theta.$$

5. Modular forms of half-integral weight

5.1. By a **weight** (of a modular form) we mean an element of $2^{-1}\mathbf{Z}$; we call it an **integral weight** if it belongs to $\mathbf{Z}$ and a **half-integral weight** otherwise. To make our formulas short, for a weight k and $\alpha \in Sp(n, \mathbf{Q})$ we hereafter put

$$(5.1a) \quad [k] = \begin{cases} k & \text{if } k \in \mathbf{Z}, \\ k - 1/2 & \text{if } k \notin \mathbf{Z}, \end{cases}$$

$$(5.1b) \quad j_\alpha^k(z) = \begin{cases} j_\alpha(z)^k & \text{if } k \in \mathbf{Z}, \\ h_\alpha(z) j_\alpha(z)^{[k]} & \text{if } k \notin \mathbf{Z}. \end{cases}$$

Here we assume that $\alpha \in P\Gamma^\theta$ when $k \notin \mathbf{Z}$. We also put

$$(5.2) \quad Gp^+(n, \mathbf{Q}) = \{\alpha \in Gp(n, \mathbf{Q}) \mid \nu(\alpha) > 0\}.$$

For k, α as in (5.1b) and a function f on $\mathfrak{H}_n$ we define a function $f||_k \alpha$ on $\mathfrak{H}_n$ by

$$(5.3a) \quad (f|_k \alpha)(z) = j_\alpha^k(z)^{-1} f(\alpha(z)).$$

This is the same as (3.1) if $k \in \mathbf{Z}$ and $\alpha \in Sp(n, \mathbf{Q})$. Notice that

$$(5.3b) \quad f|_k(-1) = (-1)^{n[k]} f.$$

Also, from Theorem 4.7(2) we see that

$$(5.3c) \quad j_\gamma^{-k}(z) = \left(\frac{-1}{\det(d_\gamma)} \right) j_\gamma^k(z)^{-1} \quad \text{if } k \notin \mathbf{Z}, \gamma \in \Gamma^\theta \text{ and } \det(d_\gamma) \notin 2\mathbf{Z}.$$

As for the analogue of formula (3.2b), from (1.7) and (4.21) we obtain

$$(5.4) \quad f|_k(\gamma\delta) = (f|_k \gamma)|_k \delta \quad \text{if } \gamma, \delta \in \Gamma^\theta \text{ and } k \notin \mathbf{Z}.$$

In fact, the last formula can be extended to the cases covered by (4.49d, e).

Suppose $k \notin \mathbf{Z}$. The symbols $j_\alpha^k(z)$ and $f|_k \alpha$ are defined with no ambiguity if $\alpha \in P\Gamma^\theta$. However, we will have to consider an arbitrary $\alpha \in Gp^+(n, \mathbf{Q})$. For that purpose we make the following convention: *Whenever we write $j_\alpha(z)^{-k} f(\alpha z)$ for $\alpha \in Gp^+(n, \mathbf{Q})$, the symbol $j_\alpha(z)^{-k}$ means any branch of the function.* There is no danger in doing so, since the statement in each case is valid with any choice of a branch.

Let Γ be a congruence subgroup of $Sp(n, \mathbf{Q})$. For an integral weight k we defined $\mathcal{M}_k(\Gamma)$ in §3.1. Suppose now k is half-integral and $\Gamma \subset \Gamma^\theta$. We then define $\mathcal{M}_k(\Gamma)$ to be the set of all holomorphic functions $f \in \mathfrak{H}_n$ such that $f|_k \gamma = f$ for every $\gamma \in \Gamma$ and $f^2 \in \mathcal{M}_{2k}$. The last condition is automatically satisfied if $n > 1$. Indeed, by Theorem 4.7(2), $f^2|_{2k} \gamma = f^2$ for every $\gamma \in \Gamma \cap \Gamma(4)$, and so $f^2 \in \mathcal{M}_{2k}$, since condition (3.4c) is unnecessary if $n > 1$. We call an element of $\mathcal{M}_k(\Gamma)$ a **modular form of weight k** with respect to Γ .

Suppose $n = 1$ and k is half-integral. Let f be a holomorphic function on $\mathfrak{H}_1$ such that $f|_k \gamma = f$ for every $\gamma \in \Gamma$. Then $f^2 \in \mathcal{M}_{2k}$ if and only if (3.4d) is satisfied with any choice of a branch of $(cz + d)^{-k}$. Indeed, for such an f satisfying (3.4d) we have $f^2|_{2k} \gamma = f^2$ for every $\gamma \in \Gamma \cap \Gamma(4)$, since $(h_\gamma(z)j_\gamma(z)^{k-1/2})^2 = j_\gamma(z)^{2k}$ for $\gamma \in \Gamma(4)$, and (3.4d) is satisfied with $(f^2, 2k)$ in place of (f, k) , and so $f^2 \in \mathcal{M}_{2k}$.

Conversely, suppose $f^2 \in \mathcal{M}_{2k}$ and $f|_k \gamma = f$ for every $\gamma \in \Gamma$. Let $\alpha \in SL_2(\mathbf{Q})$ and let $r(z)$ be a branch of $j_\alpha(z)^k$. Let $\gamma = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$ with $0 < m \in 4\mathbf{Z}$. We can find m such that $\gamma \in \Gamma \cap \alpha^{-1}\Gamma\alpha \cap \alpha^{-1}\Gamma(4)\alpha$. Put $\delta = \alpha\gamma\alpha^{-1}$ and $\kappa = 2k$. Then $\kappa \in \mathbf{Z}$, $\delta\alpha = \alpha\gamma$, and $j_\delta(\alpha z)^\kappa j_\alpha(z)^\kappa = j_{\delta\alpha}(z)^\kappa = j_{\alpha\gamma}(z)^\kappa = j_\alpha(\gamma z)^\kappa j_\gamma(z)^\kappa$, which means that $h_\delta(\alpha z)^\kappa r(z) = \pm r(\gamma z)h_\gamma(z)^\kappa$. Put $g(z) = r(z)^{-1}f(\alpha z)$. Since $\delta \in \Gamma \cap \Gamma(4)$ and $h_\gamma = 1$, we have $g(\gamma z) = r(\gamma z)^{-1}f(\alpha\gamma(z)) = r(\gamma z)^{-1}f(\delta\alpha(z)) = r(\gamma z)^{-1}h_\delta(\alpha z)^\kappa f(\alpha z) = \pm r(z)^{-1} \cdot f(\alpha z) = \pm g(z)$. Thus $g(z + m) = \pm g(z)$, and so $g(z + 2m) = g(z)$. Consequently we can put $g(z) = \sum_{\nu \in \mathbf{Z}} c(\nu)q(z)^\nu$ with $c(\nu) \in \mathbf{C}$ and $q(z) =$

$e(z/(2m))$. If $f^2 \in \mathcal{M}_\kappa$, then (3.4d) applied to f^2 implies that g^2 as a function of $q(z)$ has no singularity at $q(z) = 0$. Then the same must be true for $g(z)$, and so $c(\nu) = 0$ for $\nu < 0$. Thus f satisfies (3.4d).

We now put

$$(5.5) \quad \mathcal{M}_k = \bigcup_{N=1}^{\infty} \mathcal{M}_k(\Gamma(2N)).$$

Let k and ℓ be weights. Then for $f \in \mathcal{M}_k$ and $g \in \mathcal{M}_\ell$, we easily see that $fg \in \mathcal{M}_{k+\ell}$.

Also, $\theta(0, z; \lambda) \in \mathcal{M}_{1/2}$ for every $\lambda \in \mathcal{L}(\mathbf{Q}^n)$. This follows from (4.49a) and (4.49f) if $n > 1$. Now suppose $n = 1$. Putting $u = 0$ in (4.49g), we find that (3.4d) is satisfied with $k = 1/2$ and $f(z) = \theta(0, z; \lambda)$ as expected.

If $n = 1$, $\theta^*(z, \lambda)$ belongs to $\mathcal{M}_{3/2}$ for every $\lambda \in \mathcal{L}(\mathbf{Q}^n)$. To show this, we first note that $\theta^*(\gamma z, \lambda) = j_\gamma^{3/2}(z) \theta^*(z, \lambda)$ for γ in a suitable congruence subgroup, which follows from (4.52) and (4.49f). Since $\theta^*(z, \lambda) = (2\pi i)^{-1} \cdot (\partial\varphi/\partial u)(0, z; \lambda)$, from (4.49g) we obtain $\theta^*(\alpha z, \lambda) = j_\alpha(z)^{3/2} \theta^*(z, \mu)$, and so $\theta^*(z, \lambda)$ satisfies (3.4d) with $k = 3/2$. Thus $\theta^*(z, \lambda) \in \mathcal{M}_{3/2}$.

Let $f \in \mathcal{M}_k$ with half-integral k and $n > 1$. From (4.22a, b) we see that (3.6a) and (3.6b) hold for f with suitable M and U . Therefore we have an expansion of f of the form (3.7a) or (3.7b). Thus we can speak of the Fourier expansion of f in the case of half-integral k . If $n = 1$, we have shown that f satisfies (3.4d).

5.2. For $k \in \mathbf{Z}$ our definitions of $f\|_k \alpha$ and $\mathcal{M}_k(\Gamma)$ in Section 3 are standard, but the case of half-integral k is not so clear-cut. Indeed, for half-integral k we could have defined $f\|_k \alpha$ by $(f\|_k \alpha)(z) = h_\alpha(z)^{-2k} f(\alpha z)$ instead of (5.3a). The reason for adopting (5.3a) is that its natural generalization to the Hilbert modular case is the best definition. This of course requires a clarification, but without going into details here we refer the reader to Section 17 of the present book and [S87].

The factor of automorphy h_γ of (4.40) is different from what we introduced in [S73a], which has been employed by many researchers and which is given by

$$(5.6) \quad h'_\delta(z) = \varepsilon_d^{-1} \left(\frac{c}{d} \right) (cz + d)^{1/2}$$

for $\delta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1)$ with $c \in 4\mathbf{Z}$. For such a δ put $\gamma = \begin{bmatrix} a & 2b \\ c/2 & d \end{bmatrix}$. Then $\gamma \in \Gamma(2)$ and $h_\gamma(2z) = h'_\delta(z)$. In the present book for various reasons, we develop the theory by using h_γ of (4.40). The formulation in terms of h'_δ of (5.6) can easily be obtained by means of the equality $h_\gamma(2z) = h'_\delta(z)$.

5.3. Let us now prove Theorem 4.7(4). Let α and $r(z)$ be as in that statement. We first assume that $\alpha \in Sp(n, \mathbf{Q})$. Put $\Delta_\lambda = \{\gamma \in \Gamma^\theta \mid \lambda^\gamma = \lambda\}$

for $\lambda \in \mathcal{L}(\mathbf{Q}^n)$. By (4.49f), Δ_λ is a congruence subgroup of $Sp(n, \mathbf{Q})$. Now take λ to be the characteristic function of $\mathbf{Z}^n$ and take also μ as in (4.49g). Let $\gamma \in \Delta_\mu \cap \alpha^{-1}\Delta_\lambda\alpha$ and $\delta = \alpha\gamma\alpha^{-1}$. Then $\delta\alpha = \alpha\gamma$, $\lambda^\delta = \lambda$, $\mu^\gamma = \mu$, and $\varphi(\delta\alpha(u, z); \lambda) = h_\delta(\alpha z)\varphi(\alpha(u, z); \lambda) = h_\delta(\alpha z)r(z)\varphi(u, z; \mu)$. This equals $\varphi(\alpha\gamma(u, z); \lambda) = r(\gamma z)\varphi(\gamma(u, z); \mu) = r(\gamma z)h_\gamma(z)\varphi(u, z; \mu)$. Since $\varphi(u, z; \mu)$ is a nonzero function, we obtain $h_\delta(\alpha z)r(z) = r(\gamma z)h_\gamma(z)$. This proves Theorem 4.7(4) for $\alpha \in Sp(n, \mathbf{Q})$.

By (1.3) every element of $Gp^+(n, \mathbf{Q})$ is the product of an element of $Sp(n, \mathbf{Q})$ and an element β of the form $\beta = \text{diag}[e1_n, 1_n]$ with $0 < e \in \mathbf{Q}$. Therefore we easily see that it is sufficient to prove Theorem 4.7(4) when α is such a β ; we may even assume that $e \in \mathbf{Z}$. Thus our task is to show that $h(\beta\gamma\beta^{-1}, \beta z) = h_\gamma(z)$ for γ in a suitable congruence subgroup. Suppose $n = 1$ and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(2e)$; then $\beta\gamma\beta^{-1} = \begin{bmatrix} a & eb \\ e^{-1}c & d \end{bmatrix} \in \Gamma(2)$, and from (4.40) we see that $h(\beta\gamma\beta^{-1}, \beta z) = (\frac{e}{d})h_\gamma(z)$, and so $h(\beta\gamma\beta^{-1}, \beta z) = h_\gamma(z)$ if $d - 1 \in 4e\mathbf{Z}$, which gives the desired result.

If $n > 1$, the proof is more involved and requires some facts on the Gauss sum of a quadratic form as in (4.25). We refer the reader to [S00, Theorems 6.8 and 6.9], whose proof is given in [S00, §A2.9]. However we need Theorem 4.7(4) only for the proof of the following lemma, which we employ only when $n = 1$ in the present book.

Lemma 5.4. *Let k be an integral or a half-integral weight. Given $\alpha \in Gp^+(n, \mathbf{Q})$ and $f \in \mathcal{M}_k$, put $g(z) = j_\alpha(z)^{-k}f(\alpha z)$. Then $g \in \mathcal{M}_k$.*

For integral k this was already noted in §3.1. If k is half-integral, this is an immediate consequence of Theorem 4.7(4), and so the proof may be left to the reader.

Lemma 5.5. *Let ψ be a primitive or an imprimitive character modulo r , and let $\psi(-1) = (-1)^\mu$ with $\mu = 0$ or 1 . Put $k = (2\mu + 1)/2$ and*

$$(5.7) \quad \theta_\psi(z) = 2^{-1} \sum_{m \in \mathbf{Z}} \psi(m)m^\mu \mathbf{e}(m^2 z/2).$$

Then $\theta_\psi \in \mathcal{M}_k$ and

$$(5.8) \quad \theta_\psi \|_k \gamma = \psi(d_\gamma)\theta_\psi \quad \text{for every } \gamma \in \Gamma(2) \text{ such that } c_\gamma \in 2r^2\mathbf{Z},$$

$$(5.9) \quad \theta_\psi \|_k \begin{bmatrix} 0 & -r^{-1} \\ r & 0 \end{bmatrix} = \psi(-1)r^{-1/2}G(\psi)\theta_{\bar{\psi}} \quad \text{if } \psi \text{ is primitive.}$$

PROOF. Notice that $2^{-1} \sum_{m \in \mathbf{Z}}$ of (5.7) can be replaced by $\sum_{m=1}^\infty$ if ψ is not the principal character. If $r = 1$, then ψ is the principal character and $\theta_\psi = 2^{-1}\theta$ with θ of (4.36), and so (5.8) and (5.9) in that case follow from (4.20). Here we prove (5.8) when ψ is primitive; the general case will be settled after the proof of Lemma 7.13. Assuming that $r > 1$ and $\mu = 0$, for

$s \in \mathbf{Z}$ put $f_s(z) = \sum_{m \in \mathbf{Z}} \mathbf{e}(ms/r) \mathbf{e}(m^2 z/2)$. Then $f_s(z) = \theta(0, z; 0, s/r)$.

Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(2)$ with $c \in 2r^2 \mathbf{Z}$. By (4.15) we have

$$(5.10) \quad \begin{aligned} f_s \|_{1/2} \gamma &= \mathbf{e}(-cds^2/(2r^2)) \theta(0, z; cs/r, ds/r) \\ &= \theta(0, z; 0, ds/r) = f_{ds}. \end{aligned}$$

Now by (2.3a) we have

$$\sum_{s=1}^r \bar{\psi}(s) f_s(z) = G(\bar{\psi}) \sum_{m \in \mathbf{Z}} \psi(m) \mathbf{e}(m^2 z/2) = 2G(\bar{\psi}) \theta_{\psi}(z).$$

This combined with (5.10) shows that

$$2G(\bar{\psi}) \theta_{\psi} \|_{1/2} \gamma = \sum_{s=1}^r \bar{\psi}(s) f_s \|_{1/2} \gamma = \sum_{s=1}^r \bar{\psi}(s) f_{ds} = 2G(\bar{\psi}) \psi(d) \theta_{\psi}.$$

This proves (5.8) when $\mu = 0$, since $G(\bar{\psi}) \neq 0$ as can be seen from (2.3c). Next, suppose $\mu = 1$; let $g_s(z) = \theta^*(z; 0, s/r)$ with θ^* of (4.44). Then $k = 3/2$ and $\sum_{s=1}^r \bar{\psi}(s) g_s = 2G(\bar{\psi}) \theta_{\psi}$. Also, we see from (4.45) that $g_s \|_k \gamma = g_{ds}$. Therefore we obtain (5.8) when $\mu = 1$ in the same manner as in the case $\mu = 0$.

As for (5.9) when $\mu = 1$, from (4.45) we obtain $g_s(\iota z) = j_{\iota}^k(z) \theta^*(z; s/r, 0)$, and so

$$\begin{aligned} 2G(\bar{\psi}) \theta_{\psi}(-1/r^2 z) &= j_{\iota}^k(r^2 z) \sum_{s=1}^r \bar{\psi}(s) \theta^*(r^2 z; s/r, 0) \\ &= r^{-1} j_{\iota}^k(r^2 z) \sum_{s=1}^r \sum_{m \in \mathbf{Z}} \bar{\psi}(s) (rm + s) \mathbf{e}((rm + s)^2 z/2) = 2r^{2k-1} j_{\iota}^k(z) \theta_{\bar{\psi}}(z). \end{aligned}$$

For $\alpha = \begin{bmatrix} 0 & -r^{-1} \\ r & 0 \end{bmatrix}$ we have $j_{\alpha}^k(z) = r^k j_{\iota}^k(z)$ and so $\theta_{\psi} \|_k \alpha = r^{1/2} G(\bar{\psi})^{-1} \theta_{\bar{\psi}} = \psi(-1) r^{-1/2} G(\psi) \theta_{\bar{\psi}}$ by (2.3b, c), which gives (5.9) for $\mu = 1$. The case $\mu = 0$ can be proved by employing (4.15) instead of (4.45).

5.6. We note here an interesting special case. Take $\varphi(m) = \left(\frac{3}{m}\right)$ and put

$$(5.11) \quad \eta(z) = \theta_{\varphi}(z/12) = \sum_{m=1}^{\infty} \left(\frac{3}{m}\right) \mathbf{e}(m^2 z/24).$$

In this case $r = 12$ and $G(\varphi) = 2\sqrt{3}$ by (2.4a), and so from (5.9) we obtain $\eta \|_{1/2} \iota = \eta$. Also, $m^2 - 1 \in 24\mathbf{Z}$ for every integer m prime to 6, and so $\eta(z+1) = \mathbf{e}(1/24) \eta(z)$. Employing these relations we can easily prove

$$(5.12) \quad \eta(z) = \Delta(z)^{1/24} = \mathbf{e}(z/24) \prod_{n=1}^{\infty} (1 - \mathbf{e}(nz)),$$

where $\Delta(z) = \mathbf{e}(z) \prod_{n=1}^{\infty} (1 - \mathbf{e}(nz))^{24}$. We leave the proof to the reader. In fact, it was explained in [S07, p.19]. We also note an easy fact:

$$(5.12a) \quad \eta \|_{1/2} \gamma = \eta \quad \text{for every } \gamma \in \Gamma(1) \text{ such that } b_\gamma, c_\gamma \in 24\mathbf{Z}.$$

Lemma 5.7. *For every congruence subgroup Γ of Γ^θ and $k \in 2^{-1}\mathbf{Z}$ we have $\mathcal{M}_k(\Gamma) = \{0\}$ if $k < 0$ and $\mathcal{M}_0(\Gamma) = \mathbf{C}$.*

PROOF. Since $f^2 \in \mathcal{M}_{2k}$ for $f \in \mathcal{M}_k$, it is sufficient to treat the case $k \in \mathbf{Z}$. If $n = 1$, our assertions are well known, and so we assume that $n > 1$. We do not need this lemma for our later treatment, but we prove here that $\mathcal{M}_k(\Gamma) = \{0\}$ if $k < 0$. For $\alpha \in SL_2(\mathbf{Q})$ put $\sigma_\alpha = \begin{bmatrix} a_\alpha 1_n & b_\alpha 1_n \\ c_\alpha 1_n & d_\alpha 1_n \end{bmatrix}$; put also $p(z) = z 1_n$ for $z \in \mathfrak{H}_1$. Then $\sigma_\alpha \in Sp(n, \mathbf{Q})$, $p(z) \in \mathfrak{H}_n$, $\sigma_\alpha(p(z)) = p(\alpha(z))$, and $j(\sigma_\alpha, p(z)) = j_\alpha(z)^n$. Let $f \in \mathcal{M}_k(\Gamma)$ with a congruence subgroup Γ of $Sp(n, \mathbf{Q})$ and $0 > k \in \mathbf{Z}$. Put $\Gamma_1 = \{\alpha \in SL_2(\mathbf{Z}) \mid \sigma_\alpha \in \Gamma\}$ and $g(z) = f(p(z))$ for $z \in \mathfrak{H}_1$. Then Γ_1 is a congruence subgroup of $SL_2(\mathbf{Q})$ and $(g \|_{nk} \alpha)(p(z)) = (f \|_k \sigma_\alpha)(p(z))$, and so $g \|_{nk} \delta = g$ for $\delta \in \Gamma_1$. Moreover, since $f \|_k \sigma_\alpha \in \mathcal{M}_k$, we see that g satisfies (3.4d). Therefore $g \in \mathcal{M}_k(\Gamma_1)$, and so $g = 0$. This means that $f(p(z)) = 0$ for every $z \in \mathfrak{H}_1$. Given $w \in \mathfrak{H}_n$, as shown at the end of Section 1, we can find an element of the form $\beta = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & t a^{-1} \end{bmatrix}$ with $a \in GL_n(\mathbf{R})$ and $s \in S_n(\mathbf{R})$ such that $\beta(i 1_n) = w$. Since $GL_n(\mathbf{Q})$ and $S_n(\mathbf{Q})$ are dense in $GL_n(\mathbf{R})$ and $S_n(\mathbf{R})$, respectively, we can find an element δ of P that is in any small neighborhood of β . In other words, the set $\{\delta(i 1_n) \mid \delta \in P\}$ is dense in $\mathfrak{H}_n$. Put $f_1 = f \|_k \delta$. Then $f_1 \in \mathcal{M}_k$, and so $f_1(p(z)) = 0$, which means that $f(\delta(i 1_n)) = 0$. Since this is so for every $\delta \in P$, we obtain $f = 0$ as expected.

5.8. Let $f \in \mathcal{M}_k$ with $0 \leq k \in 2^{-1}\mathbf{Z}$. Given $\alpha \in Sp(n, \mathbf{Q})$, by Lemmas 5.4 and 3.3 we have

$$j_\alpha(z)^{-k} f(\alpha z) = \sum_{0 \leq h \in S} c_\alpha(h) \mathbf{e}(\text{tr}(h z)), \quad S = S_n(\mathbf{Q}),$$

with $c_\alpha(h) \in \mathbf{C}$. We call f a **cusp form** if $c_\alpha(h) \neq 0$ only when $h > 0$ for every α , and we denote by $\mathcal{S}_k(\Gamma)$ the set of all cusp forms contained in $\mathcal{M}_k(\Gamma)$. We put

$$(5.13) \quad \mathcal{S}_k = \bigcup_{N=1}^{\infty} \mathcal{S}_k(\Gamma(2N)).$$

From our definition and Lemma 5.4 we easily see that if $f \in \mathcal{S}_k$, then $j_\alpha(z)^{-k} f(\alpha z) \in \mathcal{S}_k$ for every $\alpha \in Gp^+(n, \mathbf{Q})$. Clearly $\mathcal{S}_0 = \{0\}$.

We note that $\theta^*(z, \lambda) \in \mathcal{S}_{3/2}$ for every $\lambda \in \mathcal{L}(\mathbf{Q})$. Indeed, since $\Gamma(1)$ is generated by ι and $\Gamma(1) \cap P_1$, from (4.52) we see that $j_\alpha(z)^{-3/2} \theta^*(\alpha z, \lambda) = \theta^*(z, \mu)$ with some $\mu \in \mathcal{L}(\mathbf{Q})$ for every $\alpha \in SL_2(\mathbf{Q})$. Clearly $\theta^*(z, \mu)$ has 0 as its constant term, and so $\theta^*(z, \lambda) \in \mathcal{S}_{3/2}$.

Lemma 5.9. *Let $f(z) = \sum_{h \in S} c(h) \mathbf{e}(\text{tr}(h z)) \in \mathcal{M}_k(\Gamma)$ with $k \in 2^{-1}\mathbf{Z}$ and a congruence subgroup Γ , where $S = S_n(\mathbf{Q})$. Put*

$$(5.14) \quad f_\rho(z) = \sum_{h \in S} \overline{c(h)} \mathbf{e}(\mathrm{tr}(hz)).$$

Then $f_\rho(z) = \overline{f(-\bar{z})}$ and $f_\rho \in \mathcal{M}_k(\varepsilon\Gamma\varepsilon^{-1})$ with $\varepsilon = \mathrm{diag}[-1_n, 1_n]$. Moreover,

$$(5.15) \quad \overline{j_\alpha^k(z)} = j_{\alpha'}^k(-\bar{z}),$$

$$(5.16) \quad f_\rho \|_k \alpha = (f \|_k \alpha')_\rho$$

for $\alpha \in Sp(n, \mathbf{Q})$, whenever $j_\alpha^k(z)$ is defined, where $\alpha' = \varepsilon\alpha\varepsilon^{-1}$.

PROOF. Clearly, $f_\rho(z) = \overline{f(-\bar{z})}$ and $\alpha' \in Sp(n, \mathbf{Q})$; also, $\alpha' \in P\Gamma^\theta$ if $\alpha \in P\Gamma^\theta$. Moreover, $-\overline{\alpha(z)} = \alpha'(-\bar{z})$, $\overline{j_\alpha(z)} = j_{\alpha'}(-\bar{z})$, and

$$(5.17) \quad f_\rho(\alpha z) = \overline{f(-\overline{\alpha z})} = \overline{f(\alpha'(-\bar{z}))}.$$

Take θ as f . Then $\theta_\rho = \theta$ and (5.17) shows that if $\alpha \in \Gamma^\theta$, then $h_\alpha(z)\theta(z) = \overline{h_{\alpha'}(-\bar{z})\theta(z)}$, and so $\overline{h_\alpha(z)} = h_{\alpha'}(-\bar{z})$, which is also true for $\alpha \in P$ because of (4.49c). Thus $\overline{h_\alpha(z)} = h_{\alpha'}(-\bar{z})$ for every $\alpha \in P\Gamma^\theta$ by (4.49d), and we obtain (5.15). Then (5.16) follows immediately from (5.15). Take α in $\varepsilon\Gamma\varepsilon^{-1}$ in (5.17). Then $\alpha' \in \Gamma$ and we obtain $f_\rho(\alpha z) = \overline{j_{\alpha'}^k(-\bar{z})f(-\bar{z})} = j_\alpha^k(z)f_\rho(z)$, and so $f_\rho \in \mathcal{M}_k(\varepsilon\Gamma\varepsilon^{-1})$ as expected. This completes the proof.

6. Holomorphic and nonholomorphic modular forms on $\mathfrak{H}_1$

6.1. Hereafter we will mainly be concerned with functions on $\mathfrak{H}_1$, and so we write simply $\mathfrak{H}$ for $\mathfrak{H}_1$. We also put

$$(6.1a) \quad GL_2^+(\mathbf{R}) = \{\alpha \in GL_2(\mathbf{R}) \mid \det(\alpha) > 0\},$$

$$(6.1b) \quad GL_2^+(\mathbf{Q}) = GL_2^+(\mathbf{R}) \cap GL_2(\mathbf{Q}),$$

$$(6.1c) \quad P = \{\alpha \in SL_2(\mathbf{Q}) \mid c_\alpha = 0\}.$$

Hereafter we work with $GL_2(\mathbf{R})$ and $\mathfrak{H}_1$; the group P is P_n (with $n = 1$) of Lemma 2.2(iii).

Formulas (1.10) and (1.16) in the case $n = 1$ take the forms:

$$(6.2) \quad d(\alpha z) = j_\alpha(z)^{-2} dz \quad \text{for every } \alpha \in SL_2(\mathbf{R}),$$

$$(6.3) \quad \mathrm{Im}(\alpha z) = |j_\alpha(z)|^{-2} \mathrm{Im}(z) \quad \text{for every } \alpha \in SL_2(\mathbf{R}).$$

Writing simply $y = \mathrm{Im}(z)$, from (6.2) and (6.3) we see that $y^{-2} dz \wedge d\bar{z}$ is invariant under $SL_2(\mathbf{R})$. Since $dz \wedge d\bar{z} = -2i dx \wedge dy$, we obtain

$$(6.4) \quad (y^{-2} dx \wedge dy) \circ \alpha = y^{-2} dx \wedge dy \quad \text{for every } \alpha \in SL_2(\mathbf{R}).$$

Thus the form $y^{-2} dx \wedge dy$ gives a measure on $\mathfrak{H}$ invariant under $SL_2(\mathbf{R})$. To make our formulas short, we put

$$(6.5) \quad \mathbf{d}z = y^{-2} dx dy,$$

and define a measure μ on $\mathfrak{H}$ by

$$(6.5a) \quad \mu(A) = \int_A \mathbf{d}z$$

for $A \subset \mathfrak{H}$. It is well known that $\mu(\Gamma(1) \backslash \mathfrak{H}) = \pi/3$, and so

$$(6.6) \quad \mu(\Gamma \backslash \mathfrak{H}) = [\Gamma(1) : \{\pm 1\}\Gamma] \pi/3 \quad \text{if } \Gamma \subset \Gamma(1).$$

Lemma 6.2. *Let $f \in \mathcal{M}_k$ and let $f(z) = \sum_{m=0}^{\infty} c_m \mathbf{e}(mz/N)$ with $c_m \in \mathbf{C}$ and $0 < N \in \mathbf{Z}$. Then the following assertions hold:*

(i) *There exists a positive number A such that $|f(z)| \leq A(1 + y^{-k})$ on the whole $\mathfrak{H}$, where $y = \text{Im}(z)$. Moreover, if f is a cusp form, then A can be chosen so that $|f(z)| \leq Ay^{-k/2}$ on the whole $\mathfrak{H}$.*

(ii) $c_m = O(m^k)$.

(iii) $c_m = O(m^{k/2})$ if f is a cusp form.

PROOF. For the proof of these facts when $k \in \mathbf{Z}$ the reader is referred to [S07, Lemmas 1.7 and 1.8]. To prove (i) when $k \notin \mathbf{Z}$, take $(f^2, 2k)$ as (f, k) in (i). Then we obtain the desired fact for (f, k) . To prove (ii) and (iii), we first note, for $0 \leq m \in \mathbf{Z}$,

$$Nc_m \exp(-2\pi my/N) = \int_0^N f(z) \mathbf{e}(-mx/N) dx.$$

Take $y = 1/m$. Then (ii) and (iii) follow from (i).

Results of the same type as the above lemma hold in the case $n > 1$. The case of cusp form is easy, but there are some nontrivial technical problems in the case of non-cusp forms; see [S00, Proposition A6.4 and formula (A6.7)] and the proof given there.

6.3. The notion of a cusp can be defined for a certain class of discrete subgroups of $SL_2(\mathbf{R})$ that includes congruence subgroups of $SL_2(\mathbf{Q})$. Since we deal only with such congruence subgroups in this book, a **cusp** is a point of $\mathbf{Q} \cup \{\infty\}$, and vice versa. Then the map $\alpha \mapsto \alpha(\infty)$ with $\alpha \in SL_2(\mathbf{Q})$ is a bijection of $SL_2(\mathbf{Q})/P$ onto $\mathbf{Q} \cup \{\infty\}$, and so $\Gamma \backslash (\mathbf{Q} \cup \{\infty\})$ is the set of Γ -equivalence classes of cusps, which is in one-to-one correspondence with $\Gamma \backslash SL_2(\mathbf{Q})/P$. We often use $P \backslash SL_2(\mathbf{Q})/\Gamma$ instead, by considering the inverse map.

For a congruence subgroup Γ of $SL_2(\mathbf{Z})$ and a weight k we denote by $C_k(\Gamma)$ the set of all C^∞ functions $f(z)$ on $\mathfrak{H}$ such that

$$(6.7) \quad f|_k \gamma = f \quad \text{for every } \gamma \in \Gamma;$$

we assume that $\Gamma \subset \Gamma^\theta$ if $k \notin \mathbf{Z}$. Then from (6.3) we see that

$$(6.8) \quad |y^{k/2} f(z)| \text{ is a } \Gamma\text{-invariant function on } \mathfrak{H}.$$

Therefore we consider more generally a C^∞ function $f(z)$ satisfying (6.8) for some Γ and k . We say that such an f is **slowly increasing** or **rapidly decreasing at every cusp** according as the following condition (6.9a) or (6.9b) is satisfied:

(6.9a) *For every $\alpha \in SL_2(\mathbf{Q})$ there exist positive constants A, B , and c depending on f and α such that*

$$|\mathrm{Im}(\alpha z)^{k/2} f(\alpha z)| < Ay^c \text{ if } y = \mathrm{Im}(z) > B.$$

(6.9b) *For every $\alpha \in SL_2(\mathbf{Q})$ and $c \in \mathbf{R}, > 0$, there exist positive constants A and B depending on f, α , and c such that*

$$|\mathrm{Im}(\alpha z)^{k/2} f(\alpha z)| < Ay^{-c} \text{ if } y = \mathrm{Im}(z) > B.$$

Notice that these are conditions on $|f|$, rather than on f . Since $SL_2(\mathbf{Q}) = \Gamma(1)P$ by Lemma 2.2(iii), we can find a finite set X such that

$$(6.10) \quad SL_2(\mathbf{Q}) = \bigsqcup_{\xi \in X} \Gamma \xi P, \quad X \subset \Gamma(1).$$

Then we easily see that condition (6.9a) or (6.9b) is satisfied for every $\alpha \in SL_2(\mathbf{Q})$ if it is satisfied for every $\alpha \in X$. Also, if f satisfies (6.9a) or (6.9b), so does $j_\beta(z)^{-k} f(\beta z)$ for every $\beta \in GL_2^+(\mathbf{Q})$.

Since $|\mathrm{Im}(\alpha z)^{k/2} f(\alpha z)|$ is invariant under $z \mapsto z+m$ with a positive integer m and the set $\{x+iy \mid 0 \leq x \leq m, B' \leq y \leq B\}$ is compact for every $B' < B$, we easily see that changing A in (6.9a, b) suitably, we can replace B in (6.9a, b) by an arbitrary positive number.

We will be considering a function $f(z, s)$ of $(z, s) \in \mathfrak{H} \times D$, also written $f_s(z)$, with a domain D in $\mathbf{C}$ such that $f_s(z)$ for each fixed s as a function of z satisfies (6.8) with the same Γ and k . If for every compact subset K of D , f_s satisfies condition (6.9a) for $s \in K$ with A, B , and c depending only on K and α , then we say that f is **slowly increasing at every cusp locally uniformly on D** (or locally uniformly in s).

Lemma 6.4. (i) *Let f be a holomorphic function on $\mathfrak{H}$ belonging to $C_k(\Gamma)$. Then $f \in \mathcal{M}_k$ if and only if f is slowly increasing at every cusp.*

(ii) *For f as in (i), $f \in \mathcal{S}_k$ if and only if f is rapidly decreasing at every cusp.*

(iii) *An element of $\mathcal{M}_k(\Gamma)$ is a cusp form if the constant term of the Fourier expansion of $j_\xi(z)^{-k} f(\xi z)$ is 0 for every ξ in the set X of (6.10).*

(iv) *Let f be an element of $C_k(\Gamma)$ that is slowly increasing at every cusp. Then there exist two positive constants A and c such that*

$$|y^{k/2} f(x+iy)| \leq A(y^c + y^{-c}) \quad \text{for every } x+iy \in \mathfrak{H}.$$

(v) If an element f of $C_k(\Gamma)$ is rapidly decreasing at every cusp, then $|\operatorname{Im}(z)^{k/2}f(z)|$ is bounded on the whole $\mathfrak{H}$.

PROOF. Given a holomorphic f satisfying (6.7), for $\alpha \in SL_2(\mathbf{Q})$ put $g(z) = j_\alpha(z)^{-k}f(\alpha z)$. Then $g \circ \gamma = \pm g$ for every $\gamma \in \alpha^{-1}\Gamma\alpha \cap P$. Now $(\alpha^{-1}\Gamma\alpha \cap P)/\{\pm 1\}$ is a cyclic group generated by an element of the form $\begin{bmatrix} 1 & t_\alpha \\ 0 & 1 \end{bmatrix}$ with $0 < t_\alpha \in \mathbf{Q}$. Therefore $g(z + 2t_\alpha) = g(z)$, and so $g(z) = \sum_{m \in \mathbf{Z}} c_\alpha(m) \mathbf{e}(mz/r_\alpha)$ with $r_\alpha = 2t_\alpha$ and $c_\alpha(m) \in \mathbf{C}$. Suppose f is slowly increasing at every cusp. Then $|g(z)| \leq Ay^c$ for $y > B$ as in (6.9a). Put $q = \mathbf{e}(z/r_\alpha)$ and $h(q) = \sum_{m \in \mathbf{Z}} c_\alpha(m) q^m$. Since $|q| = \exp(-2\pi y/r_\alpha)$, we see that $\lim_{q \rightarrow 0} qh(q) = 0$, which means that $c_\alpha(m) = 0$ for $m \leq 0$. Thus f satisfies (3.4d), and so $f \in \mathcal{M}_k$. Suppose f is rapidly decreasing at every cusp. Then $\lim_{q \rightarrow 0} h(q) = 0$, and so $c_\alpha(0) = 0$. Thus $f \in \mathcal{S}_k$.

Conversely, given $f \in \mathcal{M}_k$, let $j_\alpha(z)^{-k}f(\alpha z) = \sum_{m=0}^{\infty} c_\alpha(m) \mathbf{e}(mz/r_\alpha)$ as in (3.4d). Put $q = \mathbf{e}(z/r_\alpha)$. Then $j_\alpha^{-k}f(\alpha z) = h(q)$ with $h(q) = \sum_{m=0}^{\infty} c_\alpha(m) q^m$. Since h is a holomorphic function for $|q| < 1$, there is a constant A such that $|h(q)| \leq A$ for $|q| < 1/2$. Thus f is slowly increasing at every cusp. Next suppose $f \in \mathcal{S}_k$; then $c_\alpha(0) = 0$, and so $|h(q)| \leq A'|q|$ for $|q| < 1/2$ with a positive constant A' . Consequently, $|\operatorname{Im}(\alpha z)^{k/2}f(\alpha z)| < Ay^{k/2} \exp(-2\pi y/r_\alpha)$ if $y > B$ with a suitable B , and so f satisfies (6.9b). Assertion (iii) is clear.

To prove (iv), put

$$(6.10a) \quad T = \{x + iy \in \mathbf{C} \mid |x| \leq 1/2, y > 1/2\}.$$

Since T contains a fundamental domain for $\Gamma(1) \backslash \mathfrak{H}$, we can find a finite subset E of $\Gamma(1)$ such that

$$(6.10b) \quad \mathfrak{H} = \bigcup_{\varepsilon \in E} \Gamma \varepsilon T.$$

Then $|\operatorname{Im}(\varepsilon z)^{k/2}f(\varepsilon z)| \leq A \cdot \operatorname{Im}(z)^c$ for every $\varepsilon \in E$ and $z \in T$ with some A and c . Given $z \in \mathfrak{H}$, we can put $z = \gamma \varepsilon w$ with some $\gamma \in \Gamma$, $\varepsilon \in E$, and $w \in T$. Then $|\operatorname{Im}(z)^{k/2}f(z)| = |\operatorname{Im}(\gamma \varepsilon w)^{k/2}f(\gamma \varepsilon w)| = |\operatorname{Im}(\varepsilon w)^{k/2}f(\varepsilon w)| \leq A \cdot \operatorname{Im}(w)^c$. Let $(\gamma \varepsilon)^{-1} = \begin{bmatrix} * & * \\ r & s \end{bmatrix}$. If $r = 0$, then $\operatorname{Im}(w) = \operatorname{Im}(z)$ since $(\gamma \varepsilon)^{-1} \in \Gamma(1) \cap P$. If $r \neq 0$, then $\operatorname{Im}(w) = \operatorname{Im}(z)|rz + s|^{-2} \leq \operatorname{Im}(z)^{-1}$. Thus $|\operatorname{Im}(z)^{k/2}f(z)| \leq A(\operatorname{Im}(z)^c + \operatorname{Im}(z)^{-c})$, which proves (iv).

If f is as in (v), then $|\operatorname{Im}(\varepsilon z)^{k/2}f(\varepsilon z)| < Ay^{-c}$ for every $\varepsilon \in E$ and $z \in T$ with some A and $c > 0$. Then we easily see that $|\operatorname{Im}(\varepsilon z)^{k/2}f(\varepsilon z)|$ is bounded on T . Since $|\operatorname{Im}(z)^{k/2}f(z)|$ is a Γ -invariant function on $\mathfrak{H}$ and $\mathfrak{H} = \bigcup_{\varepsilon \in E} \Gamma \varepsilon T$, we obtain (v). This completes the proof.

6.5. Given $f, g \in C_k(\Gamma)$, we put

$$(6.11) \quad \langle f, g \rangle = \mu(\Phi)^{-1} \int_{\Phi} \bar{f} g y^k \mathbf{d}z, \quad \Phi = \Gamma \backslash \mathfrak{H}.$$

The integral over Φ is formally meaningful, since $\bar{f}gy^k$ is Γ -invariant, as can be seen from (6.3). We call $\langle f, g \rangle$ the **inner product** of f and g if the integral is convergent, in which case we easily see that the quantity of (6.11) is independent of the choice of Γ .

The integral is convergent if fg is rapidly decreasing at every cusp. Indeed, by Lemma 6.4(v), $|y^k fg|$ is bounded on the whole $\mathfrak{H}$, which implies that the integral of (6.11) is convergent, since $\int_{\Phi} y^{-2} dx dy < \infty$. We easily see that

$$(6.12) \quad \langle f \|_k \alpha, g \|_k \alpha \rangle = \langle f, g \rangle$$

for every $\alpha \in SL_2(\mathbf{Q})$. Here, if $k \notin \mathbf{Z}$, then we either assume that $\alpha \in P\Gamma^\theta$, or put $f \|_k \alpha = \kappa j_\alpha(z)^{-k} f(\alpha z)$ with $\kappa \in \mathbf{T}$ and any choice of a branch of $j_\alpha(z)^{-k}$. We also note an easy fact:

$$(6.12a) \quad t^k \langle f(tz), g(tz) \rangle = \langle f, g \rangle \quad \text{if } 0 < t \in \mathbf{Q}.$$

Lemma 6.6. *The inner product $\langle f, g \rangle$ is meaningful for every $f, g \in \mathcal{M}_{1/2}$, even when neither f nor g is a cusp form.*

PROOF. In this proof we put $k = 1/2$. Take $\Gamma \subset \Gamma(2)$ so that $f, g \in \mathcal{M}_k(\Gamma)$, and take T and E as in (6.10a, b). For each $\varepsilon \in E$ put $f_\varepsilon(z) = j_\varepsilon(z)^{-k} f(\varepsilon z)$ and $g_\varepsilon(z) = j_\varepsilon(z)^{-k} g(\varepsilon z)$ with any choice of a branch of $j_\varepsilon(z)^{-k}$. Then $f_\varepsilon, g_\varepsilon \in \mathcal{M}_k$ by Lemma 5.4, and so by Lemma 6.2(i), $|f_\varepsilon g_\varepsilon| \leq A_\varepsilon$ on T with a constant A_ε . Since $\Gamma \backslash \mathfrak{H}$ is covered by $\bigcup_{\varepsilon \in E} \varepsilon T$, we have

$$|\langle f, g \rangle| \leq \sum_{\varepsilon \in E} \int_{\varepsilon T} |fg| y^k \mathbf{d}z = \sum_{\varepsilon \in E} \int_T (|fg y^k| \circ \varepsilon) \mathbf{d}z.$$

Now $|fg y^k| \circ \varepsilon = y^k |f_\varepsilon g_\varepsilon| \leq A_\varepsilon y^k$ on T , and so

$$|\langle f, g \rangle| \leq \sum_{\varepsilon \in E} A_\varepsilon \int_T y^{k-2} dx dy = \sum_{\varepsilon \in E} A_\varepsilon \int_{1/2}^\infty y^{-3/2} dy < \infty.$$

This proves our lemma.

6.7. We put $y = \text{Im}(z)$ and view it as a real-valued function on $\mathfrak{H}$ as we have done in previous sections. For $k \in \mathbf{R}$ we define differential operators ε , δ_k , and L_k acting on C^∞ functions f on $\mathfrak{H}$ by

$$(6.13a) \quad \varepsilon f = -y^2 \partial f / \partial \bar{z},$$

$$(6.13b) \quad \delta_k f = y^{-k} (\partial / \partial z) (y^k f) = \frac{kf}{2iy} + \frac{\partial f}{\partial z},$$

$$(6.13c) \quad L_k = 4\delta_{k-2}\varepsilon = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + 2iky \frac{\partial}{\partial \bar{z}} = 4\varepsilon \delta_k - k,$$

and define also δ_k^p for $0 \leq p \in \mathbf{Z}$ inductively by

$$(6.13d) \quad \delta_k^{p+1} = \delta_{k+2p} \delta_k^p, \quad \delta_k^1 = \delta_k, \quad \delta_k^0 = 1.$$

For every $\alpha \in GL_2^+(\mathbf{R})$ these operators satisfy

$$(6.14a) \quad \varepsilon(f|_k \alpha) = (\varepsilon f)|_{k-2} \alpha,$$

$$(6.14b) \quad \delta_k(f|_k \alpha) = (\delta_k f)|_{k+2} \alpha,$$

$$(6.14c) \quad \delta_k^p(f|_k \alpha) = (\delta_k^p f)|_{k+2p} \alpha,$$

$$(6.14d) \quad L_k(f|_k \alpha) = (L_k f)|_k \alpha.$$

Here $f|_k \alpha$ for $k \notin \mathbf{Z}$ can be defined by $(f|_k \alpha)(z) = j_\alpha(z)^{-k} f(\alpha z)$ with any choice of a branch of $j_\alpha(z)^{-k}$. Then we define $(f|_{k+2p} \alpha)(z)$ with $j_\alpha(z)^{-k-2p} = j_\alpha(z)^{-k} j_\alpha(z)^{-2p}$ for every $p \in \mathbf{Z}$. From (6.13c) we easily obtain

$$(6.14e) \quad L_{k-2} \varepsilon = \varepsilon(L_k - k + 2), \quad L_{k+2} \delta_k = \delta_k(L_k + k).$$

To prove the above formulas, we first note that

$$(6.15) \quad \frac{\partial}{\partial z} f(\alpha z) = \frac{\partial f}{\partial z}(\alpha z) j_\alpha(z)^{-2}, \quad \frac{\partial}{\partial \bar{z}} f(\alpha z) = \frac{\partial f}{\partial \bar{z}}(\alpha z) \overline{j_\alpha(z)}^{-2}.$$

Employing (6.3) and (6.15), we have

$$\begin{aligned} \delta_k(f|_k \alpha) &= y^{-k} (\partial/\partial z) (y^k j_\alpha^{-k} f(\alpha z)) = y^{-k} (\partial/\partial z) (y(\alpha z)^k \overline{j_\alpha}^{-k} f(\alpha z)) \\ &= y^{-k} \overline{j_\alpha}^{-k} (\partial/\partial z) ((y^k f) \circ \alpha) = y^{-k} \overline{j_\alpha}^{-k} j_\alpha^{-2} \{(\partial/\partial z)(y^k f)\}(\alpha z) \\ &= j_\alpha^{-k-2} y(\alpha z)^{-k} \{(\partial/\partial z)(y^k f)\}(\alpha z) = (\delta_k f)|_{k+2} \alpha, \end{aligned}$$

which gives (6.14b). Formula (6.14a) can be proved in a similar way. Then (6.14c) and (6.14d) follow immediately from (6.14a, b).

From (6.2) we easily see that $y^{-2}(dx^2 + dy^2)$ is a Riemannian metric on $\mathfrak{H}$ invariant under $SL_2(\mathbf{R})$. Therefore $-L_0$ is exactly the **Laplace-Beltrami operator** with respect to this metric.

Theorem 6.8. *Let $f \in C_k(\Gamma)$ and $h \in C_{k-2}(\Gamma)$ with a congruence subgroup Γ . Then*

$$(6.16) \quad \langle f, \delta_{k-2} h \rangle = \langle \varepsilon f, h \rangle,$$

provided fh , $f \cdot \delta_{k-2} h$ and $(\varepsilon f)h$ are rapidly decreasing at every cusp.

PROOF. Replacing Γ by $\Gamma \cap \Gamma(4)$ if necessary, we may assume that $\Gamma \subset \Gamma(4)$. Then Γ has no elements of finite order other than 1. Throughout the proof, we put $y = \text{Im}(z)$. For $0 < r \in \mathbf{R}$ put

$$(6.17) \quad T_r = \{z \in \mathfrak{H} \mid y > r\}, \quad M_r = \{z \in \mathfrak{H} \mid y = r\}.$$

Take a finite subset X of $\Gamma(1)$ as in (6.10). Then $\Gamma \backslash (\mathbf{Q} \cup \{\infty\})$ is represented by $\{\xi(\infty) \mid \xi \in X\}$. Put $Q_\xi = \xi^{-1} \Gamma \xi \cap P$ for each $\xi \in X$. Then $\xi Q_\xi \xi^{-1} = \{\gamma \in \Gamma \mid \gamma \xi(\infty) = \xi(\infty)\}$. Now $\Gamma \backslash (\mathfrak{H} \cup \mathbf{Q} \cup \{\infty\})$ can be viewed as a compact Riemann surface. We can find a sufficiently large r such that the set $\xi(Q_\xi \backslash T_r)$ for $\xi \in X$ can be embedded in $\Gamma \backslash \mathfrak{H}$ without overlap. Let K be the complement of $\bigcup_{\xi \in X} \xi(Q_\xi \backslash T_r)$ in $\Gamma \backslash \mathfrak{H}$. Then K is a compact manifold with boundary, and

$$\partial K = \sum_{\xi \in X} \xi(B_\xi), \quad B_\xi = Q_\xi \setminus M_r.$$

Let φ be a Γ -invariant C^∞ 1-form on $\mathfrak{H}$. Then

$$\int_K d\varphi = \int_{\partial K} \varphi = \sum_{\xi \in X} \int_{B_\xi} \varphi \circ \xi.$$

Given f and h as in our theorem, take $\varphi = \bar{f}hy^{k-2}d\bar{z}$. Then

$$\begin{aligned} d\varphi &= (\partial/\partial z)(\bar{f}hy^{k-2})dz \wedge d\bar{z} \\ &= \{(\partial\bar{f}/\partial z)hy^{k-2} + \bar{f}(\partial h/\partial z)y^{k-2} + (-i/2)(k-2)\bar{f}hy^{k-3}\}dz \wedge d\bar{z}. \end{aligned}$$

Since $dz \wedge d\bar{z} = -2idx \wedge dy$ and $(\partial\bar{f}/\partial z) = \overline{(\partial f/\partial \bar{z})}$, we see that

$$(6.18) \quad (2i)^{-1}d\varphi = \overline{\varepsilon f} \cdot hy^{k-4}dx \wedge dy - \bar{f} \cdot (\delta_{k-2}h)y^{k-2}dx \wedge dy.$$

Thus

$$\int_K \overline{\varepsilon f} \cdot hy^{k-4}dx \wedge dy - \int_K \bar{f} \cdot (\delta_{k-2}h)y^{k-2}dx \wedge dy = (2i)^{-1} \sum_{\xi \in X} \int_{B_\xi} \varphi \circ \xi.$$

We now take the limit when $r \rightarrow \infty$. By our assumption the left-hand side converges to $\mu(\Gamma \setminus \mathfrak{H})\{\langle \varepsilon f, h \rangle - \langle f, \delta_{k-2}h \rangle\}$. We have $\varphi \circ \xi = p_\xi(z)d\bar{z}$ with $p_\xi(z) = \overline{j_\xi(z)}^{-2}(\bar{f}hy^{k-2}) \circ \xi$. Notice that $y|p_\xi(z)| = |\bar{f}hy^{k-1}| \circ \xi$. Now Q_ξ is generated by a matrix of the form $\begin{bmatrix} 1 & t_\xi \\ 0 & 1 \end{bmatrix}$ with $0 < t_\xi \in \mathbf{Q}$, and so $Q_\xi \setminus M_r$ can be identified with the line segment $\{x + ir \mid 0 \leq x \leq t_\xi\}$. Thus

$$\left| \int_{B_\xi} \varphi \circ \xi \right| \leq \int_0^{t_\xi} |p_\xi(x + ir)|dx.$$

Suppose fh is rapidly decreasing at every cusp; then $|p_\xi(x + iy)| \leq A_\xi y^{-c}$ with positive constants A_ξ and c for sufficiently large y . Therefore we obtain our theorem.

Corollary 6.9. (i) *Let Γ be a congruence subgroup and let $f \in \mathcal{S}_k$. Then $\langle f, \delta_{k-2}h \rangle = 0$ for every $h \in C_{k-2}(\Gamma)$ such that both h and $\delta_{k-2}h$ are slowly increasing at every cusp.*

(ii) *Let $f \in C_k(\Gamma)$. Suppose both f and εf are rapidly decreasing at every cusp and $L_k f = 0$. Then $f \in \mathcal{S}_k$.*

(iii) *Let $f, g \in C_k(\Gamma)$. Then under a suitable condition (see the following PROOF) we have*

$$(6.19) \quad \langle f, L_k g \rangle = \langle L_k f, g \rangle,$$

$$(6.20) \quad \langle f, L_k f \rangle \geq 0.$$

PROOF. Let f and h be as in (i). Then, by Lemma 6.4(ii), both fh and $f\delta_{k-2}h$ are rapidly decreasing at every cusp. Since f is holomorphic, we have $\varepsilon f = 0$, and so $\langle f, \delta_{k-2}h \rangle = 0$ by (6.16). This proves (i). Next let f be as

in (ii). Then $\delta_{k-2}\varepsilon f = 4^{-1}L_k f = 0$, and so $\langle \varepsilon f, \varepsilon f \rangle = \langle f, \delta_{k-2}\varepsilon f \rangle = 0$ by (6.16). Thus $\varepsilon f = 0$, which means that f is holomorphic, and so $f \in \mathcal{S}_k$ by Lemma 6.4(ii). As for (iii), we have, by (6.16), $\langle f, \delta_{k-2}\varepsilon g \rangle = \langle \varepsilon f, \varepsilon g \rangle = \langle \delta_{k-2}\varepsilon f, g \rangle$, which gives (6.19). To justify the last sequence of equalities, we need some conditions on f and g as stated in Theorem 6.8, which are often easy to verify, and so we leave the precise statements to the reader. If we take $f = g$, then $\langle f, L_k f \rangle = 4\langle \varepsilon f, \varepsilon f \rangle \geq 0$, which proves (6.20). This completes the proof.

Remark. In Corollary 6.9(ii) the condition on εf is unnecessary. This will be explained after the proof of Lemma 9.3.

Lemma 6.10. *If $f \in \mathcal{M}_k(\Gamma)$, then $\delta_k^p f$ for every $p \in \mathbf{Z}, \geq 0$, belongs to $C_{k+2p}(\Gamma)$, and is slowly increasing at every cusp. Moreover, it is rapidly decreasing at every cusp if $f \in \mathcal{S}_k$.*

PROOF. That $\delta_k^p f \in C_{k+2p}(\Gamma)$ follows from (6.14c). Let $\alpha \in \Gamma(1)$ and $g(z) = j_\alpha(z)^{-k} f(\alpha z) = \sum_{m \in \mathbf{Z}} c_\alpha(m) \mathbf{e}(mz/r_\alpha)$ as in (3.4d). By (6.14c) we have $j_\alpha(z)^{-k-2p} (\delta_k^p f)(\alpha z) = \delta_k^p g(z)$, and by induction on p we easily see that $\delta_k^p g = \sum_{\nu=0}^p a_\nu y^{-\nu} (\partial/\partial z)^{p-\nu} g$ with $a_\nu \in \mathbf{C}$, and so

$$\delta_k^p g(z) = \sum_{\nu=0}^p a_\nu y^{-\nu} \sum_{m=0}^{\infty} c_\alpha(m) (2\pi i m / r_\alpha)^{p-\nu} \mathbf{e}(mz/r_\alpha).$$

Since $c_\alpha(m) = O(m^k)$ by Lemma 6.2(iii), we can easily verify the inequality of (6.9a) for $\delta_k^p g$. If $f \in \mathcal{S}_k$, then $c_\alpha(0) = 0$, and so $\delta_k^p g$ satisfies (6.9b). Thus we obtain our lemma.

6.11. We have been discussing modular forms as functions on $\mathfrak{H}$. Instead, we can treat them as functions on $SL_2(\mathbf{R})$ or its covering. Let us explain the idea in the case $k \in \mathbf{Z}$ for simplicity. Put $K = \{\alpha \in SL_2(\mathbf{R}) \mid {}^t\alpha\alpha = 1_2\}$ and $\rho(\xi) = j_\xi^k(i)^{-1}$ for $\xi \in K$. Then ρ is a continuous homomorphism of K into $\mathbf{T}$. Given $f \in C_k(\Gamma)$, define a function $\tilde{f}$ on $SL_2(\mathbf{R})$ by $\tilde{f}(\alpha) = (f|_k \alpha)(i)$ for $\alpha \in SL_2(\mathbf{R})$. Then $\tilde{f}(\gamma\alpha) = \tilde{f}(\alpha)$ for every $\gamma \in \Gamma$ and $\tilde{f}(\alpha\xi) = \rho(\xi)\tilde{f}(\alpha)$ for every $\xi \in K$. In this way we can associate with f a function on $\Gamma \backslash G$ belonging to a representation ρ of K .

One consequence of this association is that the action of the differential operators ε and δ_k^p corresponds to that of some elements of the universal enveloping algebra $\mathfrak{U}$ of the Lie algebra of $SL_2(\mathbf{R})$. This approach gives a certain conceptual perspective, and even clarifies some technical points, especially when we deal with higher-dimensional Lie groups and symmetric spaces. In this book, however, we stay within the traditional framework of functions on $\mathfrak{H}$. The reader who is interested in the higher-dimensional cases and also in the Lie-theoretical treatment of this topic is referred to the author's articles as follows: the operators in the higher-dimensional cases are discussed in [S94],

[S00, Sections 12 and A8], and [S04, Section A1]; the connection with $\mathfrak{U}$ is explained in [S90, Section 7], and [S02, vol. IV, pp. 739–740].

In particular, (6.16) formulated on a Lie group G can be given in the form

$$(6.21) \quad \int_{\Gamma \backslash G} X f \cdot h \, d\mu = - \int_{\Gamma \backslash G} f \cdot X h \, d\mu,$$

where X is an element of the Lie algebra of G . However, in almost all papers and textbooks this is proved under the condition that f and h have compact support, or under a similar strong condition, which makes its application impractical. The fact under a much weaker condition is given in [S00, p. 287, Lemma A8.3].



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